# Particle Approximation of the Wasserstein Diffusion

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**Abstract**

We construct a system of interacting two-sided Bessel processes on the unit interval and show that the associated empirical measure process converges to the Wasserstein Diffusion [18], assuming that Markov uniqueness holds for the generating Wasserstein Dirichlet form. The proof is based on the variational convergence of an associated sequence of Dirichlet forms in the generalized Mosco sense of Kuwae and Shioya [14].

## 1 Introduction

As shown in [18], for $\beta > 0$ there exists a measure $\mathbb{P}^\beta$ and a Hunt process $\big(P_{\eta\in\mathcal{P}([0,1])}, (\mu_t)_{t\geq 0}\big)$ on $(\mathcal{P}([0,1]), \tau_w)$, the space of Borel probabilities over $[0,1]$ equipped with the weak topology, such that

i) $\mathbb{P}^\beta$ admits the formal representation $\mathbb{P}^\beta(d\mu) = \frac{1}{Z} e^{-\beta \mathrm{Ent}(\mu)} \mathbb{P}^0(d\mu)$ as a Gibbs-type measure on $\mathcal{P}([0,1])$ with the Boltzmann entropy $\mathrm{Ent}(\mu) = \int_{[0,1]} \log(d\mu/dx) d\mu$ as Hamiltonian and

ii) $\big(P_{\eta\in\mathcal{P}([0,1])}, (\mu_t)_{t\geq 0}\big)$ is a $\mathbb{P}^\beta$-symmetric diffusion on $(\mathcal{P}([0,1]), \tau_w)$ with intrinsic distance given by the quadratic Wasserstein distance $d_2^W$.

Moreover, letting denote by $(\mu_\cdot)$ the process obtained from the invariant starting distribution $\mathbb{P}^\beta$ we arrive at a solution of the following martingale problem. The initial law of $(\mu_t)_{t\geq 0}$ satisfies

$$\langle f, \mu_0 \rangle \sim \int_0^1 f(D_t^\beta) dt \quad \forall f \in C([0,1]), \tag{1}$$

where $(D_t^\beta)_{t\in[0,1]}$ is the real valued Dirichlet (or normalized Gamma) process over $[0,1]$ with parameter $\beta > 0$, and for $f \in C^2([0,1])$ with $f'(0) = f'(1) = 0$ the process

$$\begin{aligned} M_t &= \langle f, \mu_t \rangle - \beta \cdot \int_0^t \langle f'', \mu_s \rangle ds \\ &- \int_0^t \left( \sum_{I \in \mathrm{gaps}(\mu_s)} \left[ \frac{f''(I_-) + f''(I_+)}{2} - \frac{f'(I_+) - f'(I_-)}{|I|} \right] - \frac{f''(0) + f''(1)}{2} \right) ds, \end{aligned} \tag{2}$$

is a continuous martingale with quadratic variation process

$$[M]_t = 2 \int_0^t \langle (f')^2, \mu_s \rangle ds. \tag{3}$$

Here $\mathrm{gaps}(\mu)$ denotes the set of connected components in the complement of $\mathrm{spt}(\mu)$.

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Properties i) and ii) suggest to view $(\mu_t)_{t\geq 0}$ as model for a diffusing fluid when its heat flow is perturbed by a kinetically uniform random forcing. The actual construction of $(\mu_t)_{t\geq 0}$ in [18] uses abstract Dirichlet form methods without direct reference to physical intuition. $\big(P_{\eta\in\mathcal{P}([0,1])}, (\mu_t)_{t\geq 0}\big)$ is generated from the $L^2(\mathcal{P}([0,1]), \mathbb{P}^\beta)$-closure $\mathcal{E}$ of the quadratic form

$$Q(F,F) = \int_{\mathcal{P}([0,1])} \|\nabla^w F\|_\mu^2 \, \mathbb{P}^\beta(d\mu), \quad F \in \mathcal{Z}$$

on the class $\mathcal{Z} = \{F : \mathcal{P}([0,1]) \to \mathbb{R} \,|\, F(\mu) = f(\langle\phi_1,\mu\rangle, \langle\phi_2,\mu\rangle, \ldots, \langle\phi_k,\mu\rangle), f \in C_c^\infty(\mathbb{R}^k), \{\phi_i\}_{i=1}^k \subset C_c^\infty(\mathbb{R}), k \in \mathbb{N}\}$, where $\|\nabla^w F\|_\mu = \big\|(D_{|\mu}F)'(\cdot)\big\|_{L^2([0,1],\mu)}$ and $(D_{|\mu}F)(x) = \partial_{t|t=0}F(\mu + t\delta_x)$.

In this paper we aim at an approximation of $(\mu_.)$ by a sequence of interacting particle systems in order to gain insight into some of its qualitative features.

In analytic terms the Wasserstein diffusion $(\mu_.)$ solves an SPDE with nonlinear (singular) drift and non-Lipschitz multiplicative noise. It should be noted that the class of stochastic nonlinear evolution equations admitting a rigorous particle approximation appears to be rather small. Some examples of lattice systems with stochastic nonlinear hydrodynamic behaviour are reviewed in [9], the case of exchangeable diffusions is studied e.g. in [17, 13] and [3, 5] deal with stochastic nonlinear scaling limits of population models with interactive behaviour.

Given the singularity of the generator of $(\mu_.)$, here we choose an approximation by a sequence of *reversible* particle systems. This allows to use Dirichlet form methods for the passage to the limit instead of arguing along a sequence of martingale problems. For the identification of the limit we have to assume that $\mathcal{E}$ is a maximal element in the class of (not necessarily regular) Dirichlet forms on $L^2(\mathcal{P}([0,1]), \mathbb{P}^\beta)$, i.e. that *Markov uniqueness* holds for $\mathcal{E}$.

The assumption on Markov uniqueness appears in several quite similar contexts as well [12, 10]. The verification is usually difficult, in particular in a non-Gaussian infinite dimensional setting involving singular logarithmic derivatives [6]. Finally, by general principles the Markov uniqueness of $\mathcal{E}$ is weaker than the essential self-adjointness of the generator of $(\mu_t)_{t\geq 0}$ on $\mathcal{Z}$ and stronger than the well-posedness, i.e. uniqueness, of the martingale problem problem defined by (1), (2) and (3) in the class of Hunt processes on $\mathcal{P}([0,1])$, cf. [1, theorem 3.4].

# 2 Set Up and Main Result

For $N \in \mathbb{N}$ let $X_t^N = (x_t^1, \cdots, x_t^{N-1}) \in \Sigma_N := \{x \in \mathbb{R}^{N-1}, 0 \leq x^1 \leq x^2 \leq \cdots \leq x^{N-1} \leq 1\} \subset \mathbb{R}^{N-1}$ denote the ordered vector of the positions of $N-1$ particles in $[0,1]$. Define the probability measure $q_N$ on $\Sigma_N$ by

$$q_N(dx^1, \cdots, dx^{N-1}) = \frac{\Gamma(\beta)}{(\Gamma(\beta/N))^N} \prod_{i=1}^N (x^i - x^{i-1})^{\frac{\beta}{N}-1} dx^1 \ldots dx^{N-1},$$

where $x_0 = 0$ and $x_N = 1$ by convention. The $L^2(\Sigma_N, q_N)$-closure of

$$\mathcal{E}^N(f,f) = \int_{\Sigma_N} |\nabla f|^2(x) q_N(dx), \quad f \in C^\infty(\Sigma_N)$$

defines a local regular Dirichlet form, which is again denoted by $\mathcal{E}^N$. Let $(X_t^N)_{t\geq 0}$ be the associated Markov process on $\Sigma_N$, starting from the invariant distribution $q_N$ and let

$$\mu_t^N = \frac{1}{N-1} \sum_{i=1}^{N-1} \delta_{x^i_{N\cdot t}} \in \mathcal{P}([0,1]),$$

be the associated empirical measure process on $[0,1]$, considered on time scale $N \cdot t$. Then we prove the following assertion.

**Theorem 2.1.** *Assume Markov-uniqueness holds for $\mathcal{E}$, then $(\mu^N_\cdot) \overset{N\to\infty}{\Longrightarrow} (\mu_\cdot)$ in $C_{\mathbb{R}_+}\big((\mathcal{P}([0,1]),\tau_w)\big)$.*

**Remark 2.2.** A careful integration by parts for $q_N$ shows that the domain the generator $L_N$ of $\mathcal{E}^N$ contains the set of all smooth Neumann functions on $\Sigma_N$. For such $f$

$$L^N f(x) = (\frac{\beta}{N} - 1) \sum_{i=1}^{N-1} \left( \frac{1}{x^i - x^{i-1}} - \frac{1}{x^{i+1} - x^i} \right) \frac{\partial}{\partial x^i} f(x) + \Delta f(x) \quad \text{ for } x \in \mathrm{Int}(\Sigma_N).$$

Hence given initial conditions $0 < x_0^1 < x_0^2 < \cdots < x_0^{N-1} < 1$, $(X^N_\cdot)$ is the formal solution to the system of coupled Skorokhod SDEs

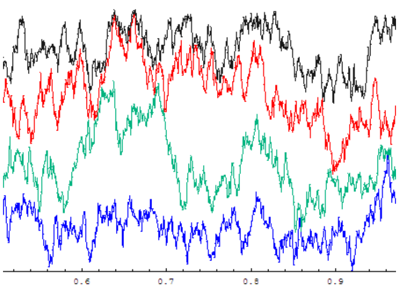
$$dx_t^i = (\frac{\beta}{N} - 1) \left( \frac{1}{x_t^i - x_t^{i-1}} - \frac{1}{x_t^{i+1} - x_t^i} \right) dt + \sqrt{2} dw_t^i + dl_t^{i-1} - dl_t^i, \quad i = 1, \cdots, N-1, \tag{4}$$

with independent real Brownian motions $\{w^i\}$ and local times $l^i$ satisfying

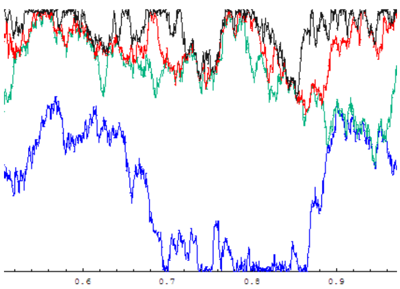
$$dl_t^i \geq 0, \quad l_t^i = \int_0^t 1\!\!1_{\{x_s^i = x_s^{i+1}\}} dl_s^i. \tag{5}$$

$(X^N_\cdot)$ may thus be considered as system of coupled two sided real Bessel processes with uniform Bessel dimension $\delta = \frac{\beta}{N}$. Similar to the real Bessel process $BES(\delta)$ with Bessel dimension $\delta < 1$, the existence of $X^N$ is not a trivial fact. By analogy one should expect that the Skorokhod-SDE defined by (4) and (5) is ill-posed, but that nevertheless $\mathcal{E}^N$ generates a Feller semigroup on $\Sigma_N$.

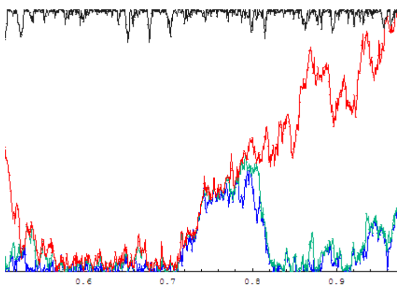
**Remark 2.3.** For simulation the dynamics of $(X^N_\cdot)$ can be approximated by $X_t^{N,\epsilon} = \mathcal{X}^{N,\epsilon}_{\lfloor t/\epsilon^2 \rfloor}$, $t \geq 0$, where $(\mathcal{X}_n^{N,\epsilon})_{n\geq 0}$ is the Markov chain on $\Sigma_N$ with transition kernel $\mu^{N,\epsilon}(x,A) = \frac{q_N(B_\epsilon(x)\cap\Sigma_N\cap A)}{q_N(B_\epsilon(x)\cap\Sigma_N)}$. Alternatively one may integrate a regularized version of the formal SDE (4) and (5). The role of the parameter $\beta$ is illustrated by following results for the case of $4 = N-1$ particles at large times, which we present by courtesy of Theresa Heeg, Bonn.



$\beta = 10$



$\beta = 1$



$\beta = 0.3$

# 3 Proof of theorem 2.1

## 3.1 Tightness

As usual we show compactness of the laws of $(\mu^N_\cdot)$ and, in a second step the uniqueness of the limit.

**Proposition 3.1.** *The sequence $(\mu^N_\cdot)$ is tight in $C_{\mathbb{R}_+}((\mathcal{P}([0,1]),\tau_w))$.*

*Proof.* According to theorem 3.7.1 in [4] it is sufficient to show that the sequence $(\langle f,\mu^N_\cdot\rangle)_{N\in\mathbb{N}}$ is tight, where $f$ is taken from a dense subset in $\mathcal{F}\subset C([0,1])$. Choose $\mathcal{F}:=\{f\in C^3([0,1])\,|, f'(0)=f'(1)=0\}$, then $\langle f,\mu^N_t\rangle = F^N(X^N_{N\cdot t})$ with

$$F^N(x)=\frac{1}{N-1}\sum_{i=1}^{N-1} f(x^i).$$

The condition $f'(0)=f'(1)=0$ implies $F^N\in\mathcal{D}(L^N)$. Moreover, for for $x\in\mathrm{Int}(\Sigma_N)$

$$\begin{aligned}N{\cdot}L^NF^N(x) &= \frac{\beta}{N-1}\sum_{i=1}^{N}\frac{f'(x^i)-f'(x^{i-1})}{x^i-x^{i-1}}\\ &+\frac{N}{N-1}\sum_{i=1}^{N-1}\left(f''(x^i)-\frac{f'(x^i)-f'(x^{i-1})}{x^i-x^{i-1}}\right)+\frac{N}{N-1}\frac{f'(x^N)-f'(x^{N-1})}{x^N-x^{N-1}},\end{aligned}$$

such that

$$|N\cdot L^NF^N(x)|\le \left\|f''\right\|_\infty\frac{N\cdot(\beta+1)}{N-1}+\left\|f'''\right\|_\infty\frac{N}{N-1}\le C(\beta,\|f\|_{C^3([0,1])}).$$

This implies a uniform in $N$ Lipschitz bound for the BV part in the Doob-Meyer decomposition of $F^N(X^N_{N\cdot})$. The process $X^N$ has continuous sample paths with square field operator $\Gamma(F,F)=L(F^2)-2F\cdot LF=|\nabla F|^2$. Hence the quadratic variation of the martingale part of $F^N(X^N_{N\cdot})$ satisfies

$$[F^N(X^N_{N\cdot})]_t-[F^N(X^N_{N\cdot})]_s = N\cdot\int_s^t|\nabla F^N|^2(X^N_s)ds=\frac{N}{(N-1)^2}\cdot\int_s^t\sum_{i=1}^{N-1}(f')^2(x^i_s)ds\le 2(t-s)\left\|f'\right\|^2_\infty.$$

Since

$$F^N(X^N_0)=\frac{1}{N-1}\sum_{i=1}^{N-1}f(D^\beta_{i/N})\to\int_0^1 f(D^\beta_s)\,ds\qquad \mathbb{Q}^\beta\text{-a.s.},$$

the law of $F^N(X^N_0)$ is convergent. Using now Aldous' tightness criterion in an appropriate version on sequences of semi-martingales the assertion follows, cf. corollary 3.6.7. in [4]. □

**Remark 3.2.** Using the symmetry of $(X^N_\cdot)$ we could have used the Lyons-Zheng decomposition for the tightness proof instead. The argument above shows the balance of first and second order parts of $N\cdot L^N$ as $N$ tends to infinity.

## 3.2 Identification of the Limit

### 3.2.1 The $\mathcal{G}$-Parameterization

In order to identify the limit of the sequence $(\mu^N_\cdot)$ we parameterize the space $\mathcal{P}([0,1])$ in terms of right continuous quantile functions, cf. [18]. The set

$$\mathcal{G} = \{g : [0,1) \to [0,1] \mid g \text{ cadlag nondecreasing}\},$$

equipped with the $L^2([0,1], dx)$ distance $d_{L^2}$ is a compact subspace of $L^2([0,1], dx)$. It is homeomorphic to $(\mathcal{P}([0,1]), \tau_w)$ by means of the map

$$\rho : \mathcal{G} \to \mathcal{P}([0,1]), \qquad g \to g_*(dx),$$

which takes a function $g \in \mathcal{G}$ to the image measure of $dx$ under $g$. The inverse map $\kappa = \rho^{-1} : \mathcal{P}([0,1]) \to \mathcal{G}$ is realized by taking the right continuous quantile function.

For technical reasons we introduce the following modification of $(\mu^N_\cdot)$ which is better behaved in terms of the map $\kappa$.

**Lemma 3.3.** *For $N \in \mathbb{N}$ define the Markov process*

$$\nu^N_t := \frac{N-1}{N}\mu^N_t + \frac{1}{N}\delta_0 \in \mathcal{P}([0,1]),$$

*then $(\nu^{N'}_\cdot)$ is convergent on $C_{\mathbb{R}_+}((\mathcal{P}([0,1]), \tau_w))$ along any subsequence $N'$ if and only if $(\mu^{N'}_\cdot)$ is. In this case both limits coincide.*

*Proof.* For any $f \in C([0,1])$ the sequence $(\langle f, \mu^{N'}_\cdot\rangle)_{N'}$ is tight if and only if the same holds true for the sequence $(\langle f, \nu^{N'}_\cdot\rangle)_{N'}$, where the limits coincide. Using Theorem 3.7.1 in [4] again, this implies $(\mu^{N'}_\cdot)$ is tight in case $(\nu^{N'}_\cdot)$ is and vice versa. Since the map $l_f : C_{\mathbb{R}_+}(\mathcal{P}([0,1])) \to C_{\mathbb{R}_+}(\mathbb{R})$, $(m_t)_{t\geq 0} \to (\langle m_t, f\rangle)_{t\geq 0}$ is continous for $f \in C([0,1])$ we conclude that the respective laws of $l_f$ on $C_{\mathbb{R}\geq 0}(\mathbb{R})$ induced by any two potential limits of $(\mu^{N'}_\cdot)$ and $(\nu^{N'}_\cdot)$ coincide. Hence those limits must in fact be identical. □

Let $(g^N_\cdot) := (\kappa(\nu^N_\cdot))$ be the process $(\nu^N_\cdot)$ in the $\mathcal{G}$-parameterization. It can also be obtained by

$$g^N_t = \iota(X^N_{N\cdot t})$$

with the imbedding $\iota = \iota^N$

$$\iota : \Sigma_N \to \mathcal{G}, \qquad \iota(x) = \sum_{i=0}^{N-1} x^i \cdot 1\!\!1_{[i/N,(i+1)/N)}.$$

Similarly, let $(g_\cdot) = (\kappa(\mu_\cdot))$ be the $\mathcal{G}$-image of the Wasserstein diffusion under the map $\kappa$ with invariant initial distribution $\mathbb{Q}^\beta$. In [18, theorem 7.5] it is shown that $(g_\cdot)$ is generated by the Dirichlet form, again denoted by $\mathcal{E}$, which is obtained as the $L^2(\mathcal{G}, \mathbb{Q}^\beta)$-closure of

$$\mathcal{E}(u,v) = \int_{\mathcal{G}} \langle \nabla u_{|g}(\cdot), \nabla v_{|g}(\cdot)\rangle_{L^2([0,1])}\, \mathbb{Q}^\beta(dg), \qquad u,v \in \mathfrak{C}^1(\mathcal{G}).$$

on the class

$$\mathfrak{C}^1(\mathcal{G}) = \{u : \mathcal{G} \to \mathbb{R} \mid u(g) = U(\langle f_1, g\rangle_{L^2}, \ldots, \langle f_m, g\rangle_{L^2}), U \in C^1_c(\mathbb{R}^m), \{f_i\}_{i=1}^m \subset L^2([0,1]), m \in \mathbb{N}\},$$

where $\nabla u_{|g}$ is the $L^2([0,1],dx)$-gradient of $u$ at $g$.

The convergence of $(\mu^N_\cdot)$ to $(\mu_\cdot)$ in $C_{\mathbb{R}_+}(\mathcal{P}([0,1]),\tau_w)$ is thus equivalent to the convergence of $(g^N_\cdot)$ to $(g_\cdot)$ in $C_{\mathbb{R}_+}(\mathcal{G},d_{L^2})$. By proposition 3.1 and lemma 3.3 $(g^N_\cdot)_N$ is a tight sequence of processes on $\mathcal{G}$. The following statement idenitifies $(g_\cdot)$ as the unique weak limit.

**Proposition 3.4.** *Let $\mathcal{E}$ be Markov-unique. Then for any $f \in C(\mathcal{G}^l)$ and $0 \le t_1 < \ldots < t_l$, $\mathbb{E}(f(g^N_{t_1},\cdots,g^N_{t_l})) \stackrel{N\to\infty}{\longrightarrow} \mathbb{E}(f(g_{t_1},\cdots,g_{t_l}))$.*

### 3.2.2 Finite Dimensional Approximation of Dirichlet Forms in Mosco Sense

Proposition 3.4 is proved by showing that the sequence of generating Dirichlet forms $N\cdot\mathcal{E}^N$ of $(g^N_\cdot)$ on $L^2(\Sigma_N,q_N)$ converges to $\mathcal{E}$ on $L^2(\mathcal{G},\mathbb{Q})$ in the generalized Mosco sense of Kuwae and Shioya, allowing for varying base $L^2$-spaces. We recall the framework developed in [14].

**Definition 3.5** (Convergence of Hilbert spaces)**.** *A sequence of Hilbert spaces $H^N$ converges to a Hilbert space $H$ if there exists a family of linear maps $\{\Phi^N : H \to H^N\}_N$ such that*

$$\lim_N \left\|\Phi^N u\right\|_{H^N} = \|u\|_H, \qquad \text{for all } u \in H.$$

*A sequence $(u_N)_N$ with $u_N \in H_N$ converges strongly to a vector $u \in H$ if there exists a sequence $(\tilde{u}_N)_N \subset H$ tending to $u$ in $H$ such that*

$$\lim_N \limsup_M \left\|\Phi^M \tilde{u}_N - u_M\right\|_{H^M} = 0,$$

*and $(u_N)$ converges weakly to $u$ if*

$$\lim_N \langle u_N, v_N\rangle_{H^N} = \langle u, v\rangle_H,$$

*for any sequence $(v_N)_N$ with $v_N \in H^N$ tending strongly to $v \in H$. Moreover, a sequence $(B_N)_N$ of bounded operators on $H^N$ converges strongly (resp. weakly) to an operator $B$ on $H$ if $B_N u_N \to Bu$ strongly (resp. weakly) for any sequence $(u_N)$ tending to $u$ strongly (resp. weakly).*

**Definition 3.6** (Mosco Convergence)**.** *A sequence $(E^N)_N$ of quadratic forms $E^N$ on $H^N$ converges to a quadratic form $E$ on $H$ in the Mosco sense if the following two conditions hold:*

**Mosco I:** *If a sequence $(u_N)_N$ with $u_N \in H^N$ weakly converges to a $u \in H$, then*

$$E(u,u) \le \liminf_N E^N(u_N,u_N).$$

**Mosco II:** *For any $u \in H$ there exists a sequence $(u_N)_N$ with $u_N \in H^N$ which converges strongly to $u$ such that*

$$E(u,u) = \lim_N E^N(u_N,u_N).$$

Extending [16] it is shown in [14] that Mosco convergence of a sequence of Dirichlet forms is equivalent to the strong convergence of the associated resolvents and semigroups. We will apply this result when $H^N = L^2(\Sigma_N,q_N)$, $H = L^2(\mathcal{G},\mathbb{Q}^\beta)$ and $\Phi^N$ is defined to be the conditional expectation operator

$$\Phi^N : H \to H^N; \qquad (\Phi^N u)(x) := \mathbb{E}(u|g_{i/N} = x_i, i = 1,\ldots,N-1).$$

However, we shall prove that the sequence $N\cdot\mathcal{E}^N$ converges to $\mathcal{E}$ in the Mosco sense in a slightly modified fashion, namely the condition (Mosco II) will be replaced by

**Mosco II':** *There is a core $K \subset \mathcal{D}(E)$ such that for any $u \in K$ there exists a sequence $(u_N)_N$ with $u_N \in \mathcal{D}(E^N)$ which converges strongly to $u$ such that $E(u,u) = \lim_N E^N(u_N, u_N)$.*

**Theorem 3.7.** *Under the assumption that $H^N \to H$ the conditions (Mosco I) and (Mosco II') are equivalent to the strong convergence of the associated resolvents.*

*Proof.* We proceed as in the proof of theorem 2.4.1 in [16]. By theorem 2.4 of [14] strong convergence of resolvents implies Mosco-convergence in the original stronger sense. Hence we need to show only that our weakened notion of Mosco-convergence also implies strong convergence of resolvents.

Let $\{R^N_\lambda, \lambda > 0\}$ and $\{R_\lambda, \lambda > 0\}$ be the resolvent operators associated with $E^N$ and $E$, respectively. Then, for each $\lambda > 0$ we have to prove that for every $z \in H$ and every sequence $(z_N)$ tending strongly to $z$ the sequence $(u_N)$ defined by $u_N := R^N_\lambda z_N \in H^N$ converges strongly to $u := R_\lambda z$ as $N \to \infty$. The vector $u$ is characterized as the unique minimizer of $E(v,v) + \lambda\langle v, v\rangle_H - 2\langle z, v\rangle_H$ over $H$ and a similar characterization holds for each $u_N$. Since for each $N$ the norm of $R^N_\lambda$ as an operator on $H^N$ is bounded by $\lambda^{-1}$, by Lemma 2.2 in [14] there exists a subsequence of $(u_N)$, still denoted by $(u_N)$, that converges weakly to some $\tilde{u} \in H$. By (Mosco II') we find for every $v \in K$ a sequence $(v_N)$ tending strongly to $v$ such that $\lim_N E^N(v_N, v_N) = E(v,v)$. Since for every $N$

$$E^N(u_N, u_N) + \lambda\langle u_N, u_N\rangle_{H^N} - 2\langle z_N, u_N\rangle_{H^N} \le E^N(v_N, v_N) + \lambda\langle v_N, v_N\rangle_{H^N} - 2\langle z_N, v_N\rangle_{H^N},$$

using the condition (Mosco I) we obtain in the limit $N \to \infty$:

$$E(\tilde{u}, \tilde{u}) + \lambda\langle \tilde{u}, \tilde{u}\rangle_H - 2\langle z, \tilde{u}\rangle_H \le E(v,v) + \lambda\langle v, v\rangle_H - 2\langle z, v\rangle_H,$$

which by the definition of the resolvent together with the density of $K \subset D(E)$ implies that $\tilde{u} = R_\lambda z = u$. This establishes the weak convergence of resolvents. It remains to show strong convergence. Let $u_N = R^N_\lambda z_N$ converge weakly to $u = R_\lambda z$ and choose $v \in K$ with the respective strong approximations $v_N \in H^N$ such that $E^N(v_N, v_N) \to E(v,v)$, then the resolvent inequality for $R^N$ yields

$$E^N(u_N, u_N) + \lambda \|u_N - z_N/\lambda\|^2_{H^N} \le E^N(v_N, v_N) + \lambda \|v_N - z_N/\lambda\|^2_{H^N}.$$

Taking the limit for $N \to \infty$, one obtains

$$\limsup_N \lambda \|u_N - z_N/\lambda\|^2_{H^N} \le E(v,v) - E(u,u) + \lambda \|v - z/\lambda\|^2_H.$$

Since $K$ is a dense subset we may now let $v \to u \in D(E)$, which yields

$$\limsup_N \|u_N - z_N/\lambda\|^2_{H^N} \le \|u - z/\lambda\|^2_H.$$

Due to the weak lower semicontinuity of the norm this yields $\lim_N \|u_N - z_N/\lambda\| = \|u - z/\lambda\|$. Since strong convergence in $H$ is equivalent to weak convergence together with the convergence of the associated norms the claim follows (cf. Lemma 2.3 in [14]). □

Proposition 3.4 will now essentially be implied by the following statement, which by the definitions above summarizes the subsequent three propositions.

**Theorem 3.8.** *Assume that $\mathcal{E}$ is Markov-unique on $L^2(\mathcal{G}, \mathbb{Q})$. Then $(N \cdot \mathcal{E}^N, H^N)$ converges to $(\mathcal{E}, H)$ along $\Phi^N$ in Mosco sense.*

**Proposition 3.9.** *$H^N$ converges to $H$ along $\Phi^N$, for $N \to \infty$.*

*Proof.* We have to show that $\left\|\Phi^N u\right\|_{H^N} \to \|u\|_H$ for each $u \in H$. Let $\mathcal{F}^N$ be the $\sigma$-Algebra on $\mathcal{G}$ generated by the projection maps $\{g \to g(i/N) \,|\, i = 1, \dots, N-1\}$. By abuse of notation we identify $\Phi^N u \in H$ with $\mathbb{E}(u|\mathcal{F}^N)$ of $u$, considered as an element of $L^2(\mathbb{Q}^\beta, \mathcal{F}^N) \subset H$. Since the measure $q_N$ coincides with the respective finite dimensional distributions of $\mathbb{Q}^\beta$ on $\Sigma_N$ we have $\left\|\Phi^N u\right\|_{H^N} = \left\|\Phi^N u\right\|_H$. Hence the claim will follow once we show that $\Phi^N u \to u$ in $H$. For the latter we use the following abstract result, whose proof can be found, e.g. in [2, lemma 1.3].

**Lemma 3.10.** *Let $(\Omega, \mathcal{D}, \mu)$ be a measure space and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ a sequence of $\sigma$-subalgebras of $\mathcal{D}$. Then $E(f|\mathcal{F}_n) \to f$ for all $f \in L^p$, $p \in [1, \infty)$ if and only if for all $A \in \mathcal{D}$ there is a sequence $A_n \in \mathcal{F}_n$ such that $\mu(A_n \Delta A) \to 0$ for $n \to \infty$.*

In order to apply this lemma to the given case $(\mathcal{G}, \mathcal{B}(\mathcal{G}), \mathbb{Q}^\beta)$, where $\mathcal{B}(\mathcal{G})$ denotes the Borel $\sigma$-algebra on $\mathcal{G}$, let $\mathcal{F}_{\mathbb{Q}^\beta} \subset \mathcal{B}(\mathcal{G})$ denote the collection of all Borel sets $F \subset \mathcal{G}$ which can be approximated by elements $F_N \in \mathcal{F}^N$ with respect to $\mathbb{Q}^\beta$ in the sense above. Note that $\mathcal{F}_{\mathbb{Q}^\beta}$ is again a $\sigma$-algebra, cf. the appendix in [2]. Let $\mathcal{M}$ denote the system of finitely based open cylinder sets in $\mathcal{G}$ of the form $M = \{g \in \mathcal{G} | g_{t_i} \in O_i, i = 1, \dots, L\}$ where $t_i \in [0,1]$ and $O_i \subset [0,1]$ open. From the almost sure right continuity of $g$ and the fact that $g_.$ is continuous at $t_1, \dots, t_L$ for $\mathbb{Q}^\beta$-almost all $g$ it follows that $M_N := \{g \in \mathcal{G} | g_{(\lceil t_i \cdot N \rceil / N)} \in O_i, i = 1, \dots, L\} \in \mathcal{F}^N$ is an approximation of $M$ in the sense above. Since $\mathcal{M}$ generates $\mathcal{B}(\mathcal{G})$ we obtain $\mathcal{B}(\mathcal{G}) \subset \mathcal{F}_{\mathbb{Q}^\beta}$ such that the assertion holds, due to lemma 3.10. □

**Remark 3.11.** It is much simpler to prove proposition 3.9 for a dyadic subsequence $N' = 2^m$, $m \in \mathbb{N}$ when the sequence $\left\|\Phi^{N'} u\right\|_{H^{N'}}$ is nondecreasing and bounded, because $\Phi^{N'}$ is a projection operator in $H$ with increasing range $\mathrm{im}(\Phi^{N'})$ as $N'$ grows. Hence, $\left\|\Phi^{N'} u\right\|_{H^{N'}}$ is Cauchy and thus

$$\left\|\Phi^{N'} u - \Phi^{M'} u\right\|_H^2 = \left\|\Phi^{N'} u\right\|_H^2 - \left\|\Phi^{M'} u\right\|_H^2 \to 0 \quad \text{for } M', N' \to \infty,$$

i.e. the sequence $\Phi^{N'} u$ converges to some $v \in H$. Since obviously $\Phi^N u \to u$ weakly in $H$ it follows that $u = v$ such that the claim is obtained from $|\left\|\Phi^{N'} u\right\|_H - \|u\|_H| \le \left\|\Phi^{N'} u - u\right\|_H$.

To simplify notation for $f \in L^2([0,1], dx)$ denote the functional $g \to \langle f, g \rangle_{L^2([0,1])}$ on $\mathcal{G}$ by $l_f$. We introduce the set $K$ of polynomials defined by

$$K = \left\{ u \in C(\mathcal{G}) \,|\, u(g) = \prod_{i=1}^n l_{f_i}^{k_i}(g),\, k_i \in \mathbb{N},\, f_i \in C([0,1]) \right\}.$$

**Corollary 3.12.** *For a polynomial $u \in K$ with $u(g) = \prod_{i=1}^n l_{f_i}^{k_i}(g)$ let $u_N := \prod_{i=1}^n \big(\Phi^N(l_{f_i})\big)^{k_i} \in H^N$, then $u_N \to u$ strongly.*

*Proof.* Let $\tilde{u}^N := \prod_{i=1}^n \big(\Phi^N(l_{f_i})\big)^{k_i} \in H$ be the respective product of conditional expectations, where as above $\Phi^N$ also denotes the projection operator on $H = L^2(\mathcal{G}, \mathbb{Q}^\beta)$. Since each of the factors $\Phi^N(l_{f_i}) \in H$ is uniformly bounded and converges strongly to $l_{f_i}$ in $L^2(\mathcal{G}, \mathbb{Q}^\beta)$, the convergence also holds true in any $L^p(\mathcal{G}, \mathbb{Q}^\beta)$ with $p > 0$. This implies $\tilde{u}^N \to u$ in $H$. Furthermore,

$$\begin{aligned}\lim_N \lim_M \left\|\Phi^M \tilde{u}_N - u_M\right\|_{H^M} &= \lim_N \lim_M \left\| \Phi^M \left( \prod_{i=1}^n \big(\Phi^N(l_{f_i})\big)^{k_i} \right) - \prod_{i=1}^n \big(\Phi^M(l_{f_i})\big)^{k_i} \right\|_H \\ &= \lim_N \left\| \prod_{i=1}^n \big(\Phi^N(l_{f_i})\big)^{k_i} - \prod_{i=1}^n l_{f_i}^{k_i} \right\|_H = 0.\end{aligned}$$

□

**Proposition 3.13** (Mosco II'). *There is a core $K \subset D(\mathcal{E})$ such that for all $u \in K$ there is a sequence $u_N \in \mathcal{D}(\mathcal{E}^N)$ converging strongly to $u \in H$ and $N \cdot \mathcal{E}^N(u_N, u_N) \to \mathcal{E}(u,u)$.*

*Proof.* It follows from the chain rule for the $L^2$-gradient operator $\nabla$ that the linear span of polynomials of the form $u(g) = \prod_{i=1}^n l_{f_i}^{k_i}(g)$ with $k_i \in \mathbb{N}$, $f_i \in C([0,1])$, $k_i \in \mathbb{N}$, is a core of $D(\mathcal{E})$. Hence it suffices to prove the claim for such $u$. Let $u_N := \prod_{i=1}^n \big(\Phi^N(l_{f_i})\big)^{k_i} \in H^N$ as above then the strong convergence of $u^N$ to $u$ is assured by corollary 3.12. From lemma 3.15 below we obtain that $\Phi^N(l_f)(X) = \langle f, g_X\rangle$. In particular

$$\big(\nabla\Phi^N(l_f)(X)\big)^i = \frac{1}{N}\cdot\big(\eta^N * f\big)(\frac{i}{N}),$$

where $\eta^N$ denotes the convolution kernel $t \to \eta^N(t) = N\cdot(1-\min(1,|N\cdot t|))$. By this the convergence of $N\cdot\mathcal{E}_N(u^N,u^N)$ to $\mathcal{E}(u,u)$ follows easily from Lebesgue's dominated convergence theorem in $L^2(\mathcal{G}\times[0,1], \mathbb{Q}^\beta \otimes dx)$. □

**Remark 3.14.** For later use we observe that for $u$ and $u_N$ as above and for $\mathbb{Q}^\beta$-a.e. $g$ we have

$$\big\|N\iota^N(\nabla u_N(g(1/N),\ldots,g((N-1)/N))) - \nabla u_{|g}\big\|_{L^2(0,1)} \to 0 \quad \text{as } N\to\infty,$$

with $\iota^N : \mathbb{R}^{N-1} \to D([0,1),\mathbb{R})$ defined as above.

**Lemma 3.15.** *For $X \in \Sigma_N$ define $g_X \in \mathcal{G}$ by*

$$g_X(t) = x_i + (N\cdot t - i)(x_{i+1}-x_i) \quad \text{if } t \in [\frac{i}{N}, \frac{i+1}{N}), \quad i = 0,\ldots,N-1,$$

*then*

$$\mathbb{E}(g|\mathcal{F}_N)(X) = g_X.$$

*Proof.* The statement is a simple consequence of the explicit formula for the finite dimensional distributions of the Dirichlet process, cf. [vRS07]. □

For the verification of Mosco I we exploit that the respective integration by parts formulas of $\mathcal{E}^N$ and $\mathcal{E}$ converge. In case of a fixed state space a similar approach is discussed in [12].

Let $T^N := \{f : \Sigma_N \to \mathbb{R}^{N-1}\}$ be equipped with the norm

$$\|f\|^2_{T^N} := \frac{1}{N}\int_{\Sigma_N} \|f(x)\|^2_{R^{N-1}}\, q_N(dx),$$

then the corresponding integration by parts formula for $q_N$ on $\Sigma_N$ reads

$$\langle \nabla u, \xi\rangle_{T^N} = -\frac{1}{N}\langle u, \operatorname{div}_{q_N}\xi\rangle_{H^N}. \tag{6}$$

To state the corresponding formula for $\mathcal{E}$ we introduce the Hilbert space of vector fields on $\mathcal{G}$ by

$$T = L^2(\mathcal{G}\times[0,1], \mathbb{Q}^\beta\otimes dx),$$

with dense subset $\Theta \subset T$

$$\Theta = \operatorname{span}\{\zeta \in T \,|\, \zeta(g,t) = w(g)\cdot\varphi(g(t)), w \in K, \varphi \in \mathcal{C}^\infty([0,1]) : \varphi(0) = \varphi(1) = 0\}.$$

The $L^2$-derivative operator $\nabla$ defines a map

$$\nabla : \mathfrak{C}^1(\mathcal{G}) \to T$$

which by [18, proposition 7.3], cf. [19], satisfies the following integration by parts formula, .

$$\langle \nabla u, \zeta \rangle_T = -\langle u, \operatorname{div}_{\mathbb{Q}^\beta} \zeta \rangle_H, \quad u \in \mathfrak{C}^1(\mathcal{G}), \zeta \in \Theta, \tag{7}$$

where, for $\zeta(g,t) = w(g) \cdot \varphi(g(t))$,

$$\operatorname{div}_{\mathbb{Q}^\beta} \zeta(g) = w(g) \cdot V_\varphi^\beta(g) + \langle \nabla w(g)(.), \varphi(g(.)) \rangle_{L^2(dx)}$$

with

$$V_\varphi^\beta(g) := V_\varphi^0(g) + \beta \int_0^1 \varphi'(g(x))dx - \frac{\varphi'(0) + \varphi'(1)}{2}$$

and

$$V_\varphi^0(g) := \sum_{a \in J_g} \left[ \frac{\varphi'(g(a+)) + \varphi'(g(a-))}{2} - \frac{\delta(\varphi \circ g)}{\delta g}(a) \right].$$

Here $J_g \subset [0,1]$ denotes the set of jump locations of $g$ and

$$\frac{\delta(\varphi \circ g)}{\delta g}(a) := \frac{\varphi\left(g(a+)\right) - \varphi\left(g(a-)\right)}{g(a+) - g(a-)} .$$

By formula (7) one can extend $\nabla$ to a closed operator on $D(\mathcal{E})$ such that $\mathcal{E}(u,u) = \|\nabla u\|_T^2$. The Markov uniqueness of $\mathcal{E}$ now implies the converse which is a characterization of $D(\mathcal{E})$ via (7).

**Lemma 3.16** (Meyers-Serrin property)**.** *Assume Markov-uniqueness holds for $\mathcal{E}$, then*

$$(\mathcal{E}(u,u))^{1/2} = \sup_{\zeta \in \Theta} \frac{\langle u, \operatorname{div}_{\mathbb{Q}^\beta} \zeta \rangle_H}{\|\zeta\|_T}. \tag{8}$$

*Proof.* We repeat the standard argument, cf. [6]. Denoting the r.h.s. of (8) by $(\hat{\mathcal{E}}(u,u))^{1/2}$ one obtains that $\hat{\mathcal{E}}$ is a Markovian extension of $\mathcal{E}$. Since $\mathcal{E}$ is assumed maximal in the class of Markovian forms it follows $\mathcal{E} = \hat{\mathcal{E}}$. □

The convergence of (6) to (7) is established by the following lemma whose prove is given below.

**Lemma 3.17.** *For $\zeta \in \Theta$ there exists a sequence of vector fields $\zeta_N : \Sigma_N \to \mathbb{R}^{N-1}$ such that $\operatorname{div}_{q_N} \zeta^N \in H^N$ converges strongly to $\operatorname{div}_{\mathbb{Q}^\beta} \zeta$ in $H$ and such that $\|\zeta^N\|_{T^N} \to \|\zeta\|_T$ for $N \to \infty$.*

**Proposition 3.18** (Mosco I)**.** *Let $\mathcal{E}$ be Markov-unique and let $u_N \in \mathcal{D}(\mathcal{E}^N)$ converge weakly to $u \in H$, then*

$$\mathcal{E}(u,u) \le \liminf_{N \to \infty} N \cdot \mathcal{E}^N(u_N, u_N).$$

*Proof.* Let $u \in H$ and $u_N \in H^N$ converge weakly to $u$. Let $\zeta \in \Theta$ and $\zeta^N$ be as in lemma 3.17, then

$$\begin{aligned} \frac{-\langle u, \operatorname{div}_{\mathbb{Q}^\beta} \zeta \rangle_H}{\|\zeta\|_T} &= \lim \frac{-\langle u_N, \operatorname{div}_{q_N} \zeta_N \rangle_{H^N}}{\|\zeta^N\|_{T^N}} \\ &= \lim N \cdot \frac{\langle \nabla u_N, \zeta_N \rangle_{T^N}}{\|\zeta^N\|_{T^N}} \le \liminf N \cdot \|\nabla u_N\|_{T^N} = \liminf \left(N \cdot \mathcal{E}^N(u_N, u_N)\right)^{1/2}, \end{aligned}$$

such that, using (8),

$$(\mathcal{E}(u,u))^{1/2} = \sup_{\zeta \in \Theta} \frac{-\langle u, \operatorname{div}_{\mathbb{Q}^\beta} \zeta \rangle_H}{\|\zeta\|_T} \le \liminf \left(N \cdot \mathcal{E}^N(u_N, u_N)\right)^{1/2}. \qquad \square$$

*Proof of lemma 3.17.* By linearity it suffices to consider the case $\zeta(g,t) = w(g)\cdot\varphi(g(t))$ with $w(g) = \prod_{i=1}^n l_{f_i}^{k_i}(g)$. Choose

$$(\zeta^N(x_1,\cdots,x_{N-1}))^i := w_N(x_1,\ldots,x_{N-1})\cdot\varphi(x_i)$$

with $w_N := \prod_{i=1}^n (\Phi^N(l_{f_i}))^{k_i}$. Then

$$\mathrm{div}_{q_N}\zeta^N = w_N\cdot V^\beta_{N,\varphi} + \langle\nabla w_N,\vec{\varphi}\rangle_{\mathbb{R}^{N-1}},$$

with

$$\vec{\varphi}(x_1,\ldots,x_{N-1}) := (\varphi(x_1),\ldots,\varphi(x_{N-1}))$$

and

$$V^\beta_{N,\varphi}(x_1,\ldots,x_{N-1}) := (\frac{\beta}{N}-1)\sum_{i=0}^{N-1}\frac{\varphi(x_{i+1})-\varphi(x_i)}{x_{i+1}-x_i} + \sum_{i=1}^{N-1}\varphi'(x_i).$$

We recall that for all bounded measurable $u:[0,1]^{N-1}\to\mathbb{R}$

$$\int_{\Sigma_N} u(x_1,\ldots,x_{N-1})\,q_N(dx) = \int_{\mathcal{G}} u(g(t_1),\ldots,g(t_{N-1}))\,\mathbb{Q}^\beta(dg),$$

with $t_i = i/N$, $i=0,\ldots,N$. Using this we get immediately

$$\begin{aligned}\left\|\zeta^N\right\|^2_{T^N} &= \frac{1}{N}\int_{\Sigma_N}\sum_{i=1}^{N-1} w_N^2(x)\,\varphi(x_i)^2\,q_N(dx) = \int_{\mathcal{G}} w_N^2(g(t_1),\ldots,g(t_{N-1}))\frac{1}{N}\sum_{i=1}^{N-1}\varphi(g(t_i))^2\,\mathbb{Q}^\beta(dg)\\ &\to \int_{\mathcal{G}} w^2(g)\int_0^1\varphi(g(s))^2\,ds\,\mathbb{Q}^\beta(dg) = \|\zeta\|^2_T\,.\end{aligned}$$

To prove strong convergence of $\mathrm{div}_{q_N}\zeta^N$ to $\mathrm{div}_{\mathbb{Q}^\beta}\zeta$, by definition we have to show that there exists a sequence $(d^N\zeta)_N\subset H$ tending to $\mathrm{div}_{\mathbb{Q}^\beta}\zeta$ in $H$ such that

$$\lim_N\limsup_M\left\|\Phi^M(d^N\zeta) - \mathrm{div}_{q_M}\zeta^M\right\|^2_{H^M} = 0.$$

The choice

$$d^N\zeta(g) := \mathrm{div}_{q_N}\zeta^N(g(t_1),\ldots,g(t_{N-1}))$$

makes this convergence trivial, once we have proven that in fact $(d^N\zeta)_N$ converges to $\mathrm{div}_{\mathbb{Q}^\beta}\zeta$ in $H$. This is carried out in the following two lemmas. □

**Lemma 3.19.** For $\mathbb{Q}^\beta$-a.s. $g$ we have

$$V^\beta_{N,\varphi}(g(t_1),\ldots,g(t_{N-1}))\to V^\beta_\varphi(g),\quad \text{as } N\to\infty,$$

and we have also convergence in $L^p(\mathcal{G},\mathbb{Q}^\beta)$, $p>1$.

*Proof.* We rewrite $V^\beta_{N,\varphi}(g(t_1),\ldots,g(t_{N-1}))$ as

$$\begin{aligned}V^\beta_{N,\varphi}(g(t_1),\ldots,g(t_{N-1})) =&\beta\sum_{i=0}^{N-1}\frac{\varphi(g(t_{i+1}))-\varphi(g(t_i))}{g(t_{i+1})-g(t_i)}(t_{i+1}-t_i)\\ &-\frac{\varphi(g(t_1))-\varphi(g(t_0))}{g(t_1)-g(t_0)} + \sum_{i=1}^{N-2}\left(\varphi'(g(t_i)) - \frac{\varphi(g(t_{i+1}))-\varphi(g(t_i))}{g(t_{i+1})-g(t_i)}\right)\\ &+\varphi'(g(t_{N-1})) - \frac{\varphi(g(t_N))-\varphi(g(t_{N-1}))}{g(t_N)-g(t_{N-1})}.\end{aligned}\tag{9}$$

Note that all terms are uniformly bounded in $g$ with a bound depending on the supremum norm of $\varphi'$ and $\varphi''$, respectively. Since the same holds for $V_\varphi^\beta(g)$ (cf. Lemma 5.1 in [vRS07]), it is sufficient to show convergence $\mathbb{Q}^\beta$-a.s. By the support properties of $\mathbb{Q}^\beta$ $g$ is continuous at $t_N = 1$, so that the last line in (9) tends to zero. Using Taylor's formula we obtain that the first term in (9) is equal to

$$\beta \sum_{i=0}^{N-1} \varphi'(g(t_i))(t_{i+1} - t_i) + \frac{1}{2} \sum_{i=0}^{N-1} \varphi''(\gamma_i)\,(g(t_{i+1}) - g(t_i))\,(t_{i+1} - t_i),$$

for some $\gamma_i \in [g(t_i), g(t_{i+1})]$. Obviously, the first term tends to $\beta \int_0^1 \varphi'(g(s))\,ds$ and the second one to zero as $N \to \infty$. Thus, it remains to show that the second line in (9) converges to

$$\sum_{a \in J_g} \left[ \frac{\varphi'(g(a+)) + \varphi'(g(a-))}{2} - \frac{\delta(\varphi \circ g)}{\delta g}(a) \right] - \frac{\varphi'(0) + \varphi'(1)}{2}. \tag{10}$$

Note that by the right-continuity of $g$ the first term in the second line in (9) tends to $-\varphi'(0)$. Let now $a_2, \ldots, a_{l-1}$ denote the $l-2$ largest jumps of $g$ on $]0,1[$. For $N$ very large (compared with $l$) we may assume that $a_2, \ldots, a_{l-2} \in ]\frac{2}{N}, 1 - \frac{2}{N}[$. Put $a_1 := \frac{1}{N}$, $a_l := 1 - \frac{1}{N}$. For $j = 1, \ldots, l$ let $k_j$ denote the index $i \in \{1, \ldots, N-1\}$, for which $a_j \in [t_i, t_{i+1}[$. In particular, $k_1 = 1$ and $k_l = N-1$. Then

$$\begin{aligned}\sum_{i \in \{k_2, \ldots, k_{l-1}\}} \varphi'(g(t_i)) - \frac{\varphi(g(t_{i+1})) - \varphi(g(t_i))}{g(t_{i+1}) - g(t_i)} &\xrightarrow[N \to \infty]{} \sum_{j=2}^{l-1} \varphi'(g(a_j-)) - \frac{\delta(\varphi \circ g)}{\delta g}(a_j) \\ &\xrightarrow[l \to \infty]{} \sum_{a \in J_g} \varphi'(g(a-)) - \frac{\delta(\varphi \circ g)}{\delta g}(a). \end{aligned} \tag{11}$$

Provided $l$ and $N$ are chosen so large that

$$|g(t_{i+1}) - g(t_i)| \le \frac{C}{l}$$

for all $i \in \{0, \ldots, N-1\} \backslash \{k_1, \ldots, k_l\}$, where $C = \sup_s |\varphi'''(s)|/6$, again by Taylor's formula we get for every $j \in \{1, \ldots, l-1\}$

$$\begin{aligned} &\sum_{i=k_j+1}^{k_{j+1}-1} \varphi'(g(t_i)) - \frac{\varphi(g(t_{i+1})) - \varphi(g(t_i))}{g(t_{i+1}) - g(t_i)} \\ &= -\sum_{i=k_j+1}^{k_{j+1}-1} \frac{1}{2}\varphi''(g(t_i))\,(g(t_{i+1}) - g(t_i)) + \frac{1}{6}\varphi'''(\gamma_i)\,(g(t_{i+1}) - g(t_i))^2 \\ &\xrightarrow[N \to \infty]{} -\frac{1}{2} \int_{a_j+}^{a_{j+1}-} \varphi''(g(s))\,dg(s) + O(l^{-2}) = -\frac{1}{2} \int_{g(a_j+)}^{g(a_{j+1}-)} \varphi''(s)\,ds + O(l^{-2}). \end{aligned}$$

Summation over $j$ leads to

$$\begin{aligned} &\sum_{j=1}^{l-1} \sum_{i=k_j+1}^{k_{j+1}-1} \varphi'(g(t_i)) - \frac{\varphi(g(t_{i+1})) - \varphi(g(t_i))}{g(t_{i+1}) - g(t_i)} \\ &\xrightarrow[N \to \infty]{} -\frac{1}{2} \sum_{j=1}^{l-1} \int_{g(a_j+)}^{g(a_{j+1}-)} \varphi''(s)\,ds + O(l^{-1}) = -\frac{1}{2} \int_0^1 \varphi''(s)\,ds + \frac{1}{2} \sum_{j=2}^{l-1} \int_{g(a_j-)}^{g(a_j+)} \varphi''(s)\,ds + O(l^{-1}) \\ &\xrightarrow[l \to \infty]{} -\frac{1}{2}(\varphi'(1) - \varphi'(0)) + \frac{1}{2} \sum_{a \in J_g} \varphi'(g(a+)) - \varphi'(g(a-)). \end{aligned}$$

Combining this with (11) yields that the second line of (9) converges in fact to (10), which completes the proof. □

Since $w_N(g(t_1),\dots,g(t_{N-1})$ converges to $w$ in $L^p(\mathcal{G},\mathbb{Q}^\beta)$, $p>0$ (cf. proof of corollary 3.12 above), the last lemma ensures that the first term of $d^N\zeta$ converges to the first term of $\mathrm{div}_{\mathbb{Q}^\beta}\zeta$ in $H$, while the following lemma deals with the second term.

**Lemma 3.20.** For $\mathbb{Q}^\beta$-a.s. $g$ we have

$$\langle \nabla w_N(g(t_1),\dots,g(t_{N-1})), \vec{\varphi}(g(t_1),\dots,g(t_{N-1}))\rangle_{\mathbb{R}^{N-1}} \to \langle \nabla w_{|g}, \varphi(g(.))\rangle_{L^2(0,1)}, \quad \text{as } N\to\infty,$$

and we have also convergence in $H$.

*Proof.* As in the proof of the last lemma it is enough to prove convergence $\mathbb{Q}^\beta$-a.s. Note that

$$\langle \nabla w_N(\vec{g}), \vec{\varphi}(\vec{g})\rangle_{\mathbb{R}^{N-1}} = N\langle \iota^N(\nabla w_N(\vec{g})), \iota^N(\vec{\varphi}(\vec{g}))\rangle_{L^2(0,1)},$$

writing $\vec{g} := (g(t_1),\dots,g(t_{N-1}))$ and using the extension of $\iota^N$ on $\mathbb{R}^{N-1}$. By triangle and Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&|\langle N\iota^N(\nabla w_N(\vec{g})), \iota^N(\vec{\varphi}(\vec{g}))\rangle_{L^2(0,1)} - \langle \nabla w_{|g}, \varphi(g(.))\rangle_{L^2(0,1)}|\\
\le&|\langle N\iota^N(\nabla w_N(\vec{g})) - \nabla w_{|g}, \iota^N(\vec{\varphi}(\vec{g}))\rangle_{L^2(0,1)}| + \langle \nabla w_{|g}, \iota^N(\vec{\varphi}(\vec{g})) - \varphi(g(.))\rangle_{L^2(0,1)}|\\
\le&\left\|N\iota^N(\nabla w_N(\vec{g})) - \nabla w_{|g}\right\|_{L^2(0,1)} \left\|\iota^N(\vec{\varphi}(\vec{g}))\right\|_{L^2(0,1)} + \left\|\nabla w_{|g}\right\|_{L^2(0,1)} \left\|\iota^N(\vec{\varphi}(\vec{g})) - \varphi(g(.))\right\|_{L^2(0,1)},
\end{aligned}$$

which tends to zero by remark 3.14 and by the definition of $\iota^N$. □

### 3.2.3 Proof of proposition 3.4

**Lemma 3.21.** *For $u\in C(\mathcal{G})$ let $u_N\in H^N$ be defined by $u_N(x) := u(\iota x)$, then $u_N\to u$ strongly. Moreover, for any sequence $f_N\in H^N$ with $f_N\to f\in H$ strongly, $u_N\cdot f_N\to u\cdot f$ strongly.*

*Proof.* Let $\tilde{u}_N\in H$ be defined by $\tilde{u}_N(g) := u(g^N)$, where $g^N := \sum_{i=1}^N g(i/N)\mathbb{1}_{[i/N,(i+1)/N)}$, then $\tilde{u}_N\to u$ in $H$ strongly. Moreover,

$$\lim_N\lim_M \left\|\Phi^M\tilde{u}_N - u_M\right\|_{H^M} = \lim_N\lim_M \left\|\Phi^M\tilde{u}_N - \tilde{u}_M\right\|_H = \lim_N \|\tilde{u}_N - u\|_H = 0,$$

where as above we have identified $\Phi^M$ with the corresponding projection operator in $L^2(\mathcal{G},\mathbb{Q}^\beta)$. For the proof of the second statement, let $H\ni\tilde{f}_N\to f$ in $H$ such that $\lim_N\limsup_M \left\|\Phi^M\tilde{f}_N - f_M\right\|_{H^M} = 0$. From the uniform boundedness of $\tilde{u}_N$ it follows that also $\tilde{u}_N\cdot\tilde{f}_N\to u\cdot f$ in $H$. In order to show $H^M\ni u_M\cdot f_M\to u\cdot f$ write

$$\left\|\Phi^M(\tilde{u}_N\cdot\tilde{f}_N) - u_M\cdot f_M\right\|_{H^M} \le \left\|\Phi^M(\tilde{u}_N\cdot\tilde{f}_N) - u_M\cdot\Phi^M(\tilde{f}_M)\right\|_{H^M} + \left\|u_M\cdot f_M - u_M\cdot\Phi^M(\tilde{f}_M)\right\|_{H^M}.$$

Identifying the map $\Phi^M$ with the associated conditional expectation operator, considered as an orthogonal projection in $H$, the claim follows from

$$\begin{aligned}
\left\|\Phi^M(\tilde{u}_N\cdot\tilde{f}_N) - u_M\cdot\Phi^M(\tilde{f}_M)\right\|_{H^M} &= \left\|\Phi^M(\tilde{u}_N\cdot\tilde{f}_N) - \tilde{u}_M\cdot\Phi^M(\tilde{f}_M)\right\|_H\\
&= \left\|\Phi^M(\tilde{u}_N\cdot\tilde{f}_N) - \Phi^M(\tilde{u}_M\cdot\tilde{f}_M)\right\|_H\\
&\le \left\|\tilde{u}_N\cdot\tilde{f}_N - \tilde{u}_M\cdot\tilde{f}_M\right\|_H
\end{aligned}$$

and

$$
\begin{aligned}
\left\| u_M \cdot f_M - u_M \cdot \Phi^M(\tilde{f}_M) \right\|_{H^M} &\leq \|u\|_\infty \left\| f_M - \Phi^M(\tilde{f}_N) \right\|_{H^M} \\
&\quad + \|u\|_\infty \left\| \Phi^M(\tilde{f}_N) - \Phi^M(\tilde{f}_M) \right\|_{H^M} \\
&= \|u\|_\infty \left\| f_M - \Phi^M(\tilde{f}_N) \right\|_{H^M} \\
&\quad + \|u\|_\infty \left\| \Phi^M(\tilde{f}_N) - \Phi^M(\tilde{f}_M) \right\|_{H} \\
&\leq \|u\|_\infty \left\| f_M - \Phi^M(\tilde{f}_N) \right\|_{H^M} \\
&\quad + \|u\|_\infty \left\| \tilde{f}_N - \tilde{f}_M \right\|_{H}
\end{aligned}
$$

such that in fact $\lim_N \limsup_M \left\| \Phi^M(\tilde{u}_N \cdot \tilde{f}_N) - u_M \cdot f_M \right\|_{H^M} = 0$. □

*Proof of proposition 3.4* It suffices to prove the claim for functions $f \in C(\mathcal{G}^l)$ of the form $f(g_1, \ldots, g_l) = f_1(g_1) \cdot f_2(g_2) \cdots f_l(g_l)$ with $f_i \in C(\mathcal{G})$. Let $P_t^N : H^N \to H^N$ be the semigroup on $H^N$ induced by $g^N$ via $\mathbb{E}_{g \cdot q_N}[f(g_t^N)] = \langle P_t^N f, g \rangle_{H^N}$. From theorem 3.8 and the abstract results in [14] it follows that $P_t^N$ converges to $P_t$ strongly, i.e. for any sequence $u^N \in H^N$ converging to some $u \in H$ strongly, the sequence $P_t^N u^N$ also strongly converges to $P_t u$. Let $f_i^N := f_i \circ \iota^N$, then inductive application of lemma 3.21 yields

$$
\begin{aligned}
&P^N_{t_l - t_{l-1}}(f_l^N \cdot P^N_{t_{l-1}-t_{l-2}}(f_{l-1}^N \cdot P^N_{t_{l-2}-t_{l-3}} \ldots f_2^N \cdot P^N_{t_1} f_1^N) \cdots) \\
&\qquad \stackrel{N \to \infty}{\longrightarrow} P_{t_l - t_{l-1}}(f_l \cdot P_{t_{l-1}-t_{l-2}}(f_{l-1} \cdot P_{t_{l-2}-t_{l-3}} \ldots f_2 \cdot P_{t_1} f_1) \cdots) \text{ strongly,}
\end{aligned}
$$

which in particular implies the convergence of inner products. Hence, using the Markov property of $g^N$ and $g$ we may conclude that

$$
\begin{aligned}
\lim_N \mathbb{E}\big(f_1(g^N_{t_1}) \ldots f_l(g^N_{t_l})\big) &= \lim_N \mathbb{E}\big(f_1^N(X^N_{t_1}) \ldots f_l^N(X^N_{t_l})\big) \\
&= \lim_N \langle 1, P^N_{t_l - t_{l-1}}(f_l^N \cdot P^N_{t_{l-1}-t_{l-2}}(f^N_{l-1} \cdot P^N_{t_{l-2}-t_{l-3}} \ldots f_2^N \cdot P^N_{t_1} f_1^N) \cdots) \rangle_{H^N} \\
&= \langle 1, P_{t_l - t_{l-1}}(f_l \cdot P_{t_{l-1}-t_{l-2}}(f_{l-1} \cdot P_{t_{l-2}-t_{l-3}} \ldots f_2 \cdot P_{t_1} f_1) \cdots) \rangle_H \\
&= \mathbb{E}\big(f_1(g_{t_1}) \ldots f_l(g_{t_l})\big).
\end{aligned}
$$

□

# 4 Appendix: On a connection to $\nabla\phi$-interface models

We conclude with a remark on a link to stochastic interface models. Consider an interface on the one-dimensional lattice $\Gamma_N := \{1, \ldots, N-1\}$. The location of the interface at time $t$ is represented by the height variables $\phi_t = \{\phi_t(x), x \in \Gamma_N\} \in \sqrt{N} \cdot \Sigma_N$ with dynamics determined by the generator $\tilde{L}^N$ defined below and with the boundary conditions $\phi_t(0) = 0$ and $\phi(N) = \sqrt{N}$ at $\partial\Gamma_N := \{0, N\}$.

$$
\tilde{L}^N f(\phi) := (\frac{\beta}{N} - 1) \sum_{x \in \Gamma_N} \left( \frac{1}{\phi(x) - \phi(x-1)} - \frac{1}{\phi(x+1) - \phi(x)} \right) \frac{\partial}{\partial \phi(x)} f(\phi) + \Delta f(\phi)
$$

for $\phi \in \mathrm{Int}(\sqrt{N} \cdot \Sigma_N)$ and with $\phi(0) := 0$ and $\phi(N) := \sqrt{N}$. $\tilde{L}^N$ corresponds to $L^N$ as an operator on $C(\sqrt{N} \cdot \Sigma_N)$ with domain

$$
\mathcal{D}(\tilde{L}^N) := \{ f \in C^2(\sqrt{N} \cdot \Sigma_N) \,|\, \tilde{L}^N f \in C(\sqrt{N} \cdot \Sigma_N) \}.
$$

Note that this system involves a non-convex interaction potential function $V$ on $(0, \infty)$ given by $V(r) = (1 - \frac{\beta}{N}) \log(r)$ and the Hamiltonian

$$H_N(\phi) := \sum_{x=0}^{N-1} V(\phi(x+1) - \phi(x)), \qquad \phi(0) := 0,\, \phi(N) := \sqrt{N}.$$

Then, the natural stationary distribution of the interface is the Gibbs measure $\mu_N$ conditioned on $\sqrt{N} \cdot \Sigma_N$:

$$\mu_N(d\phi) := \frac{1}{Z_N} \exp(-H_N(\phi)) \mathbb{1}_{\{(\phi(1),\ldots,\phi(N-1)) \in \sqrt{N}\cdot\Sigma_N\}} \prod_{x \in \Gamma_N} d\phi(x),$$

where $Z_N$ is a normalization constant. Note that $\mu_N$ is the corresponding measure of $q_N$ on the state space $\sqrt{N} \cdot \Sigma_N$. Suppose now that $(\phi_t)_{t \geq 0}$ is the stationary process generated by $\tilde{L}^N$. Then the space-time scaled process

$$\tilde{\Phi}_t^N(x) := \frac{1}{\sqrt{N}} \phi_{N^2 t}(x), \qquad x = 0, \ldots, N,$$

living on $\Sigma_N$ is associated with the Dirichlet form $N \cdot \mathcal{E}^N$. Introducing the $\mathcal{G}$-valued fluctuation field

$$\Phi_t^N(\vartheta) := \sum_{x \in \Gamma_N} \tilde{\Phi}_t^N(x) \, \mathbb{1}_{[x/N,(x+1)/N)}(\vartheta), \qquad \vartheta \in [0,1),$$

by our main result we have weak convergence for the law of the equilibrium fluctuation field $\Phi^N$ to the law of the nonlinear diffusion on $\mathcal{G}$, which is the $\mathcal{G}$-parametrization of the Wasserstein diffusion.

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