CONNECTION TIMES IN LARGE AD-HOC MOBILE NETWORKS

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Abstract. We study connectivity properties in a probabilistic model for a large mobile ad-hoc network. We consider a large number of participants of the system moving randomly, independently and identically distributed in a large domain, with a space-dependent population density of finite, positive order and with a fixed time horizon. Messages are instantly transmitted according to a relay principle, i.e., they are iteratively forwarded from participant to participant over distances smaller than the communication radius until they reach the recipient. In mathematical terms, this is a dynamic continuum percolation model.

We consider the connection time of two sample participants, the amount of time over which these two are connected with each other. In the above thermodynamic limit, we find that the connectivity induced by the system can be described in terms of the counterplay of a local, random, and a global, deterministic mechanism, and we give a formula for the limiting behaviour.

A prime example of the movement schemes that we consider is the well-known random waypoint model. Here we give a negative upper bound for the decay rate, in the limit of large time horizons, of the probability of the event that the portion of the connection time is less than the expectation.

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1. Introduction and main results

1.1. **Background and goals.** *Ad-hoc networks* consist of individuals in a given domain that communicate with each other via a relay principle: messages are forwarded from individual to individual as long as this transmission is local, until the message finally arrives at the recipient. This requires of course that the sender is *connected* with the recipient, i.e., that there is a chain of individuals connecting them such that all links are not larger than a given radius, the *transmission radius* or *communication radius*. This principle of message transmission within the system of participants, rather than via antennas or fixed wires, has a number of advantages over a firmly installed communication system; e.g., its installation is cheap, it does not require much maintaining, it can accommodate more information etc. A disadvantage is of course that the connectivity is not always fulfilled, i.e., it may be that two given individuals are not connected with each other and are therefore not able to exchange messages.

The advantages of such a type of system increase if the ad-hoc network becomes *mobile*, i.e., if all the individuals independently move around in the given region and transmit the messages at their present location, since in this case a fixed system of wires would be useless, and firmly located antennas would

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be necessary, and this may easily lead to situations of overloads in peak times. This is why mobile ad-hoc networks are increasingly in the discussion for various applications, like telecommunication, car-to-car applications for the distribution of information about the traffic situation, downloading of large data packages, and more [CBD02, CPS09, R11]. However, before one can seriously think about an introduction of a mobile ad-hoc system, one needs to know how reliable it is and how much information it can reliably transmit and how well the participants of the system are connected.

The mathematical analysis of the connectivity properties of a *mobile ad-hoc network* (usually referred to in the literature as *MANET*), is the purpose of the present paper. We discuss a natural probabilistic model and derive rigorous results about the quality of the connection in this system. Roughly speaking, in our model, a large number N of participants randomly and independently move around in a given domain $D \subset \mathbb{R}^d$ with $d \geq 2$. The movement scheme considered is quite general, but later we will discuss the prime example, the *random waypoint model (RWP)*, in detail. Each of the participants carries a device that possesses a fixed *communication radius* $2R$ (the same for everybody). The domain is so large that the individuals are distributed according to a spatial density that is of finite order, but may depend on the details of the domain (this models subareas with more or less frequent visits, like forests, lakes or public places). We assume that messages are transmitted instantly, i.e., without loss of time. Then we ask, for two fixed given participants, how large, during a given time interval $[0, T]$, the amount of time is during which they are connected, their *connection time* $\tau_T^{(N)}$. This is one of the most decisive quantities in such a system, since it measures the quality of the entire system by means of two sample participants.

The regime in which we will be working is the limit of a large number of participants, coupled with the limit of a large region such that the population density (number of participants per area unit) is of finite positive order. In the language of statistical mechanics, this is the *thermodynamic limit*. We will condition on the two sample trajectories. The connection time is obviously a complex function of the entire system, but we will be able to quantify the influence of the large number of the participants on the connectivity of the two sample participants in terms of a simple function. This function is known from the theory of *continuum percolation*, which studies connectivity through a union of randomly distributed balls. It turns out that the limiting connection time has a global, deterministic part and a local, probabilistic part, the latter of which is described in terms of the mentioned function. Furthermore, it also turns out that this limit is deterministic, given the two sample participants. This is due to one of our assumptions on the movement scheme, which requires that knowledge about the walker's location at a later time point does not fix the current location with positive probability. This assumption implies a certain independence of the locations of the totality of the walkers at any two given times and leads to a deterministic limit. This is presented in Sections 1.2 and 1.3.

From the practical point of view, a very large value of the connection time is highly desirable. This can be guaranteed by a large value of the communication radius $2R$. However, one also would like to have rules at hand that tell how large this radius must be picked in order that the connection time exceeds a certain threshold. Some general answer to this question is given in Section 1.3. We explain there that, under natural conditions, the main effect that may damage the connection are time lags that any of the two sample participants spend close to the boundary of the domain D, while, in the interior of D, the local connection quality of the system super-exponentially fast tends to the optimum for $R \to \infty$, depending on the local user density only.

Furthermore, another important question that we address is about the long-time behaviour of the connection time. More precisely, we identify the limiting fraction of the connection time by means of an ergodic theorem and estimate the probability of the unwanted event that the connection time covers only an untypically low portion of the time interval. This is an event of a downward *large deviation*, and we will show that its probability decays exponentially fast as $T \to \infty$, and we quantify an upper bound of the decay rate. For this question, we restrict to the RWP and derive some recurrence properties that may be useful also for further investigations; see Section 1.4.

The model that we consider is sometimes called a dynamic geometric random graph. Such models were analysed in a series of papers by Peres and co-workers, see [PSSS11, PSSS13], e.g. However, in contrast to our setting, they do not consider the thermodynamic limit, study different questions related to the large-time limit, and take Brownian motion or random walks as the underlying movement scheme. Our purpose is to study a more realistic model for the random movement of people.

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1.2. **Connection time of two participants in the thermodynamic limit.** Let us introduce the model; our main result here is Theorem 1.2.

We consider a system of N particles (the participants of the mobile ad-hoc network), which randomly move with time horizon $[0,T]$ within a given bounded regular domain D in $\mathbb{R}^d$. The N movements are independent and identically distributed, and the common movement scheme (path distribution) does not have to be Markovian; more precise assumptions follow below. The underlying probability measure and expectation are denoted by $\mathbb{P}$ and $\mathbb{E}$.

Later (see Section 1.4) we will be mostly interested in a particular movement scheme, namely the *random waypoint model (RWP)*. This motion dynamic is considered in information science as a realistic model for the random movement of a human being, e.g., a participant of a telecommunication system [R11, L04, BHPC04, LV06]. A brief definition of the model is as follows. The walker starts from some point, picks a random velocity and a random site (the first waypoint) and then moves with this constant velocity on a straight line to that waypoint. Having arrived there, he picks the next random velocity and random waypoint and moves there on a straight line with the second velocity. This is iterated. All the waypoints are independent and identically distributed, and the velocities as well, and the waypoints are independent from the velocities. This model is a natural extension of the classical RWP, as we admit general distributions of the waypoints and the velocities. On the other hand, we do not admit pause times that the walker spends at waypoints.

Let us proceed with a general movement scheme. We equip every walker with a fixed communication radius $2R \in (0,\infty)$. That is, there is a direct connection between any two of them if their distance is at most $2R$. Two of the N participants, located at x and y, say, are (indirectly) connected if and only if there is a sequence $x_1,\dots,x_m$ of m other participants such that all the distances between x_i and x_{i-1} are at most $2R$ for any $i=1,\dots,m+1$, where we put $x_0=x$ and $x_{m+1}=y$. In other words, the $m+2$ balls around $x_0,\dots,x_{m+1}$ with radius R have pairwise a nontrivial intersection along the chain $x_0,\dots,x_{m+1}$; in particular, there is a continuous path from x to y within their union. This is fulfilled if and only if x and y lie in the same connected component of the union of the balls of radius R centred at the N participants. In this way, we see that our model is a dynamic continuum percolation process.

It is our goal to study the thermodynamic limit of this system, i.e., we think of the volume of the domain being of order N, the number of participants, and we assume that the trajectories are coupled with N in an accordingly rescaled way. That is, the length scale is $N^{1/d}$, and the density of participants (their number per unit volume) is of finite positive order. Then it is clear that a rescaled version of this picture is better suitable for a mathematical analysis. Hence, we consider instead the equivalent situation of a fixed domain D and a fixed movement scheme (both not depending on N), and we put the communication radius equal to $2RN^{-1/d}$. We do not rescale the time interval $[0,T]$ by $N^{1/d}$, as this is a trivial change.

By $X^{(i)}=(X^{(i)}_s)_{s\in[0,T]}$ we denote the (random) trajectory of the i-th participant, i.e., a random variable taking values in the set of functions $[0,T]\to D$. We assume that (making the underlying probability space Ω explicit) the map $(s,\omega)\mapsto X_s^{(i)}(\omega)$ is measurable from $[0,T]\times\Omega$ into D. Let $B(x,r)$ denote the open ball around x with radius $r>0$. Then the set

$$D_s^{(N)}=D\cap\bigcup_{i=1}^N B\big(X_s^{(i)},RN^{-1/d}\big)$$

is the *communication zone* at time s. We introduce the notion of connectivity at time s: for $x,y\in D$ we write

$$x\overset{N}{\underset{s}{\longleftrightarrow}}y\qquad\Longleftrightarrow\qquad x\text{ and }y\text{ lie in the same component of }D_s^{(N)}. \tag{1.1}$$

We will use this notion only for $x=X_s^{(1)}$ and $y=X_s^{(2)}$. Hence, the two participants $X^{(1)},X^{(2)}$ are connected at time s if there is a polygon line from $X_s^{(1)}$ to $X_s^{(2)}$ consisting of line segments of lengths at most $2RN^{-1/d}$ with the vertices being the locations of other participants at time s. Hence, $X_s^{(1)}\overset{N}{\underset{s}{\longleftrightarrow}}X_s^{(2)}$ if and only if these two can exchange a message at time s. Note that the indicator function $\mathbb{1}\{X_s^{(1)}(\omega)\overset{N}{\underset{s}{\longleftrightarrow}}X_s^{(2)}(\omega)\}$ is jointly measurable in s and ω, since it is a polynomial function of the indicators $\mathbb{1}\{|X_s^{(i)}(\omega)-X_s^{(j)}(\omega)|\le RN^{-1/d}\}$ with $i,j\in\{1,\dots,N\}$, which are jointly measurable in s and ω.

The main object is the *connection time*

$$\tau_T^{(N)}:=\big|\big\{s\in[0,T]\colon X_s^{(1)}\overset{N}{\underset{s}{\longleftrightarrow}}X_s^{(2)}\big\}\big|=\int_0^T\mathrm{d}s\,\mathbb{1}\{X_s^{(1)}\overset{N}{\underset{s}{\longleftrightarrow}}X_s^{(2)}\}, \tag{1.2}$$

the amount of time during which these two participants are connected up to T. By the above mentioned joint measurability of the integrand, is well-defined and measurable. We will analyse the connection time in the limit $N\to\infty$.

Let us state our assumptions on the random movement of the N walkers. We write $\{f>r\}$ for short for the set $\{x\in D\colon f(x)>r\}$ and use analogous notation for similarly defined sets.

Assumption 1.1 (The movement scheme)**.** The distribution of the random path $X=(X_s)_{s\in[0,T]}$ in D satisfies the following.

(i) For any $s\in(0,T]$, the location X_s possesses a continuous Lebesgue density $f_s\colon D\to[0,\infty)$.
(ii) For any $x,y\in D$ and $s,\widetilde{s}\in(0,T]$ satisfying $s<\widetilde{s}$, we have $\mathbb{P}(X_s=x\mid X_{\widetilde{s}}=y)=0$.

Sufficient for Assumption 1.1 is the existence of a jointly continuous Lebesgue density of X_s and $X_{\widetilde{s}}$ for any $0<s<\widetilde{s}\le T$. Condition (ii) is needed for the asymptotic independence of the clusters at time s from the clusters at time $\widetilde{s}$; it allows us to neglect those walkers that define both clusters and to deal only with disjoint sets of participants that form the two clusters, see the proof of Lemma 2.3. The reason why it is stated for $s<\widetilde{s}$ is that it makes the proof of Lemma 2.3 simpler to understand. It is however possible to adapt it with the same assumption for $s>\widetilde{s}$. We leave the details of this to the reader.

We also remark that the map $(s,x)\mapsto f_s(x)$ is measurable. Indeed, by measurability of the map $(s,\omega)\mapsto X_s(\omega)$, the indicator $\mathbb{1}\{|X_s(\omega)-x|\le\epsilon\}$ is (ω,s,x)-measurable for any $\epsilon>0$. Therefore, by Fubini's theorem, its expectation is (s,x)-measurable, and by continuity of f_s, we have $f_s(x)=\lim_{\epsilon\downarrow0}\mathbb{P}(|X_s-x|\le\epsilon)/(C\epsilon^d)$, C being the volume of the unit ball in $\mathbb{R}^d$, i.e., $f_s(x)$ is a limit of (s,x)-measurable functions.

doubt that such questions would be of great interest, they would however require another approach and are not the subject of our article.

Note that we do not require the continuity of the trajectories; regularity is only required for the distributions at fixed times. Assumption 1.1 is satisfied for many diffusions in D and also for many continuous-time random walks in D. For practical reasons, we are mainly interested in the random waypoint model, see below.

We need to introduce some standard objects from (static, homogeneous) continuum percolation; see [MR96] and Section 2.1 below for general background. Let $(Z_i)_{i\in\mathbb{N}}$ be a standard Poisson point process on $\mathbb{R}^d$ with intensity $\lambda \in (0,\infty)$. We define the *percolation probability* $\overline{\theta}(\lambda, R)$ as the probability that there is a path from $B(0,R)$ to infinity that never leaves the set $U_R = \bigcup_{i\in\mathbb{N}} B(Z_i, R)$. In other words, $\overline{\theta}(\lambda, R)$ is the probability that U_R has a connected component with infinite Lebesgue measure that intersects $B(0,R)$. Connected components will be also called *clusters* in the sequel. By rescaling, it is easy to see that $\overline{\theta}(\lambda,R) = \overline{\theta}(\lambda R^d, 1)$. Furthermore, it is known that the map $\lambda \mapsto \overline{\theta}(\lambda, R)$ is increasing and that there is a $\lambda_{\rm c}(R) > 0$ such that $\overline{\theta}(\lambda,R) = 0$ for $\lambda < \lambda_{\rm c}(R)$ and $\overline{\theta}(\lambda, R) > 0$ for $\lambda > \lambda_{\rm c}(R)$. It is known that $\overline{\theta}(\cdot, R)$ is continuous outside the critical point $\lambda_{\rm c}(R)$; the continuity in this point is not known, but strongly expected. Again by rescaling, $\lambda_{\rm c}(R) = R^{-d}\lambda_{\rm c}(1)$.

The function $\overline{\theta}$ will play a crucial rôle in the asymptotic description of our model. As we will see below, the number $\overline{\theta}(\lambda, R)$ describes, in our spatially rescaled picture, the probability that, locally, a given participant belongs to the infinitely large cluster and has therefore connection over a macroscopic part of the space.

We introduce two notions of (non-random) connectedness in the domain D as follows. By 'path' we mean a continuous polygon line in D with finitely many edges, whose vertices lie in $D \cap \mathbb{Q}^d$ (with possible exception of the first and last one). For $\diamond \in \{\geq, >\}$ and $x, y \in D$, we write

$$x \overset{\diamond}{\underset{s}{\longleftrightarrow}} y \qquad \Longleftrightarrow \qquad \text{there exists a path from } x \text{ to } y \text{ within } \{f_s \diamond \lambda_{\rm c}(R)\}\,.$$

Note that the map $(x,y,s) \mapsto \mathbb{1}\{x \overset{\diamond}{\underset{s}{\longleftrightarrow}} y\}$ is measurable, as $(s,x) \mapsto f_s(x)$ is and the notion of a path involves only countably many operations; recall that we assumed D to be regular.

Furthermore, we introduce two versions of a limiting value of $\tau_T^{(N)}$. For $\diamond \in \{\geq, >\}$, define

$$\tau_T^{(\diamond)}(X^{(1)}, X^{(2)}) = \int_0^T {\rm d}s\, \mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}\overline{\theta}^{(\diamond)}\big(f_s(X_s^{(1)}), R\big)\overline{\theta}^{(\diamond)}\big(f_s(X_s^{(2)}), R\big), \tag{1.3}$$

where $\overline{\theta}^{(>)}(\lambda, R) = \overline{\theta}(\lambda-, R) = \lim_{s\uparrow\lambda}\overline{\theta}(s,R)$ and $\overline{\theta}^{(\geq)}(\lambda,R) = \overline{\theta}(\lambda+,R) = \lim_{s\downarrow\lambda}\overline{\theta}(s,R)$ are the left- and right-continuous versions of $\overline{\theta}$. Recall that these two functions coincide at least everywhere outside the critical value $\lambda_{\rm c}(R)$. Note that $\tau_T^{(\diamond)}(X^{(1)}, X^{(2)})$ is well-defined and measurable by the measurability of all the θ-functions, the joint measurabilities of $f_s(x)$ in s and x and of $X_s^{(i)}(\omega)$ in s and ω and of $\mathbb{1}\{x \overset{\diamond}{\underset{s}{\longleftrightarrow}} y\}$ in x, y and s.

Our main result is the following.

Theorem 1.2. *Fix $T > 0$ and $R > 0$, and assume that the distributions of the N i.i.d. random movements $X^{(1)}, \dots, X^{(N)}$ satisfy Assumption 1.1. Then, for almost every paths $X^{(1)}, X^{(2)}$, we have, in probability with respect to $\mathbb{P}(\cdot \mid X^{(1)}, X^{(2)})$,*

$$\tau_T^{(>)}(X^{(1)}, X^{(2)}) \leq \liminf_{N\to\infty} \tau_T^{(N)} \leq \limsup_{N\to\infty} \tau_T^{(N)} \leq \tau_T^{(\geq)}(X^{(1)}, X^{(2)}). \tag{1.4}$$

For a proof of Theorem 1.2 see Section 2; for a discussion about whether or not the limit in (1.4) exists and how it behaves for large R, see Section 1.3.

The assertion in (1.4) shows that the connectivity of the medium that is built out of $X^{(1)}, X^{(2)}, \dots, X^{(N)}$ is fully determined by just two effects: a global, deterministic one (expressed by the indicator on the event $\{X_s^{(1)} \overset{\diamond}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$ in (1.3)) and a local, stochastic one (expressed by the two θ-terms). Indeed, the two walkers at time s are connected if and only if

- their positions $x = X_s^{(1)}$ and $y = X_s^{(2)}$ are connected by a deterministic path within the supercritical region, i.e., the set $\{f_s \diamond \lambda_c(R)\}$ (with $\diamond =\geq$ for an upper bound and $\diamond => $ for a lower bound) and
- both x and y belong locally to the giant component of the static continuum percolation process with density $f_s(x)$ and $f_s(y)$, respectively, and ball radius $RN^{-1/d}$ (note that these two events are asymptotically independent).

earlier time.

1.3. **Discussion.**

1.3.1. *Does the limit in* (1.4) *exist?* Certainly, one expects that, in many cases, $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ should coincide almost surely and in (1.4) one should have a limit. This is certainly true under many additional abstract conditions. However, it is difficult to give a satisfactory sufficient condition that is both reasonably general and reasonably explicit, and therefore we abstained from that. Let us indicate where the difficulties lie.

In order to ensure coincidence of $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$, one needs a condition that ensures that connection within $\{f_s \geq \lambda_c(R)\}$ implies connection within $\{f_s > \lambda_c(R)\}$ (at least for the sites $X_s^{(1)}$ and $X_s^{(2)}$ for almost all s) and another condition that ensures that the θ-terms in (1.3) coincide for $\diamond =>$ and $\diamond =\geq$, at least for almost all s.

Some sufficient conditions of the first type are certainly easy to check in many explicit situations, where the structure of the connectivity landscape given by the density f_s is easy to control. In general, difficulties can arise if, for s in some set with positive Lebesgue measure, some components of $\{f_s > \lambda_c(R)\}$ are separated from each other by a component of $\{f_s = \lambda_c(R)\}$ that has a complicated local structure. In dimension $d = 2$, e.g., a line with some fractal structure would pose such a question. In this case, it is unclear what local properties of the separation set would imply what connectivity probabilities of the corresponding percolation process. Finding clear criteria seems to be an open problem in the study of continuum percolation. We believe that, for related reasons, one can construct situations in which $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ do not coincide, the limit in (1.4) does not exist or is random.

Sufficient conditions of the second type are, in a sense, much easier to formulate, as the function $\overline{\theta}(\cdot, R)$ is known to be continuous outside the critical point $\lambda_c(R)$, [MR96, Theorem 3.9], and therefore only times s have to be considered such that both $X_s^{(1)}$ and $X_s^{(2)}$ lie in the set $\{f_s = \lambda_c(R)\}$. In fact, in dimension $d = 2$, continuity is also known in the critical point [MR96, Theorem 4.5], such that here the θ-terms do coincide for any s. But in general dimension, continuity in the critical point is unknown. Hence, in cases where the set $\{f_s = \lambda_c(R)\}$ has a positive Lebesgue measure (which can happen only for countably many values of R), there is a positive probability that one of the two walkers belongs to its interior for a positive portion of the time, and then the θ-terms may substantially differ.

1.3.2. *Behaviour of the limit in* (1.4) *for* $R \to \infty$. From a practical point of view, installing a MANET makes sense only if the degree of connectivity in the system can be guaranteed to be extremely high, at least with high probability. Hence, it is a major goal to find sufficient conditions for a large value (i.e., close to T) of the communication time. Making the communication radius R large is certainly such a criterion, but it is also important to know how strongly this parameter influences the connectivity. Based on Theorem 1.2, we want to illustrate some partial answer to this question, i.e., we want to comment on the behaviour of the asymptotic lower bound for the connection time, $\tau_T^{(>)}$.

This lower bound consists, for any time $s \in [0, T]$, of two components: The values of $\overline{\theta}$ in the two locations of the sample trajectories, and the decision whether or not they are globally connected through the super-critical area $\{f_s > R^{-d}\lambda_c(1)\}$. An important fact (see [P91], Corollary of Theorem

3) is that $\overline{\theta}(\lambda, R)$ converges super-exponentially quickly towards 1 for $R \to \infty$, more precisely, for any $\varepsilon > 0$ and some $C_\varepsilon > 0$,

$$\overline{\theta}(\lambda, R) \geq 1 - \mathrm{e}^{-\lambda R^d |B(0,2)|(1-\varepsilon)}, \qquad \lambda R^d \geq C_\varepsilon. \tag{1.5}$$

This shows that the 'bad' event of being not connected at a given time s does predominantly not come from the $\overline{\theta}$-term, but from the non-connectivity, i.e., from the indicator on the counterevent of $\{X_s^{(1)} \overset{>}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$. It is a natural assumption that the density f_s is, for every $s \in [0, T]$, bounded away from zero in most of the domain D, except possibly close to the boundary of D and that f_s decays polynomially towards the boundary of D. Then the difference $T - \tau_T^{(>)}$ can be upper bounded by some polynomially decaying term, which depends on the time that at least one of the two walkers spends polynomially close to the boundary, and some term of the form e^{-CR^d} for the remaining time. But the time that one of the walkers spends close to the boundary of D is polynomially small in R in probability, since the density is small there. The conclusion is that bad connectivity properties of the system predominantly come from the time that the users spend close to the boundary of D, at least if the domain is homogeneously filled with users.

1.4. **Further investigations for the random waypoint model.** Let us now concentrate on the random waypoint model, which was introduced at the beginning of Section 1.2. Below we show that, under suitable conditions, the RWP is amenable to Theorem 1.2, and we study the large-T average of the connection time and long-time deviations from the mean in terms of large-deviation estimates.

We have to introduce some notation. We assume that the domain D is compact and convex. Let $(W_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. points in D, drawn from a distribution $\mathcal{W}$ on D, the *waypoint measure*. Furthermore, let $(V_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of velocities drawn from some distribution $\mathcal{V}$ on $(0, \infty)$, the *velocity measure*. The walker starts from an initial location $X_0 \in D$, heading with constant initial velocity V_1 towards the waypoint W_1 on a straight line. Having arrived at W_1, the walker immediately moves along the straight line from W_1 to W_2 with velocity V_2 and so on.

This is an extension of the classical RWP, as we admit D as any convex compact domain, $\mathcal{W}$ as any distribution on D, and $\mathcal{V}$ as any distribution on $(0, \infty)$. On the other hand, we do not admit pause times that the walker spends at waypoints, as this would destroy the validity of Assumption 1.1(ii); in fact, also the statement of Theorem 1.2 would have to be altered.

We denote by $U_n = |W_{n+1} - W_n|/V_{n+1}$ the time that it takes the walker to go from the n-th to the $(n+1)$-th waypoint. Then $T_n = U_0 + U_1 + \dots + U_{n-1}$ is the time at which the walker arrives at the n-th waypoint, W_n. We put $T_0 = 0$. Introduce the time-change $N(t) = \inf\{n \in \mathbb{N} \colon T_n > t\}$, then $W_{N(t)}$ is the waypoint that the walker is heading to at time t, $V_{N(t)}$ is his current velocity, and $T_{N(t)} - t$ is the time difference after which he arrives there. The position of the walker at time t is denoted by X_t. Then

$$X_t = W_{N(t)} + \frac{W_{N(t)-1} - W_{N(t)}}{|W_{N(t)-1} - W_{N(t)}|} V_{N(t)}(T_{N(t)} - t). \tag{1.6}$$

We define all these processes as right-continuous. Note that the location process $X = (X_t)_{t\in[0,\infty)}$ is not Markov, but the process

$$Y = (Y_t)_{t\in[0,\infty)} = \Big(X_t, W_{N(t)}, V_{N(t)}\Big)_{t\in[0,\infty)} \tag{1.7}$$

is a continuous-time Markov process on the state space $\mathcal{D} = D \times D \times [v_-, v_+]$.

We need to assume some regularity. Throughout the paper, we assume that the waypoint measure $\mathcal{W}$ and the velocity measure $\mathcal{V}$ possess continuous Lebesgue densities on D and on some interval $[v_-, v_+] \subset (0, \infty)$, respectively. In particular, the velocities are bounded away from 0 and from ∞.

We now check that we can apply Theorem 1.2 to the RWP.

Lemma 1.3 (The RWP satisfies Assumption 1.1)**.** *We initialise the RWP by drawing $W_0 \in D$ and a velocity V_0 from some distributions on D respectively on $[v_-, v_+]$ having continuous densities, such that all the random variables W_0, W_1, V_0 are independent, and put $X_0 = W_0$ and X_t as in* (1.6)*. Then the RWP satisfies Assumption 1.1.*

Proof. We first show that Assumption 1.1(i) is satisfied. Indeed, fix $s \in (0, \infty)$ and note that, on the event $\{s \leq T_1\}$,

$$X_s = X_0 + sV_1 \frac{W_1 - W_0}{|W_1 - W_0|},$$

which has obviously a continuous density, since W_0, V_1 and W_1 have and are independent. On the event $\{T_j < s \leq T_{j+1}\}$ with $j \in \mathbb{N}$, we represent

$$X_s = W_j + (s - T_j)V_{j+1} \frac{W_{j+1} - W_j}{|W_{j+1} - W_j|},$$

which also has a continuous density, since W_j, V_{j+1} and W_{j+1} have and are independent (and T_j is a continuous function of them). Hence, $X_s \mathbb{1}\{T_j < s \leq T_{j+1}\}$ has a continuous density. Summing on $j \in \mathbb{N}_0$, we also see by use of Dini's theorem that also X_s has a continuous density.

Let us now verify Assumption 1.1(ii). For any $x \in D$, $\mathbb{P}(X_s = x | X_{\widetilde{s}} = y) = 0$ is clear on the event $\bigcup_{j\in\mathbb{N}}\{s \leq T_j < \widetilde{s}\}$, since there was a change of direction between time s and $\widetilde{s}$. On the counterevent, $\bigcup_{j\in\mathbb{N}_0}\{T_j < s < \widetilde{s} \leq T_{j+1}\}$, we have

$$\begin{aligned}\mathbb{P}(X_s = x | X_{\widetilde{s}} = y) &= \mathbb{P}\Big(V_{j+1} = \frac{|X_{\widetilde{s}} - x|}{\widetilde{s} - s}, \ \frac{W_{j+1} - W_j}{|W_{j+1} - W_j|} = \frac{X_{\widetilde{s}} - x}{|X_{\widetilde{s}} - x|} \Big| X_{\widetilde{s}} = y\Big) \\ &\leq \mathbb{P}\Big(V_{j+1} = \frac{|y - x|}{\widetilde{s} - s} \Big| X_{\widetilde{s}} = y\Big) = 0\end{aligned}$$

because the speed is independent from the location and has a continuous density. □

1.4.1. *Long-time limit.* Let us consider the long-time behaviour of $\tau_T^{(\diamond)} = \tau_T^{(\diamond)}(X^{(1)}, X^{(2)})$ defined in (1.3) for $\diamond \in \{>, \geq\}$ for the RWP. We will show in Section 3.1 that the RWP is Harris ergodic and in particular possesses an invariant distribution, towards which it converges as the time grows to infinity. In particular, the distribution of the location of the RWP, X_t, converges in total variation sense towards a probability measure μ_* on D, and it has a continuous Lebesgue density $f_* \colon D \to [0, \infty)$. However, it is not so easy to deduce convergence of $\frac{1}{T}\tau_T^{(\diamond)}$ from this, and we are not able to do so in all cases. For $\diamond \in \{>, \geq\}$, introduce

$$p_*^{(\diamond)} = \int_D \mu_*(\mathrm{d}x) \int_D \mu_*(\mathrm{d}y) \, \mathbb{1}\{x \overset{\diamond}{\underset{*}{\longleftrightarrow}} y\} \overline{\theta}^{(\diamond)}(f_*(x), R) \overline{\theta}^{(\diamond)}(f_*(y), R) \in [0, 1], \tag{1.8}$$

where $\overset{\diamond}{\underset{*}{\longleftrightarrow}}$ denotes connectedness within the set $\{f_* \diamond \lambda_{\mathrm{c}}(R)\}$. Then $p_*^{(\diamond)}$ is a measure for connectedness of two independent sites in D drawn from the limiting distribution of X_t. Furthermore, introduce

$$\tau_T^{(\diamond,*)} = \int_0^T \mathrm{d}s \, \mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{*}{\longleftrightarrow}} X_s^{(2)}\} \overline{\theta}^{(\diamond)}\big(f_*(X_s^{(1)}), R\big) \overline{\theta}^{(\diamond)}\big(f_*(X_s^{(2)}), R\big), \tag{1.9}$$

the special case of $\tau_T^{(\diamond)}$ for all the random waypoint walkers starting in the invariant distribution.

Lemma 1.4 (Ergodic limit)**.** *Let $X^{(1)}$ and $X^{(2)}$ be two independent copies of X. Then for $\diamond \in \{>, \geq\}$,*

$$\lim_{T\to\infty} \frac{1}{T} \tau_T^{(\diamond,*)}(X^{(1)}, X^{(2)}) = p_*^{(\diamond)}, \qquad \textit{almost surely and in } L^1(\mathbb{P}), \tag{1.10}$$

We will give a proof of this lemma in Section 3.3; it is based on a time-discrete Markov chain that is introduced in Section 1.4.2.

Remark 1.5. *The previous result is stated with the trajectories of the walkers started from the invariant state. In general, it is not clear if* $\frac{1}{T}\tau_T^{(\diamond)}$ *converges towards* $p_*^{(\diamond)}$*. Indeed, the critical point is the convergence of* $\mathbb{1}\{x \overset{\diamond}{\underset{s}{\longleftrightarrow}} y\}$ *towards* $\mathbb{1}\{x \overset{\diamond}{\underset{*}{\longleftrightarrow}} y\}$ *for* $x, y \in D$ *as* $s \to \infty$*, which is not true in many counterexamples, as one can easily find. However one can check that, under the additional assumption that* $f_s \to f_*$ *as* $s \to \infty$ *uniformly in* D*, then, in probability,*

$$\limsup_{T\to\infty} \frac{1}{T}\tau_T^{(\geq)}(X^{(1)}, X^{(2)}) \leq p_*^{(\geq)}, \qquad \text{and} \qquad \liminf_{T\to\infty} \frac{1}{T}\tau_T^{(>)}(X^{(1)}, X^{(2)}) \geq p_*^{(>)}. \tag{1.11}$$

We remark here that, in cases where the limit in (1.4) exists, we expect that the limits $T \to \infty$ and $N \to \infty$ can also be interchanged without changing the value, i.e.,

$$p_*^{(>)} = \lim_{N\to\infty} \lim_{T\to\infty} \frac{1}{T}\tau_T^{(N)}.$$

Indeed, in the limit $T \to \infty$, the ergodic theorem leads to the average connection probability for two out of N i.i.d. sites drawn from the invariant distribution, and then the identification of the limit $N \to \infty$ follows from Theorem 1.2, applied to the RWP starting in the invariant distribution. We decided to leave the details of the proof to the reader.

1.4.2. *Large-T deviations.* In our next result, we describe the downward deviations of $\tau_T^{(>,*)}(X^{(1)}, X^{(2)})$, more precisely, the probability of the event $\{\tau_T^{(>,*)} \leq Tp\}$ for $p \in (0, p_*^{(>)})$, in the limit $T \to \infty$. This is certainly an interesting question, since one would like to effectively bound the probability of the unwanted event of being connected over less than the average portion in the long-time limit. We show that this probability decays even exponentially fast, and we give an explicit bound for the decay rate. Because of (1.4), such a bound for $\tau_T^{(>,*)}$ (rather than for $\tau_T^{(\geq,*)}$) gives a useful upper deviation bound for $\tau_T^{(N)}$. We write $\mathbb{P}_*$ for the probability measure of the RWP if both copies $Y^{(1)}$ and $Y^{(2)}$ start from the invariant distribution.

Theorem 1.6. *For any* $p \in (0, p_*^{(>)})$,

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}_*(\tau_T^{(>,*)} \leq Tp) < 0. \tag{1.12}$$

The proof of Theorem 1.6 is in Section 3.4. It describes an explicit upper bound for the left-hand side of (1.12) in terms of a variational problem. The main novelty lies in the proof, which describes the probability in question in terms of an interesting Markov chain with nice properties, such that the theory of large deviations may be applied in a standard way. This Markov chain is an object of independent interest, as it may serve also for other long-time investigations of the model, as well as for computer simulations.

2. Proof of Theorem 1.2

In this section, we prove our first main result, Theorem 1.2. As a preparation, we first summarise in Section 2.1 all relevant available information about continuum percolation. In Section 2.2 we find the limit of the expectation of the connection time, and in Section 2.3 we finish the proof.

2.1. **Static continuum percolation.** Let us collect some facts from (static) continuum percolation, see [MR96] or [P03]. Throughout the paper we assume that $d \geq 2$. Let $(Z_i)_{i\in\mathbb{N}}$ be a Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$. Fix a radius $R > 0$ and consider the union U_R of the balls $B(Z_i, R)$ over $i \in \mathbb{N}$. We say that two sites $x, y \in \mathbb{R}^d$ are *connected* if they belong to the same connected component of U_R. Connected components of U_R are called *clusters*. By $\mathcal{C}(x)$ we denote the cluster that contains $x \in \mathbb{R}^d$. The *percolation probability* $\theta(\lambda, R)$ is defined as the probability that $\mathcal{C}(0)$ has infinite Lebesgue measure, which we phrase that 0 is connected with ∞. By scaling,

$\theta(\lambda, R) = \theta(\lambda R^d, 1)$. There is a *critical threshold* $\lambda_c(R) = R^{-d}\lambda_c(1)$, defined by $\theta(\lambda, R)$ being 0 for $\lambda < \lambda_c(R)$ and positive for $\lambda > \lambda_c(R)$. Another characterisation of the critical threshold is that $|\mathcal{C}(0)| = \infty$ with positive probability for $\lambda > \lambda_c(R)$ and $|\mathcal{C}(0)| < \infty$ with probability 1 for $\lambda < \lambda_c(R)$. In the supercritical case, there exists, with probability one, a unique cluster with infinite Lebesgue measure, which we call $\mathcal{C}_\infty$. In the subcritical case, there is no cluster with infinite Lebesgue measure, almost surely, and the random variable $|\mathcal{C}(0)|$ has finite exponential moments. The map $\lambda \mapsto \theta(\lambda, R)$ is continuous in any point, with a possible exception at the critical point, $\lambda_c(R)$ [S97, Theorem 1.1]. The continuity at the critical point is an open question, but is widely conjectured to be true. For numerical estimations we refer to [QZ07].

Actually, it is not θ that we will work with in our model, for the following reason. Certainly, the points Z_i play the rôle of the locations of the participants in our telecommunication system. It will turn out that a given participant located at Z_i is well connected with the main part of the system if $B(Z_i, R)$ has a non-trivial intersection with $\mathcal{C}_\infty$; it is not necessary that Z_i itself belongs to $\mathcal{C}_\infty$. Hence, we will be working with a slightly different notion of percolation: Define $\overline{\theta}(\lambda, R)$ as the probability that the ball $B(0, R)$ is connected with ∞, i.e., has a non-trivial intersection with $\mathcal{C}_\infty$. Obviously, $\theta \leq \overline{\theta}$, and $\overline{\theta}$ shares the above mentioned properties with θ, however with possibly different numerical values. In particular, $\overline{\theta}$ possesses the same scaling properties, and is an increasing function of λ, and is positive above some threshold and zero below. One can also easily check that the percolation threshold is the same for the two definitions, and that the proof of the continuity for the usual definition extends to this definition.

2.2. **Limiting expectation of the connection time.** We fix $T > 0$ for the remainder of the section. In the following, we abbreviate

$$\mathbb{P}_{1,2}(\cdot) = \mathbb{P}\big(\cdot \,|\, X^{(1)}, X^{(2)}\big) \qquad \text{and} \qquad \mathbb{E}_{1,2}[\,\cdot\,] = \mathbb{E}\big[\cdot \,|\, X^{(1)}, X^{(2)}\big].$$

Use (1.2) and Fubini's theorem to see that

$$\mathbb{E}_{1,2}[\tau_T^{(N)}] = \int_0^T \mathrm{d}s\, \mathbb{P}_{1,2}\Big(X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\Big).$$

We are going to approximate the event $\{X_s^{(1)} \overset{N}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\}$ by the event that $X_s^{(1)}$ and $X_s^{(2)}$ are separated from each other, but connected through either $\{f_s > \lambda_c(R)\}$ or through $\{f_s \geq \lambda_c(R)\}$ and belong locally to the macroscopic part of the communication zone. More precisely, for $s \in [0, T]$, $\delta > 0$ and $N \in \mathbb{N}$, we introduce the events

$$G_{N,s,\delta}^{(i)} = \Big\{X_s^{(i)} \overset{N}{\underset{s}{\longleftrightarrow}} \partial\big[X_s^{(i)} + (-\delta/2, \delta/2)^d\big]\Big\}, \qquad i \in \{1, 2\},$$

that $X_s^{(i)}$ and at least some point of the boundary of the $\delta/2$-box around $X_s^{(i)}$ lie in the same connected component of the union of the $RN^{-1/d}$-balls around $X_s^{(1)}, \dots, X_s^{(N)}$.

Note that $G_{N,s,\delta}^{(i)}$ only depends on the walkers within the δ-box around $X_s^{(i)}$.

We will give bounds for the connection time $\tau_T^{(N)}$ in terms of

$$\tau_T^{(N,\delta,\diamond)}(X^{(1)}, X^{(2)}) = \int_0^T \mathrm{d}s \prod_{i=1}^2 \mathbb{1}_{G_{N,s,\delta}^{(i)}} \mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| \geq 3\delta\} \mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\},$$

in the limit $N \to \infty$, followed by $\delta \downarrow 0$. We will use $\tau_T^{(N,\delta,>)}$ as a lower bound and $\tau_T^{(N,\delta,\geq)}$ as an upper bound for $\tau_T^{(N)}$. Recall the quantities $\tau_T^{(\diamond)}$ defined in (1.3), which will serve as limiting objects of $\tau_T^{(N,\delta,\diamond)}$.

Proposition 2.1 (Limiting expectation of $\tau_T^{(N)}$)**.** *Let the distributions of the N i.i.d. walkers satisfy Assumption 1.1(i). Then, for $\mathbb{P}$-almost all $X^{(1)}$ and $X^{(2)}$, provided that R is chosen such that $\int_0^t \mathrm{d}s\, \mathbb{1}\{f_s(X_s^{(i)}) = \lambda_c(R)\} = 0$, for $i = 1, 2$,*

(i)

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{E}_{1,2}\big(\tau_T^{(N)}-\tau_T^{(N,\delta,\geq)}\big)^+ \;=\; 0, \tag{2.1}$$

$$\liminf_{\delta\downarrow 0}\liminf_{N\to\infty}\mathbb{E}_{1,2}\big(\tau_T^{(N)}-\tau_T^{(N,\delta,>)}\big)^- \;=\; 0. \tag{2.2}$$

(ii) *For any* $\diamond\in\{>,\geq\}$,

$$\lim_{\delta\downarrow 0}\lim_{N\to\infty}\mathbb{E}_{1,2}[\tau_T^{(N,\delta,\diamond)}]=\tau_T^{(\diamond)}(X^{(1)},X^{(2)}). \tag{2.3}$$

The main step in the proof is the following.

Lemma 2.2. *Let the distributions of the N i.i.d. walkers satisfy Assumption 1.1(i). Then, for $\mathbb{P}$-almost all $X^{(1)}$ and $X^{(2)}$, for almost any $s\in[0,T]$ and on the event $\{f_s(X_s^{(1)})\neq\lambda_{\rm c}(R)\}\cap\{f_s(X_s^{(2)})\neq\lambda_{\rm c}(R)\}\cap\{X_s^{(1)}\neq X_s^{(2)}\}$,*

(i)

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}_{1,2}\Big[\Big(X_s^{(1)}\xleftrightarrow[s]{N}X_s^{(2)}\Big)\Big\backslash\Big(G_{N,s,\delta}^{(1)}\cap G_{N,s,\delta}^{(2)}\cap\{X_s^{(1)}\xleftrightarrow[s]{\geq}X_s^{(2)}\}\Big)\Big] \;=\; 0, \tag{2.4}$$

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}_{1,2}\Big[\Big(G_{N,s,\delta}^{(1)}\cap G_{N,s,\delta}^{(2)}\cap\{X_s^{(1)}\xleftrightarrow[s]{>}X_s^{(2)}\}\Big)\Big\backslash\Big(X_s^{(1)}\xleftrightarrow[s]{N}X_s^{(2)}\Big)\Big] \;=\; 0. \tag{2.5}$$

(ii)

$$\begin{aligned}\overline{\theta}(f_s(X_s^{(1)})-,R)\overline{\theta}(f_s(X_s^{(2)})-,R)&\leq\liminf_{\delta\downarrow 0}\liminf_{N\to\infty}\mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)}\cap G_{N,s,\delta}^{(2)}\Big)\\&\leq\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)}\cap G_{N,s,\delta}^{(2)}\Big)\\&\leq\overline{\theta}(f_s(X_s^{(1)})+,R)\overline{\theta}(f_s(X_s^{(2)})+,R).\end{aligned} \tag{2.6}$$

Proof. Fix s and let us abbreviate $x=X_s^{(1)}$ and $y=X_s^{(2)}$. Under $\mathbb{P}_{1,2}$, only the sites $X_s^{(3)},\dots,X_s^{(N)}$ are random (in fact, they are i.i.d. with density f_s), but the notion of connectedness and components induced by the point process refer to *all* the balls $B(X_s^{(i)},RN^{-1/d})$ with $i=1,2,\dots,N$.

Let us prove (ii). First we consider the case that $f_s(x)<\lambda_{\rm c}(R)$ or $f_s(y)<\lambda_{\rm c}(R)$, in which case the events $\{X_s^{(1)}\xleftrightarrow[s]{>}X_s^{(2)}\}$ and $\{X_s^{(1)}\xleftrightarrow[s]{\geq}X_s^{(2)}\}$ are not fulfilled. Without loss of generality, let us assume that $f_s(x)<\lambda_{\rm c}(R)$. Choose $\delta>0$ so small that the δ-box around x does not contain y and that $f_s<\lambda_{\rm c}(R)$ within that box. We apply [P95, Proposition 2] for $\varepsilon=\delta/4$ and obtain that, with $\mathbb{P}_{1,2}$-probability tending to 1 as $N\to\infty$, any connected component of $\bigcup_{i=3}^N B(X_s^{(i)},RN^{-1/d})$ in this cube has a diameter bounded from above by ε. In particular, with $\mathbb{P}_{1,2}$-probability tending to 1, x is not connected with the boundary of the cube $x+(-\delta,\delta)^d$. Therefore (2.6) is trivial, as all terms are zero.

To prove (2.6) in the remaining case $f_s(x)\geq\lambda_{\rm c}(R)$ and $f_s(y)\geq\lambda_{\rm c}(R)$, we show now that the two events $G_{N,s,\delta}^{(1)}$ and $G_{N,s,\delta}^{(2)}$ are asymptotically independent with $\mathbb{P}_{1,2}$-probabilities tending to $\overline{\theta}(f_s(x),R)$ and $\overline{\theta}(f_s(y),R)$, respectively. Let μ_s denote the measure with density f_s. Indeed, first note that, for every sufficiently large N such that the ball diameter $2RN^{-1/d}$ is less than the distance between $x+(-\delta,\delta)^d$ and $y+(-\delta,\delta)^d$. Hence, the positions of the points falling in $x+(-\delta,\delta)^d$ and $y+(-\delta,\delta)^d$ are independent, conditionally on their numbers. These two numbers are binomially distributed with parameters N and $\mu_s(x+(-\delta,\delta)^d))$ and $\mu_s(y+(-\delta,\delta)^d)$, respectively. Therefore, by the law of large numbers, they stochastically dominate, with $\mathbb{P}_{1,2}$-probability tending to 1, the Poisson law with parameters $N(\mu_s(x+(-\delta,\delta)^d)-\eta(2\delta)^d)$ and $N(\mu_s(y+(-\delta,\delta)^d)-\eta(2\delta)^d)$, respectively, for any $\eta>0$. Note that the events $G_{N,s,\delta}^{(1)}$ and $G_{N,s,\delta}^{(2)}$ are monotonic in the intensity, i.e., their $\mathbb{P}_{1,2}$-probability is not larger than the $\mathbb{P}_{1,2}$-probability of the same event under continuum percolation in $x+(-\delta,\delta)^d$ and $y+(-\delta,\delta)^d$ with intensity parameters $f_s(x)-2\eta$ and $f_s(y)-2\eta$, respectively, and ball diameter $RN^{-1/d}$. Since we are now considering Poisson point processes, the events are independent. Their respective probabilities converge towards $\overline{\theta}(f_s(x)-2\eta,R)$ and $\overline{\theta}(f_s(y)-2\eta,R)$. Since this is true for any η, we can use the continuity of $\overline{\theta}(\cdot,R)$, to obtain the lower bound in (2.6). The upper bound is proved in a similar manner, using that $\overline{\theta}(\lambda)$ is the limiting probability that the origin is connected with the boundary of a centred cube for diverging radius. This finishes the proof of (ii).

In order to show (i), we are going to decompose into four separate cases. First we consider the case that $f_s(x)<\lambda_c(R)$ or $f_s(y)<\lambda_c(R)$. As before, let us assume that $f_s(x)<\lambda_c(R)$. With $\mathbb{P}_{1,2}$-probability tending to 1, x is not connected with the boundary of the cube $x+(-\delta,\delta)^d$ and therefore neither with y, by the previous argument. This proves (2.4) and (2.5) in this case.

In the second part of the proof, we assume that x and y belong to the same component of $\{f_s>\lambda_c(R)\}$, in which case both events $\{X_s^{(1)}\overset{>}{\underset{s}{\longleftrightarrow}}X_s^{(2)}\}$ and $\{X_s^{(1)}\overset{\geq}{\underset{s}{\longleftrightarrow}}X_s^{(2)}\}$ are fulfilled. Pick some auxiliary parameter $\eta>0$ that is smaller than $f_s(x)-\lambda_c(R)$ and smaller than $f_s(y)-\lambda_c(R)$. Now, using the continuity of f_s in accordance with Assumption 1.1(i), pick $\delta>0$ so small that $x+(-\delta,\delta)^d$ and $y+(-\delta,\delta)^d$ have positive distance and that f_s takes values in $[f_s(x)-\eta,f_s(x)+\eta]$ in $x+(-\delta,\delta)^d$ and values in $[f_s(y)-\eta,f_s(y)+\eta]$ in $y+(-\delta,\delta)^d$ and such that there exists a set of the form $U=\bigcup_{i=0}^m 2\delta z_i+[-\delta,\delta]^d$ in $\{f_s>\lambda_c(R)\}$ with $m\in\mathbb{N}$, $z_1,\dots,z_m\in\mathbb{Z}^d$ such that z_i and z_{i-1} are nearest neighbours for any $i=1,\dots,m$ and $x+(-\delta,\delta)^d\subset U$ and $y+(-\delta,\delta)^d\subset U$ and $f_s>\lambda_c(R)$ inside U. That this is possible is easy to see by elementary continuity and compactness arguments. Since U is a compact subset of $\{f_s>\lambda_c(R)\}$, the density f_s is even bounded away from $\lambda_c(R)$ on U.

Let $\mathcal{C}_{x,\delta}^{(s,N)}$ and $\mathcal{C}_{y,\delta}^{(s,N)}$, respectively, denote the largest component of the union of the $RN^{-1/d}$-balls around the points $X_s^{(1)},\dots,X_s^{(N)}$ which lie in $x+(-\delta,\delta)^d$, respectively in $y+(-\delta,\delta)^d$. According to [P95, Proposition 3], with $\mathbb{P}_{1,2}$-probability tending to 1 as $N\to\infty$, these are the only ones in the respective boxes whose size (measured in terms of the number of i such that $X_s^{(i)}$ belongs to it) is of order N, and they are also uniquely determined by requiring their diameter of positive order. In particular, as $N\to\infty$, the probability of the symmetric difference between the events $\{x\in\mathcal{C}_{x,\delta}^{(s,N)}\}$ and $G_{N,s,\delta}^{(1)}$ (respectively $\{y\in\mathcal{C}_{y,\delta}^{(s,N)}\}$ and $G_{N,s,\delta}^{(2)}$) goes to zero. By [P95, Proposition 4], such a unique cluster, $\mathcal{C}_U^{(s,N)}$ also exists for the set U. Hence, with $\mathbb{P}_{1,2}$-probability tending to 1, both $\mathcal{C}_{x,\delta}^{(s,N)}$ and $\mathcal{C}_{y,\delta}^{(s,N)}$ belong to $\mathcal{C}_U^{(s,N)}$. This implies that with probability tending to 1 as $N\to\infty$, the symmetric difference between the event $\{x\overset{N}{\underset{s}{\longleftrightarrow}}y\}$ and the event $G_{N,s,\delta}^{(1)}\cap G_{N,s,\delta}^{(2)}$ goes to zero, which implies (2.5) and (2.4).

In the third case, we have $f_s(x)>\lambda_c(R)$ and $f_s(y)>\lambda_c(R)$, and $x\overset{\geq}{\underset{s}{\longleftrightarrow}}y$, but not $x\overset{>}{\underset{s}{\longleftrightarrow}}y$, in which case (2.5) is trivial, as the event inside the probability is empty. To prove (2.4), it is enough to see that, deterministically, the existence of a path between x and y implies $G_{N,s,\delta}^{(1)}$ and $G_{N,s,\delta}^{(2)}$.

In the fourth case, we have $f_s(x)>\lambda_c(R)$ and $f_s(y)>\lambda_c(R)$, but not $x\overset{\geq}{\underset{s}{\longleftrightarrow}}y$. Here, (2.5) is again trivial, as the event inside the probability is empty. To prove (2.4) it is enough to check that, with probability tending to 1, x and y are not connected in the union of the $RN^{-1/d}$-balls around the points $X_s^{(1)},\dots,X_s^{(N)}$. Here it is intuitively clear that any path between x and y has to cross a non-trivial zone where $f_s<\lambda_c(R)$ and that this disconnects x and y in the limit. Let us give a proof.

First we argue that there is a (deterministic) compact set $\Gamma\subset D$ and $\varepsilon,\gamma>0$ such that $\Gamma\subset\{f_s\leq\lambda_c(R)-\varepsilon\}$ and every path connecting x and y passes through Γ for at least γ space units. Indeed, since $x\overset{\geq}{\underset{s}{\longleftrightarrow}}y$ does not hold, x and y lie in disjoint components of $\{f_s\geq\lambda_c(R)\}$. Hence, both these components have a positive distance η to the remainder of $\{f_s\geq\lambda_c(R)\}$, since these three sets are compact and mutually disjoint. Abbreviate

$$\Gamma_\alpha = \{z \in D\colon \mathrm{dist}(z, \{f_s \geq \lambda_{\mathrm{c}}(R)\}) \geq \alpha\}, \qquad \alpha > 0,$$

and pick $\Gamma = \Gamma_{\eta/16}$. Then every path from x to y passes at least a distance $\gamma = \eta - 2\eta/16 = 7\eta/8$ through Γ. By continuity of f_s, this set Γ is compact and is contained in $\{f_s \leq \lambda_{\mathrm{c}}(R) - \varepsilon\}$ for some $\varepsilon > 0$.

Second, we argue that, with $\mathbb{P}_{1,2}$-probability tending to 1 as $N \to \infty$, any connected component of $\bigcup_{i=3}^N B(X_s^{(i)}, RN^{-1/d})$ in Γ has a diameter at most $\gamma/2$. Indeed, consider the neighbourhood $\widetilde{\Gamma} = \Gamma_{\eta/32}$ of Γ, then, for N sufficiently large, the connected components inside Γ do not depend on the configuration outside $\widetilde{\Gamma}$. By continuity of f_s, on $\widetilde{\Gamma}$, the function f_s is still bounded away from $\lambda_{\mathrm{c}}(R)$, say it is bounded from above by $\lambda_{\mathrm{c}}(R) - \widetilde{\varepsilon}$ for some $\widetilde{\varepsilon} > 0$. We upper bound the probability of having any connected component inside $\widetilde{\Gamma}$ of diameter bigger than $\gamma/2$ against the same probability under the homogeneous Poisson point process with intensity parameter $\lambda_{\mathrm{c}}(R) - \widetilde{\varepsilon}/2$ on some cube that contains $\widetilde{\Gamma}$ (see the above argument). Now, as this intensity parameter is subcritical, this probability tends to 0 as $N \to \infty$.

Now we finish the proof of (2.4) and (2.5) in the fourth case. Indeed, the existence of a connection from x to y through $\bigcup_{i=1}^N B(X_s^{(i)}, RN^{-1/d})$ implies the existence of at least one connected component of this set in Γ of diameter at least γ, since any path from x to y passes at least a distance γ through Γ. But, as we saw in the second step, the probability of this existence tends to 0 as $N \to \infty$.

□

Proof of Proposition 2.1. Observe that

$$\begin{aligned}
&\mathbb{E}_{1,2}\big(\tau_T^{(N)} - \tau_T^{(N,\delta,\geq)}\big)^+ \\
&\leq \int_0^T \mathrm{d}s\,\Big(\mathbb{P}_{1,2}\Big[\Big(X_s^{(1)} \xleftrightarrow[s]{N} X_s^{(2)}\Big)\Big\backslash\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)} \cap \{X_s^{(1)} \xleftrightarrow[s]{\geq} X_s^{(2)}\}\Big)\Big]\mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| > 3\delta\} \\
&\qquad \cdot \mathbb{1}\{f_s(X_s^{(1)}) \neq \lambda_{\mathrm{c}}(R)\}\mathbb{1}\{f_s(X_s^{(2)}) \neq \lambda_{\mathrm{c}}(R)\} \\
&\qquad + \mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| < 3\delta\} + \mathbb{1}\{f_s(X_s^{(1)}) = \lambda_{\mathrm{c}}(R)\} + \mathbb{1}\{f_s(X_s^{(2)}) = \lambda_{\mathrm{c}}(R)\}\Big).
\end{aligned}$$

Hence, by (2.4),

$$\limsup_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{E}_{1,2}\big(\tau_T^{(N)} - \tau_T^{(N,\delta,\geq)}\big) \leq \int_0^T \mathrm{d}s\,\mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| = 0\},$$

according to our assumption on R. Note that, almost surely, $\int_0^T \mathrm{d}s\,\mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| = 0\} = 0$, since $X_s^{(1)}$ and $X_s^{(2)}$ are independent with density f_s for any $s \in [0,T]$. Hence, the proof of (2.1) is finished. The proof of (2.2) is done in the same way using (2.5). Hence, part (i) is proved.

Now we turn to the proof of (ii).

Note that our assumptions exclude that $f_s(X_s^{(i)}) = \lambda_{\mathrm{c}}(R)$ outside a set of measure zero. Therefore this does not appear in the integral. Furthermore, $\overline{\theta}$ is continuous except maybe for $\lambda_{\mathrm{c}}(R)$. Therefore for almost every s, (2.6) reformulates to

$$\lim_{\delta\downarrow 0}\liminf_{N\to\infty}\mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big) = \lim_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big) = \overline{\theta}(f_s(X_s^{(1)}), R)\overline{\theta}(f_s(X_s^{(2)}), R).$$

Thus (ii) follows by Lebesgue's theorem. □

2.3. **Finish of the proof.** The second main step in proving Theorem 1.2 is the following lemma. Recall that $\mathbb{P}_{1,2}$ denotes the conditional distribution given $X^{(1)}$ and $X^{(2)}$.

Lemma 2.3 ($\tau_T^{(N,\delta,\diamond)}$ is asymptotically deterministic). *Let the distributions of the N i.i.d. walkers satisfy Assumption 1.1(i) and (ii). Then, for any $\diamond \in \{>, \geq\}$, for almost every paths $X^{(1)}, X^{(2)}$, the difference $\tau_T^{(N,\delta,\diamond)} - \mathbb{E}_{1,2}[\tau_T^{(N,\delta,\diamond)}]$ vanishes as $N \to \infty$, followed by $\delta \downarrow 0$, in $\mathbb{P}_{1,2}$-probability, provided that R is chosen such that $\int_0^T \mathrm{d}s\, \mathbb{1}\{f_s(X_s^{(i)}) = \lambda_{\mathrm{c}}(R)\} = 0$ for $i = 1, 2$.*

Proof. The claimed convergence follows, by Chebyshev's inequality, from the fact that the $\mathbb{P}_{1,2}$-variance of $\tau_T^{(N,\delta,\diamond)}$ vanishes. Writing $\mathbb{V}_{1,2}$ for the $\mathbb{P}_{1,2}$-variance, this is equal to

$$\begin{aligned}\mathbb{V}_{1,2}(\tau_T^{(N,\delta,\diamond)}) = \int_0^T \mathrm{d}s \int_0^T \mathrm{d}\widetilde{s}\, &\mathbb{1}\{|X_s^{(1)} - X_s^{(2)}| > 3\delta\}\mathbb{1}\{X_s^{(1)} \overset{\diamond}{\underset{s}{\longleftrightarrow}} X_s^{(2)}\} \\ &\cdot \mathbb{1}\{|X_{\widetilde{s}}^{(1)} - X_{\widetilde{s}}^{(2)}| > 3\delta\}\mathbb{1}\{X_{\widetilde{s}}^{(1)} \overset{\diamond}{\underset{\widetilde{s}}{\longleftrightarrow}} X_{\widetilde{s}}^{(2)}\} \\ \cdot \Big[\mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)} \cap G_{N,\widetilde{s},\delta}^{(1)} \cap G_{N,\widetilde{s},\delta}^{(2)}\Big) &- \mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big)\mathbb{P}_{1,2}\Big(G_{N,\widetilde{s},\delta}^{(1)} \cap G_{N,\widetilde{s},\delta}^{(2)}\Big)\Big].\end{aligned} \tag{2.7}$$

We now show, for any $s \neq \widetilde{s}$, that the limit superior of the term in the last line is not positive. This finishes the proof by Lebesgue's theorem.

We abbreviate $x = X_s^{(1)}$ and $\widetilde{x} = X_{\widetilde{s}}^{(1)}$ and $y = X_s^{(2)}$ and $\widetilde{y} = X_{\widetilde{s}}^{(2)}$. Without loss of generality, we assume that $s < \widetilde{s}$, $x \neq y$ and $\widetilde{x} \neq \widetilde{y}$. Furthermore we also may and will assume that $x \overset{\geq}{\underset{s}{\longleftrightarrow}} y$ and $\widetilde{x} \overset{\geq}{\underset{\widetilde{s}}{\longleftrightarrow}} \widetilde{y}$. Without loss of generality, all the four terms $f_s(x), f_s(y), f_{\widetilde{s}}(\widetilde{x})$ and $f_{\widetilde{s}}(\widetilde{y})$ are larger than $\lambda_{\mathrm{c}}(R)$. Let, as in the proof of Lemma 2.2, $\mathcal{C}_{x,\delta}^{(s,N)}$ denote the biggest component of the union of the $RN^{-1/d}$-balls around $X_s^{(1)}, X_s^{(2)}, \dots, X_s^{(N)}$ within $x + (-\delta, \delta)^d$, analogously for $y, \widetilde{s}, \widetilde{x}$ and $\widetilde{y}$.

We recall from the proof of Lemma 2.2 that the probability of the symmetric difference between $G_{N,t,\delta}^{(i)}$ and the event $\{X_t^{(i)} \in \mathcal{C}_{X_t^{(i)},\delta}^{(t,N)}\}$, $i = 1, 2$ and $t = s, \widetilde{s}$, tends to 0 as N goes to infinity, followed by $\delta \downarrow 0$. This reduces the problem to showing that

$$\begin{aligned}\limsup_{\delta\downarrow 0}\limsup_{N\to\infty} \Big[&\mathbb{P}_{1,2}\big(x \in \mathcal{C}_{x,\delta}^{(s,N)}, y \in \mathcal{C}_{y,\delta}^{(s,N)}, \widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)}, \widetilde{y} \in \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)}\big) \\ &- \mathbb{P}_{1,2}\Big(G_{N,s,\delta}^{(1)} \cap G_{N,s,\delta}^{(2)}\Big)\mathbb{P}_{1,2}\Big(G_{N,\widetilde{s},\delta}^{(1)} \cap G_{N,\widetilde{s},\delta}^{(2)}\Big)\Big] \leq 0.\end{aligned} \tag{2.8}$$

We pick $\delta > 0$ smaller than $\frac{1}{3}\min\{|x-y|, |\widetilde{x}-\widetilde{y}|\}$. Let us give some heuristic explanation of the following argument. To get (2.8), we only have to prove that, with probability tending to 1 as $N \to \infty$, the partial clusters $\mathcal{C}_{x,\delta}^{(s,N)} \cup \mathcal{C}_{y,\delta}^{(s,N)}$, and $\mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)} \cup \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)}$, depend only on two disjoint sub-collections of $X^{(3)}, \dots, X^{(N)}$ or at least on sub-collections with a small overlap. What we mean precisely here is that the density of the walkers in $\mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)} \cup \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)}$ is roughly the same if we remove those points that were in $\mathcal{C}_{x,\delta}^{(s,N)} \cup \mathcal{C}_{y,\delta}^{(s,N)}$. Therefore we need Assumption 1.1(ii) to describe the position of the walkers at time s, given their position at time $\tilde{s}$. In more technical terms, it says the following. By $\mathcal{B}(D)$ we denote the Borel σ-field on D. Let a version of the conditional distribution of X_s given $X_{\widetilde{s}} = y$ be given, i.e., a Markov kernel $K_{s,\widetilde{s}}\colon D \times \mathcal{B}(D) \to \mathcal{B}(D)$ such that, almost surely, $\mathbb{P}(X_s \in A \mid X_{\widetilde{s}} = y) = K_{s,\widetilde{s}}(y, A)$ for any $A \in \mathcal{B}(D)$. Then we require that $K_{s,\widetilde{s}}(y, \{x\}) = 0$ for any $x \in D$. Indeed, this assumption implies that, for any $y \in D$,

$$\lim_{\delta\downarrow 0} \mathbb{P}(X_s \in B(x,\delta) \mid X_{\widetilde{s}} = y) = \lim_{\delta\downarrow 0} K_{s,\widetilde{s}}(y, B(x,\delta)) = K_{s,\widetilde{s}}(y, \{x\}) = 0. \tag{2.9}$$

Since the probability on the left-hand side is continuous in y and monotonous in δ, the convergence is even uniform in $y \in D$, according to Dini's theorem. Hence, we can multiply this term with $f_{\widetilde{s}}(y)$, integrate over $y \in D$ and interchange this integration with the limit $\delta \downarrow 0$. Now we can see heuristically the statement as follows. According to a large-N ergodic theorem, there are only of order $N\delta^{2d}$ walkers that are at time s in $B(x,\delta)$ and at time $\widetilde{s}$ in $B(\widetilde{x},\delta)$, analogously with y and $\widetilde{y}$. Hence, among all the $\asymp N\delta^d$ walkers present in $B(\widetilde{x},\delta)$ at time $\widetilde{s}$, those ones who were in $B(x,\delta)$ at time s are negligible for small δ. This implies the claimed asymptotic independence.

Let us turn to the proof. We need to introduce a bit of notation. For $A \subset \{1,\dots,N\}$, we write $\mathcal{C}_{x,\delta}^{(s,A)}$ for the largest cluster in the δ-box around x that is built out of all the $X_s^{(i)}$ with $i \in A$ only. We put

$$A_s^{(N)} = \{i \in \{1,\dots,N\} \colon X_s^{(i)} \notin B(x,\delta) \cup B(y,\delta)\}.$$

Now we use the triangle inequality to bound

$$\begin{aligned}
&\mathbb{P}_{1,2}\Big(x \in \mathcal{C}_{x,\delta}^{(s,N)}, y \in \mathcal{C}_{y,\delta}^{(s,N)}, \widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)}, \widetilde{y} \in \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)}\Big) \\
&\quad\le \mathbb{P}_{1,2}\Big(x \in \mathcal{C}_{x,\delta}^{(s,N)}, y \in \mathcal{C}_{y,\delta}^{(s,N)}, \widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},A_s^{(N)})}, \widetilde{y} \in \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},A_s^{(N)})}\Big) \\
&\qquad + \mathbb{P}_{1,2}\Big(\widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)} \setminus \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},A_s^{(N)})}\Big) + \mathbb{P}_{1,2}\Big(\widetilde{y} \in \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},N)} \setminus \mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},A_s^{(N)})}\Big).
\end{aligned} \tag{2.10}$$

Since $\mathcal{C}_{x,\delta}^{(s,N)}$ and $\mathcal{C}_{y,\delta}^{(s,N)}$ depend only on the $X_s^{(i)}$ with i in the complement of $A_s^{(N)}$, the first two events in the first term on the right-hand side are independent from the last two events. Lemma 2.2(ii) and the continuity of $\overline{\theta}(\cdot,R)$ imply that the probability of the intersection of the first two events converges towards $\overline{\theta}(f_s(x),R)\overline{\theta}(f_s(y),R)$. Note that the particles that the point processes $\mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},A_s^{(N)})}$ and $\mathcal{C}_{\widetilde{y},\delta}^{(\widetilde{s},A_s^{(N)})}$ puts are given by trajectories that do not visit any of the two balls $B(x,\delta)$ and $B(y,\delta)$ at time s; more precisely, they are picked according to the density

$$f_{\widetilde{s}}^{(s,\delta)}(z) = \mathbb{P}(X_s \notin B(x,\delta) \cup B(y,\delta), X_{\widetilde{s}} \in \mathrm{d}z)/\mathrm{d}z = K_{s,\widetilde{s}}\big(z, (B(x,\delta) \cup B(y,\delta))^{\mathrm{c}}\big)\, f_{\widetilde{s}}(z). \tag{2.11}$$

Hence, the probability of the intersection of the last two events converges towards $\overline{\theta}(f_{\widetilde{s}}^{(s,\delta)}(\widetilde{x}),R)\overline{\theta}(f_{\widetilde{s}}^{(s,\delta)}(\widetilde{y}),R)$.

A glance at (2.11) shows that $f_{\widetilde{s}}^{(s,\delta)}(z)$ converges, as $\delta \downarrow 0$, for any $z \in D$, towards $\mathbb{P}(X_s \neq x, X_s \neq y, X_{\widetilde{s}} \in \mathrm{d}z)/\mathrm{d}z$, which is, by Assumption 1.1(i) (or also by (ii)), equal to $f_{\widetilde{s}}(z)$. Since $f_{\widetilde{s}}(\widetilde{x})$ and $f_{\widetilde{s}}(\widetilde{y})$ are larger than the critical value, we may use continuity of $\overline{\theta}$.

All together, we have that the first term of the right-hand side of (2.10) converges, as $N \to \infty$ followed by $\delta \downarrow 0$, towards

$$\overline{\theta}(f_s(x),R)\overline{\theta}(f_s(y),R)\overline{\theta}(f_{\widetilde{s}}(\widetilde{x}),R)\overline{\theta}(f_{\widetilde{s}}(\widetilde{y}),R). \tag{2.12}$$

Furthermore, Assumption 1.1(ii) also implies that

$$\limsup_{N\to\infty} \mathbb{P}_{1,2}\Big(\widetilde{x} \in \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)} \setminus \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},A_s^{(N)})}\Big) \tag{2.13}$$

vanishes as $\delta \downarrow 0$. Indeed, we know that $\mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},A_s^{(N)})} \subset \mathcal{C}_{\widetilde{x},\delta}^{(\widetilde{s},N)}$, therefore the above limit superior is equal to $\overline{\theta}(f_{\widetilde{s}}(\widetilde{x})) - \overline{\theta}(f_{\widetilde{s}}^{(s,\delta)}(\widetilde{x}))$. Hence, the convergence of $f_{\widetilde{s}}^{(s,\delta)}$ and the continuity of $\overline{\theta}$ give the result. We proceed analogously for the last term in (2.10) and get that the limit superior as $N \to \infty$ and $\delta \downarrow 0$ of the left-hand side of (2.10) is not larger than the expression in (2.12). Now use Lemma 2.2(ii) for the second term in (2.8) to see that from this the desired assertion follows. ☐

Proof of Theorem 1.2. First note that both assertions of (1.4) easily follow from Proposition 2.1, in conjunction with Lemma 2.3, provided that R is chosen such that

$$\int_0^T \mathrm{d}s\, \mathbb{1}\{f_s(X_s^{(i)}) = \lambda_{\mathrm{c}}(R)\} = 0 \qquad \text{for } i = 1,2. \tag{2.14}$$

Furthermore, note that, almost surely, (2.14) holds for almost all R. Indeed, this follows from

$$\mathbb{E}\Big(\int_0^\infty \mathrm{d}R \int_0^T \mathrm{d}s\, \mathbb{1}\{f_s(X_s^{(i)}) = \lambda_{\mathrm{c}}(R)\}\Big) = \int_0^T \mathrm{d}s \int_D \mathrm{d}x\, f_s(x) \int_0^\infty \mathrm{d}R\, \mathbb{1}\{f_s(x) = R^{-d}\lambda_{\mathrm{c}}(1)\} = 0.$$

Hence, for a given (random) exceptional R, we pick sequences $(R_k)_{k\in\mathbb{N}}$ and $(R'_k)_{k\in\mathbb{N}}$ such that $R_k \downarrow R$ and $R'_k \uparrow R$ and R_k and R'_k satisfy (2.14) for any k in place of R. Since $\tau_T^{(N)}$ is an increasing function of R, we may estimate it from above and below by replacing R with R_k and R'_k, respectively, and applying Proposition 2.1 and Lemma 2.3 with these. This yields (1.4) with $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ replaced by their versions for R replaced with R_k and with R'_k, respectively.

The only thing that we need to do is to show the right-uppersemicontinuity of the map $R \mapsto \tau_T^{(\geq)}$ and the left-lowersemicontinuity of the map $R \mapsto \tau_T^{(>)}$. To show these, note that $\overline{\theta}^{(\geq)}(\cdot, R) = \overline{\theta}(R^d \cdot +, 1)$ is right-continuous and $\overline{\theta}^{(>)}(\cdot, R) = \overline{\theta}(R^d \cdot -, 1)$ is left-continuous. Furthermore, for any $x, y \in D$ and any $s \in [0, T]$, the map $R \mapsto \mathbb{1}\{x \xleftrightarrow[s]{\geq} y\}$ is right-uppersemicontinuous, and the map $R \mapsto \mathbb{1}\{x \xleftrightarrow[s]{>} y\}$ is left-lowersemicontinuous. The latter assertion is quite easy to see; let us show the former. Assume that, for all $\varepsilon > 0$, x and y are connected through the set $\{f_s \geq \lambda_{\mathrm{c}}(R + \varepsilon)\}$. Recall that $\lambda_{\mathrm{c}}(R) = R^{-d}\lambda_{\mathrm{c}}(1)$ is decreasing in R. If x and y were not connected through the set $\{f_s \geq \lambda_{\mathrm{c}}(R)\}$, then they would lie in different components of this set. By compactness, these components have a positive distance to each other. Hence, there is a hyperplane in D through the complement of $\{f_s \geq \lambda_{\mathrm{c}}(R)\}$ that separates these two components. Since this hyperplane is compact, f_s assumes a maximum on it, which is strictly smaller than $\lambda_{\mathrm{c}}(R)$. Hence, every curve from x to y must cross this hyperplane, i.e. must pass a point with an f_s-value bounded away from $\lambda_{\mathrm{c}}(R)$. This means that, for some sufficiently small $\varepsilon > 0$, x and y are not connected through $\{f_s \geq \lambda_{\mathrm{c}}(R + \varepsilon)\}$. Hence, $\limsup_{\varepsilon\downarrow 0} \mathbb{1}\{x \xleftrightarrow[s]{\geq, R+\varepsilon} y\} \leq \mathbb{1}\{x \xleftrightarrow[s]{\geq, R} y\}$, where we wrote $\xleftrightarrow[s]{\geq, R}$ for connectedness through the set $\{f_s \geq \lambda_{\mathrm{c}}(R)\}$. Using Lebesgue's theorem shows the claimed continuity properties of $\tau_T^{(\geq)}$ and $\tau_T^{(>)}$ in R and finishes the proof of Theorem 1.2. □

3. Long-time investigations for the random waypoint model

In this section, we prove Lemma 1.4 and Theorem 1.6, that is, we restrict ourselves to the random waypoint model (RWP) introduced in Section 1.4 and study the long time behaviour of the limiting connection time both in terms of an ergodic theorem and a large-deviations result. First we prove in Section 3.1 the convergence of the RWP to its invariant distribution. The proof of Lemma 1.4 is based on a certain discrete-time Markov chain, whose ergodic and mixing properties are derived in Section 3.2. The proof then follows in Section 3.3. Finally, we prove Theorem 1.6 in Section 3.4.

3.1. **Recurrence and ergodicity of the RWP.** Since we want to study long-time properties of the connection time, we will need recurrence and ergodic properties of the RWP, which we provide in this section. For the special case of $\mathcal{W}$ being the uniform distribution on D, most of our results in this section are already contained in [LV06], but our Proposition 3.2 below also contains a statement on convergence in total variation, which will be important in Lemma 3.4 below. For the reader's convenience, we provide all necessary proofs; they are independent of [LV06], but use different variants of the Markov renewal theorem available in the literature.

The trajectory is divided into *trips*, by which we mean the parts from leaving a waypoint to arriving at the next one. $\mathbb{P}^{(0)}$ and $\mathbb{E}^{(0)}$ denote probability and expectation if the process starts at time 0 at the beginning of a trip at the zeroth of the waypoints, i.e., if the initial waypoint W_0 has distribution $\mathcal{W}$.

In [LV06, Theorem 6], another variant of Y is considered, and it is argued that that process possesses a unique invariant distribution. Projecting on our first coordinate, the location of the walker, the distribution of X in equilibrium is given by the formula

$$\mu_*(\mathrm{d}x) = \frac{1}{Z}\int_0^1 \mathrm{d}s\, \mathbb{E}^{(0)}\Big(\frac{V_1}{|W_1 - W_0|}; W_0 + s(W_1 - W_0) \in \mathrm{d}x\Big), \tag{3.1}$$

where Z is a normalisation. It turns out below that this formula persists also for a general waypoint measure. In particular μ_* has a continuous density. We refer in particular to [L04] for a general methodology to describe this measure. See [BW02, Section 5] and [HLV06, Section III and IV] for explicit formulas, approximations and simulations for special cases of domains D and waypoint measures $\mathcal{W}$, like uniform distributions on rectangles and balls.

For the sake of illustration, we give an explicit value in $d = 2$ in the simplest case where the domain is the unit disk, the waypoint measure $\mathcal{W}$ is the uniform measure on it and the velocity is chosen to be constant. In this case, the density of the waypoint location in the invariant distribution is given by

$$f_*(x) = \frac{45}{64\,\pi}(1 - |x|^2)\int_0^\pi \sqrt{1 - |x|^2\cos^2(\varphi)}\,\mathrm{d}\varphi, \qquad x \in B(0,1).$$

An approximation with a mean square error ≤ 0.0065 and an absolute error ≤ 0.067 is given by $f_*(x) = \frac{2}{\pi}(1 - |x|^2)$, see [QZ07] and [BW02, eq. (18)].

In the following, we give detailed proofs for ergodic properties of the RWP, based on the Markov renewal theorem in the form provided by [K74]. Alternative proofs could be based on the form given in [LV06, Theorem 6].

We first show that the sequence of the trips is positive Harris recurrent. More precisely, we consider the sequence $\mathcal{T} = (\mathcal{T}_n)_{n\in\mathbb{N}} = (W_{n-1}, W_n, V_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$. Since $(W_n)_{n\in\mathbb{N}_0}$ and $(V_n)_{n\in\mathbb{N}}$ are independent i.i.d. sequences, $\mathcal{T}$ is obviously a Markov chain. Furthermore, it is also easy to see that $\mathcal{T}$ is positive Harris recurrent, since it satisfies

$$\mathbb{P}_y(\mathcal{T}_n \in A) = \mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}(A), \qquad n \geq 2, y \in \mathcal{D}, A \subset \mathcal{D} \text{ mb.}, \tag{3.2}$$

where we wrote $\mathbb{P}_y$ for the probability measure under which the walker starts from $Y(0) = y$. We use this to prove the convergence of Y_t introduced in (1.7). The proof goes in two step. The first one (see Lemma 3.1) applies the Markov renewal theorem using the fact that Y_t is a time change of $\mathcal{T}$ and gives a good understanding and a description of the limit law (in particular it states the existence of an invariant distribution with finite mass). However, as we will see, this approach only gives weak convergence. In a second step we use Harris recurrence (see Proposition 3.2) to obtain convergence in total variation. Of course it is then easy to check that the convergence has to be towards the same limit. By $\mathbb{P}_\alpha$ we denote the probability measure under which the process $(Y_t)_{t\in[0,\infty)}$ starts from the distribution α.

Lemma 3.1. *For any bounded continuous function $g\colon \mathcal{D}\times\mathbb{R}^+ \to \mathbb{R}^+$, and for any $y \in \mathcal{D}$,*

$$\lim_{t\to\infty}\mathbb{E}_y[g(\mathcal{T}_{N(t)}, T_{N(t)} - t)] = \frac{1}{\mathbb{E}[U_1]}\int_{\mathcal{D}}\mathbb{P}_{\mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}}[\mathcal{T}_1 \in \mathrm{d}z, U_1 \in \mathrm{d}\lambda]\int_0^\lambda g(z,s)\,\mathrm{d}s. \tag{3.3}$$

Proof. We apply [K74, Theorem 1], which immediately implies the assertion, noting that the measure ψ in [K74] is indeed equal to $\mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}$ by [K74, Lemma 2]. That is, we only have to check the validity of Conditions I.1-4 of [K74].

Conditions I.1 and I.2 are trivial here, while Condition I.3 is the usual non-lattice assumption. It states that there is a non-lattice sequence $(\zeta_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{R}$ such that, for each $\nu \in \mathbb{N}$ and $\delta > 0$, there exists some $y \in \mathcal{D}$, such that, for every $\epsilon > 0$, there exists a measurable set A with positive $\mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}$-measure, integers m_1, m_2 and $\tau \in \mathbb{R}$ such that, for $x \in A$,

$$\mathbb{P}_x[d(\mathcal{T}_{m_1}, y) < \epsilon, |T_{m_1} - \tau| \leq \delta] > 0 \qquad \text{and} \qquad \mathbb{P}_x[d(\mathcal{T}_{m_2}, y) < \epsilon, |T_{m_2} - \tau - \zeta_\nu| \leq \delta] > 0, \tag{3.4}$$

d being the usual Euclidean distance on $\mathcal{D}$.

We will prove this assumption with an arbitrary $y = (w_0, w_1, v_1)$ inside the support of $\mathcal{W} \otimes \mathcal{W} \otimes \mathcal{V}$, not depending on ν nor on δ, and with $A = \{x \in \mathcal{D} \colon d(x, y) < \epsilon\}$, where we assumed without loss of generality that $2\epsilon v_-^{-1} + \mathrm{diam}(D)\epsilon v_-^{-2} < \delta/3$. Furthermore, we put $\tau := |w_1 - w_0|/v_1$ and pick any non-lattice sequence $(\zeta_\nu)_{\nu\in\mathbb{N}}$ inside the support of $\tau + |w_0 - w_1|/V_1$. Furthermore, put $m_1 = 1$ and $m_2 = 3$. By continuity of the densities of $\mathcal{W}$ and $\mathcal{V}$, the $\mathcal{W} \otimes \mathcal{W} \otimes \mathcal{V}$-measure of A is positive. Putting $x = (w_0', w_1', v_1') \in A$ and denoting by $T_1(x) = |w_1' - w_0'|/v_1'$ the (deterministic) value of T_1 starting from x, we see that

$$|T_1(x) - \tau| \le \frac{|w_1' - w_0' - (w_1 - w_0)|}{v_1'} + |w_1 - w_0|\Big|\frac{1}{v_1} - \frac{1}{v_1'}\Big| \le \frac{2\epsilon}{v_-} + \frac{\mathrm{diam}(D)\epsilon}{v_-^2} < \frac{\delta}{3}. \tag{3.5}$$

Noting that $\mathcal{T}_1 = x$ with $\mathbb{P}_x$-probability one, we see that the first part of (3.4) is satisfied; the probability is even equal to one.

Now we turn to the proof of the second. Keep $x \in A$ fixed. Recall that $T_n = U_0 + U_1 + \cdots + U_{n-1}$ and that $U_n = |W_{n+1} - W_n|/V_n$ for any n. Note that, under $\mathbb{P}_x$, $\mathcal{T}_3$ has distribution $\mathcal{W} \otimes \mathcal{W} \otimes \mathcal{V}$, and therefore $\mathbb{P}_x(d(\mathcal{T}_3, y) < \epsilon) = \mathcal{W} \otimes \mathcal{W} \otimes \mathcal{V}(A) > 0$. On the event $\{d(\mathcal{T}_3, y) < \epsilon\}$, with $\mathbb{P}_x$-probability one, (3.5) shows that $|U_0 - \tau| < \delta/3$, and a the same calculation with x replaced by $\mathcal{T}_3$ shows that $|U_2 - \tau| < \delta/3$. By our choice of ζ_ν and by continuity of the densities of $\mathcal{W}$ and $\mathcal{V}$, we easily see that the event $\{|U_1 + \tau - \zeta_\nu| \le \delta/3\}$ has positive $\mathbb{P}_x$-probability on $\{d(\mathcal{T}_3, y) < \epsilon\}$, since

$$|U_1 + \tau - \zeta_\nu| \le \frac{|W_2 - W_1 - (w_0 - w_1)|}{v_-} + \Big|\frac{|w_0 - w_1|}{V_2} - (\zeta_\nu - \tau)\Big| \le \frac{2\epsilon}{v_-} + \Big|\frac{|w_0 - w_1|}{V_2} - (\zeta_\nu - \tau)\Big|,$$

and the probability (with respect to V_2) to have the last term smaller than $\mathrm{diam}(D)\epsilon v_-^{-2}$ is positive. Since

$$|T_3 - \tau - \zeta_\nu| = |U_0 + U_1 + U_2 - \tau - \zeta_\nu| \le |U_0 - \tau| + |U_1 + \tau - \zeta_\nu| + |U_2 - \tau|,$$

we now see that also the last condition in (3.4) is satisfied.

Condition I.4 states that, for any $x \in \mathcal{D}$, $\delta > 0$, there exists $r_0(x, \delta) > 0$ such that for any measurable function $f \colon \mathcal{D}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}_0} \to \mathbb{R}$, and for all y with $d(y, x) < r_0(x, \delta)$,

$$\mathbb{E}_x\big[f\big((\mathcal{T}_i)_{i\in\mathbb{N}}, (U_i)_{i\in\mathbb{N}_0}\big)\big] \le \mathbb{E}_y\big[\lim_{n\to\infty} \sup\big\{f\big((t_i)_{i\in\mathbb{N}}, (u_i)_{i\in\mathbb{N}_0}\big) \colon d(t_i, \mathcal{T}_i) + |u_i - U_i| < \delta \text{ for } i \le n\big\}\big] + \delta \sup|f|. \tag{3.6}$$

This assumption is in general difficult to prove, but here things are simple, as $\mathcal{T}_i$ and U_i are independent of the starting point for $i \ge 3$. We can do the following coupling: write $x = (w_0^{(x)}, w_1^{(x)}, v_1^{(x)})$ and $y = (w_0^{(y)}, w_1^{(y)}, v_1^{(y)})$. We draw a sequence of i.i.d. waypoints and speeds $(W_i, V_i)_{i\ge2}$ according to $\mathcal{W} \otimes \mathcal{V}$. Define, for $z \in \{x, y\}$,

$$W_0^{(z)} = w_0^{(z)}, \qquad W_1^{(z)} = w_1^{(z)}, \qquad V_1^{(z)} = v_1^{(z)}, \qquad (W_i^{(z)}, V_i^{(z)})_{i\ge2} = (W_i, V_i)_{i\ge2}, \tag{3.7}$$

and put $\mathcal{T}_i^{(z)} = (W_{i-1}^{(z)}, W_i^{(z)}, V_i^{(z)})$. It is then clear that $(\mathcal{T}_i^{(z)})_{i\in\mathbb{N}}$ is a realisation of $(\mathcal{T}_i)_{i\in\mathbb{N}}$ under $\mathbb{P}_z$ and that for any $i \ge 3$, $\mathcal{T}_i^{(x)} = \mathcal{T}_i^{(y)}$. We saw in the verification of Condition I.3 that, if $d(x, y) < r$, then with obvious notation,

$$d(\mathcal{T}_i^{(x)}, \mathcal{T}_i^{(y)}) < r, \qquad d(U_i^{(x)}, U_i^{(y)}) < r\Big(\frac{2}{v_-} + \frac{\mathrm{diam}(D)}{v_-^2}\Big).$$

Taking $r_0(\delta)$ such that both right-hand sides are $< \delta$, immediately gives Condition I.4. □

Using (1.6), we easily derive the above mentioned weak convergence of X_t towards μ_* identified in (3.1), as X_t may be written as an explicit continuous function of $\mathcal{T}_{N(t)}$ and $T_{N(t)} - t$. We now give a refined result, using the notion of Harris recurrence for continuous-time Markov chains. First note that the process

$$\mathcal{Y} = (\mathcal{Y}_t)_{t\in[0,\infty)} = \Big(\mathcal{T}_{N(t)}, \frac{T_{N(t)} - t}{U_{N(t)-1}}\Big)_{t\in[0,\infty)}$$

is a continuous-time Markov chain on $\mathcal{D} \times [0,1]$ with right-continuous paths. The second component of $\mathcal{Y}$ runs from 0 to 1 with linear speed between the arrival times at the waypoints. It is also easy to express Y_t as a continuous functional of $\mathcal{Y}_t$.

Proposition 3.2. *$(\mathcal{Y}_t)_{t\in[0,\infty)}$ is a strongly aperiodic Harris recurrent chain, and its distribution converges in total variation towards the unique invariant distribution. As a consequence, the convergence in Lemma 3.1 is true for any measurable bounded function g. Furthermore, an ergodic theorem holds for $(\mathcal{Y}_t)_{t\in[0,\infty)}$.*

Proof. We use the characterization of Harris recurrence given in [KM94, Theorem 1], with the measure ν given by $\mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}\otimes\lambda$, where λ is the Lebesgue measure on $[0,1]$. It is easy to see that any set A with positive ν-measure will be hit by the process $(\mathcal{Y}_t)_{t\in[0,\infty)}$. Indeed, without loss of generality, we can assume that A is a product set. By independence it will certainly happen that one of the $\mathcal{T}_n$ will fall into the $\mathcal{D}$-component of A. Then as $\frac{T_{N(t)}-t}{U_{N(t)-1}}$ visits all of $[0,1]$ between two waypoints, it follows that also A will be hit by $\mathcal{Y}$, implying Harris recurrence.

This implies in particular the existence of a unique (up to multiplicative constants) invariant measure. It is not difficult to check that this measure has to be the one appearing in Lemma 3.1, up to the normalisation. In particular, it has finite total mass. As a consequence, $\mathcal{Y}$ is strongly Harris recurrent. We also have that this process has spread-out cycles, in the sense of [A03, p. 202]. In fact, the hitting times of any set under any starting point are spread out. Indeed, the first hitting times might be deterministic (if the initial condition implies that the set is hit during the first travel of the walker), but then one can easily check that, due to the existence of a density for the speed, the hitting times also have a continuous density. Therefore, using [A03, Proposition VII.3.8], this implies convergence in total variation of $\mathcal{Y}_t$ towards its invariant distribution. The ergodic theorem can be found in [A03, Proposition VII.3.7]. □

Note that, at this point, it would be possible to use the above result to get a simple proof of Lemma 1.4. However, we would like to present a different proof, as we need to introduce the important discrete-time Markov chain $(Z_j)_{j\in\mathbb{N}_0}$, that will be useful for the sequel. This proof can be found in Section 3.3.

3.2. **Recurrence and mixing properties of Z.** In this section, we introduce an important tool for our proofs of Lemma 1.4 and Theorem 1.6, a discrete-time Markov chain $Z = (Z_j)_j$ that registers the locations, waypoints and velocities of two independent RWPs at all the times at which one of them arrives at a new waypoint. In this section, we study recurrence and the mixing properties of this chain, in Sections 3.3 and 3.4 we will use it to derive the long-time average and large-deviations properties of the connection time. For proving just the former of the two results in Lemma 1.4, some straight-forward ergodic arguments would be also sufficient, however, we will need the identification of the ergodic limit in terms of the Markov chain Z in order to prove the large-deviations result in Theorem 1.6. We show that $(Z_k)_{k\in\mathbb{N}_0}$ is a time-homogeneous, ψ-mixing and Harris ergodic Markov chain. It is an object of independent interest, as it may serve also for other long-time investigations of the model, as well as for computer simulations.

The Markov chain Z is defined as follows. We consider the times $0 \le S_1 < S_2 < \dots$ at which any of the two walkers arrives at his waypoint. Formally, $S_0 = 0$ and

$$S_j = \inf\Big\{t > S_{j-1} \colon W^{(1)}_{N^{(1)}(t)} \neq W^{(1)}_{N^{(1)}(S_{j-1})} \text{ or } W^{(2)}_{N^{(2)}(t)} \neq W^{(2)}_{N^{(2)}(S_{j-1})}\Big\}, \qquad j\in\mathbb{N}, \tag{3.8}$$

where the superscripts (1) and (2) mark the two walkers. Put

$$Z_j = \big(Y^{(1)}(S_j), Y^{(2)}(S_j)\big) = \Big(\big(X^{(1)}_{S_j}, W^{(1)}_{N^{(1)}(S_j)}, V^{(1)}_{N^{(1)}(S_j)}\big), \big(X^{(2)}_{S_j}, W^{(2)}_{N^{(2)}(S_j)}, V^{(2)}_{N^{(2)}(S_j)}\big)\Big) \in \mathcal{D}^2, \quad j \in \mathbb{N}_0. \tag{3.9}$$

That is, $Z = (Z_j)_{j\in\mathbb{N}_0}$ is the trace-Markov chain of two independent copies of the RWP, observed at the times at which any of the two arrives at a waypoint; it is a time-change of $(Y^{(1)}, Y^{(2)})$. It is easy to see that $(Z_j)_j$ is a time-homogeneous Markov chain on $\mathcal{D}^2$. This chain does not explicitly record the location of the random walker at any fixed time, but the time that passes between the waypoint arrivals can be deduced from the information contained in Z. Hence, it is well-suitable for deducing asymptotic assertions for long time. First we derive a mixing property, which will later be used for the large-deviations principle.

Lemma 3.3. *The sequence $(Z_j)_j$ is ψ-mixing under any starting distribution, i.e.,*

$$\lim_{k\to\infty} \sup_{A\in\mathcal{F}_0^0, B\in\mathcal{F}_k^\infty} \Big|\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)\mathbb{P}(B)} - 1\Big| = 0,$$

where $\mathcal{F}_m^k := \sigma(Z_m, \dots, Z_k)$.

Proof. Introduce the event

$$E_k = \Big\{\exists l, m \in \{1, \dots, k-1\} : W_0^{(1)} \neq W^{(1)}_{N^{(1)}(S_l)} \neq W^{(1)}_{N^{(1)}(S_k)} \text{ and } W_0^{(2)} \neq W^{(2)}_{N^{(2)}(S_m)} \neq W^{(2)}_{N^{(2)}(S_k)}\Big\}$$

that both walkers choose at least two new waypoints by time S_k. Then, conditional on E_k, any $A \in \mathcal{F}_0^0$ and $B \in \mathcal{F}_k^\infty$ are independent. Indeed, on the event E_k, A depends on $X_0^{(1)}, W_1^{(1)}, V_1^{(1)}, X_0^{(2)}, W_1^{(2)}, V_1^{(2)}$ only, while B depends only on the variables $W_l^{(1)}, V_l^{(1)}, W_l^{(2)}, V_l^{(2)}$ for some $l \geq 3$ and on $X^{(1)}_{S_l}, X^{(2)}_{S_l}$ with $l \geq 2$; note that, for $i \in \{1, 2\}$, $X^{(i)}_{S_l}$ is a function of $W^{(i)}_{N^{(i)}(S_l)}, W^{(i)}_{N^{(i)}(S_l)-1}$ and $V^{(i)}_{N^{(i)}(S_l)}$ only, and $N^{(i)}(S_l) \geq 3$ on E_k. Using the independence of A and B on E_k, a small calculation yields that

$$\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)\mathbb{P}(B)} = \frac{\mathbb{P}(E_k|A)\mathbb{P}(E_k|B)}{\mathbb{P}(E_k)} + \mathbb{P}(E_k^{\rm c}|A\cap B)\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)\mathbb{P}(B)}.$$

Hence, the assertion follows from

$$\lim_{k\to\infty} \sup_{A\in\mathcal{F}_0^0, B\in\mathcal{F}_k^\infty} \mathbb{P}(E_k^{\rm c}|A\cap B) = 0. \tag{3.10}$$

We show now that (3.10) holds. The event $E_k^{\rm c}$ splits into the event that the first walker has chosen not more than one new waypoint by time S_k, but the second has chosen at least $k-1$ new waypoints, and the same event with first and second walker reversed. Let us only look at the first of these two events. On this event, the time S_k is not larger than $2\mathrm{diam}(D)/v_-$, since a choice of a new waypoint is done after $\mathrm{diam}(D)/v_-$ time units at the latest, since all ways are no longer than $\mathrm{diam}(D)$ and all velocities are no less than v_-. Since the time that passes between the second walker picks his $(j-1)$-st and the j-th waypoint is $|W_j^{(2)} - W_{j-1}^{(2)}|/V_j^{(2)}$, we have that its sum over $j \in \{1, \dots, k-1\}$ is not larger than $2\mathrm{diam}(D)/v_-$. Hence, on this event we have

$$\sum_{j=1}^{k-1} \big|W_j^{(2)} - W_{j-1}^{(2)}\big| \leq 2\frac{v_+}{v_-}\mathrm{diam}(D).$$

Leaving out the summands for $j = 1$ and $j = k-1$, this remaining sum is still upper bounded by the right-hand side, and it does not depend on Z_0 nor on $Z_k, Z_{k+1}, \dots$. Hence, the probability for this sum being smaller than the right-hand side is an upper bound for the half of $\mathbb{P}(E_k^{\rm c}|A\cap B)$ that we are considering, and it does not depend on A nor on B. Since the right-hand side is constant and since the waypoints are not deterministic, the probability for this event tends to 0 as $k \to \infty$. This shows that (3.10) holds and ends the proof. □

The following lemma says that Z is Harris recurrent, has a unique invariant distribution and is non-lattice, which is summarised by saying that it is Harris ergodic. In particular, it satisfies an ergodic theorem, i.e., for any bounded measurable function f, the averages $\frac{1}{N}\sum_{i=1}^N f(Z_i)$ converge almost surely to the integral of f with respect to the invariant distribution.

Lemma 3.4. *The chain Z is Harris ergodic.*

Proof. Harris recurrence of Z is equivalent to the existence of a nontrivial σ-finite measure φ such that Z is φ-recurrent, see [A03, Cor. VII.3.12]. Therefore we have to show that there exists some σ-finite measure φ such that every measurable set $F \subset \mathcal{D}^2$ with $\varphi(F) > 0$ is recurrent.

We denote the invariant measure of the process $(Y_t^{(1)})_{t\in[0,\infty)}$ by γ. Define $\varphi = \gamma \otimes \mathcal{W} \otimes \mathcal{W} \otimes \mathcal{V}$, which is obviously σ-finite. Let $F \subset \mathcal{D}^2$ be measurable with $\varphi(F) > 0$. We are going to show that the hitting time of F is almost surely finite. Note that $\varphi(F) > 0$ implies, by Fubini's theorem, that, for some $\epsilon > 0$, the set $\widetilde{F}$ of all T satisfying $\int \mathbb{1}_F((Y,T))\,\gamma(\mathrm{d}Y) > \epsilon$ has positive $\mathcal{W} \otimes \mathcal{W} \otimes \mathcal{V}$ measure.

First consider the sequence $(n_k)_{k\in\mathbb{N}_0}$ of times at which the second walker arrives at a waypoint, that is, $(S_{n_k})_{k\in\mathbb{N}_0} = (T_k^{(2)})_{k\in\mathbb{N}_0}$. The first component of the process $(Z_{n_k})_{k\in\mathbb{N}_0}$ is a RWP sampled at times which are given by an independent renewal process, and the second component has the same law as $(\mathcal{T}_{k+1})_{k\in\mathbb{N}_0}$. According to (3.2) and [A03, Cor. VII.3.12] the second component is $(\mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V})$-positive recurrent. In particular there exists a subsequence $(\widetilde{n}_k)_k$ of $(n_k)_k$ such that the second component of $Z_{\widetilde{n}_k}$ belongs to $\widetilde{F}$ for any $k \in \mathbb{N}_0$. Also $(S_{\widetilde{n}_k})_{k\in\mathbb{N}_0}$ is a transient Markov renewal process, independent of $Y^{(1)}$.

Now conditioning on the second component process, $Y^{(2)}$, $(Y^{(1)}_{S_{\widetilde{n}_k}})_{k\in\mathbb{N}_0}$ is given by sampling the, by Proposition 3.2 Harris ergodic, process $Y^{(1)}$ at a deterministic, sequence of times that increase to infinity. Still conditioning on $Y^{(2)}$, the event that $Z_{\widetilde{n}_k} \in F$ has probability asymptotically lower bounded by ϵ. It is then obvious by ergodicity that this event will occur infinitely often.

According to [A03, Cor. VII.3.12], this proves Harris recurrence of $(Z_n)_{n\in\mathbb{N}}$, and in particular the existence of a unique invariant measure, [A03, Thm. VII.3.5]. Now as we want positive Harris recurrence, we are going to show that this measure is finite.

Note that the previous arguments, together with the ergodic theorem [A03, Prop. VII.3.7] give that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \mathbb{1}_{\{Z_{\widetilde{n}_k}\in F\}} = \gamma(F^{(1)}) > 0.$$

Note that $n_k/k \to 2$, since the arrival times of $Y^{(1)}$ and $Y^{(2)}$ are disjoint and have asymptotically the same distribution. Hence, since $\mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}(F^{(2)})$ is equal to the probability that $Y^{(2)}$ hits $F^{(2)}$, we have $\widetilde{n}_k/k \to 2/\mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}(F^{(2)})$ by the ergodic theorem. Noting the symmetry in the two components, we see that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N \mathbb{1}_{\{Z_k\in F\}} = \frac{1}{2}\big(\gamma\otimes\mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}(F) + \mathcal{W}\otimes\mathcal{W}\otimes\mathcal{V}\otimes\gamma(F)\big).$$

Since the right-hand side is a probability measure in F, $(Z_n)_{n\in\mathbb{N}}$ is positive Harris recurrent. Note that we proved the ergodic theorem in the course of the proof, as well as gave an explicit form for the invariant measure.

We also see from this proof that the sequence of hitting times of F is non-lattice, since the sequence $(\widetilde{n}_k)_{k\in\mathbb{N}}$ is non-lattice, because $(n_k)_{k\in\mathbb{N}}$ is non-lattice. □

3.3. **Longtime average of the connection time.** Here we give a proof of the ergodic limit in Lemma 1.4 using the Markov chain Z defined in (3.9). As we mentioned above, a simpler proof can be done using ergodic theory, but we will later need the representation of the ergodic limit in terms of Z. We saw in Section 3.2 that $(Z_k)_{k\in\mathbb{N}_0}$ is a time-homogeneous, ψ-mixing and Harris ergodic Markov chain on $\mathcal{D}^2$. In this section we prove the ergodic limit in Lemma 1.4, giving an explicit formula for the limit $p_*^{(>)}$. The main object in the proof of Theorem 1.6 in Section 3.4 is the empirical pair measure of Z, for which a large-deviation principle is known to hold.

We are going to express $\tau_T^{(\diamond,*)}$ in terms of Z. To this end, we define, for any $z_k = \big((x_k^{(1)},w_k^{(1)},v_k^{(1)});(x_k^{(2)},w_k^{(2)},v_k^{(2)})\big)\in\mathcal{D}^2$,

$$M^{(1)}(z_1,z_2) = \frac{|x_2^{(1)}-x_1^{(1)}|}{v_2^{(1)}}, \tag{3.11}$$

$$F_\diamond(z_1,z_2) = \int_0^1 \mathrm{d}s\,\overline{\theta}^{(\diamond)}\big(f_*(p_1(s)),R\big)\overline{\theta}^{(\diamond)}\big(f_*(p_2(s)),R\big)\mathbb{1}\{p_1(s)\overset{\diamond}{\underset{*}{\longleftrightarrow}}p_2(s)\}, \tag{3.12}$$

where $p_i(s)=sx_2^{(i)}+(1-s)x_1^{(i)}$, $s\in[0,1]$, denotes the path of the i-th walker from $x_1^{(i)}$ to $x_2^{(i)}$. Then $M^{(1)}$ is the time that elapses while the two walkers move from one waypoint arrival to the next one, and $F_\diamond$ describes the proportion of time that the two are connected with each other on that way.

Recalling (3.9), we have, for any $n\in\mathbb{N}$,

$$S_n=\sum_{j=1}^n(S_j-S_{j-1})=\sum_{j=1}^n\frac{|X_{S_j}^{(1)}-X_{S_{j-1}}^{(1)}|}{V_{N^{(1)}(S_j)}}=\sum_{j=1}^n M^{(1)}(Z_{j-1},Z_j). \tag{3.13}$$

Now we express $\tau_T^{(\diamond,*)}$ for T replaced by the waypoint arrival time. For any $n\in\mathbb{N}$, we have

$$\begin{aligned}\tau_{S_n}^{(\diamond,*)} &= \sum_{j=1}^n\int_{S_{j-1}}^{S_j}\mathrm{d}s\,\overline{\theta}^{(\diamond)}(f_*(X_s^{(1)}),R)\overline{\theta}^{(\diamond)}(f_*(X_s^{(2)}),R)\mathbb{1}\{X_s^{(1)}\overset{\diamond}{\underset{*}{\longleftrightarrow}}X_s^{(2)}\}\\ &=\sum_{j=1}^n(S_j-S_{j-1})\int_0^1\mathrm{d}s\,\overline{\theta}^{(\diamond)}(f_*(p_1(s)),R)\overline{\theta}^{(\diamond)}(f_*(p_2(s)),R)\mathbb{1}\{p_1(s)\overset{\diamond}{\underset{*}{\longleftrightarrow}}p_2(s)\}\\ &=\sum_{j=1}^n M^{(1)}(Z_{j-1},Z_j)F_\diamond(Z_{j-1},Z_j),\end{aligned} \tag{3.14}$$

where $p_i(s)=X_{S_{j-1}}^{(i)}+s(X_{S_j}^{(i)}-X_{S_{j-1}}^{(i)})$.

Now the proof of Lemma 1.4 is quite obvious. According to [A03, Th. VII.3.6], based on Lemma 3.4, implies that the distribution of Z_k converges towards its invariant distribution, which we want to call π. Hence, (Z_{j-1},Z_j) converges to its invariant distribution $\pi\otimes P$, where we wrote $P\colon\mathcal{D}\times\mathcal{F}\to[0,1]$ for its transition kernel, writing $\mathcal{F}$ for the σ algebra on $\mathcal{D}$. This convergence is in total variation sense. Since $M^{(1)}$ and $F_\diamond$ are bounded and measurable, we have that

$$\lim_{n\to\infty}\frac1n S_n=\int M^{(1)}\,\mathrm{d}(\pi\otimes P)\qquad\text{and}\qquad\lim_{n\to\infty}\frac1n\tau_{S_n}^{(\diamond,*)}=\int M^{(1)}F_\diamond\,\mathrm{d}(\pi\otimes P).$$

Pick $n_T=\sup\{n\in\mathbb{N}\colon S_n\le T\}$, then it is easy to see that $\frac1T n_T\to 1/\int M^{(1)}\,\mathrm{d}(\pi\otimes P)$ as $T\to\infty$, almost surely and in probability. It is only an exercise to prove that the above limits are also true if n is replaced by n_T. Furthermore, it is also easy to see that $\frac1T(\tau_T^{(\diamond,*)}-\tau_{S_{n_T}}^{(\diamond,*)})$ vanishes almost surely and in probability as $T\to\infty$. Hence, we have

$$p_*^{(\diamond)}=\lim_{T\to\infty}\frac1T\tau_T^{(\diamond,*)}=\frac{\int M^{(1)}F_\diamond\,\mathrm{d}(\pi\otimes P)}{\int M^{(1)}\,\mathrm{d}(\pi\otimes P)}. \tag{3.15}$$

This ends the proof of Lemma 1.4 with the identification of the limit $p_*^{(\diamond)}$ as the right-hand side of (3.15).

3.4. **Proof of Theorem 1.6.** Now we turn to the proof of Theorem 1.6, i.e., we prove the upper bound for the downwards deviations of the normalised connection time, $\frac{1}{T}\tau_T^{(>,*)}$, for the RWP in the limit $T \to \infty$. Let us abbreviate $\tau_T^{(>,*)}$ by τ_T. We are going to give an explicit upper bound for the probability of the event $\{\tau_T \le Tp\}$ for any $p \in (0, p_*^{(>)})$. In order to formulate it, we need to introduce some more notation, which mostly stems from the theory of large deviations. See [DZ10] for more about this theory.

As a consequence of Lemma 3.3, also $(Z_{j-1}, Z_j)_{j\in\mathbb{N}}$ is a ψ-mixing and bounded Markov chain. As a nice consequence, we now have a large-deviation principle (LDP) for the empirical pair measure of the Z_n, defined as

$$Q_n := \frac{1}{n}\sum_{j=1}^{n} \delta_{(Z_{j-1},Z_j)} \in \mathcal{M}_1(\mathcal{D}\times\mathcal{D}), \tag{3.16}$$

see [BD96], Theorem 1 under the mixing condition (S) and the remark on page 554, which states that ψ-mixing implies (S). The rate function in [BD96, Theorem 1] is given by

$$I(Q) = \sup_{f\in B(\mathcal{D}^2,\mathbb{R})} \Big\{ \int_{\mathcal{D}^2} Q(\mathrm{d}x,\mathrm{d}y) f(x,y) - \Lambda(f) \Big\},$$

where $\Lambda(f) = \lim_{n\to\infty} \frac{1}{n}\log \mathbb{E}_*\big[\mathrm{e}^{\sum_{j=1}^n f(Z_{j-1},Z_j)}\big]$, and $B(\mathcal{D}^2,\mathbb{R})$ is the set of all bounded, Borel measurable functions on $\mathcal{D}^2$ to $\mathbb{R}$. We denote by $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D})$ the set of probability measures Q on $\mathcal{D}\times\mathcal{D}$ whose two marginals coincide. We denote any of the two marginals of such a Q by $\overline{Q}$, i.e., $\overline{Q}(A) = Q(A\times\mathcal{D}) = Q(\mathcal{D}\times A)$ for $A\in\mathcal{B}(\mathcal{D})$. Now we use [DZ10], Theorem 6.5.2 for the state space $\Sigma = \mathcal{D}^2$ and then Theorem 6.5.12 for $k=1$ to identify the rate function as

$$I(Q) = H(Q \mid \overline{Q}\otimes P) = \int_{\mathcal{D}}\int_{\mathcal{D}} Q(\mathrm{d}x,\mathrm{d}y)\log\frac{Q(\mathrm{d}x,\mathrm{d}y)}{\overline{Q}(\mathrm{d}x)P(x,\mathrm{d}y)} \text{if } Q \ll \overline{Q}\otimes P, \tag{3.17}$$

and $I(Q) = \infty$ otherwise, for $Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D})$.

Explicitly, the LDP states that the level sets $\{Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D})\colon I(Q)\le c\}$ are compact for any $c\in\mathbb{R}$, and that we have the estimates

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_*(Q_n\in F) \le -\inf_F I \qquad\text{and}\qquad \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}_*(Q_n\in G)\ge -\inf_G I,$$

for any closed, respectively open, subset F and G of $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D})$.

Theorem 1.6 follows from the following theorem. We now prefer the notation $\langle f, P\rangle$ for the integral of a function f with respect to a measure P. We recall from (3.15) that $p_*^{(>)} = \langle M^{(1)}F_>, \pi\otimes P\rangle / \langle M^{(1)}, \pi\otimes P\rangle$, where π is the invariant distribution of Z.

Theorem 3.5. *For any* $p\in(0,p_*^{(>)})$,

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}_*(\tau_T\le Tp)\le -\chi_p, \tag{3.18}$$

where

$$\chi_p = \inf\Big\{\frac{I(Q)}{\langle M^{(1)},Q\rangle}\colon Q\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{D}\times\mathcal{D}), \frac{\langle M^{(1)}F_>,Q\rangle}{\langle M^{(1)},Q\rangle}\le p\Big\}. \tag{3.19}$$

Moreover, the infimum is attained, and χ_p *is positive.*

The term $\langle M^{(1)}, Q\rangle$ is the average time that elapses between two subsequent arrivals at waypoints, if the two walkers move in such a way that the distribution of the location, velocity and next waypoint at two subsequent such arrivals is given by Q, and $\langle M^{(1)}F_>, Q\rangle$ is the average portion of connection time on such a way, and $I(Q)$ is the negative rate of the probability that the two follow that strategy Q per number of waypoints. Hence, the upper bound in (3.18) is intuitive and can be interpreted. Note that $F_>$ is lower semicontinuous, as the indicator of connectedness of two points through $\{f_* > \lambda_{\rm c}(R)\}$ is a countable sum of indicators of open sets. However, in general $F_>$ may not be upper semicontinuous. This makes it questionable whether or not also the lower bound in (3.18) holds, since the map $Q \mapsto \langle Q, M^{(1)}F_>\rangle$ is in general not continuous.

Proof of Theorem 3.5. That the infimum in (3.19) is attained is easily seen as follows. By lower semicontinuity of $F_>$ and a result of Fatou-type (see, e. g. [DZ10, Theorem D.12]), the map $Q \mapsto \langle Q, M^{(1)}F_>\rangle$ is also lower semicontinuous. Since also I is lower semicontinuous and has compact level sets and the map $Q \mapsto \langle Q, M^{(1)}\rangle$ is continuous, it easily follows that the infimum in (3.19) is even a minimum.

Now we argue that χ_p is positive. Indeed, the only minimiser of I on $\mathcal{M}_1^{(\rm s)}(\mathcal{D}\times\mathcal{D})$ is the measure $\pi\otimes P$, where we recall that π is the invariant distribution of Z and P its transition kernel. To see this, note that, for any Q satisfying $I(Q) = 0$, we have $Q({\rm d}x, {\rm d}y) = \overline{Q}({\rm d}x)P(x, {\rm d}y)$ by the equality discussion in Jensen's inequality, and from the marginal property it follows that $\overline{Q}$ is invariant for P, i.e., equal to π by uniqueness of the invariant distribution for the chain Z. Hence, also the only minimiser of $Q \mapsto I(Q)/\langle M^{(1)}, Q\rangle$ is $\pi\otimes P$, and it satisfies $p_*^{(>)} = \langle M^{(1)}F_>, \pi\otimes P\rangle/\langle M^{(1)}, \pi\otimes P\rangle$, see below (3.15). Therefore, it is not contained in the admissibility set on the right of (3.19) and is therefore not equal to its minimiser. Hence, χ_p is positive.

Now we prove (3.18). We are going to express the time T and the variable τ_T in terms of integrals over Q_n. First the time. We write $Z_j = \Big(\big(X^{(1)}_{S_j}, W^{(1)}_{N^{(1)}(S_j)}, V^{(1)}_{N^{(1)}(S_j)}\big), \big(X^{(2)}_{S_j}, W^{(2)}_{N^{(2)}(S_j)}, V^{(2)}_{N^{(2)}(S_j)}\big)\Big)$. From (3.13) and (3.14) we have, for any $n\in\mathbb{N}$,

$$S_n = n\langle M^{(1)}, Q_n\rangle \qquad \text{and} \qquad \tau_{S_n} = n\langle M^{(1)}F_>, Q_n\rangle,$$

recalling the definition of $M^{(1)}$ and of $F_>$ in (3.11), where $p_i(s) = X^{(i)}_{S_{j-1}} + s(X^{(i)}_{S_j} - X^{(i)}_{S_{j-1}})$. Hence, we can already give a heuristic proof of Theorem 3.5 as follows. The LDP for $(Q_n)_{n\in\mathbb{N}}$ roughly says that $\mathbb{P}_*(Q_n \approx Q) \approx {\rm e}^{-nI(Q)}$ for any strategy $Q\in\mathcal{M}_1^{(\rm s)}(\mathcal{D}^2)$. Taking n such that $T\approx S_n$, we have that $n\approx T/\langle M^{(1)}, Q_n\rangle$ and $\tau_T/T \approx \langle M^{(1)}F_>, Q_n\rangle/\langle M^{(1)}, Q_n\rangle$. Hence, we should have

$$\begin{aligned}\mathbb{P}_*(\tau_T \le pT) &\approx \mathbb{P}_*\Big(\langle M^{(1)}F_>, Q_n\rangle/\langle M^{(1)}, Q_n\rangle \le p\Big)\\ &\approx \exp\Big(-n\inf\Big\{I(Q)\colon Q\in\mathcal{M}_1^{(\rm s)}(\mathcal{D}^2), \frac{\langle M^{(1)}F_>, Q\rangle}{\langle M^{(1)}, Q\rangle} \le p\Big\}\Big)\\ &\approx {\rm e}^{-T\chi_p},\end{aligned}$$

with χ_p as defined in Theorem 3.5. The main difficulty in making this line of argument rigorous lies in the randomness of n.

Let us now give a rigorous proof of the upper bound in (3.18). Fix $p\in(0, p_*^{(>)})$ and pick a large auxiliary parameter K and a small one, $\delta > 0$. First we distinguish all the n no larger than KT such that $T\approx S_n$:

$$1 \le \sum_{n=\lfloor T/L\rfloor}^{\lfloor KT\rfloor} \mathbb{1}\{S_n \le T < S_{n+1}\} + \mathbb{1}\{T \ge S_{\lfloor KT\rfloor+1}\}.$$

On the first event, $\{S_n \le T < S_{n+1}\}$, we have,

$$\tau_T \ge \tau_{S_n} = n\langle Q_n, M^{(1)}F_>\rangle \ge (T-L)\frac{\langle M^{(1)}F_>, Q_n\rangle}{\langle M^{(1)}, Q_n\rangle} \ge T(1-\delta)\frac{\langle M^{(1)}F_>, Q_n\rangle}{\langle M^{(1)}, Q_n\rangle},$$

where the last inequality is true for all sufficiently large T (depending only on δ and L), which we want to assume from now.

Observe that $M^{(1)}$ is bounded from above by $L = \operatorname{diam}(D)/v_-$, with probability 1 with respect to Q for any $Q \in \mathcal{M}_1^{(s)}(\mathcal{D}^2)$, since D is bounded and all velocities are at least v_-. Hence, we have $S_j - S_{j-1} \le L$ for any $j \in \mathbb{N}$ and therefore also $0 < S_n/n \le L$ for any $n \in \mathbb{N}$. Therefore, the indicator on the event $\{S_n \le T < S_{n+1}\}$ can be upper bounded as

$$\mathbb{1}\{S_n \le T < S_{n+1}\} \le \mathbb{1}\{T - L \le S_n \le T\} \le \mathbb{1}\Big\{(1-\delta)\frac{T}{n} \le \langle M^{(1)}, Q_n\rangle \le \frac{T}{n}\Big\}.$$

This implies the upper bound

$$\mathbb{P}_*(\tau_T \le pT) \le \sum_{n=\lfloor T/L\rfloor}^{\lfloor KT\rfloor} \mathbb{P}_*\Big(\frac{\langle M^{(1)}F_>, Q_n\rangle}{\langle M^{(1)}, Q_n\rangle} \le \frac{p}{1-\delta}, (1-\delta)\frac{T}{n} \le \langle M^{(1)}, Q_n\rangle \le \frac{T}{n}\Big) + \mathbb{P}_*(T \ge S_{\lfloor KT\rfloor+1}).$$

The last term is an error term, as we will show later that

$$\lim_{K\to\infty} \limsup_{T\to\infty} \frac{1}{T}\log \mathbb{P}_*(T \ge S_{\lfloor KT\rfloor+1}) = -\infty. \tag{3.20}$$

Now we cut the sum over n into pieces of length $T\varepsilon$, where $\varepsilon > 0$ is a small auxiliary parameter:

$$\sum_{n=\lfloor T/L\rfloor}^{\lfloor KT\rfloor} = \sum_{i=1+\lfloor 1/L\varepsilon\rfloor}^{\lfloor K/\varepsilon\rfloor} \sum_{(i-1)T\varepsilon < n \le iT\varepsilon}.$$

For fixed i and $(i-1)T\varepsilon < n \le iT\varepsilon$, we can estimate, for any large T,

$$\mathbb{P}_*\Big(\frac{\langle M^{(1)}F_>, Q_n\rangle}{\langle M^{(1)}, Q_n\rangle} \le \frac{p}{1-\delta}, (1-\delta)\frac{T}{n} \le \langle M^{(1)}, Q_n\rangle \le \frac{T}{n}\Big) \le \mathbb{P}_*(Q_n \in A_i), \tag{3.21}$$

where

$$A_i = \Big\{Q \in \mathcal{M}_1^{(s)}(\mathcal{D}^2) \colon \frac{\langle M^{(1)}F_>, Q\rangle}{\langle M^{(1)}, Q\rangle} \le \frac{p}{1-\delta}, \frac{1-\delta}{i\varepsilon} \le \langle M^{(1)}, Q\rangle \le \frac{1}{(i-1)\varepsilon}\Big\}.$$

Recall that $F_>$ is lower semicontinuous. By [DZ10, Theorem D.12], the map $Q \mapsto \langle Q, M^{(1)}F_>\rangle$ is also lower semicontinuous. Hence, A_i is closed in the weak topology. Now we apply the upper bound in the above mentioned LDP, to obtain, as $T \to \infty$,

$$\sup_{(i-1)T\varepsilon < n \le iT\varepsilon} \mathbb{P}_*(Q_n \in A_i) \le \mathrm{e}^{-T\widetilde{\chi}_p(\delta,\varepsilon)}\, \mathrm{e}^{o(T)},$$

where

$$\begin{aligned}
\widetilde{\chi}_p(\delta,\varepsilon) &= (i-1)\varepsilon \inf\Big\{I(Q) \colon Q \in A_i\Big\} \\
&= (i-1)\varepsilon \inf\Big\{I(Q) \colon Q \in \mathcal{M}_1^{(s)}(\mathcal{D}^2), \frac{\langle M^{(1)}F_>, Q\rangle}{\langle M^{(1)}, Q\rangle} \le \frac{p}{1-\delta}, \frac{1-\delta}{i\varepsilon} \le \langle M^{(1)}, Q\rangle \le \frac{1}{(i-1)\varepsilon}\Big\} \\
&\ge \inf\Big\{I(Q)\Big(\frac{1-\delta}{\langle M^{(1)}, Q\rangle} - \varepsilon\Big) \colon Q \in \mathcal{M}_1^{(s)}(\mathcal{D}^2), \frac{\langle M^{(1)}F_>, Q\rangle}{\langle M^{(1)}, Q\rangle} \le \frac{p}{1-\delta}, \\
&\qquad\qquad \frac{1-\delta}{i\varepsilon} \le \langle M^{(1)}, Q\rangle \le \frac{1}{(i-1)\varepsilon}\Big\} \\
&\ge \inf\Big\{I(Q)\Big(\frac{1-\delta}{\langle M^{(1)}, Q\rangle} - \varepsilon\Big) \colon Q \in \mathcal{M}_1^{(s)}(\mathcal{D}^2), \frac{\langle M^{(1)}F_>, Q\rangle}{\langle M^{(1)}, Q\rangle} \le \frac{p}{1-\delta}\Big\} \\
&=: \chi_p(\delta,\varepsilon).
\end{aligned}$$

It is easy to see that $\lim_{\varepsilon\downarrow 0,\delta\downarrow 0}\chi_p(\delta,\varepsilon)=\chi_p$ as defined in (3.19). Hence, the upper bound in (3.18) is proved, subject to (3.20), which we prove now.

Note that $\{S_n\colon n\in\mathbb{N}_0\}=\{T_n^{(1)}\colon n\in\mathbb{N}_0\}\cup\{T_n^{(2)}\colon n\in\mathbb{N}_0\}$, where $T_n^{(i)}$ denotes the arrival time of the i-th walker at the n-th waypoint. The j-th step $U_j^{(1)}$ of the first of these processes is the duration of the first walker's travel from the j-th to the $j+1$-st waypoint. Hence,

$$\mathbb{P}_*(T\geq S_{\lfloor KT\rfloor+1})\leq 2\mathbb{P}_*(T\geq T^{(1)}_{\lfloor KT/2\rfloor+1})\leq 2\mathbb{P}_*(T\geq \widetilde{T}^{(1)}_{\lfloor KT/4\rfloor}),$$

where $\widetilde{T}_n^{(1)}=\sum_{j=0}^{n-1}U_{2j}^{(1)}$ denotes the random walk consisting of the even steps only. Hence, we are now looking at downwards deviations of the random walk $(\widetilde{T}_n^{(1)})_{n\in\mathbb{N}}$, whose steps $U_{2j}^{(1)}$ are i.i.d. with support in $[0,L]$. Therefore, Cramér's theorem yields

$$\begin{aligned}\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}_*(\widetilde{T}^{(1)}_{\lfloor KT/4\rfloor}\leq T)&\leq\frac{K}{4}\limsup_{T\to\infty}\frac{1}{KT/4}\log\mathbb{P}_*\Big(\widetilde{T}^{(1)}_{\lfloor KT/4\rfloor}\leq\frac{4}{K}\lfloor KT/4\rfloor\Big)\\&\leq-\frac{K}{4}\sup_{\lambda<0}\Big(\lambda\frac{4}{K}-\log\mathbb{E}_*[\mathrm{e}^{\lambda U_0^{(1)}}]\Big)=-\sup_{\lambda<0}\Big(\lambda-\frac{K}{4}\log\mathbb{E}_*[\mathrm{e}^{\lambda U_0^{(1)}}]\Big).\end{aligned}$$

Note that the essential infimum of $U_0^{(1)}$ is equal to zero, as we assumed that the waypoint measure has a continuous density. Indeed, if the waypoint walker stands in his waypoint, with probability 1 there is a nontrivial ball around the location in which the waypoint measure has a positive density, and therefore arbitrarily small travels to the next waypoint have a positive probability.

Hence, $\log\mathbb{E}_*[\mathrm{e}^{\lambda U_0^{(1)}}]=o(|\lambda|)$ as $\lambda\to-\infty$, and therefore it is possible to pick a sequence $\lambda_K\to\infty$ as $K\to\infty$ such that $\lambda_K-\frac{K}{4}\log\mathbb{E}_*[\mathrm{e}^{\lambda_K U_0^{(1)}}]\to\infty$ as $K\to\infty$. This implies that (3.20) holds and finishes the proof of Theorem 3.5.

□

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