THE FREE ENERGY OF THE INTERACTING BOSE GAS: A VARIATIONAL DESCRIPTION WITH LOOPS AND INTERLACEMENTS

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(15 April 2026)

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Abstract: We consider the interacting Bose gas in the thermodynamic limit in a large box in $\mathbb{R}^d$ at positive temperature $1/\beta \in (0,\infty)$ with particle density $\sim \rho \in (0,\infty)$. We follow a path-integral approach and adopt from [ACK11] a description of the free energy in terms of the *Brownian loop soup*, a Poisson point process consisting of Brownian bridges, also called loops or cycles. It is the objective of this paper to derive, for any values of β and ρ, a formula for the limiting free energy with explicit control on the particle numbers in the short and in the long loops. The latter are presumed to play the role of the condensate, according to Feynman's [Fe53] famous, vague suggestion, and they turn into *random interlacements* (bi-infinite, locally finite random processes in $\mathbb{R}^d$) in our formula. In [ACK11] there was no concept that could describe the long loops; only small ρ could be handled successfully.

In the present paper we represent the limiting free energy in terms of a variational formula, ranging over the set of all stationary point processes with loops and with interlacements, having each a given particle density, and minimizing the sum of the interaction energy and a characteristic entropy term. The latter is a new kind of a *specific relative entropy density* with respect to the reference process of loops (the Brownian loop soup), together with an independent Markov kernel describing collections of path shreds in large boxes. In $d \geq 3$, the latter can be seen as a projection of the *Brownian interlacement Poisson point process with β-spacing*. Our proof tool box comes from large-deviation theory, both for the derivation of the formula for the free energy and for the proof of the existence of the specific relative entropy.

The limiting variational formula has good compactness properties and may serve as a starting point to future deeper investigations, as it describes the interacting Bose gas in much smoother and more well-behaved terms. We give various approximations of that formula, in particular we show that it is equal to a version without interlacements. A brief discussion about the conjectured relation to the famous *Bose–Einstein condensation* phenomenon is given as well.

MSC 2020. 60F10; 60J65; 82B10; 81S40

Keywords and phrases. Interacting Bose gas, free energy, path-integral analysis, Brownian loop soup, thermodynamic limit, Brownian bridges, random marked spatial point processes, long loops and short loops, Brownian interlacements, specific relative entropy density, large deviations, empirical stationary measure, variational formula.

Contents

1. Introduction

1.1. **Background, ansatz and purpose.** In this paper, we study the interacting Bose gas in the thermodynamic limit, a well-known quantum model from statistical physics that is of particularly high interest since decades because of the conjectured phase transition called *Bose–Einstein condensation (BEC)*. Its introduction is in terms of the symmetrised trace of the exponential of an interacting Hamilton operator, and its crucial properties are formulated in terms of spectral properties of the convolution operator formed out of the corresponding one-particle density matrix. Notably, the usual notion of a phase transition as the existence of a non-analyticity of the free energy as a function of temperature (or of the particle density) is not the core of the definition of BEC, but a much finer property called *off-diagonal long-range order (ODLRO)* of the mentioned symmetrised operator. The validity or non-validity of ODLRO is a widely open problem of mathematical physics, and one is currently far from an understanding, in spite of many phenomenological ansatzes. The main conjecture, which is also in good coincidence with the famous experiments from 1995, is that ODLRO (and therefore BEC) is valid in dimensions $d \geq 3$ at sufficiently small temperature but not at other temperatures and not in other dimensions. In this paper, we are not going to study ODLRO and refer the reader for this question to the vast literature (e.g., a concise literature survey in Section 1.7 of [KVZ25]). Observe that the free (i.e., non-interacting) case is much simpler and has been thoroughly understood, since all spatial properties easily wash out, and the model drastically simplifies. Furthermore several rescaled regimes have been successfully worked out as well [FKSS20a]. However, the interacting case in the thermodynamic limit is still widely open, and this is that we concentrate on here.

Even though the phenomenon of ODLRO cannot be decided by looking at the limiting free energy alone, a comprehensive understanding of the free energy is obviously likely to contribute a lot to the understanding of BEC. Therefore we concentrate in the present paper on understanding the limiting free energy in a comprehensive and detailed way. We are doing this from the view point of a *random loop ensemble* using a *path-integral analysis*. Random-loop presentations of the interacting Bose gas are known at least since the 1950s and were first considered a mathematical tool only, but Feynman [Fe53] raised the, by now famous, vague advice to think about the loop length distribution as an order parameter that might have some physical relevance. This idea was phenomenologically taken up occasionally by mathematical physicists with strong probability background, and it inspired several investigations of related models without any physical relevance, but triggered also phenomenological discussions about possible relations between BEC and the loop length statistics. See Section 3.3 for a thorough survey on the literature on path-integral approaches to the Bose gas. In short words, the rough idea was coined that BEC should have much to do (if not even being equivalent to) with the occurrence of a macroscopic part of the particles in long loops. In several simplified models this conjecture was confirmed, but there were also counter-examples found. For the most important case of an interacting Bose gas in the thermodynamic limit, the question of the relation between BEC and macroscopic occupation of long loops has not been settled at all, even more, there is currently no mathematical tool box that might be able to attack this question.

The present paper changes this situation. We introduce a mathematical path-integral framework for describing all loops, distinguishing long and short ones, in the interacting random loop ensemble in the thermodynamic limit. The starting point is, as in [ACK11], a representation in terms of a Poisson point process (PPP) on loops, a version of the *Brownian loop soup.* In this paper, we carry out a mesoscopic decomposition procedure of the macrobox into smaller subboxes, a large-deviation ansatz, and arrive at a variational description of the limiting free energy, which is our main result. The state spaces that arise along this way are spaces of *point processes with marks*, which comprehensively describe the entire configuration of loops. However, in order to describe also long loops in the limit, we need to enlarge the point process space. One innovative aspect (a novelty with respect to [ACK11]) is that the long loops are described in term of point processes of *interlacements*, which are bi-infinite and locally finite functions $\mathbb{R} \to \mathbb{R}^d$. Observe that such processes cannot be seen as marked point processes in $\mathbb{R}^d$. This extended probability space is difficult to handle, since it consists of very non-local quantities, which induce far-reaching correlations. The variational description is in terms of the minimization of energy (pair interaction between all the legs) plus entropy (negative exponential rate of probabilities) for point processes of loops and interlacements with fixed given particle density in loops and in interlacements, respectively. The interlacement-part of the entropy, which also a novelty of the present paper, is relative with respect to an explicit reference point process kernel. Interestingly, this kernel is, in $d \geq 3$, reminiscent of a local version of Sznitman's *Brownian interlacement PPP*, see the Appendix.

The arising variational description has good compactness properties, and the minimizing point process of loops and interlacements has the interpretation of the typical, the 'optimal', spatial distribution of all the particles of the Bose gas, organised in local and in global structures. Decisive is then the question for what values of the particle density the minimizer has a non-trivial interlacement-part. This is obviously a mathematically sound formulation of the distinction between the occurrence and non-occurrence of long Feynman-cycles. We conjecture that the interlacement particles directly relate (if they are not even equal) to the particles in the condensate. Conjecturally, the occurrence of interlacements will in future work turn out to be equivalent to the occurrence of BEC (i.e., ODLRO) and, in dimension $d \geq 3$, it will turn out occur for all large particle densities only. In the present paper, we leave both questions open, but we believe that the mathematical framework that we develop offers good chances to answer them in future work.

We introduce the interacting Bose gas in Section 1.2, present the starting point of our investigations (a rewrite in terms of the Brownian loop soup) in Section 1.3 and explain our ansatz in Section 1.4. A quick survey on the remainder of the paper is at the end of Section 1.4.

1.2. **The interacting Bose gas.** We consider an interacting bosonic many-body system in a large box in $\mathbb{R}^d$ at positive temperature $1/\beta \in (0,\infty)$ with fixed particle density $\rho \in (0,\infty)$ in the thermodynamic limit. To start with, denote by

$$\mathcal{H}_N^{(\Lambda,\mathrm{bc})} = -\sum_{i=1}^N \Delta_i + \sum_{1\leq i<j\leq N} v(x_i - x_j), \qquad x_1,\dots,x_N \in \Lambda, \tag{1.1}$$

the N-particle operator in a box $\Lambda \subset \mathbb{R}^d$ with some boundary condition 'bc' (later we will consider periodic and Dirichlet zero boundary conditions) with kinetic energy and pair interaction given by a measurable interaction functional $v\colon \mathbb{R}^d \to [0,\infty]$; see Assumption (V) in Section 2.3 for more specific assumptions.

As we are interested in *bosons*, we project the operator $\mathcal{H}_N^{(\Lambda,\mathrm{bc})}$ on the set of symmetric, i.e., permutation invariant, wave functions. Furthermore, we consider the particle system at positive temperature $1/\beta \in (0,\infty)$. The main object of the study of this paper is the *partition function* of the system, that is, the trace of the operator $\mathrm{e}^{-\beta\mathcal{H}_N^{(\Lambda,\mathrm{bc})}}$ in Λ with symmetrisation:

$$Z_N^{(\mathrm{bc})}(\beta,\Lambda) = \mathrm{Tr}_+^{(\Lambda,\mathrm{bc})}(\mathrm{e}^{-\beta\mathcal{H}_N^{(\Lambda,\mathrm{bc})}}), \tag{1.2}$$

where the index $+$ denotes this symmetrisation, i.e., the projection on the set of permutation invariant wave functions. We will introduce the *particle density* $\rho \in (0,\infty)$ (the number of particles per unit volume), fix a centred box Λ_N of volume $\sim N/\rho$, and study the *limiting free energy per unit volume* in the *thermodynamic limit*,

$$\mathrm{f}(\beta,\rho) = -\frac{1}{\beta}\lim_{N\to\infty}\frac{1}{|\Lambda_N|}\log Z_N^{(\mathrm{bc})}(\beta,\Lambda_N). \tag{1.3}$$

Under suitable conditions on the interaction potential, the existence of this limit is well-known (see, e.g., [Rue69, Thm. 3.58]) for many decades and also the fact that it does not depend on the boundary condition (at least for periodic and Dirichlet zero conditions). However, an explicit or even interpretable formula is still lacking (with the exception of [ACK11, Cor. 1.3] for small values of ρ; see Section 3.3). It is the main purpose of this paper to provide some. It will be of the form (see Corollary 2.7)

$$\mathrm{f}(\beta,\rho) = \frac{1}{\beta}\inf\{F_U(P) + \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)\colon P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S}), \mathfrak{N}_U^{(\ell)}(P) = \rho\},$$

where $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ is the set of stationary point processes of loops and interlacements in $\mathbb{R}^d$, F_U is the interaction energy in the unit box $U = [-\frac{1}{2},\frac{1}{2}]^d$, $\mathrm{h}^{(\mathcal{L},\mathcal{S})}$ is an entropy functional introduced in Theorem 2.3, and $\mathfrak{N}_U^{(\ell)}$ registers the particle density. Our actual main result, Theorem 2.5, in combination with Corollary 2.6, even makes the distinction between short and long loops rigorously visible; actually we will show that the short loops lead to the $\mathcal{L}$-part and the long loops give rise to the $\mathcal{S}$-part. This gives a mathematically rigorous meaning to what is often called the *Feynman cycles*. Our proof is flexible enough to express several other characteristic quantities of the interacting Bose gas, like the density of loops having a given length or with various other properties.

It is the goal of this line of research to employ that formula as a platform for deeper analysis of the interacting Bose gas in view of the famous Bose–Einstein phase transition. A natural next step would be to derive a proof for the conjecture that, in dimension $d \geq 3$, for every large particle density, the minimising point process cannot contain loops only but must have a non-trivial interlacement part. We are not going to do that in this paper and devote future work to that. See Section 2.4 for more about that.

We will keep the inverse temperature $\beta \in (0,\infty)$ fixed for the entire paper and therefore omit it in most of the notation.

1.3. **Rewrite in terms of the Brownian loop soup.** It is known since decades that the partition function $Z_N^{(\mathrm{bc})}(\beta,\Lambda)$ can be written in terms of the famous *Feynman–Kac formula* in terms of an ensemble of a random number of Brownian bridges (cycles) in Λ with total sum of lengths equal to N and a pair-interaction between any two legs of the cycles. Such presentation is the starting point of our analysis, and we are going to introduce a version of it that was found in [ACK11], based on the classical text [G70]. This formula is in terms of a marked interacting Poisson point process of Brownian cycles, sometimes called the Brownian loop soup. We need to introduce some notation, which will be crucial for the entire paper.

For $k \in \mathbb{N}$, let $\mathcal{C}_k$ denote the space of continuous functions $f\colon [0, k\beta] \to \mathbb{R}^d$, and $\mathcal{C} := \bigcup_{k\in\mathbb{N}} \mathcal{C}_k$ the union of all these spaces. Let us write $\mathcal{C}_k^{(\circlearrowleft)} = \{f \in \mathcal{C}_k : f(k\beta) = f(0)\}$ for the set of loops in $\mathcal{C}_k$, and $\mathcal{C}^{(\circlearrowleft)} = \bigcup_{k\in\mathbb{N}} \mathcal{C}_k^{(\circlearrowleft)}$ for their union. The *length* of $f \in \mathcal{C}_k$ is $\ell(f) := k$. We call all the k sites $f(0), f(\beta), f(2\beta), \dots, f((\ell(f)-1)\beta)$ the *particles* of f, hence $\ell(f)$ is the number of particles of f. On $\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)}$ we are going to introduce our crucial reference measure, a Poisson point process (PPP) with a certain intensity measure. This process and variants thereof are often called the *Brownian loop soup*. The canonical *Brownian bridge measure* from x to y in time $k\beta$ in the box Λ with boundary condition 'bc' is defined by

$$\mu_{x,y}^{(k,\Lambda,\mathrm{bc})}(A) = \frac{\mathbb{P}_x^{(k,\Lambda,\mathrm{bc})}(B \in A; B_{k\beta} \in \mathrm{d}y)}{\mathrm{d}y}, \qquad A \subset \mathcal{C}_k^{(\circlearrowleft)} \text{ measurable.} \tag{1.4}$$

Here, $B = (B_t)_{t\in[0,k\beta]}$ is a Brownian motion in $\mathbb{R}^d$ with generator $\Delta^{(\Lambda,\mathrm{bc})}$, starting from x under $\mathbb{P}_x^{(k,\Lambda,\mathrm{bc})}$; the latter is the Brownian motion in Λ on the time interval $[0, k\beta]$ with this boundary condition. We will be considering periodic boundary condition or Dirichlet zero boundary condition and will consider boxes Λ only.

We now introduce a variant of Dirichlet zero boundary conditions for the bridge measure that will later make the analysis less cumbersome, but does not come from any generator with any boundary condition. We require that every particle is in Λ, i.e., we restrict to the event $\{B_{l\beta} \in \Lambda\ \forall l \in \{0, 1, 2, \dots, k\}\}$. This can be seen as Dirichlet zero boundary condition for the Gaussian random walk $(B(0), B(\beta), B(2\beta), \dots, B(k\beta))$ plus interpolation with a free Brownian bridge between the times $0, \beta, 2\beta, \dots, k\beta$. It will turn out to be a useful condition, since it perfectly harmonises with the crucial operator $\Pi_\Lambda^{(\mathcal{L})}$ that we will introduce in Section 4.1. We refer to this condition as *particle-boundary condition*, and we will write 'per', '0' and 'par', respectively, for each of these three boundary conditions. For each of these conditions, $\mu_{x,y}^{(k,\Lambda,\mathrm{bc})}$ is a regular Borel measure on $\mathcal{C}_k^{(\circlearrowleft)}$ with total mass equal to

$$\mu_{x,y}^{(k,\Lambda,\mathrm{bc})}(\mathcal{C}_k^{(\circlearrowleft)}) = g_{k\beta}^{(\Lambda,\mathrm{bc})}(x,y) = \frac{\mathbb{P}_x^{(k,\Lambda,\mathrm{bc})}(B_{k\beta} \in \mathrm{d}y)}{\mathrm{d}y}. \tag{1.5}$$

If we do not have any boundary condition (and no set Λ), then we write $\mu_{x,y}^{(k)}$ for the canonical Brownian bridge measure from x to y in the time interval $[0, k\beta]$.

We denote by $\mathcal{M}_{\mathbb{N}_0}(\mathcal{X})$ the set of simple point measures on a Polish space $\mathcal{X}$. By $\mathcal{L} = \mathcal{M}_{\mathbb{N}_0}(\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)})$ we denote the set of simple point measures $\omega = \sum_{x\in\zeta} \delta_{(x,f_x)}$ with $f_x(0) = x$ on $\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)}$, which we also see as the set of all marked point processes in $\mathbb{R}^d$ with marks in $\mathcal{C}^{(\circlearrowleft)}$. We introduce a reference probability measure on $\mathcal{L}$, which is a *Poisson point process* in $\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)}$:

Definition 1.1 (The reference loop Poisson point process)**.** *For any $\Lambda \Subset \mathbb{R}^d$ and any boundary condition* $\mathrm{bc} \in \{\mathrm{per}, 0, \mathrm{par}\}$*, we write $\mathsf{Q}^{(\Lambda,\mathrm{bc})}$ for the distribution of the Poisson point process (PPP) ω_{P} on $\Lambda \times \mathcal{C}^{(\circlearrowleft)}$ with intensity measure*

$$\nu_\Lambda^{(\mathrm{bc})}(\mathrm{d}x, \mathrm{d}f) := \sum_{k\in\mathbb{N}} \frac{1}{k}\, \mathrm{d}x \otimes \mu_{x,x}^{(k,\Lambda,\mathrm{bc})}(\mathrm{d}f) \qquad x \in \Lambda, f \in \mathcal{C}^{(\circlearrowleft)}. \tag{1.6}$$

For Λ replaced by $\mathbb{R}^d$, we write Q for the PPP on $\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)}$ with intensity measure $\nu(\mathrm{d}x, \mathrm{d}f) = \sum_{k\in\mathbb{N}} \frac{1}{k} \mathrm{d}x \otimes \mu_{x,x}^{(k)}(\mathrm{d}f)$.

The process in Definition 1.1 was introduced to the study of the Bose gas in [ACK11]. A close cousin of it was already used for the study of other phenomena in statistical mechanics (e.g., conformal invariance in dimension two) in [LW04] under the name *Brownian loop soup*. Here we rely on the recent adaptation in [KVZ25] and refer proofs to Appendix A there. The projection on the loop part, i.e., $\{f_x : x \in \zeta\}$, is what is called sometimes the Brownian loop soup in the inner sense.

Denote by

$$\mathfrak{N}_\Lambda(\omega) = |\zeta \cap \Lambda| \qquad \text{and} \qquad \mathfrak{N}_\Lambda^{(\ell)}(\omega) = \sum_{x\in\zeta\cap\Lambda} \ell(f_x), \qquad \Lambda \subset \mathbb{R}^d, \tag{1.7}$$

the number of points of ω in Λ (i.e., the number of loops that start in Λ), and the total number of particles of ω attached to points in Λ, respectively.

We introduce a functional on $\mathcal{L}$ that expresses the interaction between any two particles of loops that start within Λ. For $\omega \in \mathcal{L}$ and $\Lambda, \widetilde{\Lambda} \Subset \mathbb{R}^d$, we define $\Phi_{\Lambda,\widetilde{\Lambda}} \colon \mathcal{L} \to [0,\infty]$ by

$$\Phi_{\Lambda,\widetilde{\Lambda}}(\omega) = \sum_{x\in\zeta\cap\Lambda, y\in\zeta\cap\widetilde{\Lambda}} \mathfrak{V}(f_x, f_y), \qquad \omega \in \mathcal{L}, \tag{1.8}$$

where

$$\mathfrak{V}(f_x, f_y) = \frac{1}{2} \sum_{i=1}^{\ell(f_x)} \sum_{j=1}^{\ell(f_y)} \mathbb{1}_{\{(x,i)\neq(y,j)\}} V(f_{x,i}, f_{y,j}), \tag{1.9}$$

and

$$V(f,g) = \int_0^\beta v(f(s) - g(s))\,\mathrm{d}s, \tag{1.10}$$

and $f_i(\cdot) = f((i-1)\beta + \cdot)|_{[0,\beta]}$ is the i-th leg of a function $f \in \mathcal{C}$.

We write $\mu(f) = \int f(\omega)\,\mu(\mathrm{d}\omega) = \langle \mu, f\rangle = \langle f, \mu\rangle$ for the integral of a function f with respect to a measure μ whenever this makes sense. The following is Proposition 1.1 in [ACK11].

Proposition 1.2 (The Bose gas in terms of the marked Poisson point process)**.** *Fix $\beta \in (0,\infty)$. Let $v\colon [0,\infty) \to (-\infty,\infty]$ be measurable and bounded from below, and let $\Lambda \subset \mathbb{R}^d$ be measurable with finite volume (assumed to be a torus for periodic boundary condition). Then, for any $N \in \mathbb{N}$, and* $\mathrm{bc} \in \{\mathrm{per}, 0\}$,

$$Z_N^{(\mathrm{bc})}(\Lambda) = \mathrm{e}^{\nu_\Lambda^{(\mathrm{bc})}(\Lambda\times\mathcal{C}^{(\circlearrowleft)})} \mathsf{Q}^{(\Lambda,\mathrm{bc})}\big[\mathrm{e}^{-\Phi_{\Lambda,\Lambda}} \mathbb{1}\{\mathfrak{N}_\Lambda^{(\ell)} = N\}\big]. \tag{1.11}$$

That is, up the non-random term $\nu_\Lambda^{(\mathrm{bc})}(\Lambda \times \mathcal{C}^{(\circlearrowleft)})$, the partition function is equal to the expectation over the Boltzmann factor $\mathrm{e}^{-\Phi}$ of a marked Poisson point process in Λ, restricted to a fixed total number of particles. See Figure 1 for an illustration.

From [ACK11, Lemma 3.12] we already know that the first term on the right-hand side of (1.11) satisfies, for any $\rho \in (0,\infty)$,

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \nu_{\Lambda_N}^{(\mathrm{bc})}(\Lambda_N \times \mathcal{C}^{(\circlearrowleft)}) = (4\pi\beta)^{-d/2} \sum_{k\in\mathbb{N}} k^{-1-d/2} =: \overline{q}, \tag{1.12}$$

where Λ_N is the centred box with volume $\sim N/\rho$ (it was shown there for periodic and Dirichlet boundary condition only, but it is easy to see that it holds also for particle boundary condition). Hence, it appears natural to formulate all our main assertions about the interacting Bose gas in terms of the transformed probability measure

$$\mathrm{d}\widehat{\mathsf{Q}}^{(\Lambda,\mathrm{bc})} = \frac{\mathrm{e}^{-\Phi_{\Lambda,\Lambda}}}{\widehat{Z}^{(\mathrm{bc})}(\Lambda)}\,\mathrm{d}\mathsf{Q}^{(\Lambda,\mathrm{bc})}, \qquad \text{where } \widehat{Z}^{(\mathrm{bc})}(\Lambda) = \mathsf{Q}^{(\Lambda,\mathrm{bc})}[\mathrm{e}^{-\Phi_{\Lambda,\Lambda}}], \tag{1.13}$$

and we do that for all three boundary conditions (periodic, zero, particle). Note that $\widehat{\mathsf{Q}}^{(\Lambda,\mathrm{bc})}$ is a probability measure on the set of configurations with an arbitrary number of particles; there is no condition on $\mathfrak{N}_{\Lambda^{(\ell)}}$.

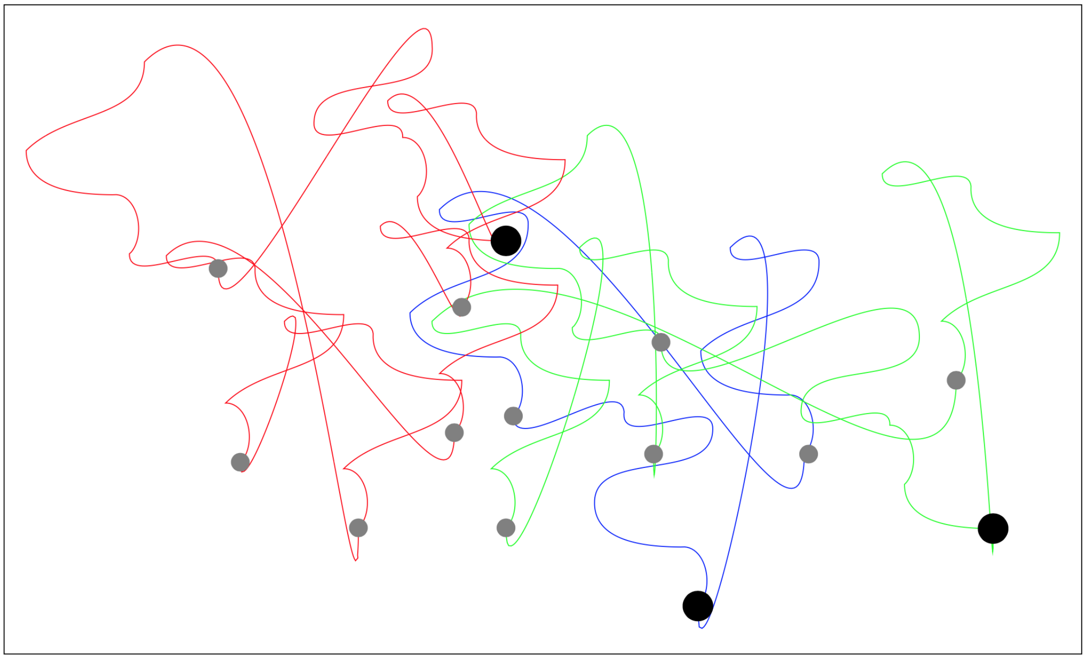

FIGURE 1. Illustration (taken from [CJK23]) of a realisation of the Bose gas with 14 particles (grey and black bullets), organised in three Brownian bridges, attached to three Poisson points (black bullets). The red cycle has six particles, the blue and green ones each four.

1.4. **Purpose of this paper.** The main purpose of the present paper is to identify, in the thermodynamic limit, the limiting free energy with distinction between long and short loops. The most immediate such distinction would be to call all particles in loops with lengths $> L$ the *L-condensate*, and to register its cardinality and to quantify its contribution to the partition function in the limit as $N \to \infty$, followed by $L \to \infty$. See Figure 2 for an illustration of a configuration with one long loop.

However, we are going to rely on a slightly different distinction, which distinguishes the loops according to space criteria rather than length criteria, which will come in handy in our proofs and is also dictated by the presence of interactions, which are spatially defined. Furthermore, this new concept of distinction between short and long loops opens up the possibility of suitable methods to cut the long loops into pieces ("shreds") and to embed these pieces later into infinitely long trajectories.

Fix $\rho \in (0, \infty)$ and a centred box Λ_N in $\mathbb{R}^d$ with volume $|\Lambda_N| \sim N/\rho$. Fix $R \in \mathbb{N}$ and write $W = [-R, R]^d$ and $W_z = z + W$ for $z \in 2R\mathbb{Z}^d$. We assume N and R to be such that for suitable $Z_{N,R} \subset 2R\mathbb{Z}^d$ we have that $\Lambda_N = \bigcup_{z \in Z_{N,R}} W_z$ is the (up to boundaries, disjoint) union of the subboxes W_z with $z \in Z_{N,R}$. (Here, R may depend on N and converges to R as $N \to \infty$; we will sometimes not indicate this dependence in the notation.) We will consider N particles in Λ_N.

We will call a loop *R-crossing* if it has particles in at least two different ones of these subboxes. Now we consider the number

$$\mathfrak{N}_\Lambda^{(\ell,R)}(\omega) = \sum_{x \in \zeta \cap \Lambda} \ell(f_x) \mathbb{1}\{f_x \text{ is } R\text{-crossing}\}, \qquad \omega = \sum_{x \in \zeta} \delta_{(x, f_x)}, \tag{1.14}$$

of particles in R-crossing loops and take this as a substitute of the number of particles in long loops. We call the particles in R-crossings the *R-condensate*, and $\frac{1}{|\Lambda_N|} \mathfrak{N}_\Lambda^{(\ell,R)}(\omega)$ the *R-condensate density.*

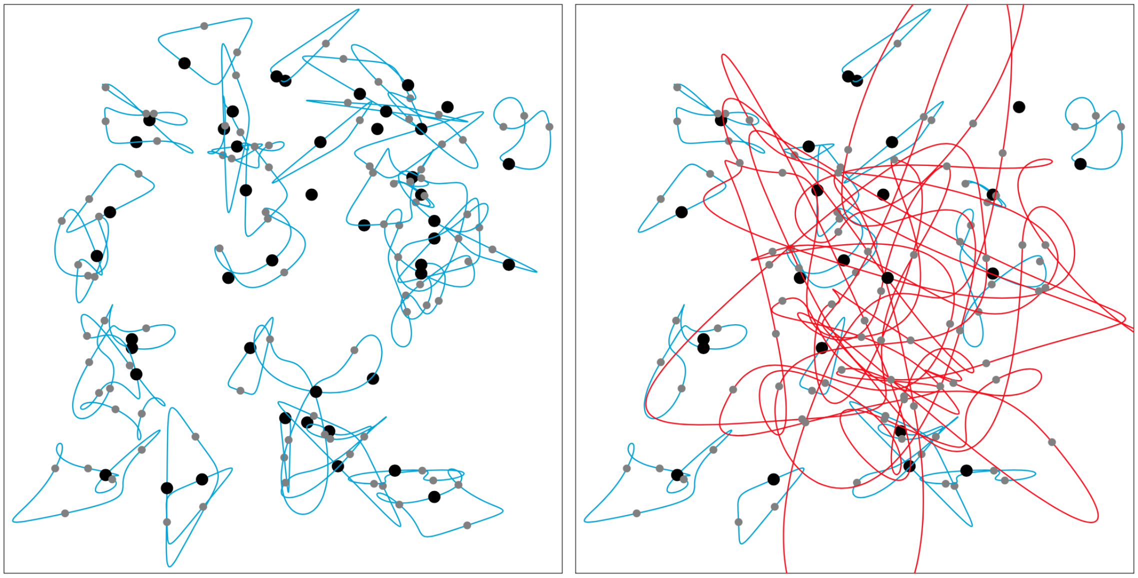

FIGURE 2. Illustration (taken from [CJK23]) of two realisations of a Bose gas: left with low density with only short loops, and right with higher density with one additional long loop (red).

The number of all the other particles is denoted by

$$\mathfrak{N}_\Lambda^{(\ell,\neg R)}(\omega) = \sum_{x\in\zeta\cap\Lambda} \ell(f_x)\mathbb{1}\{f_x \text{ is not } R\text{-crossing}\}, \tag{1.15}$$

i.e., the number of particles in loops whose particles are entirely contained in one of the subboxes $z+W$. Note that $\mathfrak{N}_\Lambda^{(\ell,R)} + \mathfrak{N}_\Lambda^{(\ell,\neg R)} = \mathfrak{N}_\Lambda^{(\ell)}$.

The main object of interest is the pair $(\mathfrak{N}_{\Lambda_N}^{(\ell,R)}, \mathfrak{N}_{\Lambda_N}^{(\ell,\neg R)})$ in the limit $N\to\infty$, followed by $R\to\infty$. We will formulate our main findings in Theorem 2.5 in terms of a large-deviation principle for this pair (properly normalised) under the measure $\widehat{\mathtt{Q}}^{(\Lambda,\mathrm{bc})}$ defined in (1.13).

See Section 3.1 for an explanation that this concept of R-crossing loops is equivalent to the more familiar concept of long loops.

Here is a survey of the remainder of the paper: In Section 2 we present our main results, in Section 3 we make various comments about the impact and background of our results and the history of the problem, and we survey our proof strategy, in Section 4 we begin our proof by introducing the necessary framework for point processes, introducing our main tool, a crucial empirical measure of the loop-shred configurations in subboxes of a fixed radius R, and rewriting the free energy in terms of it. In Section 5, we examine the distribution of a crucial part of our empirical measure, the configurations of the entry-exit sites of the shreds into the subboxes. The necessary large-deviation setting is developed and applied in Section 6. For making the limit as $R\to\infty$ afterwards, we introduce in Section 8 the crucial specific relative entropy density. Finally, we finish the proof of our main theorem (the description of the free energy in terms of a variational formula) in Section 9. As preparations, some crucial, but technical properties of the formula are proved in Section 7. In the Appendix, Section A, we explain the connection of our description with the Brownian interlacement Poisson point process by Sznitman.

2. Main results

In this section we describe our main results. In Section 2.3 we formulate our result on the identification of the constrained free energy with full control on the number of particles in R-crossing loops and in the others (i.e., in long and short loops). This formula will be a variational formula that plays in the world of stationary processes of points with loops as marks, superposed with a point process of interlacements, i.e., infinitely long double-sided locally finite trajectories. In the formula, such processes minimise the sum of an energy term plus an entropy term, subject to a fixed particle density in loops and in interlacements, respectively. The underlying loop space was already introduced in Section 1.3, but we also need to introduce the interlacement space, which we do in Section 2.1. The crucial and novel specific relative entropy density term is introduced in Section 2.2. Then we formulate our main result on the free energy in Section 2.3. The arising characteristic variational formula contains a lot of information about the limiting behaviour of the interacting Bose gas; this is presented in Section 2.4.

2.1. **The interlacement space and the shred spaces.** We are going to introduce the space on which we will describe the long loops and their particles in the limit as $N \to \infty$. Define

$$\mathcal{C}_\infty = \Big\{ g \in \mathcal{C}(\mathbb{R} \to \mathbb{R}^d) \colon \{g(k\beta) \colon k \in \mathbb{Z}\} \text{ is locally finite} \Big\}, \tag{2.1}$$

where by $\mathcal{C}(\mathbb{R} \to \mathbb{R}^d)$ we denote the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$. We will always tacitly identify functions in $\mathcal{C}_\infty$ as equal if one of them is the time-shift of the other by an integer multiple of β; hence strictly speaking we are dealing with equivalence classes modulo time shifts in $\beta\mathbb{Z}$.

Remark 2.1 (Interlacements) The term 'interlacements' (in plural) is often used for the range $\bigcup_{g\in\Gamma_{\rm B}} g(\mathbb{R})$ of a certain Poisson point process $\varpi^{({\rm B})} = \sum_{g\in\Gamma_{\rm B}} \delta_g$ of bi-infinite Brownian motions in $\mathbb{R}^d$ with $d \geq 3$ (see Remark 2.2), but this set will play no role in this paper, as we will have to work always with the entire trajectories g. Therefore, we will take the freedom to call every element of $\mathcal{C}_\infty$ an *interlacement*. $\Diamond$

We call the sites $g(k\beta)$ the *particles* of the interlacement $g \in \mathcal{C}_\infty$ and the pieces $g_k = g|_{[\beta k, \beta(k+1)]}$ the *legs* of g. We consider the space

$$\mathcal{S} = \Big\{ \varpi = \sum_{g\in\Gamma} \delta_g \colon \Gamma \subset \mathcal{C}_\infty, \varpi \text{ is simple}, \{g(k\beta) \colon k \in \mathbb{Z}, g \in \Gamma\} \text{ is locally finite} \Big\} \tag{2.2}$$

of all simple locally finite point processes with points in $\mathcal{C}_\infty$. The set $\{g(k\beta) \colon k \in \mathbb{Z}, g \in \Gamma\}$ is the set of all particles of ϖ. For $W \Subset \mathbb{R}^d$,

$$\mathfrak{N}_W^{(\ell)}(\varpi) = \sum_{g\in\Gamma} \ell_W(g), \qquad \text{where } \ell_W(g) = \sum_{k\in\mathbb{Z}} \mathbb{1}_W(g(k\beta)), \tag{2.3}$$

is the number of particles that ϖ has in W. The maximal pieces $g|_{[\beta k_1, \beta k_2]}$ with $k_1, k_2 \in \mathbb{Z}$ and $g(\beta k_1), g(\beta(k_1+1)), g(\beta(k_1+2)), \dots, g(\beta(k_2-1)) \in W$ (and hence $g(\beta(k_1)-1)) \notin W$ and $g(\beta k_2) \notin W$) are the W*-shreds* of g; see Section 4.2, in particular (4.9). Then $\mathcal{C}_W$ denotes the set of all W-shreds, and $\mathcal{S}_W = \mathcal{M}_{\mathbb{N}_0}(\mathcal{C}_W)$ denotes the set of locally finite point measures of all the W-shreds. On $\mathcal{S}$ we consider the topology that is generated by a family of certain projection ('shredding') operators $\Pi_W^{(\mathcal{S})} \colon \mathcal{S} \to \mathcal{S}_W$ with $W \Subset \mathbb{R}^d$, which we spell out in (2.10). Let us already mention that these shredding operators cut the interlacements $g \in \mathcal{C}_\infty$ into a point process of all the W-shreds, but never cut legs. Let $\mathcal{M}_1^{({\rm s})}(\mathcal{S})$ denote the set of shift-invariant probability measures on $\mathcal{S}$. On this set we consider the weak topology that is generated by the shredding operators. For details, see Section 4.2.

We now introduce a crucial Brownian reference measure on the set $\mathcal{S}_W$. Let $B = (B_t)_{t\in[0,\infty)}$ be a standard Brownian motion in $\mathbb{R}^d$ that starts from $x \in \mathbb{R}^d$ under $\mathbb{P}_x$. We denote by

$$q_{x,y}^{(W,l)}({\rm d}g) = \mathbb{P}_x\big((B_t)_{t\in[0,\beta l]} \in {\rm d}g \mid B_\beta, B_{2\beta}, \dots, B_{(l-1)\beta} \in W, B_{l\beta} = y\big) \in \mathcal{M}_1(\mathcal{C}_W) \tag{2.4}$$

the distribution of a Brownian shred from $x \in W$ to $y \in W^{\rm c}$ with $l \in \mathbb{N}$ particles, and by

$$\mathtt{K}_W(\mu, \mathrm{d}\varpi) = \bigotimes_{i\in I} q_{x_i,y_i}^{(W,l_i)}(\mathrm{d}g_i) \in \mathcal{M}_1(\mathcal{S}_W), \qquad \mu = \sum_{i\in I} \delta_{(x_i,l_i,y_i)}, \varpi = \sum_{i\in I} \delta_{g_i}, \tag{2.5}$$

the conditional distribution of a Brownian W-shred configuration given the point process $\mu \in \mathcal{M}_{\mathbb{N}_0}(W \times \mathbb{N} \times W^{\rm c})$ of their entry and exit sites and their lengths (certainly under the assumption that $g_i(0) = x_i$ and $g_i(l_i\beta) = y_i$ and $g_i \in \mathcal{C}_{l_i}$ for all $i \in I$, and I is a finite index set).

Remark 2.2 (Brownian interlacements PPP) In dimension $d \geq 3$, the kernel $\mathtt{K}_W$ appears as a regular version of the conditional distribution of a variant of the well-known Brownian interlacement Poisson point process; see Section A. More precisely, given that the entry and exit sites and lengths of all the W-shreds of this PPP are given as μ, $K_W(\mu,\cdot)$ is the conditional distribution of the point process of the W-shreds in this PPP, see Lemma A.4. This works only in $d \geq 3$ because of the transience; in $d \leq 2$, this PPP would be uninteresting; it would not even be locally finite. $\Diamond$

2.2. **Our first result: A new specific relative entropy density.** In our description of the free energy of the interacting Bose gas in Theorem 2.5 below, a crucial ingredient will be a new specific relative entropy density with respect to the superposition of the reference measure introduced in Definition 1.1 and the family of kernels defined in (2.5). Here we introduce it as our first main result. By $H(\mu \mid \nu)$ we denote the relative entropy of a finite measure μ with respect to another one, ν, on a Polish space $\mathcal{X}$, defined by

$$H(\mu \mid \nu) = \nu(\mathcal{X}) - \mu(\mathcal{X}) + \int_{\mathcal{X}} \mu(\mathrm{d}x) \log \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x), \tag{2.6}$$

if $\mu \ll \nu$, and otherwise $H(\mu \mid \nu) = \infty$. A characterisation that is often helpful if μ and ν are probability measures is the formula

$$H(\mu \mid \nu) = \sup_f \Big(\langle f, \mu\rangle - \log\langle \mathrm{e}^f, \nu\rangle \Big), \qquad \mu,\nu \in \mathcal{M}_1(\mathcal{X}), \tag{2.7}$$

where the supremum is on measurable and bounded functions $f\colon \mathcal{X} \to \mathbb{R}$ (the supremum is approximately attained in $f = \log \frac{\mathrm{d}\mu}{\mathrm{d}\nu}$).

It is well-known, see [G11, Ch. 15] and [GZ93], that there is a notion of a *specific relative entropy density* with respect to the loop reference process $\mathtt{Q}$, defined as

$$\mathrm{h}(P \mid \mathtt{Q}) = \lim_{R\to\infty} \frac{1}{|W_R|} H\big(\Pi_{W_R}(P) \mid \Pi_{W_R}(\mathtt{Q})\big), \qquad P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}), W_R = [-R,R]^d, \tag{2.8}$$

for all shift-invariant probability measures P on the set $\mathcal{L}$ of simple marked point measures with loops as marks; here Π_W denotes the usual projection to the loops with starting site in $W \Subset \mathbb{R}^d$. The function h is introduced in the general frame of point processes with marks; and the shape or any properties of the marks play hardly any role in this theory.

In the present paper, we are going to work on the extended space $\mathcal{L} \times \mathcal{S}$, which we conceive as the set $\mathcal{M}_{\mathbb{N}_0}([\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)}] \times \mathcal{C}_\infty)$ of simple point measures on the set of loops and interlacements in $\mathbb{R}^d$. Let $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ denote the set of all shift-invariant probability measures on $\mathcal{L}\times\mathcal{S}$. We write the elements of $\mathcal{L}$ as $\omega = \sum_{x\in\zeta} \delta_{(x,f_x)}$ and the elements of $\mathcal{S}$ as $\varpi = \sum_{g\in\Gamma} \delta_g$.

We make a distinction between loops whose particles are contained entirely in one of the subboxes $z + W_R$ with $z \in 2R\mathbb{Z}^d$ and those whose particles are not; the latter loops are called R-crossings, see Section 1.4. Hence, adapted notions of specific relative entropies need to be introduced. For the first (the loops whose particles are contained in some $z + W$ with $z \in 2R\mathbb{Z}^d$), we need to introduce the *restricted* projection operator of loops,

$$\Pi_W^{(\mathcal{L})}(\omega) = \sum_{x\in\zeta\colon\, f_x(k\beta)\in W\ \forall k\in[\ell(f_x)]} \delta_{(x,f_x)} \in \mathcal{L}_W = \mathcal{M}_{\mathbb{N}_0}(W \times \mathcal{C}_W^{(\circlearrowleft)}), \tag{2.9}$$

which restricts to all the loops that have all their particles in W (we used the notation $[n]=\{1,\dots,n\}$). We wrote $\mathcal{M}_{\mathbb{N}_0}(\mathcal{X})$ for the set of all point measures on points in some space $\mathcal{X}$ and $\mathcal{C}_W^{(\circlearrowleft)}$ for the set of loops with all particles in W. (See Section 4.1 for more explanations.) Note that the legs $f_{x,i}$ start and end in W; but do not have to stay in W; they can travel arbitrarily far away from W.

For describing the second kind of loops (the R-crossing ones), we introduce the shredding operator

$$\Pi_W^{(\mathcal{S})}(\omega)=\sum_{x\in\zeta\colon\{f_x(k\beta)\colon k\in[\ell(f_x)]\}\not\subset W}\ \sum_{g\in\Gamma_W(f_x)}\delta_g\in\mathcal{S}_W=\mathcal{M}_{\mathbb{N}_0}(\mathcal{C}_W), \tag{2.10}$$

where $\Gamma_W(f_x)$ is the set of W-shreds of the loop f_x, and $\mathcal{C}_W$ is the set of all W-shreds. (See Section 4.2 for more details and explanations.) Analogously, the operator $\Pi_W^{(\mathcal{S})}(\varpi)$ is defined also for point measures $\varpi=\sum_{g\in\Gamma}\delta_g$ of interlacements. By $\Pi_W(\omega,\varpi)$ we denote the sum (i.e., the superposition) $\Pi_W^{(\mathcal{L})}(\omega)+\Pi_W^{(\mathcal{S})}(\varpi)$. Another important operator is the boundary-shred operator

$$\partial\Pi_W^{(\mathcal{S})}(\varpi)=\sum_{g\in\Gamma}\sum_{g'\in\Gamma_W(g)}\delta_{(g'(\beta),\ell(g'),g'(\ell(g')\beta))}\in\mathcal{T}_W=\mathcal{M}_{\mathbb{N}_0}(W\times\mathbb{N}\times W^{\mathrm{c}}), \tag{2.11}$$

which registers initial and terminal sites and particle numbers of the W-shreds $(g'(\beta),\dots,g'(\ell(g')\beta))$ with $g'(0)\in W^{\mathrm{c}}$ and $g'(\beta),\dots,g((\ell(g')-1)\beta)\in W$ and $g'(\ell(g')\beta)\in W^{\mathrm{c}}$. For probability measures P on loop or shred space and any of these operators Π, we denote by $\Pi(P)$ the image measure $P\circ\Pi^{-1}$ of P under Π. Hence, for $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$, the measure $\Pi_W(P)$ is the distribution of all the loops of P that have all particles in W, superposed with all the W-shreds that come from interlacements or from the other loops; *a priori* the W-shreds cannot be distinguished as to whether they come from a loop or from a shred.

As it is standard in measure theory, the product of a measure μ on a measurable space $\mathcal{X}$ with a kernel K from $\mathcal{X}$ into another measurable space $\mathcal{Y}$ is defined by

$$\mu\otimes K(\mathrm{d}(x,y))=\mu(\mathrm{d}x)\,K(x,\mathrm{d}y). \tag{2.12}$$

We introduce the relative entropy

$$J_W(\xi)=\frac{1}{|W|}H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi\,\big|\,\Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes[\partial\Pi_W^{(\mathcal{S})}(\xi)\otimes\mathtt{K}_W]\big)\in[0,\infty],\qquad W\Subset\mathbb{R}^d,\xi\in\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W). \tag{2.13}$$

In our next result, we introduce the main entropy function of this paper,

$$\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)=\lim_{R\to\infty}J_{W_R}(\Pi_{W_R}(P)),\qquad P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S}),\text{with } W_R=[-R,R]^d. \tag{2.14}$$

This can be seen as a kind of specific relative entropy density with respect to the $\Pi_W^{(\mathcal{L})}$-projections of the reference loop process $\mathtt{Q}$ and the family of Brownian W-shred reference kernels $\mathtt{K}_W$ defined in (2.5); observe that the shred-part of the reference measure comes from the boundary-shred distribution of P. This means that the loop-part reference measure has a particle density, but the shred-part inherits it from P.

We need to introduce also some particle-counting operators. Let $\mathfrak{N}_W^{(\ell,\mathcal{L})}$ and $\mathfrak{N}_W^{(\ell,\mathcal{S})}$ denote the functionals that count the particles in W of the loop point process and of the shred point process, respectively (recall (1.7) and (2.3)):

$$\mathfrak{N}_W^{(\ell,\mathcal{L})}(\omega,\varpi)=\mathfrak{N}_W^{(\ell)}(\omega)=\sum_{x\in\zeta\cap W}\ell(f_x)\qquad\text{and}\qquad\mathfrak{N}_W^{(\ell,\mathcal{S})}(\omega,\varpi)=\mathfrak{N}_W^{(\ell)}(\varpi)=\sum_{g\in\Gamma}\ell_W(g), \tag{2.15}$$

and put $\mathfrak{N}_W^{(\ell)}(\omega,\varpi)=\mathfrak{N}_W^{(\ell,\mathcal{L})}(\omega,\varpi)+\mathfrak{N}_W^{(\ell,\mathcal{S})}(\omega,\varpi)$ for the total particle number in W. Note that $\mathfrak{N}_W^{(\ell,\mathcal{L})}$ counts particles in all loops that start from W (regardless whether the particle lies in W or not), while $\mathfrak{N}_W^{(\ell,\mathcal{S})}$ counts only particles that lie in W coming from any interlacement. (See Remark 2.11 for an alternate way of counting particles.)

Theorem 2.3 (Joint specific relative entropy density)**.** *If $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ satisfies $\langle P, \mathfrak{N}_U^{(\ell)}\rangle < \infty$ (with $U = [-\frac{1}{2}, \frac{1}{2}]^d$ the unit box), then the limit in (2.14) exists in $[0,\infty]$, and it satisfies*

$$J_{W_R}(\Pi_{W_R}(P)) \le \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P), \qquad R \in \mathbb{N}. \tag{2.16}$$

Furthermore, the map $P \mapsto \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$ is lower semi-continuous and convex, and it is also concave (i.e., even affine). Moreover, for any sequence $(K_{R_N})_{N\in\mathbb{N}}$ of compact sets $K_{R_N} \subset \mathcal{T}_{W_{R_N}}$ such that $R_N \to \infty$ as $N \to \infty$, the restricted level set

$$\bigcap_{N\in\mathbb{N}} \big\{P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S}) \colon \partial\Pi^{(\mathcal{S})}_{W_{R_N}}(P) \in K_{R_N}, \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P) \le c, \langle P, \mathfrak{N}_U^{(\ell)}\rangle \le c\big\} \tag{2.17}$$

is compact for any $c \in \mathbb{R}$.

The proof is in Section 8. While establishing $\mathrm{h}(P \mid \mathtt{Q})$ relies on superadditivity, our proof of (2.14) relies on probabilistic arguments in connection with large-deviation estimates for a crucial empirical measure.

We call the function $\mathrm{h}^{(\mathcal{L},\mathcal{S})}$ the *specific relative entropy density* with respect to the superposition of the Brownian loop soup and the family of Brownian shred kernels $\mathtt{K}_W$. The reference measure in (2.14) samples with $\Pi_W^{(\mathcal{L})}(\mathtt{Q})$ the loops and then, independently, with $\partial\Pi_W^{(\mathcal{S})}(P)$ the boundary shred configuration and then with $\mathtt{K}_W$ the shred interior configuration in W.

2.3. **Our main result: an extended variational description.** We are heading towards a formulation of our main result. We work on the space $\mathcal{L}\times\mathcal{S}$ of pairs of point processes of points with loops as marks and point processes of interlacements. Recall that we identify this set as the set of all superpositions of point processes of loops and interlacements.

We need to introduce an interaction functional, decomposed into its different loop-interlacement components. For this, we need notation for interactions between any two measurable sets $W, \widetilde{W} \subset \mathbb{R}^d$. For $(\omega, \varpi) = (\sum_{x\in\zeta}\delta_{(x,f_x)}, \sum_{g\in\Gamma}\delta_g) \in \mathcal{L}\times\mathcal{S}$ we put

$$F^{(\mathcal{L}\mathcal{L})}_{W,\widetilde{W}}(\omega,\varpi) = \sum_{x\in\zeta\cap W}\sum_{y\in\zeta\cap\widetilde{W}}\sum_{i=1}^{\ell(f_x)}\sum_{j=1}^{\ell(f_y)} \mathbb{1}_{(x,i)\neq(y,j)} V(f_{x,i}, f_{y,j}), \tag{2.18}$$

$$F^{(\mathcal{L}\mathcal{S})}_{W,\widetilde{W}}(\omega,\varpi) = \sum_{x\in\zeta\cap W}\sum_{g\in\Gamma}\sum_{i=1}^{\ell(f_x)}\sum_{k\in\mathbb{Z}} \mathbb{1}_{\widetilde{W}}(g(k\beta)) V(f_{x,i}, g_{k+1}), \tag{2.19}$$

$$F^{(\mathcal{S}\mathcal{S})}_{W,\widetilde{W}}(\omega,\varpi) = \sum_{g,g'\in\Gamma}\sum_{k,k'\in\mathbb{Z}} \mathbb{1}_{(g,k)\neq(g',k')}\mathbb{1}_W(g(k\beta))\mathbb{1}_{\widetilde{W}}(g'(k'\beta)) V(g_{k+1}, g'_{k'+1}), \tag{2.20}$$

where we write the legs of g here as $g_k(\cdot) = g((k-1)\beta + \cdot)|_{[0,\beta]}$, i.e., the $(k+1)$-st leg starts at the particle $g(k\beta) = g_{k+1}(0)$. In words, $F^{(\mathcal{L}\mathcal{L})}_{W,\widetilde{W}}$ is the interaction between any two different legs of loops that start in W and $\widetilde{W}$, respectively, (regardless where the leg starts), $F^{(\mathcal{L}\mathcal{S})}_{W,\widetilde{W}}$ is the one between legs of loops that start in W and legs starting in $\widetilde{W}$ of any interlacement, and $F^{(\mathcal{S}\mathcal{S})}_{W,\widetilde{W}}$ is the one between any two legs that start in W, respectively in $\widetilde{W}$, of any (one of two) interlacement(s). Then we define the total interaction within W as

$$F_{W,W} = \frac{1}{2}F^{(\mathcal{L}\mathcal{L})}_{W,W} + F^{(\mathcal{L}\mathcal{S})}_{W,W} + \frac{1}{2}F^{(\mathcal{S}\mathcal{S})}_{W,W}, \tag{2.21}$$

(since every pair of legs should appear precisely once and because of the possibility of double-counting, we need to be a bit cautious).

For the formulation of Theorem 2.5, we need the total interaction between the unit box $U=[-\frac12,\frac12]^d$ and $\mathbb{R}^d$:

$$F_U=\frac12\big[F^{(\mathcal{L}\mathcal{L})}_{U,U}+2F^{(\mathcal{L}\mathcal{S})}_{U,U}+F^{(\mathcal{S}\mathcal{S})}_{U,U}\big]+F^{(\mathcal{L}\mathcal{L})}_{U,U^{\rm c}}+F^{(\mathcal{L}\mathcal{S})}_{U,U^{\rm c}}+F^{(\mathcal{L}\mathcal{S})}_{U^{\rm c},U}+F^{(\mathcal{S}\mathcal{S})}_{U,U^{\rm c}}. \tag{2.22}$$

In words, this is the interaction between all legs of loops that start in U and all the other legs plus the interaction between all interlacement-legs that start in U and all the others (without double-counting). See Remark 2.11 for an alternative way of counting the interaction between U and the rest.

We are going to work on the following assumption on the interaction potential:

Assumption (V). *The interaction potential $v\colon\mathbb{R}^d\to[0,\infty)$ is bounded and continuous and has compact support. Furthermore, there exists $C>0$ such that, for any box $W\Subset\mathbb{R}^d$ containing $[-1,1]^d$,*

$$\sum_{1\le i<j\le k}v(x_i-x_j)\ge C\left(\frac{k^2}{|W|}-k\right),\qquad k\in\mathbb{N},x_1,\dots,x_k\in W. \tag{2.23}$$

The last part of Assumption (V) is called *superstability* of v [Rue69, §3.2.9].

Remark 2.4 (Assumptions on the interaction) As far as we are aware of, there is no canonical choice of an interaction function v from a physicist's point of view. Details of v might be motivated by physical or chemical properties of the matter that is to be modelled, but we have no particular choice in mind. We tried to keep v general and put the assumptions of measurability, boundedness of v and of the support just for convenience to keep the technicalities in the proofs low. In fact, several lengthy technicalities in the proofs in [ACK11] are due to our decision to allow v and its support to be unbounded there, but do not appear in the present paper. However, in our proof we need that the effect of v is sufficiently repulsive, which motivates (2.23). $\Diamond$

The following is our main result; it identifies the limiting free energy in the thermodynamic limit with fixed mass in the R-crossings in the limit $R\to\infty$. Recall the definition of the number $\mathfrak{N}^{(\ell,R)}_\Lambda(\omega)$ of particles in R-crossing loops from (1.14) and the number $\mathfrak{N}^{(\ell,\neg R)}_{\Lambda_N}$ of the remaining particles from (1.15).

Theorem 2.5 (Explicit formula for constrained free energy)**.** *Assume $\beta,\rho\in(0,\infty)$ to be fixed and suppose that v satisfies Assumption (V). Fix a sequence of centred boxes $\Lambda_N\subset\mathbb{R}^d$ such that $\lim_{N\to\infty}|\Lambda_N|/N=1/\rho$. We partition (up to boundary elements) $\Lambda_N=\bigcup_{z\in Z_{N,R}}W_z$ into the sub-boxes $W_z=z+W=z+[-R,R]^d$ with $Z_{N,R}\subset 2R\mathbb{Z}^d$. Then, in the limit as $N\to\infty$, followed by $R\to\infty$, the pair $\frac{1}{|\Lambda_N|}(\mathfrak{N}^{(\ell,R)}_{\Lambda_N},\mathfrak{N}^{(\ell,\neg R)}_{\Lambda_N})$ satisfies under the transformed measure $\widehat{\mathtt{Q}}^{(\Lambda_N,{\rm bc})}$ defined in (1.13) a large-deviations principle on the set $\{(\rho_1,\rho_2)\in[0,\infty)^2\colon\rho_1+\rho_2=\rho\}$ with continuous and convex rate function $(\rho_1,\rho_2)\mapsto\chi(\rho_1,\rho_2)-\overline{\chi}(\rho)$, where $\overline{\chi}(\rho)=\inf_{\rho_1+\rho_2=\rho}\chi(\rho_1,\rho_2)$ and*

$$\chi(\rho_1,\rho_2):=\inf\Big\{{\rm h}^{(\mathcal{L},\mathcal{S})}(P)+\langle F_U,P\rangle\colon P\in\mathcal{M}_1^{({\rm s})}(\mathcal{L}\times\mathcal{S}),\langle\mathfrak{N}^{(\ell,\mathcal{L})}_U,P\rangle=\rho_1,\langle\mathfrak{N}^{(\ell,\mathcal{S})}_U,P\rangle=\rho_2\Big\}. \tag{2.24}$$

The proof of Theorem 2.5 comes in three steps in Sections 4.5 (preparation), 6.4 (making $N\to\infty$) and 9.1 (making $R\to\infty$) for the upper bound, and Sections 4.6, 6.5 and 9.2 for the lower bound. A survey on our proof strategy is in Section 3.2. The variational formula in (2.24) has good compactness properties. Indeed, as we formulate in Lemma 2.12 below, the infimum in (2.24) may be restricted to some compact set, and χ is continuous.

We also obtain the same LDP for the numbers of particles in long and short loops, which are more natural quantities from the viewpoint of Feynman's proposal to look at loop lengths as order parameters:

Corollary 2.6 (LDP for particle numbers in long and short loops)**.** *In the situation of Theorem 2.5, the pair* $\frac{1}{|\Lambda_N|}(\mathfrak{N}^{(\ell)}_{\Lambda_N,>L},\mathfrak{N}^{(\ell)}_{\Lambda_N,\le L})$ *satisfies the same LDP as the pair* $\frac{1}{|\Lambda_N|}(\mathfrak{N}^{(\ell,R)}_{\Lambda_N},\mathfrak{N}^{(\ell,\neg R)}_{\Lambda_N})$*, where* $\mathfrak{N}^{(\ell)}_{\Lambda_N,\le L}$ *and* $\mathfrak{N}^{(\ell)}_{\Lambda_N,>L}$ *are the numbers of particles in loops of length* $\le L$ *and* $>L$*, respectively.*

This follows from Lemma 3.2 below, where we show that these two pairs of counting variables are exponentially equivalent to each other in the limit $N\to\infty$, followed by $R\to\infty$, respectively $L\to\infty$.

For the general theory of (the probabilities of) large deviations, see [DZ10]. The way how the large-deviations principle (LDP) of Theorem 2.5 is to be understood (and how it will be proved) is that, for any $\rho_1,\rho_2\in[0,\infty)$,

$$\lim_{\delta\to 0}\limsup_{R\to\infty}\limsup_{N\to\infty}\frac{1}{|\Lambda_N|}\log\Big[\widehat{Z}^{(\mathrm{bc})}(\Lambda_N)\widehat{\mathtt{Q}}^{(\Lambda_N,\mathrm{bc})}\Big(\Big|\frac{1}{|\Lambda_N|}\mathfrak{N}^{(\ell,R)}_{\Lambda_N}-\rho_1\Big|\le\delta,\Big|\frac{1}{|\Lambda_N|}\mathfrak{N}^{(\ell,\neg R)}_{\Lambda_N}-\rho_2\Big|\le\delta\Big)\Big]$$
$$\le -\chi(\rho_1,\rho_2), \tag{2.25}$$

and a corresponding lower bound with lim sup twice replaced by lim inf. As a consequence, the minimiser(s) (ρ_1,ρ_2) play the decisive role when further analysing the limiting Bose gas. As a first partial result, we can identify the free energy defined in (1.3) in the following way (recall (1.12)).

Corollary 2.7 (Variational description of the free energy)**.** *In the situation of Theorem 2.5,*

$$\beta\,\mathrm{f}(\beta,\rho)=-\overline{q}+\inf_{\rho_1,\rho_2\ge 0\colon\,\rho_1+\rho_2=\rho}\chi(\rho_1,\rho_2)=-\overline{q}+\overline{\chi}(\rho), \tag{2.26}$$

where

$$\overline{\chi}(\rho)=\inf\Big\{\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)+\langle F_U,P\rangle\colon P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S}),\langle P,\mathfrak{N}_U^{(\ell,\mathcal{L})}+\mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle=\rho\Big\}. \tag{2.27}$$

We leave the proof of this corollary to the reader. In (2.29) below it turns out that in (2.27), the interlacements are not necessary, i.e., the value does not change if we restrict P to having only loops.

Remark 2.8 (Interpretation of Theorem 2.5) The limiting Bose gas with density parameters ρ_1 and ρ_2 is characterised as a random superposition of a loop point process and an interlacement point process. The particles of the latter ones represent the condensate with condensate density ρ_2 per unit volume, and the loop part contains the remaining particles with density ρ_1. This random loop-interlacement process is stationary and has strong correlations between the loop- and the interlacement part via the interaction. The process is characterised by minimality of the sum of energy (the interaction) and entropy with respect to the reference PPP-process $\mathtt{Q}$ and the family of kernels $\mathtt{K}_W$ under the constraint of having these two densities. Any minimiser P in (2.27) has the interpretation of the effective spatial distribution of all the bosons in space, organised in loops and interlacements, of the interacting Bose gas in the thermodynamic limit with total particle density ρ. ◊

Remark 2.9 (More refined assertions) Using Lemma 3.2 below, it is also no problem to derive an LDP for more quantities, like the joint distribution over $k\in\mathbb{N}$ of the number of particles in loops of length k (without any reference to the decomposition of Λ_N into R-subboxes) or the number of such loops. The corresponding rate function that arises is clear; one needs to restrict the variational problem to those P that have on an average the requested density of k-loops. Examples of proofs for such statements in a different context can be found in [AKP21] and [AKLP23]. ◊

Remark 2.10 (Approximations of $\chi(\rho_1,\rho_2)$.) In the course of the proof of Theorem 2.5, we are forced to establish some technical approximations of the minimizers P in the variational formula $\chi(\rho_1,\rho_2)$ in (2.24). In particular, in Lemma 7.6, we approximate P with ergodic measures that are concentrated on uniformly bounded loop/interlacement configurations with some nice additional properties, and in Lemma 2.12, we compactify the set of Ps on which we minimize. ◊

Remark 2.11 (Counting particle numbers and interaction in U) Instead of counting particle numbers in loops that start from U, one can also count particles in U in loops starting at any place. More precisely, if we define $\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{L})}(\omega)=\sum_{x\in\zeta}\sum_{i=1}^{\ell(f_x)}\mathbb{1}\{f_x((i-1)\beta)\in U\}$, then one can easily see that $\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{L})}$ has the same expectation under any $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ as $\mathfrak{N}_U^{(\ell,\mathcal{L})}$. Indeed,

$$\begin{aligned}\langle P,\mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle&=\int P(\mathrm{d}\omega)\sum_{x\in\zeta}\sum_{i=1}^{\ell(f_x)}\mathbb{1}\{x\in U\}\sum_{z\in\mathbb{Z}^d}\mathbb{1}\{f_x((i-1)\beta)\in z+U\}\\&=\sum_{z\in\mathbb{Z}^d}\int P(\mathrm{d}\omega)\sum_{x\in\zeta}\sum_{i=1}^{\ell(f_x)}\mathbb{1}\{f_x((i-1)\beta)\in U\}\mathbb{1}\{x\in -z+U\}=\langle P,\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{L})}\rangle.\end{aligned}$$

Analogously, one can see that the expected interaction between legs whose loop starts in U and all other legs in loops is the same as the expected interaction between legs that start in U (regardless where the loop starts) and all other legs in loops, with an analogous statement about interaction between loops and interlacements. More precisely, defining

$$\widetilde{F}_{W,\widetilde{W}}^{(\mathcal{L}\mathcal{L})}(\omega,\varpi)=\sum_{x,y\in\zeta}\sum_{i=1}^{\ell(f_x)}\sum_{j=1}^{\ell(f_y)}\mathbb{1}\{f_x((i-1)\beta))\in W\}\mathbb{1}\{f_y((j-1)\beta)\in\widetilde{W}\}\mathbb{1}_{(x,i)\neq(y,j)}V(f_{x,i},f_{y,j}),$$

one sees in the same way that $\langle P,\frac12 F_{U,U}^{(\mathcal{L}\mathcal{L})}+F_{U,U^{\mathrm{c}}}^{(\mathcal{L}\mathcal{L})}\rangle=\langle P,\frac12\widetilde{F}_{U,U}^{(\mathcal{L}\mathcal{L})}+\widetilde{F}_{U,U^{\mathrm{c}}}^{(\mathcal{L}\mathcal{L})}\rangle$ for any $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$. Consequently, F_U defined in (2.22) has the same expectation under P as the total interaction between any leg that starts in U (regardless where its loop starts, if it is in a loop) and any other leg. $\Diamond$

2.4. **The variational formula and BEC.** In this section, inspired by the results of [CJK23, Section 2], we analyse the variational formula χ in (2.24) with respect to its analytic properties (existence and properties of minimisers), depending on the parameters ρ_1 and ρ_2. On the technical side, it will be nice to know that the infimum may be restricted to the set

$$K=\big\{P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})\colon \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)\leq L,\langle P,(\mathfrak{N}_U^{(\ell)})^{5/4}\rangle\leq L\big\},\tag{2.28}$$

with a sufficiently large $L\in(0,\infty)$ (recall that $U=[-\frac12,\frac12]^d=W_{1/2}$). This, together with the compactness property of Theorem 2.3, makes the domain of the variational formula compact, and it gives us a good control on the two particle densities.

Lemma 2.12 (Regularity properties of χ)**.** *Suppose that the interaction potential v satisfies Assumption (V), in particular* (2.23)*. Then the following holds.*

(1) *The set K in* (2.28) *is compact, and for any ρ_1,ρ_2, the infimum in the definition* (2.24) *of $\chi(\rho_1,\rho_2)$ may be restricted to K, not changing its value for any sufficiently large L. Furthermore, the maps $P\mapsto\langle P,\mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle$ and $P\mapsto\langle P,\mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle$ are continuous on K.*
(2) *The maps $[0,\infty)^2\ni(\rho_1,\rho_2)\mapsto\chi(\rho_1,\rho_2)$ and $[0,\infty)\ni\rho\mapsto\overline{\chi}(\rho)$ are convex. Furthermore, for any ρ_1, respectively $\rho_2\in[0,\infty)$, $\lim_{\rho_2\to\infty}\chi(\rho_1,\rho_2)=\infty$ and $\lim_{\rho_1\to\infty}\chi(\rho_1,\rho_2)=\infty$.*
(3) *The map $\rho_1\mapsto\chi(\rho_1,0)$ is continuous at $\rho_1=0$ with $\chi(0,0)=\chi^{(v=0)}(0,0)=\overline{q}=(4\pi\beta)^{-d/2}\zeta(1+d/2)$ and $\partial_{\rho_1}\chi(\rho_1,0)|_{\rho_1=0}=-\infty$.*
(4) *For any $\rho_1,\rho_2\in[0,\infty)$, there is at least one minimiser P in the formula for $\chi(\rho_1,\rho_2)$ on the right-hand side of* (2.24)*.*
(5) *For any $\rho\in[0,\infty)$, there is at least one minimiser P in the formula for $\overline{\chi}(\rho)$ on the right-hand side of* (2.27)*.*

The proof of Lemma 2.12 is in Section 7.1. As easy consequences of Lemma 2.12, the set of minimising P's is convex and compact, and χ and $\overline{\chi}$ are continuous and have minimisers (ρ_1,ρ_2) respectively ρ.

Now we prove that every value of energy plus entropy that can be achieved with loops and interlacements can also be achieved with only loops, with the same total particle density. In particular, the free energy can be expressed in terms of configurations that have no interlacements, for any arbitrarily large particle density.

Lemma 2.13 (Interlacements are not necessary)**.** *Suppose that the interaction potential v satisfies Assumption (V). Then $\chi(\rho_1, \rho_2) \geq \chi(\rho_1 + \rho_2, 0)$ for any $\rho_1, \rho_2 \in [0, \infty)$.*

The proof of Lemma 2.13 is in Section 7.3. In particular, in the variational formula $\overline{\chi}(\rho)$ describing the free energy according to Corollary 2.7, we see now that

$$\overline{\chi}(\rho) = \inf \Big\{ \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P) + \langle F_U, P\rangle : P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}), \langle P, \mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle = \rho \Big\}, \qquad \rho \in [0, \infty), \tag{2.29}$$

where we conceive P as an element of $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L} \times \mathcal{S})$ with empty interlacement part. Note that its entropy $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$ has a shred-part that is *a priori* non-trivial, but Lemma 7.7 says that the expected particle number in shreds is negligible.

Hence, on the level of the value of the free energy, it is not possible to distinguish between optimal configuration with loops *and* interlacements or just with loops alone. In [ACK11], using totally different tools, a variant of (2.29) was proved for a range of ρ's that is bounded from above; instead of $\mathrm{h}^{(\mathcal{L})}$, the function h from (2.8) appeared.

Remark 2.14 (Existence of minimizers?) For the variational problem $\chi(\rho_1, \rho_2)$ in (2.24) there is a relatively simple argument that it has always minimizer(s) (see Lemma 2.12(1)), which is based on the set K and Assumption (V) and the compactness property of Theorem 2.3.

However, there is no argument like that for the formula for $\overline{\chi}(\rho)$ in (2.29). Indeed, if we use the topology on $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L} \times \mathcal{S})$ that is induced by the weak topologies defined by all the $\Pi_W^{(\mathcal{L})}$ and $\Pi_W^{(\mathcal{S})}$ with $W \Subset \mathbb{R}^d$ (as we do in the proofs), then $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L})$ is not a closed subset of $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L} \times \mathcal{S})$, since it is possible to approach loop/interlacement configurations having non-trivial interlacement part with pure loop configurations, which is a kind of condensation effect. On the other hand, if one is based only on the topology defined by all the $\Pi_W^{(\mathcal{L})}$ with $W \Subset \mathbb{R}^d$, then the particle density in loops is not a continuous function, since this quantity needs the shreds of the loops (see the proof of Lemma 7.1). $\Diamond$

Let us therefore introduce the critical threshold between existence and non-existence of interlacements in minimizing configurations:

$$\begin{aligned} \rho_{\mathrm{c}} &= \sup \big\{ \rho \in [0, \infty) : \exists \text{ minimiser } P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L} \times \mathcal{S}) \text{ in (2.27) with } \Pi^{(\mathcal{S})}(P) = 0 \big\} \\ &= \sup \big\{ \rho \in [0, \infty) : \text{(2.29) has a minimizer} \big\} \in [0, \infty]. \end{aligned} \tag{2.30}$$

We conjecture the following:

- For $\rho \leq \rho_{\mathrm{c}}$, (2.29) has a minimizer, and for $\rho > \rho_{\mathrm{c}}$ it has not.
- $\rho \mapsto \overline{\chi}(\rho)$ is real-analytic in $(0, \rho_{\mathrm{c}})$ and in $(\rho_{\mathrm{c}}, \infty)$, but not in any larger interval.
- $\rho_{\mathrm{c}} < \infty$ for $d \geq 3$, but $\rho_{\mathrm{c}} = \infty$ in $d \leq 2$.

Note that these statements are true in the free case (i.e., for $v = 0$); see Remark 2.15. Proofs of these conjectures for some choices of v (for example, those that satisfy our Assumption (V)) may be within reach using the new variational framework that was introduced in this work. In some technical proofs in this paper (see the proofs of Lemmas 2.13 and 7.6 in Sections 7.3 and 7.5), we were able to develop strategies that make it possible to compare different loop/interlacement configurations with each other by rewiring shreds in large boxes outside the box. See the short descriptions of the two proofs at the beginnings of Sections 7.3 and 7.5. The variational description in terms of entropy plus energy in large boxes made it possible to show that every configuration can be approximated by loop-configurations, by just rewiring the interlacement-shreds outside a large box in such a way that only loops appear.

However, variants that could lead to a proof of finiteness of $\rho_{\rm c}$ in $d\geq 3$ seem to require much deeper ideas. One needs to show that, if ρ is sufficiently large, any configuration that has only loops is not optimal. One idea is to remove a non-vanishing loop part from the large box and replace it by an interlacement part of the same particle density and to show that the entropy goes down by more than the energy, but Lemma 7.7 says that this cannot be done just be rewiring all the shred ends; they concern too few particles. Another idea would be to execute some few manipulations on the loop configuration in the interior of the large box and a rewiring outside the box, such that interlacements appear and the energy is hardly changed, but the entropy drops down significantly, but it is not clear that such a strategy exists. Therefore more extensive manipulations have to be carried out, which we leave to future work.

However, our conjectures extend much further: we think that $\rho_{\rm c}$ will turn out to be the threshold for the occurrence of BEC, i.e., for the validity of off-diagonal long-range order (ODLRO) for the symmetrised and normalized exponential of the Hamiltonian in (1.1), but proving this seems to require much more work than we can currently indicate. Nevertheless, the framework introduced here is likely to serve as a platform for attacking proofs for such a statement.

Remark 2.15 (The free gas) Let us briefly discuss BEC in the *free Bose gas*, where no interaction is present, i.e., $v=0$. Here it is known since long that $\rho_{\rm c}=(4\pi\beta)^{-d/2}\zeta(d/2)$ with ζ the Riemann zeta function. In this case, we easily see that a minimiser of the formula in (2.27) is given as

$$P_\rho=\begin{cases}\mathtt{Q}^{(\rho)}\otimes\delta_{\underline{0}}, & \text{if } \rho\leq\rho_{\rm c},\\ \mathtt{Q}\otimes\mathtt{R}^{(u_\rho,\beta)}, & \text{if }\rho>\rho_{\rm c},\end{cases}\tag{2.31}$$

where $\underline{0}$ is the empty interlacement point process, and $\mathtt{Q}^{(\rho)}$ is the marked Poisson process as in Definition 1.1 with $q_k=\frac{1}{k}(4\pi\beta k)^{-d/2}$ for $k\in\mathbb{N}$ replaced by $m_k^{(\rho)}=q_k{\rm e}^{\alpha_\rho k}$ with the unique suitable $\alpha_\rho\in(-\infty,0]$ such that $\sum_{\in\mathbb{N}}km_k^{(\rho)}=\rho$. Furthermore, in $d\geq 3$ (only in this case the second case is not empty), $\mathtt{R}^{(u_\rho,\beta)}$ is the Brownian interlacement PPP introduced in Section A, and the density parameter u_ρ is picked in such a way that the expected number of particles of $\mathtt{R}^{(u_\rho,\beta)}$ in U is equal to $\rho-\rho_{\rm c}$. One can write (2.31) in one line by writing $\mathtt{Q}^{(0)}=\mathtt{Q}$ and noting that $\alpha_\rho=0$ and $m^{(\rho)}=q$ for $\rho\geq\rho_{\rm c}$, and putting $\mathtt{R}^{(u_0,\beta)}=\delta_{\underline{0}}$. Indeed, using the notation in (6.13), we see that

$$\begin{aligned}J_W(\Pi_W(P_\rho))&=\frac{1}{|W|}H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\Pi_W^{(\mathcal{L})}(\mathtt{Q}^{(\rho)})\otimes\Pi_W^{(\mathcal{S})}(\mathtt{R}^{(u_\rho,\beta)})\,\big|\,\Pi_W^{(\mathcal{L})}(\mathtt{Q}^{(\rho\wedge\rho_{\rm c})})\otimes\Pi_W^{(\mathcal{S})}(\mathtt{R}^{(u_\rho,\beta)})\big)\\&\to H(m^{(\rho)}\mid q)\qquad\text{as } W\uparrow\mathbb{R}^d,\end{aligned}$$

and can deduce that (2.27) drastically reduces to the formula

$$\overline{\chi}^{(v=0)}(\rho)=\inf\Big\{H(m|q)\colon m\in[0,\infty)^{\mathbb{N}},\sum_{k\in\mathbb{N}}km_k=\rho\Big\},\tag{2.32}$$

where $H(m|q)=\sum_k(q_k-m_k+m_k\log\frac{m_k}{q_k})$ is the relative entropy of the sequence m with respect to q and to find the unique minimiser $m^{(\rho)}$ via a standard Lagrange minimization. See [BKM24] for a more direct way to show that $-\beta{\rm f}(\beta)$ is equal to the right-hand side of (2.32).

In words, the limiting non-interacting Bose gas puts Brownian loops of length k independently with density $m_k^{(\rho)}$ and adds, if their total particle mass has not reached $\rho_{\rm c}$, additionally and independently a Brownian interlacement process with the appropriate density. The formula in (2.32) appeared in several earlier works on the free Bose gas, e.g., in [A08]. See also [V23] for a proof of the emergence of the above minimiser (i.e., the one on point-process level) in the thermodynamic limit as a limit distribution. $\Diamond$

3. Discussion

In this section, we discuss some aspects and explain some background. In Section 3.1 we show that our concept of R-crossing loops is equivalent to the usual idea of long loops. In Section 3.2 we concisely describe our strategy to prove Theorem 2.5, and in Section 3.3 we give an account on the literature on the thermodynamic limit of the Bose gas in connection with path-integral analysis.

3.1. **R-crossings versus long loops.** In the partition function, we want to distinguish between condensate particles and the others. In the loop soup representation of the Bose gas, the former should be the ones in 'long loops'. The separation line between long and short requires an additional parameter, $L \in \mathbb{N}$: one can call the L-condensate all the particles in loops with lengths $> L$. This requires taking another limiting procedure, namely $L \to \infty$, after having taken $N \to \infty$. In our work, we decided to stick to another definition of the condensate, namely we call the R-condensate the totality of all the particles which form part of loops that are an *R-crossing*, i.e., that have particles in more than one of the subboxes $z + [-R, R]^d$ with $z \in 2R\mathbb{Z}^d$. Again, after taking the limit as $N \to \infty$, one needs to take the limit as $R \to \infty$.

In this section, we point out that these two notions are asymptotically the same under the measure $\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{bc})}$ defined in (1.13). That is, we will show that the number of particles in R-crossings and the number of particles in loops of lengths $> L$ are exponentially equivalent under $\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{bc})}$ in the limit $N \to \infty$, followed by $R \to \infty$, respectively $L \to \infty$. For this purpose, let, for $\omega = \sum_{x\in\zeta} \delta_{(x,f_x)} \in \mathcal{L}$,

$$\mathfrak{N}^{(\ell,R)}_{\Lambda,\leq L}(\omega) = \sum_{x\in\zeta\cap\Lambda} \ell(f_x)\mathbb{1}\{f_x \text{ is } R\text{-crossing}\}\mathbb{1}\{\ell(f_x) \leq L\} \tag{3.1}$$

denote the number of particles in loops that are R-crossing and have length $\leq L$, and let

$$\mathfrak{N}^{(\ell,\neg R)}_{\Lambda,>L}(\omega) = \sum_{x\in\zeta\cap\Lambda} \ell(f_x)\mathbb{1}\{f_x \text{ is not } R\text{-crossing}\}\mathbb{1}\{\ell(f_x) > L\} \tag{3.2}$$

be the number of particles in loops that are not R-crossings, but longer than L. The following lemma shows that these two particle numbers are exponentially negligible if $L = L_R^{(\leq)}$ respectively $L = L_R^{(>)}$ are chosen properly, tending to infinity as $R \to \infty$. This implies that our notion of condensate via R-crossings is equivalent to the usual notion via the length for the interacting Bose gas.

We remark that one half of this property (namely, the negligibility of $\mathfrak{N}^{(\ell,\neg R)}_{\Lambda,>L}(\omega)$) highly depends on the interaction. Indeed, the fact that long loops that stay within one of the boxes $z + W_R$ with $z \in 2R\mathbb{Z}^d$ are suppressed does not come from combinatorics of numbers of loops, but only from the fact that their self-interaction is large, using the superstability of the potential v. Therefore, as a pre-step, we first estimate the expectation of $\mathrm{e}^{-\Phi_{\Lambda,\Lambda}}$ from above for any Brownian loop that lies in some subbox $z + W_R$ with $z \in Z_{N,R}$ and has a large length. For this, we are going to use (2.23) in Assumption (V) in order to obtain a good lower bound for the self-interaction of this loop. Using this is not so immediate, since such a loop has only all its particles in W_R (without loss of generality, we put $z = 0$), but each leg may travel far away and evade the interaction. For future reference, we isolate the necessary part of the statement as follows.

Lemma 3.1 (Upper bound for expected path interaction). *Suppose that Assumption (V) holds. Then there is a $C \in (0,\infty)$ (only depending on d, v or β) such that, for any box W of diameter $R \in \mathbb{N}$ that is contained in the box $\Lambda \subset \mathbb{R}^d$, and for any $k \in \mathbb{N}$ that is sufficiently large (depending on R),*

$$\mathbb{E}\big[\mathrm{e}^{-\Phi_{\Lambda,\Lambda}}\big] \leq \mathrm{e}^{-Ck^{3/2}R/\sqrt{|W|}}, \qquad x_1,\dots,x_k \in W, \tag{3.3}$$

where $\mathbb{E} = \bigotimes_{i=1}^k \mathbb{P}_{x_i}$ is with respect to k independent Brownian motions with time interval $[0,\beta]$, starting from $x_1,\dots,x_k \in W$ and having any of the three boundary conditions in Λ, and $\Phi_{\Lambda,\Lambda}$ is the interaction between all the k motions.

Proof. Without loss of generality, we put $W = W_R = [-R, R]^d$, as the boundary conditions will not be used anyway in our proof. We use C as a generic constant that depends only on d, v or β and may change its value from appearance to appearance.

Even though all starting sites of the k motions lie in W, we get a useful bound only for the pairs of those legs that stay, say, in the larger box W_{2R}, at least for some part of the time interval $[0, \beta]$. For a vector f of k legs $f_1, \dots, f_k \in \mathcal{C}_1$, let $L_\gamma(f) = \#\{i \in [k] \colon f_i([0, \gamma]) \subset W_{2R}\}$ denote the number of those legs who stay for the first γ time units in W_{2R}. We will be using (2.23) only in the way that the self-intersection of f satisfies the lower bound

$$\sum_{1 \le i < j \le k} \int_0^\beta v\big(|f_i(s) - f_j(s)|\big)\, \mathrm{d}s \ge C\gamma\Big(\frac{m^2}{|W_R|} - m\Big), \qquad m, R \in \mathbb{N} \text{ if } m = L_\gamma(f). \tag{3.4}$$

For a vector B of k independent motions $B_1, \dots, B_k$ with starting sites $x_1, \dots, x_k$, we lower bound $\mathrm{e}^{-\Phi_{\Lambda,\Lambda}}$ against the exponential of the negative left-hand side of (3.4) with $f_i = B_i$ and hence against the exponential of the negative right-hand side, and we split the expectation according to the values of $L_\gamma(B) \in \{0, 1, \dots, k\}$. Note that the probability of $\{L_\gamma(B) = m\}$ is not larger than $\mathrm{e}^{-(k-m)CR^2/\gamma}$, since $k - m$ of them travel a distance $\ge R$ during the time interval $[0, \gamma]$. Using (3.4) on this event, we obtain

$$\begin{aligned} \mathbb{E}\big[\mathrm{e}^{-\Phi_{\Lambda,\Lambda}}\big] \le \sum_{m=0}^k \mathrm{e}^{-C\gamma(\frac{m^2}{|W_R|} - m) - (k-m)CR^2/\gamma} &= \mathrm{e}^{-CkR^2/\gamma} \sum_{m=0}^k \mathrm{e}^{Cm(\gamma + R^2/\gamma)} \mathrm{e}^{-Cm^2\gamma/|W_R|} \\ &\le \mathrm{e}^{-CkR^2/\gamma}(k+1)\mathrm{e}^{\frac{C}{4}\gamma|W_R|(1+R^2/\gamma^2)^2}, \end{aligned} \tag{3.5}$$

where in the last step we estimated against the maximum over m, namely taking $m = \frac{|W_R|}{2}(1 + R^2/\gamma^2)$. Now, assuming that k is much larger than $|W_R|R^2$, we pick $\gamma = R\sqrt{|W_R|/k} \in [0, \beta]$ and estimate, for sufficiently large k, the second exponent as follows.

$$\frac{C}{4}\gamma|W_R|(1 + R^2/\gamma^2)^2 \le \frac{C}{3}\gamma\,|W_R|\frac{R^4}{\gamma^4} = \frac{C}{3}k^{3/2}\frac{R}{\sqrt{|W_R|}}.$$

Hence, adapting the value of C and assuming that k is large enough (only depending on R) we obtain that the right-hand side of (3.5) is not larger than $\exp\{-Ck^{3/2}R/\sqrt{|W_R|}\}$. □

Now we prove that the concepts of R-crossing loops and long loops are equivalent. Recall the definition of $\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{bc})}$ in (1.13).

Lemma 3.2 (Equivalence of condensate notions)**.** *Suppose that v satisfies Assumption (V). Fix $\rho \in (0, \infty)$ and let Λ_N be the centered box in $\mathbb{R}^d$ with volume $|\Lambda_N|/N \to 1/\rho$ as $N \to \infty$. Then there are diverging sequences $(L_R^{(>)})_{R\in\mathbb{N}}$ and $(L_R^{(\le)})_{R\in\mathbb{N}}$ such that, for any $\varepsilon \in (0, 1)$,*

$$\limsup_{R\to\infty} \limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{bc})}\big(\mathfrak{N}^{(\ell,R)}_{\Lambda_N, \le L_R^{(\le)}} > \varepsilon|\Lambda_N|\big) \quad = \quad -\infty, \tag{3.6}$$

$$\limsup_{R\to\infty} \limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{bc})}\big(\mathfrak{N}^{(\ell,\neg R)}_{\Lambda_N, > L_R^{(>)}} > \varepsilon|\Lambda_N|\big) \quad = \quad -\infty. \tag{3.7}$$

Proof. We put $W_R = [-R, R]^d$ and assume that Λ_N is equal to the union of $z + W_R$ over $z \in Z_{N,R} = \{z \in 2R\mathbb{Z}^d \colon z + W_R \subset \Lambda_N\}$. Possibly we need to admit that $R = R_N$ depends on N with $R_N \to R$, but we suppress this from notation.

We first prove (3.6). This will not use the interaction, but only combinatorics and large deviations for the Poisson distribution. Note that $\mathfrak{N}^{(\ell,R)}_{\Lambda_N, \le L} = \sum_{k=1}^L kX_k$ where the X_k are independent and Poisson-distributed with parameter

$$\frac{1}{k}\int_{\Lambda_N} \mathrm{d}x\, \mu_{x,x}^{(k,\Lambda_N,\mathrm{bc})}(\tau_R \le k);$$

here, $\tau_R = \inf\{k \in \mathbb{N}\colon B_{k\beta} \notin z_{B_0} + W_R\}$, and $z_{B_0} \in 2R\mathbb{Z}^d$ is picked such that $B_0 \in z_{B_0} + W_R$. The random variable $\sum_{k=1}^L X_k$ is Poisson-distributed with parameter $|\Lambda_N| a_N(R)$, where $a_N(R) = \frac{1}{|\Lambda_N|}\sum_{k=1}^L \frac{1}{k}\int_{\Lambda_N} \mathrm{d}x\, \mu_{x,x}^{(k,\Lambda_N,\mathrm{bc})}(\tau_R \leq k)$. Hence, we can estimate, as $N \to \infty$,

$$
\begin{aligned}
\widehat{\mathtt{Q}}^{(\Lambda_N,\mathrm{bc})}\big(\mathfrak{N}^{(\ell,R)}_{\Lambda_N,\leq L} > \varepsilon|\Lambda_N|\big) &\leq \widehat{\mathtt{Q}}^{(\Lambda_N,\mathrm{bc})}\Big(\sum_{k=1}^L X_k > \frac{\varepsilon}{L}|\Lambda_N|\Big)\\
&\leq \frac{1}{\widehat{Z}_N^{(\mathrm{bc})}(\Lambda_N)} \exp\Big(-|\Lambda_N|(1+o(1))\big[a_N(R) - \tfrac{\varepsilon}{L} + \tfrac{\varepsilon}{L}\log\tfrac{\varepsilon}{L a_N(R)}\big]\Big),
\end{aligned}
\tag{3.8}
$$

according to Cramér's theorem for the sum of $\asymp |\Lambda_N|$ i.i.d. Poisson-distributed random variables with parameter $a_N(R)$ (we dropped the interaction). Let us estimate $a_N(R)$. We are going to show that $\limsup_{N\to\infty} a_N(R) \leq O(1/R)$ as $R \to \infty$. We are going to use C as a generic constant $\in (0,\infty)$, depending only on d or on β and possibly changing its value from appearance to appearance. We will use the estimate $\mu_{x,x}^{(k,\Lambda_N,\mathrm{bc})}(\tau_R \leq k) \leq C k^{-d/2}$ for any $x \in \Lambda_N$ and any $k \in \mathbb{N}$, and we use that $\mu_{x,x}^{(k,\Lambda_N,\mathrm{bc})}(\tau_R \leq k) \leq C\mathrm{e}^{-Cm^2/k}$ for any $x \in \Lambda_N$ that is further away from $\bigcup_{z\in Z_{N,R}} \partial(z+W_R)$ than m (use that for the Brownian motion with time interval $[0,k\beta]$ it has probability $\leq C\mathrm{e}^{-Cm^2/k}$ to terminate at distance $\geq m$ away from its starting site). This gives, for any choice of $m_1,\dots,m_L \in (0,\infty)$,

$$
\begin{aligned}
a_N(R) &= \frac{1}{|\Lambda_N|}\sum_{z\in Z_{N,R}}\sum_{k=1}^L \frac{1}{k}\int_{z+W_R} \mathrm{d}x\, \mu_{x,x}^{(k,\Lambda_N,\mathrm{bc})}(\tau_R \leq k)\\
&\leq C\frac{1}{|W_R|}\sum_{k=1}^L \frac{1}{k}\Big[\int_{W_{R-m_k}} \mathrm{e}^{-Cm_k^2/k}\,\mathrm{d}x + k^{-d/2}\big(|W_R| - |W_{R-m_k}|\big)\Big]\\
&\leq C\sum_{k=1}^L \frac{1}{k}\Big[\mathrm{e}^{-Cm_k^2/k} + k^{-d/2}\frac{m_k}{R}\Big].
\end{aligned}
$$

We pick now $m_k = \sqrt{\frac{k}{C}\log(kR)}$ and obtain the upper bound (recall that $d \geq 3$)

$$
a_N(R) \leq C\Big[\frac{1}{R} + \sum_{k=1}^L k^{-(d+1)/2}\frac{\sqrt{\log(kR)}}{R}\Big] \leq \frac{C}{R}\sqrt{\log R}.
$$

We now put $L = L_R^{(\leq)} = \log\log R$. Substituting this in the large-deviation rate on the right-hand side of (3.8), we see that, as $R \to \infty$,

$$
a_N(R) - \tfrac{\varepsilon}{L} + \tfrac{\varepsilon}{L}\log\tfrac{\varepsilon}{L a_N(R)} \geq o(1) + \frac{\varepsilon}{\log\log R}\log\Big(\frac{\varepsilon R}{(\log\log R)\, C\sqrt{\log R}}\Big) \to \infty,
$$

which implies (3.6).

Now we prove (3.7). Here we use the superstability of v, according to Assumption (V). Observe that $\mathfrak{N}^{(\ell,\neg R)}_{\Lambda_N,>L} = \sum_{k=L+1}^N k\widetilde{X}_k$, where the $\widetilde{X}_k$ are independent Poisson-distributed random variables with parameter

$$
\frac{1}{k}\int_{\Lambda_N} \mathrm{d}x \mu_{x,x}^{(k,\Lambda_N,\mathrm{bc})}(\tau_R > k).
$$

(Using the fact alone that this is $\leq \frac{1}{k}|\Lambda_N|\mathrm{e}^{-Ck/R^2}$ for some $C \in (0,\infty)$ and all k, R does not help in this proof, as one sees when trying it. Therefore, we employ the self-interaction of each such loop.)

Lemma 3.1 implies that the expected exponential interaction of each of the $\widetilde{X}_k$ Brownian loops of length k is not larger than $\mathrm{e}^{-C_R k^{3/2}}$ for some C_R, if k is large, only depending on R. Pick L_R so large that $L_R^{1/2}C_R\to\infty$ as $R\to\infty$. Therefore, for any $a\in(0,\infty)$ and any $L\in\mathbb{N}$ that is larger than L_R, using the exponential Chebyshev inequality,

$$\begin{aligned}\widehat{Z}_N^{(\mathrm{bc})}(\Lambda_N)\widehat{\mathsf{Q}}^{(\Lambda,\mathrm{bc})}\big(\mathfrak{N}_{\Lambda_N,>L}^{(\ell,\neg R)}>\varepsilon|\Lambda_N|\big)&\leq \mathrm{e}^{-a\varepsilon|\Lambda_N|}\prod_{k>L}\mathbb{E}\big[\mathrm{e}^{ak\widetilde{X}_k-C_Rk^{3/2}\widetilde{X}_k}\big]\\&\leq \mathrm{e}^{-a\varepsilon|\Lambda_N|}\prod_{k>L}\mathbb{E}\big[\mathrm{e}^{k\widetilde{X}_k[a-C_RL^{1/2}]}\big].\end{aligned}\tag{3.9}$$

We pick now $a=a_R$ such that $1\ll a_R\ll L_R^{1/2}C_R$, then the exponent in the expectation is negative for all large enough R, and the product is ≤ 1. This implies (3.7). □

3.2. **Our strategy.** We are going to describe our strategy to prove the LDP in Theorem 2.5. This strategy is very comprehensive and in particular provides a number of results that are also vital for the proof of Theorem 2.3. Recall that we decompose the centred box Λ_N with volume $\sim N/\rho$ regularly into all the subboxes $W_z=z+W_R$ with $W_R=[-R,R]^d=W$ and $z\in Z_{N,R}\subset 2R\mathbb{Z}^d$. We have to find the large-N exponential asymptotics of the restricted partition function

$$\widehat{Z}_{N,R,\delta}^{(\mathrm{bc})}(\Lambda_N,\rho_1,\rho_2)=\mathsf{Q}^{(\Lambda_N,\mathrm{bc})}\Big[\mathrm{e}^{-\Phi_{\Lambda_N,\Lambda_N}}\mathbb{1}\{|\tfrac{1}{|\Lambda_N|}\mathfrak{N}_{\Lambda_N}^{(\ell,\neg R)}-\rho_1|\leq\delta\}\mathbb{1}\{|\tfrac{1}{|\Lambda_N|}\mathfrak{N}_{\Lambda_N}^{(\ell,R)}-\rho_2|\leq\delta\}\Big],$$

followed by making $R\to\infty$ and then $\delta\downarrow 0$.

For each $z\in Z_{N,R}$, we distinguish the loops that are entirely contained in W_z, and the W_z-shreds of all the other loops (the R-crossing loops). Hence, we have a loop-shred configuration in the subboxes W_z. We are going to neglect all the interaction between different subboxes. Therefore, with the shift-operator θ_z that satisfies $\theta_z(z)=0$, we can describe the entire system in terms of the crucial empirical measure

$$\Xi_{N,R}(\omega)=\frac{1}{\#Z_{N,R}}\sum_{z\in Z_{N,R}}\delta_{(\theta_z(\Pi_{W_z}^{(\mathcal{L})}(\omega)),\theta_z(\Pi_{W_z}^{(\mathcal{S})}(\omega)))},$$

which is a probability measure on the set $\mathcal{L}_W\times\mathcal{S}_W$ of loop-shred configurations in W; see the notation in Section 2.2. (We identify $\mathcal{L}_W\times\mathcal{S}_W$ with the set $\mathcal{L}_W\oplus\mathcal{S}_W$ of superpositions of loop configurations and shred configurations.) Neglecting the interaction between any two different ones of the subboxes, we will show in Section 4.5 (upper bound) and 4.6 (lower bound), respectively, that the partition function can be roughly be written as

$$\begin{aligned}\widehat{Z}_{N,R,\delta}^{(\mathrm{bc})}(\Lambda_N,\rho_1,\rho_2)\approx\mathsf{Q}^{(\Lambda_N,\mathrm{bc})}\Big[&\mathrm{e}^{-|\Lambda_N|\frac{1}{|W_R|}\langle F_{W_R,W_R},\Xi_{N,R}\rangle}\\&\mathbb{1}\{\tfrac{1}{|W_R|}\langle\mathfrak{N}_{W_R}^{(\mathcal{L},\ell)},\Xi_{N,R}\rangle\approx\rho_1\}\mathbb{1}\{\tfrac{1}{|W_R|}\langle\mathfrak{N}_{W_R}^{(\mathcal{S},\ell)},\Xi_{N,R}\rangle\approx\rho_2\}\Big],\end{aligned}$$

where $F_{W,W}$ is the interaction between W-loops and W-shreds as defined in (2.18) – (2.20) and (2.21).

We need to understand the large-N properties of the distribution of $\Xi_{N,R}$ under $\mathsf{Q}^{(\Lambda_N,\mathrm{bc})}$. Observe that, by the virtue of the Poisson point process, the loop part is independent of the shred part, and the loop parts are essentially (modulo boundary effects from the boundary of the large box Λ_N) i.i.d. over z. However, the shred parts are by far not i.i.d. But luckily, we have the following conditional independence property: given the empirical measure of boundary configurations,

$$\partial\Xi_{N,R}^{(\mathcal{S})}=\frac{1}{\#Z_{N,R}}\sum_{z\in Z_{N,R}}\delta_{(\theta_z(\partial\Pi_{W_z}^{(\mathcal{S})}(\omega)))},$$

these shred configurations $\Pi_{W_z}^{(\mathcal{S})}(\omega)$, $z\in Z_{N,R}$, are independent, and the boundary configuration contains all information about their distribution that is necessary to establish an LDP for $\Xi_{N,R}$ with an explicit rate function (see below). In Section 5.2 we derive the lower bound

$$\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathsf{Q}^{(\Lambda_N,\mathrm{bc})}(\partial\Xi_{N,R}\approx\psi)\geq-\gamma_R(\psi),\qquad\psi\in\mathcal{T}_W,$$

where $\gamma_R(\psi)$ will turn out in Section 5.3 to be small in the limit $R\to\infty$ if ψ is picked carefully (for the upper bound we can essentially take $\gamma_R(\psi)=0$). Indeed, we will need certain ergodic properties of ψ to derive the asymptotics and certain concentration properties of it in order to show that $\gamma_R(\psi)$ is small. The crucial point is that the question whether or not the event $\{\partial\Xi_{N,R}\approx\psi\}$ has positive probability for all large N or not is highly difficult to decide, since this requires knowledge about very long-range properties of the loop configuration; the long loops have global properties; and by far not any collection of loop-shred configurations in many subboxes can come from suitable global loop ensembles. This tricky problem is taken care of by a clever argument involving an ergodicity property and the ergodic theorem.

Moreover, the contribution to the large-deviation probabilities coming from $\{\partial\Xi^{(\mathcal{S})}_{N,R}=\psi\}$ can in Section 6.1 be explicitly be expressed using a large-deviation principle of the form

$$\frac{1}{|\Lambda_N|}\log \mathtt{Q}(\Xi_{N,R}\approx\xi|\partial\Xi_{N,R}=\psi)\approx -J^{(\psi)}_{W_R}(\xi),\qquad N\to\infty.$$

Here $J^{(\psi)}_W$ is the entropy term that appears in the definition of $\mathrm{h}^{(\mathcal{L},\mathcal{S})}$ with $\partial\Pi^{(\mathcal{S})}_W(\xi)$ replaced by ψ,

$$J^{(\psi)}_W(\xi)=\frac{1}{|W|}H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi\,\big|\,\Pi^{(\mathcal{L})}_W(\mathtt{Q})\otimes[\psi\otimes\mathtt{K}_W]\big).$$

For the upper bound in the proof of Theorem 2.5, we just optimize over all ψ that have the required expected particle number in the shreds, $\rho_2|W_R|$, and for the lower bound we restrict to a ψ of the form $\psi=\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)$ for a ξ, whose properties we still can refine. Using the above large-deviation principle and Varadhan's lemma, we will in Sections 6.4 (upper bound) and 6.5 (lower bound) arrive at

$$\begin{aligned}\frac{1}{|\Lambda_N|}\log\widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N,\rho_1,\rho_2)\approx-\inf\Big\{&J_{W_R}(\xi)+\frac{1}{|W_R|}\langle F_{W_R,W_R},\xi\rangle\colon\xi\in\mathcal{M}_1(\mathcal{L}_{W_R}\times\mathcal{S}_{W_R}),\\&\tfrac{1}{|W_R|}\langle\mathfrak{N}^{(\mathcal{L},\ell)}_{W_R},\xi\rangle\approx\rho_1,\tfrac{1}{|W_R|}\langle\mathfrak{N}^{(\mathcal{L},\ell)}_{W_R},\xi\rangle\approx\rho_2\Big\}.\end{aligned}$$

On the way to this, a number of technical problems are to be handled, as is usual in large-deviation theory. Crucial ones of these obstacles come from topological technicalities like missing compactness and missing upper semi-continuity in the proof of the upper bound and missing openness and lower semi-continuity in the proof of the lower bound; therefore both proofs have to separately be approached and require many separate steps. Another general issue that renders the proof lengthy and technical is the fact that Brownian motions are infinitely flexible with positive probability and need to be controlled and enclosed in bounded areas in various senses, in order to gain a clear control on the various influences on the interaction energy and particle numbers, in particular in view of the decomposition into the boxes $z+W$.

The continuities of the energy functional and of the two particle number functionals is not clear *a priori* and needs to be achieved by restrictions to suitable events, both for the proof of the upper and the lower bound. In Section 6.3, using Assumption (V), we introduce two events that we need to insert such that their complement is negligible and on these sets, the two particle numbers are continuous. In Sections 4.6 and 5.3, we restrict to an event on which only loop and shred configurations with several particular properties appear, namely only legs of bounded spread, only configuration with locally bounded particle numbers everywhere and only shreds with bounded stretches. This requires the usage of several cut-off parameters, which will be sent to ∞ at the very end of the proof. In Section 7.4 (after a preparation in Section 7.2, where we have a closer look at the various terms in the entropy J_{W_R}) we control the most involved part, the entropy J_{W_R}, when replacing an arbitrary ξ by some measure that satisfies all these cut-off properties, and additionally the ergodicity property mentioned above. This is used in Section 7.5 to control the value of the variational problem above when doing this change.

In order to carry out the limit as $R \to \infty$ in the above variational formula, the most important point is of course the existence of the limit of J_{W_R} (Theorem 2.3), which we prove in Section 8, after some preparations in the other sections. In the finish of the proof of the upper bound in Theorem 2.5, we need to embed the above variational formula on R-subboxes into a limiting variational formula on gobal stationary loop-interlacement configurations in the entire $\mathbb{R}^d$. Here we first use some tool from large-deviation theory (see Proposition 6.6(4)) to extend a minimizing ξ to arbitrarily large boxes without noticeably enlarging the entropy. Afterwards, we use an extension theorem in the spirit of Carathéodory's theorem to extend it to a global configuration on the entire $\mathbb{R}^d$; see Proposition 4.1. In Sections 9.1 (upper bound) and 9.2 (lower bound) all these arguments are put together and the proof of Theorem 2.5 is finished.

The proof of Theorem 2.3 in Section 8 and several crucial arguments in other sections relies on various properties and estimations for the entropy functional J_{W_R}, which we derive in Section 6.2. We remark that we carry out the necessary estimates with the help of large-deviation arguments, see Proposition 6.6. In particular, we derive a crucial estimate of the form

$$J_{W_R}(\Pi_{W_R}(P)) \leq J_{W_{mR}}(\Pi_{W_{mR}}(P)) + o_{mR}(1), \qquad m \to \infty.$$

This replaces the ubiquitous super-additivity argument that is usually employed for establishing the existence of a specific relative entropy (actually we did not succeed in proving sub-additivity and believe that this property does not hold).

3.3. **Literature survey.** The study of quantum gases, in particular the Bose gas and its statistical mechanics and condensation, is a huge fascinating subject that provides many challenging questions and involves a lot of mathematical ansatzes and toolboxes, see [PS01, PS03] for extensive summaries.

Unlike many investigations on the interacting Bose gas in the mathematical-physics literature, the present paper follows the ansatz of a *path-integral analysis*. This goes back to the vague suggestion by Feynman in [Fe53] that Brownian bridges and the statistics of their lengths might be taken as an order parameter. To our best knowledge, there is no result in the mathematical physics literature on deeper properties of the free energy of the interacting Bose gas in the thermodynamic limit, hence we will restrict here to reviewing only ansatzes with path-integral analysis.

Technical foundations where laid by Ginibre [G70]. There are not many rigorous works yet that follow this advice, starting with phenomenological discussions in [U06] and discussions of the relation between long loops and condensate in [Sü93, Sü02] for some simplified models, including the free Bose gas. More recently, [FKSS20a] conceived the rescaled interaction of the Brownian loops in $d = 4$ as a regularisation as the intersection local time as a possible ansatz for deriving Φ^4-theories.

The Brownian-bridge ansatz is greatly strengthened by combining it with a spatial point-process ansatz and a Poisson point process description and additionally with a large deviation approach that is sometimes called a level-3 approach. An early version of such a Poisson point process, the Brownian loop soup was introduced by Lawler and Werner [LW04], however with a different purpose (not for the Bose gas), namely for making conformal invariance rigorously provable in models from two-dimensional statistical mechanics. The idea of an LDP theory for marked point processes like the Bose gas was implicitly present in papers by Georgii/Zessin in the 1990s [GZ93, G94], and it was combined with the desire to find criteria for the equivalence of ensembles for interacting marked Poisson point processes. Further works (see, e.g, [Raf09, NPZ13, Z22, BDM24]) concentrated on the construction of such processes in the entire space $\mathbb{R}^d$ and to discuss them from the view point of classical Gibbs point theory as was laid down in the famous monograph [G11].

A recent achievement in the tradition of constructing Gibbs measures of marked point processes is [BDM24], where the interacting Bose gas is achieved as such a process with the legs as marks: *nota bene*, the legs as members of $\mathcal{C}_1$, not the loops. The formation of an ensemble of interacting loops is part of the interaction term, which renders this interaction term highly complicated and discontinuous. In [BDM24], an infinite-space Gibbs measure is constructed for any value of ρ and β; however it is rather difficult to decide whether infinite long loops appear in this measure.

For the problem of analysing the limiting free energy of the Bose gas using Brownian bridges, the crucial Poisson description of Proposition 1.2 was introduced in [ACK11]. It was used a number of times in the last years for answering various partial questions. In the non-interacting case, an LDP with identification of the phase transition in terms of the rate function was derived in [A08]. A proof for the phase transition of Bose–Einstein condensation (that is, for off-diagonal long range order (ODLRO), the generally acknowledged criterion for BEC) in terms of Brownian bridges was given recently in [KVZ25], and for a (still non-interacting) mean-field version in [BKV24]. In [BKM24] interactions only within the same loop were admitted in the gas and a related kind of condensation phase transition was proved in connection with the famous self-avoiding walk problem.

As it comes now to interacting Bose gases, the most immediate predecessor paper is [ACK11], which is based on the PPP-description of Proposition 1.2 and applies large-deviation arguments based on the ansatz from [GZ93] for the *empirical stationary process*, $\mathcal{R}_\Lambda(\omega_{\rm P}) = \frac{1}{|\Lambda|}\int_\Lambda \delta_{\theta_x(\omega_{\rm P})}\,{\rm d}x$, a kind of continuous variant of the main tool of the present paper, $\Xi_{N,R}$. A weaker variant of Corollary 2.7 was derived for all small enough particle densities ρ, but without any attempt to describe long loops, since the topological difficulties could not be resolved there and any kind of concept was missing for long loops. Another important predecessor of the current work is [CJK23], which develops and demonstrates the strategy that we use in the proof of Theorem 2.5 for a simpler model, where the loops are replaced by boxes with random and unbounded sizes.

A different ansatz was followed in [QT23], which gives the – to the best of my knowledge, first – proof of the existence of a loop of length $\asymp N$ in an interacting spatial ensemble of random loops of the type of an interacting Bose gas. More precisely, it is shown that the expectation of the length of the loop that contains the origin is not smaller than a constant times the volume of the box. This formidable result has been achieved by transferring the well-known reflection-positive ansatz to this setting; indeed, it was found a way to employ some clever correlation inequalities about the behaviour of the system under reflections. However, there are a number of caveats: the class of interactions is restricted by some unnatural conditions that come from some steps in the Fourier setting, so far the technique could be done only in the time- and space-discrete setting, and only for periodic boundary condition. Removing each of these three shortcomings seems to require serious new input.

A Poisson point process of interlacements was introduced by Sznitman [Sz10] for simple random walks and in [Sz13] for Brownian motions, however without making any connection to Bose gases. A version of this process can be seen as lurking in the background of our description of the Bose gas in Theorem 2.5 for $d \geq 3$, see Section A. The idea of using such a process for the description of the condensate of the Bose gas was first introduced to the literature in [AFY21], however without any relation to any limits or interactions or free energies. There, the version for Gaussian random walks was constructed (alternatively to what we do in Section A), and some properties were examined. Also the independent superposition of the corresponding loop soup and the Gaussian interlacement PPP was introduced and studied. In [V23], it is shown that the simple-random-walk loop soup converges, in the thermodynamic limit with free boundary condition, towards the independent superposition of the same process (with critical density) and the simple-random walk interlacement PPP, see Remark 2.15. In an interacting Bose gas (however, with interactions only involving loop lengths, no spatial details) in the spatially continuous setting, in [DV24] convergence of the gas towards the Brownian loop soup, superposed with the Brownian interlacement PPP, was proved.

Remark 3.3 (LDP for empirical processes) As we explained, an important ingredient of the proof in the present paper is a kind of large-deviation principle for the (discrete version of the) empirical stationary field of the Poisson loop process. This is remarkably close to the main object of [Sz23], but has also decisive differences. Let us comment on that here.

Indeed, [Sz23] studies the stationary empirical field $\mathcal{L}_\Lambda(\varpi_{\rm B}) = \frac{1}{|\Lambda|}\int_\Lambda {\rm d}x\, \delta_{\theta_x(\varpi_{\rm B})}$ of the Brownian interlacement PPP introduced in [Sz13], more precisely, the probability of large deviations of certain linear test functions $\langle F, \mathcal{L}_\Lambda(\varpi_{\rm B})\rangle$. This process $\mathcal{L}_\Lambda$ is the interlacement counterpart of $\mathcal{R}_\Lambda$ above and it is the continuous version of the shred part of our empirical process $\Xi_{N,R}$ (which can be analogously defined for interlacements in place of R-crossing loops). Under certain conditions on the test function F, [Sz23] proves the upper bound of the large-N exponential rate of the probabilities of upward deviations of $\langle F, \mathcal{L}_\Lambda(\varpi_{\rm B})\rangle$ in terms of an explicit variational formula that shows some reminiscences of the famous Donsker–Varadhan LDP rate function. (See [CN23] for the corresponding lower bound.) In principle, this kind of result seems to be close to what is also needed for the description of the Bose gas (if we neglect the fact that Sznitman's process is based on the interlacement process rather than the long-loop part of the Brownian loop soup). But however, the precise assumptions made in [Sz23] make it impossible to apply the results of [Sz23] to the interacting Bose gas.

The surprise is that the LDP in [Sz23] is on capacity scale $|\Lambda_N|^{1-\frac{2}{d}}$ rather than on volume scale $|\Lambda_N|$, the scale on which we are working in this paper. This seems to be due to the fact that the description of the most likely behaviour of the interlacements to meet the event $\{\langle F, \mathcal{L}_\Lambda(\varpi_{\rm B})\rangle > C\}$ for some large C comes from an upscaling of an event in a box of a diverging radius (i.e., from some mesoscopic scale), while we have to describe all the details of the interaction on the microscopic scale. Equivalently, one can say that the description of the Bose gas needs a finer scale of details and is therefore adequately described on volume scale. $\Diamond$

4. Scanning the configuration with finite windows

In this section we introduce a framework for describing the loop configuration in the box Λ_N in terms of an empirical mixture of the local loop and shred configurations that we see in all the shifts of the window $[-R,R]^d$ that scans Λ_N in regular steps of size $2R$.

In Sections 4.1 and 4.2 we introduce the loop space consisting of marked point processes, and the shred space consisting of point measures on shreds. In Section 4.3 we combine the two spaces and give a Carathéodory-like result on the existence of stationary probability measures on the space of loops and interlacements with given marginals on projections on boxes. Our main tool, the mentioned empirical measure $\Xi_{N,R}$ of the loop and shred configurations in the subboxes, is introduced in Section 4.4. In Sections 4.5 and 4.6 we rewrite the constrained partition function in terms of $\Xi_{N,R}$ and derive preliminary upper and lower bounds towards a proof of Theorem 2.5.

Let us recall and introduce some basic notation. We denote by $\mathcal{C}_k$ the set of all continuous functions $[0,\beta k]\to\mathbb{R}^d$ and write $\mathcal{C}=\bigcup_{k\in\mathbb{N}}\mathcal{C}_k$. Each $\mathcal{C}_k$ is equipped with the topology induced by the supremum norm. On $\mathcal{C}$ we use the topology within each $\mathcal{C}_k$, i.e., a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ converges towards some $f\in\mathcal{C}$ if there is some k such that $f\in\mathcal{C}_k$ and $f_n\in\mathcal{C}_k$ for all n large enough, and $f_n\to f$ in the supremum norm on $\mathcal{C}_k$. Hence, we do not assign a distance to function pairs with different lengths. We introduce the loop space

$$\mathcal{C}^{(\circlearrowleft)}=\bigcup_{k\in\mathbb{N}}\mathcal{C}_k^{(\circlearrowleft)},\qquad\text{where}\qquad \mathcal{C}_k^{(\circlearrowleft)}=\{f\in\mathcal{C}_k\colon f(0)=f(k\beta)\}. \tag{4.1}$$

Since loops are cyclic, two elements of $\mathcal{C}^{(\circlearrowleft)}$ are considered equal if they are time-shifts by a time in $\beta\mathbb{Z}$ of each other; hence we conceive $\mathcal{C}_k^{(\circlearrowleft)}$ actually as a quotient space. By default, if nothing else is said, we write an $f\in\mathcal{C}_k^{(\circlearrowleft)}$ as $f\colon[0,k\beta]\to\mathbb{R}^d$. We call the pieces $f|_{[j\beta,(j+1)\beta]}$ with $j\in\mathbb{N}_0$ the *legs* of f and the sites $f(0), f(\beta), f(2\beta),\dots,f((k-1)\beta)$ the *particles* of f. The number of particles of f is denoted by $\ell(f)=k$. Furthermore, we denote by $\mathcal{C}_\infty$ the set of all continuous functions $f\colon\mathbb{R}\to\mathbb{R}^d$ satisfying

$\lim_{t\to-\infty}|f(t)| = \lim_{t\to\infty}|f(t)| = \infty$. Such functions have only finitely many particles in W for any compact set $W \subset \mathbb{R}^d$.

4.1. **Point processes with loops as marks.** We introduce notation for point processes in $\mathbb{R}^d$ with marks in $\mathcal{C}^{(\circlearrowleft)}$ and provide some remarks on their topologies.

We consider the space $\mathcal{L} = \mathcal{M}_{\mathbb{N}_0}(\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)})$ of simple point processes on $\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)}$, more precisely, the set of marked point processes in $\mathbb{R}^d$ with marks in $\mathcal{C}^{(\circlearrowleft)}$. That is, an element $\omega \in \mathcal{L}$ can be represented in terms of a point process ζ on $\mathbb{R}^d$ as

$$\omega = \sum_{x\in\zeta} \delta_{(x,f_x)}, \qquad \text{where } f_x \in \mathcal{C}^{(\circlearrowleft)} \text{ with } f_x(0) = f_x(\beta\ell(f_x)) = x, \tag{4.2}$$

and such that $\omega(A \times \mathcal{C}^{(\circlearrowleft)}) < +\infty$, for any bounded measurable $A \subset \mathbb{R}^d$. We see $\mathcal{L}$ as the set of superpositions $\sum_{k\in\mathbb{N}} \sum_{x\in\zeta_k} \delta_{(x,f_x)}$, where $\zeta_k = \{x \in \zeta : \ell(f_x) = k\}$. We equip $\mathcal{L}$ with the usual evaluation σ-algebra, i.e., the one that is generated by all the maps $\omega \mapsto \Pi_W(\omega) = \sum_{x\in\zeta\cap W} \delta_{(x,f_x)}$ with $W \Subset \mathbb{R}^d$. Let $\mathcal{M}_1(\mathcal{L})$ denote the set of probability measures on $\mathcal{L}$, and $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L})$ the subset of shift-invariant probability measures on $\mathcal{L}$, i.e., the set of those $P \in \mathcal{M}_1(\mathcal{L})$ such that $P \circ \theta_x^{-1} = P$ for any $x \in \mathbb{R}^d$; by θ_x we denote the shift operator by $x \in \mathbb{R}^d$ satisfying $\theta_x(x) = 0$, acting on vectors, subsets of $\mathbb{R}^d$, functions in $\mathbb{R}^d$, measures on $\mathbb{R}^d$ and on point processes in $\mathbb{R}^d$ with and without marks.

For any $W \Subset \mathbb{R}^d$, consider the projection on all the loops that start in W and have all their particles in W:

$$\Pi_W^{(\mathcal{L})}(\omega) = \sum_{x\in\zeta\cap W} \delta_{(x,f_x)} \mathbb{1}\{f_x(k\beta) \in W\ \forall k \in [\ell(f)]\}. \tag{4.3}$$

We denote by $\mathcal{L}_W$ the image of $\mathcal{L}$ under this projection, i.e., the set $\mathcal{L}_W = \mathcal{M}_{\mathbb{N}_0}(W \times \mathcal{C}_W^{(\circlearrowleft)})$ of all point measures in $\mathbb{R}^d$ with marks that are in the set

$$\mathcal{C}_W^{(\circlearrowleft)} = \bigcup_{k\in\mathbb{N}} \big\{f \in \mathcal{C}_k^{(\circlearrowleft)} : f(i\beta) \in W\ \forall i \in [k]\big\} \tag{4.4}$$

of loops whose particles are entirely contained in W. Observe the difference to the usual projection Π_W, which restricts to all loops that start in W but may have particles outside W. Furthermore, observe that the loops in $\mathcal{C}_W^{(\circlearrowleft)}$ may also leave W on the way from particle to particle. It is clear that $\Pi_W^{(\mathcal{L})}$ is indeed a projection, that is, $\Pi_{\widetilde{W}}^{(\mathcal{L})} \circ \Pi_W^{(\mathcal{L})} = \Pi_W^{(\mathcal{L})} \circ \Pi_{\widetilde{W}}^{(\mathcal{L})} = \Pi_W^{(\mathcal{L})}$ if $W \subset \widetilde{W} \Subset \mathbb{R}^d$. A significant difference to Π_W is that, for two disjoint sets $W, \widetilde{W} \Subset \mathbb{R}^d$, the image $\mathcal{L}_{W\cup\widetilde{W}}$ consists of *three* parts: $\mathcal{L}_W$, $\mathcal{L}_{\widetilde{W}}$ and the set of simple point measures with loops that have particles both in W and in $\widetilde{W}$, but nowhere elsek.

The topology on $\mathcal{L}_W$ that we use is the $w^\#$-topology, which is induced by test integrals against continuous and bounded functions $\mathcal{M}_{\mathbb{N}_0}(W \times \mathcal{C}_W^{(\circlearrowleft)}) \to \mathbb{R}$ that vanish outside some ball. Here we define a ball of radius r around some $(x, f_x) \in W \times \mathcal{C}_k^{(\circlearrowleft)}$ as the set of $(y, g) \in W \times \mathcal{C}_k^{(\circlearrowleft)}$ such that $|x-y| + \|f_x - g\|_\infty < r$ and $g(0) = x$, and a ball around some configuration $\sum_{x\in\zeta} \delta_{(x,f_x)}$ is the union of these balls around the points of the configuration. This turns $\mathcal{L}_W$ itself into a Polish space, according to [DVJ08, Prop. 9.1.IV]. According to [DVJ08, Exercise 11.1.7], convergence of point measures $\sum_{x\in\zeta_n\cap W} \delta_{(x,f_x^{(n)})}$ towards a point measure $\sum_{x\in\zeta\cap W} \delta_{(x,f_x)}$ as $n \to \infty$ is equivalent to having $\#\xi_n = \#\xi$ for all large n and pointwise convergence of the points $(x, f^{(n)})$ of ξ_n in $W \times \mathcal{C}^{(\circlearrowleft)}$ towards the ones of ξ as $n \to \infty$ in some assignment. We denote this topology on $\mathcal{L}_W$ by $\mathcal{G}_W^{(\mathcal{L})}$ and the induced topology on $\mathcal{L}$ by $\mathcal{F}_W^{(\mathcal{L})} = (\Pi_W^{(\mathcal{L})})^{-1}(\mathcal{G}_W^{(\mathcal{L})})$.

On the set $\mathcal{M}_1(\mathcal{L}_W)$ of probability measures on $\mathcal{L}_W$, we will be using the weak topology, i.e., the one that is induced by test integral against bounded and continuous functions $\mathcal{L}_W \to \mathbb{R}$. According to [DVJ08, Theorem 11.1.VII], for probability measures $\xi, \xi_1, \xi_2, \dots$ on $\mathcal{L}_W$, the weak convergence $\xi_n \Longrightarrow \xi$ is true if and only if the finite-dimensional distributions of ξ_n converge towards the one of ξ, i.e., if for any $k \in \mathbb{N}$ and any bounded measurable sets $A_1, \dots, A_k \subset \mathcal{L}_W$, we have that $(\xi_n(A_1), \dots, \xi_n(A_k))$ converges weakly towards $(\xi(A_1), \dots, \xi(A_k))$.

We write $f(P)$ for the image measure $P \circ f^{-1}$. On $\mathcal{M}_1(\mathcal{L})$, we will consider the weak topology induced by all the projections $\Pi_W^{(\mathcal{L})}$ with $W \Subset \mathbb{R}^d$, i.e., $P_n \Longrightarrow P$ if and only if $\Pi_W^{(\mathcal{L})}(P_n) \Longrightarrow \Pi_W^{(\mathcal{L})}(P)$ for any $W \Subset \mathbb{R}^d$.

Let us also mention that the weak topology on the set $\mathcal{M}(\mathcal{X})$ of all finite measures on a Polish space $\mathcal{X}$ is induced by the Prokhorov metric

$$\mathrm{d}(\mu,\nu) = \inf\big\{\varepsilon \geq 0\colon \mu(F) \leq \nu(F^\varepsilon), \nu(F) \leq \mu(F^\varepsilon) \text{ for all closed } F \subset \mathcal{X}\big\}, \qquad \nu,\mu \in \mathcal{M}(\mathcal{X}), \tag{4.5}$$

where $F^\varepsilon = \{y \in \mathcal{X}\colon \mathrm{d}(y,F) < \varepsilon\}$ denotes the open ε-neighbourhood of F [DVJ03, Sect. A.2.5]. This turns $\mathcal{M}(\mathcal{X})$ into a Polish space as well [DVJ03, A2.5.III].

4.2. **Shredding paths.** This section is the heart of our ansatz that we introduce in this paper. It is a chopping procedure on path level, which cuts, for each compact set $W \subset \mathbb{R}^d$, any loop that has not all its particles in W or any infinitely long path into the pieces that run within W and collects all these pieces in terms of a point measure. In this way, we introduce a kind of projection operator $\Pi_W^{(\mathcal{S})}$ that has many properties similarly to Π_W or to $\Pi_W^{(\mathcal{L})}$ and can be used to induce a natural topological structure on the set $\mathcal{S}$ of point measures on $\mathcal{C}_\infty$.

For any $f \in \mathcal{C}^{(\circlearrowleft)}$, we introduce the stopping times

$$\begin{aligned} \tau_W(f) &= \inf\{k \in \mathbb{N}_0\colon f(k\beta) \in W\} \in \mathbb{N}_0 \cup \{\infty\}, &(4.6)\\ \sigma_W(f) &= \inf\{k \in \mathbb{N}\colon k > \tau_W(f), f(k\beta) \in W^{\mathrm{c}}\}. &(4.7)\end{aligned}$$

An analogous definition is used for $f \in \mathcal{C}_\infty$. For $f \in \mathcal{C}_k^{(\circlearrowleft)}$ or $f \in \mathcal{C}_\infty$, we write the times of the successive entries of $(f(i\beta))_{i\in\{0,1,\dots,k\}}$, respectively $(f(i\beta))_{i\in\mathbb{Z}}$ to W and exits from W as

$$-\infty < \tau_W^{(1)}(f) < \sigma_W^{(1)}(f) < \tau_W^{(2)}(f) < \sigma_W^{(2)}(f) < \dots < \infty;$$

a more formal definition of these stopping times may be given in terms of appropriate time-shifts of $\tau_W(f)$ and $\sigma_W(f)$. Then each restriction $f|_{[(\tau_W^{(i)}-1)\beta, \sigma_W^{(i)}\beta]}$, shifted in time to obtain a function $f^{(i)}\colon [0, (\sigma_W^{(i)} - \tau_W^{(i)})\beta] \to \mathbb{R}^d$, is called a *$W$-shred* of f; it is a member of the *W-shred space*

$$\mathcal{C}_W = \bigcup_{k\in\mathbb{N}} \big\{g \in \mathcal{C}_k\colon g(0), g(\beta), g(2\beta), \dots, g((k-1)\beta) \in W, g(k\beta) \in W^{\mathrm{c}}\big\}. \tag{4.8}$$

(We ignore at this point the fact that part of the characterisation of $g \in \mathcal{C}_W$ is that $g(-\beta) \in W^{\mathrm{c}}$.) We denote by $\mathcal{M}_{\mathbb{N}_0}(\mathcal{X})$ the set of all simple point measures in a measurable set $\mathcal{X}$. Then we define

$$\mathcal{S}_W = \mathcal{M}_{\mathbb{N}_0}(\mathcal{C}_W) \qquad \text{and} \qquad \Pi_W^{(\mathcal{S})}\colon \mathcal{C}^{(\circlearrowleft)} \cup \mathcal{C}_\infty \to \mathcal{S}_W, \qquad \Pi_W^{(\mathcal{S})}(f) = \sum_i \delta_{f^{(i)}}. \tag{4.9}$$

All topological assertions that we spelled out in Section 4.1 for $\mathcal{L}_W$ hold also for $\mathcal{S}_W$ analogously. In particular, a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of point measures in $\mathcal{S}_W$ converges to some $\Gamma \in \mathcal{S}_W$ if and only if, for all sufficiently large n, $|\Gamma_n| = |\Gamma|$, and the W-shreds of Γ_n converge to the ones of Γ in some assignment. If $\mathcal{G}_W^{(\mathcal{S})}$ denotes the corresponding Borel σ-field, then $(\mathcal{S}_W, \mathcal{G}_W^{(\mathcal{S})})$ is a measurable space.

Introduce $\ell_W(g) = \sum_i \mathbb{1}_W(g((i-1)\beta))$, the number of particles of $g \in \mathcal{C}^{(\circlearrowleft)} \cup \mathcal{C}_\infty$ in W. Observe that $g \in \mathcal{C}_\infty$ is characterised by the requirement that $\ell_W(g) < \infty$ for any compact $W \subset \mathbb{R}^d$. We also introduce the space

$$\mathcal{S} = \Big\{\varpi = \sum_{g\in\Gamma} \delta_g \in \mathcal{M}_{\mathbb{N}_0}(\mathcal{C}_\infty)\colon \sum_{g\in\Gamma} \ell_W(g) < \infty \text{ for all compact } W \subset \mathbb{R}^d\Big\} \tag{4.10}$$

of simple point measures of doubly-infinite continuous functions, such that the cloud has only finitely many particles in any compact subset of $\mathbb{R}^d$.

We also use the symbol $\Pi_W^{(\mathcal{S})}$ for the shredding operation on point process level: for W, $\widetilde{W}$ compact such that $W\subset\widetilde{W}$, define the point process of W-shreds derived from ω by

$$\Pi_W^{(\mathcal{S})}\colon \mathcal{S}\cup\mathcal{S}_{\widetilde{W}}\to\mathcal{S}_W,\qquad \Pi_W^{(\mathcal{S})}(\omega)=\sum_{f\in\omega}\sum_i \delta_{f^{(i)}},\qquad \text{where } f^{(i)} \text{ are the } W\text{-shreds of } f. \tag{4.11}$$

We sometimes write $\Pi^{(\mathcal{S})}_{\widetilde{W}\to W}\colon \mathcal{S}_{\widetilde{W}}\to\mathcal{S}_W$ for this map in order to indicate the domain. It is easy to see that the shredding operators are consistent in the sense that

$$W\subset\widetilde{W}\qquad\Longrightarrow\qquad \Pi_W^{(\mathcal{S})}\circ\Pi_{\widetilde{W}}^{(\mathcal{S})}=\Pi_W^{(\mathcal{S})}. \tag{4.12}$$

On the set $\mathcal{S}$, we introduce a measurable structure by taking $\mathcal{F}^{(\mathcal{S})}$ as the σ-field such that all the shredding operators $\Pi_W^{(\mathcal{S})}$ are measurable, i.e.,

$$\mathcal{F}^{(\mathcal{S})}=\sigma\big(\mathcal{F}_W^{(\mathcal{S})}\colon W\subset\mathbb{R}^d \text{ compact}\big),\qquad \text{where}\quad \mathcal{F}_W^{(\mathcal{S})}=(\Pi_W^{(\mathcal{S})})^{-1}(\mathcal{G}_W^{(\mathcal{S})}).$$

On the set $\mathcal{M}_1(\mathcal{S})$ we introduce the weak topology induced by all the $\Pi_W^{(\mathcal{S})}$ with $W\Subset\mathbb{R}^d$: A sequence $(P_n)_{n\in\mathbb{N}}$ of probability measures on $\mathcal{S}$ converges towards some $P\in\mathcal{M}_1(\mathcal{S})$ if and only if $\Pi_W^{(\mathcal{S})}(P_n)$ (which is by definition equal to the image measure $P_n\circ(\Pi_W^{(\mathcal{S})})^{-1}$) converges weakly to $\Pi_W^{(\mathcal{S})}(P)$ for any $W\Subset\mathbb{R}^d$. Then $(\mathcal{S},\mathcal{F})$ is also a Polish space with metric

$$\mathrm{d}(\Gamma_1,\Gamma_2)=\sum_{R\in\mathbb{N}}2^{-R}\frac{\mathrm{d}_{W_R}\big(\Pi_{W_R}^{(\mathcal{S})}(\Gamma_1),\Pi_{W_R}^{(\mathcal{S})}(\Gamma_2)\big)}{1+\mathrm{d}_{W_R}\big(\Pi_{W_R}^{(\mathcal{S})}(\Gamma_1),\Pi_{W_R}^{(\mathcal{S})}(\Gamma_2)\big)},\qquad W_R=[-R,R]^d, \tag{4.13}$$

where we lifted $\Pi_{W_R}^{(\mathcal{S})}$ to $\mathcal{S}$ by putting $\Pi_{W_R}^{(\mathcal{S})}(\Gamma)=\sum_{g\in\Gamma}\Pi_{W_R}^{(\mathcal{S})}(g)$. (Here d_W denotes some metric on $\mathcal{S}_W$ that induces the topology on $\mathcal{S}_W$.)

Now we introduce a kind of boundary-shredding operation as follows. Define the operator that maps each W-shred g satisfying $g(0),g(\beta),g(2\beta),\dots,g((k-1)\beta)\in W$ and $g(k\beta)\in W^{\mathrm{c}}$ on the triple of its initial and terminal condition and its length:

$$\partial\Pi_W^{(\mathcal{S})}\colon \mathcal{C}_W\to W\times\mathbb{N}\times W^{\mathrm{c}},\qquad \partial\Pi_W^{(\mathcal{S})}(g)=(g(0),l,g(k\beta)). \tag{4.14}$$

We write

$$\begin{aligned}\mathcal{T}_W&=\mathcal{M}_{\mathbb{N}_0}(W\times\mathbb{N}\times W^{\mathrm{c}})\\ &=\Big\{\mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}\colon I \text{ finite}, (x_i,l_i,y_i)\in W\times\mathbb{N}\times W^{\mathrm{c}} \text{ pairwise distinct } \forall i\in I\Big\}\end{aligned}$$

for the set of all finite simple counting measures on $W\times\mathbb{N}\times W^{\mathrm{c}}$. We use the same symbol for the projection operation on shred configurations:

$$\partial\Pi_W^{(\mathcal{S})}\colon \mathcal{S}_W\to\mathcal{T}_W,\qquad \partial\Pi_W^{(\mathcal{S})}\Big(\sum_g\delta_g\Big)=\sum_g\delta_{\partial\Pi_W^{(\mathcal{S})}(g)}, \tag{4.15}$$

and we extend the domain of this operator to $\mathcal{S}_{\widetilde{W}}\cup\mathcal{S}$ by putting $\partial\Pi_W^{(\mathcal{S})}(\varpi)=\partial\Pi_W^{(\mathcal{S})}(\Pi_W^{(\mathcal{S})}(\varpi))$ for $\varpi\in\mathcal{S}_{\widetilde{W}}\cup\mathcal{S}$ for compact sets $W\subset\widetilde{W}\subset\mathbb{R}^d$. Again, all topological assertions of Section 4.1 about $\mathcal{L}_W$ hold analogously for $\mathcal{T}_W$.

Note that $\mu=\partial\Pi_W^{(\mathcal{S})}(\varpi)$ contains all information about the number of particles of the shred configuration $\varpi=\sum_{g\in\Gamma}\delta_g$ in W, when we define

$$\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})}(\mu)=\sum_{i\in I}l_i\qquad \text{for } \mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}; \tag{4.16}$$

then $\mathfrak{N}_W^{(\ell,\mathcal{S})}(\varpi)=\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})}(\mu)$ and $\langle\xi,\mathfrak{N}_W^{(\ell,\mathcal{S})}\rangle=\langle\psi,\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})}\rangle$ for $\xi\in\mathcal{M}_1(\mathcal{S}_W)$ and $\psi=\partial\Pi_W^{(\mathcal{S})}(\xi)\in\mathcal{M}_1(\mathcal{T}_W)$.

Let us also mention the following tightness criterion [DVJ08, Proposition 11.1.VI]: A subset K of $\mathcal{M}_1(\mathcal{T}_W)$ is tight if for any ball $B\subset W\times\mathbb{N}\times W^{\mathrm{c}}$ and for any $\varepsilon>0$, there is $M>0$ such that $\psi(\{\mu\in\mathcal{T}_W\colon\mu(\overline{B})>M\})<\varepsilon$ for any $\psi\in K$. In particular, the set $\{\psi\in\mathcal{M}_1(\mathcal{T}_W)\colon\langle\psi,\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})}\rangle\le\rho_2\}$

is relative compact for any $\rho_2 \in (0,\infty)$, since $\mu(\overline{B}) \leq |I| \leq \sum_{i\in I} l_i = \mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W}(\mu)$. (However, closedness of this set does not follow, since the map $\psi \mapsto \langle \psi, \mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W}\rangle$ is *a priori* not continuous.)

4.3. **An extension result for loop-interlacement configurations.** We need to introduce also some specifics of the space of the joint loop-shred configurations in boxes, respectively loop-interlacement configurations in $\mathbb{R}^d$. In Proposition 4.1 we prove an existence theorem in the spirit of Kolmogorov's theorem in this setting.

We conceive the loop-interlacement configuration space

$$\mathcal{L}\times\mathcal{S} = \mathcal{M}_{\mathbb{N}_0}\big((\mathbb{R}^d\times\mathcal{C}^{(\circlearrowleft)})\cup\mathcal{C}_\infty\big) \tag{4.17}$$

as the superposition point measure space with loops and interlacements. We conceive the shredding operator $\Pi^{(\mathcal{S})}_W$ defined in (4.9) as an operator $\Pi^{(\mathcal{S})}_W\colon \mathcal{L}\times\mathcal{S}\to\mathcal{S}_W$ and the loop-restriction operator $\Pi^{(\mathcal{L})}_W$ defined in (4.3) as an operator $\Pi^{(\mathcal{L})}_W\colon \mathcal{L}\times\mathcal{S}\to\mathcal{L}_W$. Note that $\Pi^{(\mathcal{L})}_W$ can produce something non-trivial only from $\mathcal{L}$ (not from shreds or from interlacements), while $\Pi^{(\mathcal{S})}_W$ can cut both loops and shreds into shreds. Then the operator

$$\Pi_W\colon \mathcal{L}\times\mathcal{S}\to\mathcal{L}_W\times\mathcal{S}_W,\qquad \text{defined by } \Pi_W(\omega,\varpi) = \Pi^{(\mathcal{L})}_W(\omega)+\Pi^{(\mathcal{S})}_W(\varpi), \tag{4.18}$$

is the superposition operator of $\Pi^{(\mathcal{S})}_W$ and $\Pi^{(\mathcal{L})}_W$, when we conceive $\mathcal{L}_W\times\mathcal{S}_W$ as $\mathcal{M}_{\mathbb{N}_0}((W\times\mathcal{C}^{(\circlearrowleft)}_W)\cup\mathcal{C}_W)$. On $\mathcal{S}_W$ and $\mathcal{L}_W$, respectively, we have the natural sigma fields $\mathcal{G}^{(\mathcal{S})}_W$ and $\mathcal{G}^{(\mathcal{L})}_W$, respectively, that are generated by the vague topology. On $\mathcal{L}\times\mathcal{S}$ we consider the sigma field $\mathcal{F}$ that is generated by $\mathcal{F}_W = \sigma((\Pi^{(\mathcal{S})}_W)^{-1}(\mathcal{G}^{(\mathcal{S})}_W)\cup(\Pi^{(\mathcal{L})}_W)^{-1}(\mathcal{G}^{(\mathcal{L})}_W))$ with $W=W_R=[-R,R]^d$ for all $R\in\mathbb{N}$, i.e., $\mathcal{F}=\sigma(\bigcup_{R\in\mathbb{N}}\mathcal{F}_{W_R})$. The definition of the projection operator $\Pi_{\widetilde{W}\to W}\colon \mathcal{L}_{\widetilde{W}}\times\mathcal{S}_{\widetilde{W}}\to\mathcal{L}_W\times\mathcal{S}_W$ for $W\subset\widetilde{W}\Subset\mathbb{R}^d$ is clear.

The following proposition will be enormously helpful in the proof of properties of the specific relative entropy density in the proof of Theorem 2.3 and in the proof of the upper bound for the free energy of the interacting Bose gas (Theorem 2.5). It is an extension theorem for consistent families of probability measures on loop/shred configurations in boxes. It is in the spirit of the famous extension theorem by Kolmogorov, but this theorem seems not applicable, since we are not working with product spaces, and the objects on which the extension is a measure (interlacements) are global and have a long range. Hence, we need to redo its proof (actually, the proof of Ionescu-Tulcea's theorem, the 'discrete' variant) with the help of Carathéodory's theorem.

Proposition 4.1 (Loop-interlacements point process extension)**.** *Let* $(\xi^{(W_R)})_{R\in\mathbb{N}}$ *be a family of probability measures* $\xi^{(W_R)}\in\mathcal{M}_1(\mathcal{L}_{W_R}\times\mathcal{S}_{W_R})$*, such that* $\Pi_{W_{R+1}\to W_R}(\xi^{(W_{R+1})})=\xi^{(W_R)}$ *for any* $R\in\mathbb{N}$*, i.e.,* $(\xi^{(W_R)})_{R\in\mathbb{N}}$ *is consistent. Then there exists a unique* $P\in\mathcal{M}_1(\mathcal{L}\times\mathcal{S})$ *such that* $\Pi_{W_R}(P)=\xi^{(W_R)}$ *for all* $R\in\mathbb{N}$*.*

Proof. We want to apply Carathéodory's theorem to the algebra $\mathcal{F}^{(0)}=\bigcup_{R\in\mathbb{N}}\mathcal{F}_{W_R}$ and to the map $P_0\colon\mathcal{F}^{(0)}\to[0,1]$ that is defined by

$$P_0(A)=\xi^{(W_R)}(B),\qquad A\in\Pi^{-1}_{W_R}(\mathcal{G}_{W_R}), \tag{4.19}$$

where $B\in\mathcal{G}_{W_R}$ is the unique set such that $A=\Pi^{-1}_{W_R}(B)$. The fact that $P_0\colon\mathcal{F}^{(0)}\to[0,1]$ is well-defined follows from the consistency in the same way as in the proof of Ionescu-Tulcea's theorem. It is also not difficult to show that P_0 is a content, that is, finitely additive and satisfies $P_0(\varnothing)=0$. It also obviously satisfies $\Pi_W(P_0)=\xi^{(W)}$ for any $W\Subset\mathbb{R}^d$.

We would like to extend P_0 from $\mathcal{F}^{(0)}$ to a measure P on $\mathcal{F}$, which finishes the proof. According to the uniqueness part of Carathéodory's theorem, this P is unique, and according to its existence part, for showing the existence it is sufficient to show that P_0 is continuous in the empty set. Hence, assume that $(A_m)_{m\in\mathbb{N}}$ is a sequence in $\mathcal{F}^{(0)}$ that decreases, as $m\to\infty$, to $A=\bigcap_{m\in\mathbb{N}}A_m\in\mathcal{F}^{(0)}$. We assume that $\liminf_{m\to\infty}P_0(A_m)>0$ and need to show that A is not empty.

Without loss of generality, we can assume that $A_m = \Pi_{W_m}^{-1}(B_m)$ for some $B_m \in \mathcal{G}_{W_m}$.

We can easily extend the consistency relation $\Pi_{W_{R+1}\to W_R}(\xi^{(W_{R+1})}) = \xi^{(W_R)}$ for all $R \in \mathbb{N}$ to the consistency relation $\Pi_{W_{n+m}\to W_n}(\xi^{(W_{n+m})}) = \xi^{(W_n)}$ for any $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. For $W \subset \widetilde{W} \Subset \mathbb{R}^d$, we can write $\mathcal{L}_{\widetilde{W}} \times \mathcal{S}_{\widetilde{W}} = (\mathcal{L}_W \times \mathcal{S}_W) \times (\mathcal{L} \times \mathcal{S})_{\widetilde{W}\setminus W}$, where

$$\big(\mathcal{L} \times \mathcal{S}\big)_{\widetilde{W}\setminus W} = \mathcal{M}_{\mathbb{N}_0}\big(\big[(\widetilde{W} \times \mathcal{C}^{(\circlearrowleft)}_{\widetilde{W}}) \cup \mathcal{C}_{\widetilde{W}}\big] \setminus \big[(W \times \mathcal{C}^{(\circlearrowleft)}_{W}) \cup \mathcal{C}_{W}\big]\big).$$

Since $\mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$ is a Polish space for any $W \Subset \mathbb{R}^d$, there are Markov kernels $K_{n,m}$ from $\mathcal{L}_{W_n} \times \mathcal{S}_{W_n}$ into $(\mathcal{L} \times \mathcal{S})_{W_{n+m}\setminus W_n}$ such that $\xi^{(W_{n+m})} = \xi^{(W_n)} \otimes K_{n,m}$ (with $K_{n,0} = \mathbb{1}_{B_n}$).

We are going to construct now $(\omega^{(m)}, \varpi^{(m)}) \in B_m$ for $m \in \mathbb{N}$ such that $(\omega^{(m-1)}, \varpi^{(m-1)}) = \Pi_{W_m\to W_{m-1}}(\omega^{(m)}, \varpi^{(m)})$. For this, define $h_{n,m}\colon \mathcal{L}_{W_n} \times \mathcal{S}_{W_n} \to [0,1]$ by putting

$$h_{n,m}(\omega_{W_n}, \varpi_{W_n}) = K_{n,m}\big((\omega_{W_n}, \varpi_{W_n}), (B_{n+m})^{(1)}_{(\omega_{W_n},\varpi_{W_n})}\big),$$

where

$$(B_{n+m})^{(1)}_{(\omega_{W_n},\varpi_{W_n})} = \big\{(\omega^{(n)}_{W_{n+m}}, \varpi^{(n)}_{W_{n+m}}) \in (\mathcal{L} \times \mathcal{S})_{W_{n+m}\setminus W_n} \colon \\ (\omega_{W_n} + \omega^{(n)}_{W_{n+m}}, \varpi_{W_n} + \varpi^{(n)}_{W_{n+m}}) \in B_{n+m}\big\}$$

is the W_n-cut of B_{n+m}. It is relatively easy to show that, for any $n \in \mathbb{N}$ and any $(\omega_{W_n}, \varpi_{W_n})$, the sequence $(h_{n,m}(\omega_{W_n}, \varpi_{W_n}))_{m\in\mathbb{N}_0}$ is non-increasing, since $B_m \subset \Pi^{-1}_{W_m\to W_{m-1}}(B_{m-1})$ and hence $(B_{n+m+1})^{(1)}_{(\omega_{W_n},\varpi_{W_n})} \subset (B_{n+m})^{(1)}_{(\omega_{W_n},\varpi_{W_n})}$.

We construct now recursively a sequence of pairs $(\omega_{W_n}, \varpi_{W_n}) \in B_n \subset \mathcal{L}_{W_n} \times \mathcal{S}_{W_n}$ such that $\lim_{m\to\infty} h_{n,m}(\omega_{W_n}, \varpi_{W_n}) > 0$. Because of monotonicity in m and $h_{n,0} = \mathbb{1}_{B_n}$, this finishes the proof of the existence of a sequence $(\omega^{(m)}, \varpi^{(m)}) \in B_m$ for $m \in \mathbb{N}$ such that $(\omega^{(m-1)}, \varpi^{(m-1)}) = \Pi_{W_m\to W_{m-1}}(\omega^{(m)}, \varpi^{(m)})$.

First, we consider $n = 1$. For $m \in \mathbb{N}_0$, we have

$$\int h_{1,m}(\omega_{W_1}, \varpi_{W_1})\, \xi^{(W_1)}(\mathrm{d}(\omega_{W_1}, \varpi_{W_1})) = \xi^{(W_m)}(B_m) = P_0(A_m).$$

By our assumption that $P_0(A_m)$ has a non-zero limit, there is a $(\omega_{W_1}, \varpi_{W_1}) \in B_1$. Now we make the induction step from n to $n+1$. A calculation shows that, for any $m \in \mathbb{N}_0$,

$$\int_{(\mathcal{L}\times\mathcal{S})_{W_{n+m}\setminus W_n}} h_{n+1,m}\big((\omega_{W_n}, \varpi_{W_n}), (\omega^{(n)}_{W_{n+m}}, \varpi^{(n)}_{W_{n+m}})\big)\, K_{n,1}\big((\omega_{W_n}, \varpi_{W_n}), \mathrm{d}(\omega^{(n)}_{W_{n+m}}, \varpi^{(n)}_{W_{n+m}})\big) \\ = h_{n,m+1}(\omega_{W_n}, \varpi_{W_n}).$$

Hence, the sequence $(h_{n+1,m}((\omega_{W_n}, \varpi_{W_n}), (\omega^{(n)}_{W_{n+m}}, \varpi^{(n)}_{W_{n+m}}))_{m\in\mathbb{N}}$ is a non-increasing sequence of functions in the second argument that does not vanish as $m \to \infty$. Therefore, there is $(\omega_{W_{n+1}}, \varpi_{W_{n+1}}) \in B_{n+1}$ whose projection on W_n is equal to $(\omega_{W_n}, \varpi_{W_n})$.

Now that we have a sequence of $(\omega^{(n)}, \varpi^{(n)}) \in B_n \subset \mathcal{L}_{W_n} \times \mathcal{S}_{W_n}$ such that $(\omega^{(n-1)}, \varpi^{(n-1)}) = \Pi_{W_n\to W_{n-1}}(\omega^{(n)}, \varpi^{(n)})$ for any $n \in \mathbb{N}$; hence we can construct an element $(\omega, \varpi) \in \mathcal{L} \times \mathcal{S}$ such that $\Pi_{W_n}(\omega, \varpi) = (\omega^{(n)}, \varpi^{(n)})$ for any $n \in \mathbb{N}$. Therefore, $(\omega, \varpi) \in A_n$ for any n, i.e., it lies in A, which ends the proof. First we construct $\widetilde{\omega}$ by simply extending all the $\omega^{(n)}$ from point measures in $W_n \times \mathcal{C}^{(\circlearrowleft)}_{W_n}$ to point measures in $\mathbb{R}^d \times \mathcal{C}^{(\circlearrowleft)}$. The construction of ϖ and of $\omega - \widetilde{\omega}$ goes as follows. Start with some of the points, $f_1 \in \mathcal{C}_{W_1}$ of $\varpi^{(1)}$ with some time interval $[T_1^-\beta, T_1^+\beta]$ and integers T_1^-, T_1^+. In $\varpi^{(2)}$, there must be a point $f_2 \in (W_2 \times \mathcal{C}^{(\circlearrowleft)}_{W_2}) \cup \mathcal{C}_{W_2}$ that extends f_1 (possibly also involving some other points of $\varpi^{(1)}$) to a continuous function with time interval $[T_2^-\beta, T_2^+\beta]$. If $f_2 \in W_2 \times \mathcal{C}^{(\circlearrowleft)}_{W_2}$, then we add $\delta_{(f_2(0), f_2)}$ to $\widetilde{\omega}$. Otherwise we keep f_2 as a W_2-shred and proceed. Recall that f_1 satisfies $f_1(T_1^-\beta) \in W_1^{\mathrm{c}}$, since it is a W_1-shred. Inductively, one can go on and find, for any $m \in \mathbb{N}$, either a W_m-loop $f_m \in W_m \times \mathcal{C}^{(\circlearrowleft)}_{W_m}$, which we add to $\widetilde{\omega}$, or a W_m-shred $f_m \in \mathcal{C}_{W_m}$ in $\varpi^{(m)}$ that extends f_{m-1} and has a time interval

$[T_m^-\beta, T_m^+\beta]$ with integers T_m^-, T_m^+, and $T_m^- \leq T_{m-1}^-$ and $T_m^+ \geq T_{m-1}^+$ and $f_m((T_m^- - 1)\beta) \in W_m^{\rm c}$ and $f_m(T_m^+\beta) \in W_m^{\rm c}$. Since $\lim_{m\to\infty} T_m^- = -\infty$ and $\lim_{m\to\infty} T_m^+ = \infty$ and $\lim_{m\to\infty} f_m((T_m^- - 1)\beta) = \infty$ and $\lim_{m\to\infty} f_m(T_m^+\beta) = \infty$, we have constructed a continuous function $f \in \mathcal{C}_\infty$, the extension of all the f_m that are W_m-shreds. This we take as a point of ϖ. Repeat the same procedure with all the other points of $\varpi^{(1)}$ (if there are some left), otherwise with $\varpi^{(2)}$ and so on. The sum of all the delta measures on W_m loops that we obtained during the entire procedure is defined as $\omega - \widetilde{\omega}$, and this defines ω. □

4.4. **The empirical measure of the loop/shred configurations.** We now introduce the main tool in our proof of Theorem 2.5 and explain the first steps of our strategy. The main object is the empirical measure of all the loop/shred configurations in all the R-subboxes within the macrobox Λ_N. We are in the situation of Section 1.4: we fix $R \in \mathbb{N}$ and write $W = [-R, R]^d$ and $W_z = z + W$ for $z \in 2R\mathbb{Z}^d$ and fix $\rho \in (0,\infty)$ and a centered box Λ_N with volume $\sim N/\rho$ as $N \to \infty$. We decompose $\Lambda_N = \bigcup_{z\in Z_{N,R}} W_z$ for some suitable $Z_{N,R} \subset 2R\mathbb{Z}^d$ (possibly $R = R_N$ depends on N and converges to R; we suppress this in the notation). Note that $\# Z_{N,R} \sim |\Lambda_N|/|W_R| \sim \frac{N}{\rho(2R)^d}$ as $N \to \infty$.

We introduce now our main tool in the proof of Theorem 2.5: for any $\omega \in \mathcal{L}$, we define

$$\Xi_{N,R}^{(\omega)} = \frac{1}{\# Z_{N,R}} \sum_{z\in Z_{N,R}} \delta_{(\theta_z(\Pi_{W_z}^{(\mathcal{L})}(\omega)),\theta_z(\Pi_{W_z}^{(\mathcal{S})}(\omega))} \in \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W), \tag{4.20}$$

where θ_z is the shift-operator such that $\theta_z(W_z) = W$. (Recall the operators $\Pi_W^{(\mathcal{L})}$ and $\Pi_W^{(\mathcal{S})}$ from (4.3) and (4.11).) This is the empirical measure of all the loop/shred configurations of ω in the subboxes, shifted to the origin.

Let us briefly recall our plan from Section 3.2 to prove Theorem 2.5: for $\omega = \omega_{\rm P}$, the marked reference PPP of Definition 1.1,

(1) rewrite the partition function $\widehat{Z}_N$ in terms of an integral over $\Xi_{N,R}^{(\omega_{\rm P})}$,
(2) find a large deviation principle for $\Xi_{N,R}^{(\omega_{\rm P})}$ as $N \to \infty$,
(3) use Varadhan's lemma to express the large-N exponential rate of the partition function as a variational formula on the space $\mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$ with $W = [-R, R]^d$,
(4) make $R \to \infty$ in that formula to arrive at χ defined in (2.24).

For making the first step, we will present upper and lower bounds in Sections 4.5 and 4.6, respectively. The second step will be done in Section 6.1, and the third in Sections 6.4 and 6.5. In the fourth step in Section 9, we are employing a new relative entropy density that we introduce in Section 8. For various reasons, the proof of the upper and the lower bound will have substantial differences.

It will be convenient to distinguish the the loop part and the shred part of $\Xi_{N,R}$,

$$\Xi_{N,R}^{(\omega_{\rm P},\mathcal{L})} = \Pi_W^{(\mathcal{L})}(\Xi_{N,R}^{(\omega_{\rm P})}) = \frac{1}{\# Z_{N,R}} \sum_{z\in Z_{N,R}} \delta_{\theta_z(\Pi_{W_z}^{(\mathcal{L})}(\omega_{\rm P}))} \in \mathcal{M}_1(\mathcal{L}_W), \tag{4.21}$$

$$\Xi_{N,R}^{(\omega_{\rm P},\mathcal{S})} = \Pi_W^{(\mathcal{S})}(\Xi_{N,R}^{(\omega_{\rm P})}) = \frac{1}{\# Z_{N,R}} \sum_{z\in Z_{N,R}} \delta_{\theta_z(\Pi_{W_z}^{(\mathcal{S})}(\omega_{\rm P}))} \in \mathcal{M}_1(\mathcal{S}_W). \tag{4.22}$$

These two are independent under the reference PPP, since the first one comes from the part of the PPP that has no R-crossings, and the second from the R-crossing part of the PPP, which are independent. The second observation is that the loop part is an empirical measure of independent objects, whose distributions are basically identical, modulo the boundary conditions, whose influence will be shown to vanish in the limit as $N \to \infty$.

Now, as for the shred part, it is obvious that the subconfigurations in the R-subboxes are very far from independent, which presents a major obstacle in the analysis of its distribution. However, the third observation is that the configurations of the shreds in the subboxes are independent if the statistics of their starting sites, lengths and terminal sites are fixed, i.e., conditional on the boundary-shred empirical measure

$$\partial\Xi_{N,R}^{(\omega_{\rm P},\mathcal{S})}=\partial\Pi_W^{(\mathcal{S})}(\Xi_{N,R}^{(\omega_{\rm P})})=\frac{1}{\#Z_{N,R}}\sum_{z\in Z_{N,R}}\delta_{\theta_z(\partial\Pi_{W_z}^{(\mathcal{S})}(\omega_{\rm P}))}\in\mathcal{M}_1(\mathcal{T}_W). \tag{4.23}$$

4.5. **Upper bound.** We are going to derive an upper bound for the partition function in terms of a expectation with respect to the empirical measure $\Xi_{N,R}$. We need to estimate it in such a way that it is decomposed into a product of the contributions over all the disjoint subboxes. For this, we restrict the interaction to the sum of (slightly modified) interactions within the subboxes. In the proof of the upper bound, since the interaction potential v is nonnegative, we are in the lucky situation that we can simply drop all the interaction between different subboxes and that we can drop also the probabilistic cost of the behaviour of the boundary configurations of the shreds.

Assume the situation and notation described in Section 4.4. As one sees from (2.25), we want to find an upper bound for

$$\widehat{Z}_{N,R,\delta}^{(\rm bc)}(\Lambda_N,\rho_1,\rho_2)=\mathtt{Q}^{(\Lambda_N,{\rm bc})}\big[{\rm e}^{-\Phi_{\Lambda_N,\Lambda_N}}\mathbb{1}\{|\tfrac{1}{|\Lambda_N|}\mathfrak{N}_{\Lambda_N}^{(\ell,\neg R)}-\rho_1|\le\delta\}\mathbb{1}\{|\tfrac{1}{|\Lambda_N|}\mathfrak{N}_{\Lambda_N}^{(\ell,R)}-\rho_2|\le\delta\}\big]. \tag{4.24}$$

The main result of this section is the following.

Lemma 4.2 (Upper bound). *For any $N,R\in\mathbb{N}$, $\delta\in(0,1)$, $\rho_1,\rho_2\in[0,\infty)$,*

$$\begin{aligned}\widehat{Z}_{N,R,\delta}^{(\rm bc)}(\Lambda_N,\rho_1,\rho_2)\le\mathtt{Q}^{(\Lambda_N,{\rm bc})}\Big[&\exp\Big\{-N_R|W|\langle\Xi_{N,R}^{(\omega_{\rm P})},\tfrac{1}{|W|}F_{W,W}\rangle\Big\}\\&\times\mathbb{1}\Big\{\langle\Xi_{N,R}^{(\omega_{\rm P})},\tfrac{1}{|W|}\mathfrak{N}_W^{(\ell,\mathcal{L})}\rangle\in\overline{\mathcal{B}}_{2\delta}(\rho_1)\Big\}\mathbb{1}\Big\{\langle\partial\Xi_{N,R}^{(\omega_{\rm P},\mathcal{S})},\tfrac{1}{|W|}\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})}\rangle\in\overline{\mathcal{B}}_{2\delta}(\rho_2)\Big\}\Big],\end{aligned} \tag{4.25}$$

where $N_R=\#Z_{N,R}$, and $F_{W,W}$ is defined in (2.21), and the particle-counting operators are defined in (2.15).

Proof. We are going to express $\widehat{Z}_{N,R,\delta}^{(\rm bc)}(\Lambda_N,\rho_1,\rho_2)$ in terms of the subconfigurations

$$\omega_{W_z}=\Pi_{W_z}^{(\mathcal{L})}(\omega)=\sum_{x\in W_z\colon f_x(i\beta)\in W_z\forall i}\delta_{(x,f_x)}\qquad\text{and}\qquad\varpi_{W_z}=\Pi_{W_z}^{(\mathcal{S})}(\omega)=\sum_{g\in\Gamma_z}\delta_g\qquad\text{with } z\in Z_{N,R}, \tag{4.26}$$

where we wrote $\Gamma_z=\Gamma_z(\omega)$ for the set of W_z-shreds of ω. Recall that we write $g\colon[0,\ell(g)\beta]\to\mathbb{R}^d$ for a W-shred g with $g(0),g(\beta),g(2\beta),\dots,g((\ell(g)-1)\beta)\in W$ and $g(\ell(g)\beta)\in W^{\rm c}$, and its i-th leg is $g_i=g|_{[(i-1)\beta,i\beta]}$ for $i\in[\ell(g)]$. Then we estimate, for all sufficiently large N,

$$\begin{aligned}&\mathbb{1}\{|\tfrac{1}{|\Lambda_N|}\mathfrak{N}_{\Lambda_N}^{(\ell,\neg R)}-\rho_1|\le\delta\}\mathbb{1}\{|\tfrac{1}{|\Lambda_N|}\mathfrak{N}_{\Lambda_N}^{(\ell,R)}-\rho_2|\le\delta\}\\&\qquad\le\mathbb{1}\Big\{\frac{1}{N_R}\sum_{z\in Z_{N,R}}\frac{1}{|W|}\mathfrak{N}_{W_z}^{(\ell,\mathcal{L})}(\omega_{W_z})\in\overline{\mathcal{B}}_{2\delta}(\rho_1)\Big\}\mathbb{1}\Big\{\frac{1}{N_R}\sum_{z\in Z_{N,R}}\frac{1}{|W|}\mathfrak{N}_{W_z}^{(\ell,\mathcal{S})}(\varpi_{W_z})\in\overline{\mathcal{B}}_{2\delta}(\rho_2)\Big\},\end{aligned} \tag{4.27}$$

where we write $\mathcal{B}_\delta(x)=(x-\delta,x+\delta)$ and $\overline{\mathcal{B}}_\delta(x)$ for its closure, for any $x\in\mathbb{R}$ and $\delta>0$.

Recall the interaction defined in (1.8). We drop all interactions between subboxes W_z and $W_{\widetilde z}$ for different $z,\widetilde z\in Z_{N,R}$ and obtain a lower bound for the entire interaction. Recall the notation from (1.10) and from (2.18)–(2.20). We extend (2.19) and (2.20) from $\varpi\in\mathcal{S}$ to $\varpi_W=\sum_{g\in\Gamma}\delta_g\in\mathcal{S}_W$, i.e., from $\Gamma\subset\mathcal{C}_\infty$ to $\Gamma\subset\mathcal{C}_W$, in an obvious way. Then we estimate

$$\Phi_{\Lambda_N,\Lambda_N}(\omega)\ge\sum_{z\in Z_{N,R}}F_{W,W}(\theta_z(\omega_{W_z}),\theta_z(\varpi_{W_z})), \tag{4.28}$$

where $F_{W,W}(\omega_W,\varpi_W)=\frac12F_{W,W}^{(\mathcal{L}\mathcal{L})}(\omega_W)+F_{W,W}^{(\mathcal{L}\mathcal{S})}(\omega_W,\varpi_W)+\frac12F_{W,W}^{(\mathcal{S}\mathcal{S})}(\varpi_W)$.

This implies the following for all sufficiently large N:

$$\widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N,\rho_1,\rho_2)\le \mathtt{Q}^{(\Lambda_N,\mathrm{bc})}\Big[\mathbb{1}\Big\{\frac{1}{N_R}\sum_{z\in Z_{N,R}}\frac{1}{|W|}\mathfrak{N}^{(\ell,\mathcal{L})}_{W_z}\in\overline{\mathcal{B}}_{2\delta}(\rho_1)\Big\} \\ \times\mathbb{1}\Big\{\frac{1}{N_R}\sum_{z\in Z_{N,R}}\frac{1}{|W|}\mathfrak{N}^{(\ell,\mathcal{S})}_{W_z}\in\overline{\mathcal{B}}_{2\delta}(\rho_2)\Big\}\prod_{z\in Z_{N,R}}\mathrm{e}^{-F_{W,W}(\theta_z(\omega_{W_z}),\theta_z(\varpi_{W_z}))}\Big].\tag{4.29}$$

We now write the right-hand side of (4.29) in terms of the empirical measure $\Xi^{(\omega_{\mathrm{P}})}_{N,R}$ from (4.20) and its boundary-shred projection $\partial\Xi^{(\omega_{\mathrm{P}},\mathcal{S})}_{N,R}$ from (4.23) as follows. Recall (4.16) for writing the particle number in shreds as a functional of $\partial\Xi^{(\omega_{\mathrm{P}},\mathcal{S})}_{N,R}$. From this, the assertion follows. □

For later developments in Section 6, when we will take the limit as $N\to\infty$, it will be convenient to work exclusively with particle boundary condition, which was introduced below (1.4). Therefore, let us further estimate from above, until we arrive at a similar expectation with bc $=$ par. For 'bc' being Dirichlet zero boundary conditions, this is trivial, since zero boundary conditions in particular satisfy particle boundary conditions. For periodic boundary conditions, however, we have to work harder for this point; this will be done in Sections 5 and at the beginning of Section 6.1.

At the beginning of Section 6.1 we will make some technical preparations for applying a suitable large-deviation principle in the limit as $N\to\infty$ to the right-hand side of (4.25). A suitable formulation of this LDP will be in Corollary 6.5. After inserting in Section 6.3 some events that induce some necessary compactness and control on the particle number and interaction, we will apply the LDP in Section 6.4, resulting in an R- and δ-depending variational formula. This will be used in Section 9.1 to make $R\to\infty$ and $\delta\downarrow 0$, and to finish the proof of the upper bound in Theorem 2.5.

4.6. **Lower bound.** We are going to derive a lower bound for $\widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N,\rho_1,\rho_2)$ defined in (4.24) for any sufficiently large N. The main result of this section is Lemma 4.3 below. Our lower bound there will be analogous to the upper bound in Lemma 4.2, but will require much more restrictions in the integration area.

In general, we assume the situation and notation described in Section 4.4. We will have to neglect all the interactions between loops and shreds in different subboxes W_z, such that the contributions will be independent over the subboxes. For this, we need to restrict the amount of particles in the boundary area of W_z for any $z\in Z_{N,R}$. We will do that in a much more comprehensive way than what is needed here, in order to prepare for future steps in Section 6.5, where we will take the limit as $N\to\infty$. That is, we will impose here that, in each subbox W_z, all the legs (of loops and of shreds) have a *spread* $\|f\|_{\mathrm{sp}}=\max_{t\in[0,\beta]}|f(t)-f(0)|$ that is $<M$, and in each ball of radius $\mathfrak{r}$ (to be introduced), the particle number is $\le T$ for some large M,T. Furthermore, we need to restrict the global configuration close to the boundary of Λ_N in order to achieve particle boundary condition.

We are going to use $C\in(0,\infty)$ as a generic constant that depends only on d, β or v (which we neglect) and may change its value from appearance to appearance. If it may depend on additional parameters, then these are appended as indices.

We need to restrict to configurations in

$$\mathcal{L}^{(M)}=\Big\{\sum_{x\in\zeta}\delta_{(x,f_x)}\in\mathcal{L}\colon \|f_{x,i}\|_{\mathrm{sp}}<M\ \forall x\in\zeta,\forall i\Big\},\qquad M\in(1,\infty),\tag{4.30}$$

where we recall that f_i is the i-th leg of the loop f. We will write $\mathcal{L}^{(M)}_W=\mathcal{L}_W\cap\mathcal{L}^{(M)}$. We will make a change of measure from the original PPP $\mathtt{Q}$ (see Definition 1.1) to the PPP $\mathtt{Q}_M$ with intensity measure

$$\nu^{(\mathrm{par})}_{\Lambda_N,M}(\mathrm{d}x,\mathrm{d}f)=\sum_{k\in\mathbb{N}}\frac{1}{k}\mathrm{d}x\otimes\mu^{(k,M,\Lambda_N,\mathrm{par})}_{x,x}(\mathrm{d}f),\tag{4.31}$$

where $\mu_{x,x}^{(k,M,\Lambda_N,\mathrm{par})}$ is the restriction of $\mu_{x,x}^{(k,\Lambda_N,\mathrm{par})}$ to the set of loops of length k whose legs have spread $< M$. The W_R-shred-configurations under $\mathtt{Q}_M$ lie in the set

$$\mathcal{S}^{(M)} = \Big\{ \sum_{g\in\Gamma} \delta_g \in \mathcal{S} \colon \|g_i\|_{\mathrm{sp}} < M \ \forall g \in \Gamma, i \in \mathbb{Z} \Big\}. \tag{4.32}$$

We also need a notation for the number of all particles in $W \Subset \mathbb{R}^d$, regardless where their loop or shred comes from:

$$\widetilde{\mathfrak{N}}_W^{(\ell)}(\omega,\varpi) = \sum_{x\in\zeta}\sum_i \mathbb{1}\{f_{x,i}(0)\in W\} + \sum_{g\in\Gamma}\sum_i \mathbb{1}\{g_i(0)\in W\}, \qquad \omega\in\mathcal{L}, \varpi\in\mathcal{S}. \tag{4.33}$$

With a parameter $\mathfrak{r} \in (0,1)$, we decompose (up to the boundaries) W into the boxes $z + W_{\mathfrak{r}}$ with $z \in W \cap 2\mathfrak{r}\mathbb{Z}^d$. With positive parameters L, K, T, define the events

$$A^{(\mathcal{L})}_{W;M,L,K} \;=\; \bigcap_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d} \Big\{ \omega \in \mathcal{L}_W^{(M,\le L)} \colon \forall l \in [L] \colon |\zeta_{z,l}| \le K \Big\}, \tag{4.34}$$

$$A^{(\mathcal{S})}_{W;M,\mathfrak{r},T} \;=\; \bigcap_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d} \Big\{ \varpi \in \mathcal{S}_W^{(M)} \colon \widetilde{\mathfrak{N}}^{(\ell)}_{z+W_{\mathfrak{r}}}(\varpi) \le T \Big\}, \tag{4.35}$$

where $\zeta_{z,l}$ is the set of initial points of l-length loops of ω in $z + W_{\mathfrak{r}}$. That is, the loop configurations in $A^{(\mathcal{L})}_{W;M,L,K}$ have only legs with spread $< M$, only loop lengths $\le L$, and there are, in any of the subboxes $z + W_{\mathfrak{r}}$, no more l-length loops than K, while in $A^{(\mathcal{S})}_{W;M,\mathfrak{r},T}$ we impose a uniform upper bound for the number of particles in any of the subboxes $z + W_{\mathfrak{r}}$.

We will later restrict the integration with respect to the PPP ω_{P} to the event

$$A_{W;\Theta} = A^{(\mathcal{L})}_{W;M,L,K} \times A^{(\mathcal{S})}_{W;M,\mathfrak{r},T}, \qquad \text{with } \Theta = (M, L, K, \mathfrak{r}, T). \tag{4.36}$$

More precisely, we will restrict to the event $\bigcap_{z\in Z^\circ_{N,R}} \{(\omega_{W_z}, \varpi_{W_z}) \in A_{W_z;\Theta}\}$, where we introduce the slightly smaller (than Λ_N) box $\Lambda'_N = \bigcup_{z\in Z^\circ_{N,R}} W_z$, and $Z^\circ_{N,R}$ is the set of all $z \in Z_{N,R}$ that have no neighbour in $Z^{\mathrm{c}}_{N,R}$ (where we adopt the usual neighbourhood relation in $\mathbb{Z}^d$, upscaled by a factor of $2R$). Let us introduce the conditional version of $\mathtt{Q}_M^{(\Lambda'_N,\mathrm{par})}[\,\cdot\,|\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R} = \psi]$ given this event, i.e., given that $(\omega_{W_z}, \varpi_{W_z}) \in A_{W_z;\Theta}$ for all $z \in Z^\circ_{N,R}$, which we denote by $\mathtt{Q}_{W;\Theta}^{(\Lambda'_N,\mathrm{par})}[\,\cdot\,|\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R} = \psi]$. The slightly modified empirical measure is denoted

$$\Xi^{(\circ,\omega)}_{N,R} = \frac{1}{\# Z^\circ_{N,R}} \sum_{z\in Z^\circ_{N,R}} \delta_{\left(\theta_z(\omega^{(\mathcal{L})}_{W_z}),\theta_z(\varpi^{(\mathcal{S})}_{W_z})\right)} \in \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W) \tag{4.37}$$

and $\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R}$ is the boundary-shred projection of $\Xi^{(\circ,\omega)}_{N,R}$ analogously to (4.23). We also need the function

$$p_{W;M,\mathfrak{r},T}(\mu) = \mathtt{K}_W\big(\mu; A^{(\mathcal{S})}_{W;M,\mathfrak{r},T}\big), \qquad \mu \in \mathcal{T}_W. \tag{4.38}$$

As usual, we write $W = W_R = [-R, R]^d$ for short.

Lemma 4.3 (Lower bound)**.** *Fix $\delta \in (0,1)$ and $\rho_1, \rho_2 \in [0,\infty)$. Then, for any $M, L, K, T, \mathfrak{r} \in (0,\infty)$, there are constants $c_M \in (0,1)$ satisfying $\lim_{M\to\infty} c_M = 1$ and $C_{M,L,K,T} \in (0,\infty)$ (depending only on M respectively only on M, L, K, T) such that, for any $N, R \in \mathbb{N}$ such that $R > M$, and for any*

$\psi\in\mathcal{M}_1(\mathcal{T}_W)$ (possibly depending on N) such that the following conditional expectation is well-defined,

$$
\begin{aligned}
\widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N,\rho_1,\rho_2)\geq{}& \mathrm{e}^{-\frac{1}{R}C_{M,L,K,T}|\Lambda_N|}\mathrm{e}^{-o(|\Lambda_N|)}c_M^{|\Lambda_N|}\big[\Pi_W^{(\mathcal{L})}(\mathtt{Q})(A^{(\mathcal{L})}_{W;M,L,K})\big]^{|\Lambda_N'|/|W|}\mathrm{e}^{\frac{|\Lambda_N'|}{|W|}\langle\log p_{W;M,\mathfrak{r},T},\psi\rangle}\\
&\times\mathtt{Q}_M^{(\Lambda_N',\mathrm{par})}\big(\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R}=\psi\big)\\
&\times\mathtt{Q}_{W;\Theta}^{(\Lambda_N',\mathrm{par})}\Big[\exp\Big\{-|\Lambda_N'|\langle\Xi^{(\circ,\omega_{\mathrm{P}})}_{N,R},\tfrac{1}{|W|}F_{W,W}\rangle\Big\}\\
&\qquad\times\mathbb{1}\Big\{\langle\Xi^{(\circ,\omega_{\mathrm{P}})}_{N,R},\tfrac{1}{|W|}\mathfrak{N}_W^{(\ell,\mathcal{L})}\rangle\in\mathcal{B}_{\delta/3}(\rho_1)\Big\}\mathbb{1}\Big\{\langle\psi,\tfrac{1}{|W|}\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})}\rangle\in\mathcal{B}_{\delta/3}(\rho_2)\Big\}\Big|\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R}=\psi\Big].
\end{aligned}
\tag{4.39}
$$

Proof. In the expectation in (4.24) that defines $\widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N,\rho_1,\rho_2)$ we insert the indicator on the event that the configuration lies in $\mathcal{L}^{(M)}$. Furthermore, we now require that the configuration ω_{Λ_N} has no particle in the boundary area $\Lambda_N\setminus\Lambda_N'$. This implies that the restricted configuration $\omega_{\Lambda_N'}$ satisfies particle-boundary conditions in Λ_N' (which was introduced below (1.4)), since no loop reaches $\partial\Lambda_N$ (because of $M<R$) and this holds for both, Dirichlet zero and periodic boundary condition in Λ_N. The event of having no particles in $\Lambda_N\setminus\Lambda_N'$ is the intersection of the two independent events that no loops start in $\Lambda_N\setminus\Lambda_N'$ (whose probability is $\geq\mathrm{e}^{-o(|\Lambda_N|)}$, since the volume of $\Lambda_N\setminus\Lambda_N'$ is $o(|\Lambda_N|)$) and the event that no shred terminates in $\Lambda_N\setminus\Lambda_N'$, which is absorbed in the expectation with respect to particle-boundary conditions in Λ_N'. Hence, we will restrict to an expectation on configurations in Λ_N' instead of Λ_N, which will be asymptotically lead to no changes in the limit as $N\to\infty$.

Note that

$$
\lim_{N\to\infty}\frac{1}{|\Lambda_N|}\big([\nu^{(\mathrm{par})}_{\Lambda_N}-\nu^{(\mathrm{par})}_{\Lambda_N,M}](\Lambda_N\times\mathcal{C}^{(\circlearrowleft)})\big)=\sum_{k\in\mathbb{N}}\frac{1}{k}\mu_{0,0}^{(k)}(\mathcal{C}_k^{(\circlearrowleft)}\setminus\mathcal{C}_{k,M}^{(\circlearrowleft)})\to 0\qquad\text{as }M\to\infty,
$$

where $\mathcal{C}_{k,M}^{(\circlearrowleft)}$ denotes the set of loops of length k whose legs have spread $<M$. Furthermore, the number of loops in the event in (4.24) is $\leq O(|\Lambda_N|)$; hence the change of measure from $\mathtt{Q}^{(\Lambda_N',\mathrm{par})}$ to $\mathtt{Q}_M^{(\Lambda_N',\mathrm{par})}$ yields, as a lower bound a factor of $c_M^{|\Lambda_N|}$ with some $c_M\in(0,1)$ satisfying $c_M\to1$ as $M\to\infty$.

Summarizing, this transition to the M-restricted configurations in Λ_N' instead of Λ_N leads to the bound

$$
\widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N,\rho_1,\rho_2)\geq\mathrm{e}^{o(|\Lambda_N|)}c_M^{|\Lambda_N|}\mathtt{Q}_M^{(\Lambda_N',\mathrm{par})}\Big[\mathrm{e}^{-\Phi_{\Lambda_N',\Lambda_N'}}\mathbb{1}\{|\tfrac{1}{|\Lambda_N'|}\mathfrak{N}_{\Lambda_N'}^{(\ell,\neg R)}-\rho_1|\leq\tfrac{\delta}{2}\}\mathbb{1}\{|\tfrac{1}{|\Lambda_N'|}\mathfrak{N}_{\Lambda_N'}^{(\ell,R)}-\rho_2|\leq\tfrac{\delta}{2}\}\Big].
\tag{4.40}
$$

Our next step is to restrict to the event $\{\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R}=\psi\}$ for some (later particularly chosen) $\psi\in\mathcal{M}_1(\mathcal{T}_W)$, where $\Xi^{(\circ,\omega)}_{N,R}$ is defined in (4.37).

Here we will pick ψ (possibly depending on N) such that this event has positive probability, and hence we write (4.40) as

$$
\begin{aligned}
\widehat{Z}_{N,R,\delta}{}^{(\mathrm{bc})}(\Lambda_N,\rho_1,\rho_2)\geq{}&\mathrm{e}^{o(|\Lambda_N|)}c_M^{|\Lambda_N|}\mathtt{Q}_M^{(\Lambda_N',\mathrm{par})}\big(\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R}=\psi\big)\\
&\times\mathtt{Q}_M^{(\Lambda_N',\mathrm{par})}\Big[\mathrm{e}^{-\Phi_{\Lambda_N',\Lambda_N'}}\mathbb{1}\{|\tfrac{1}{|\Lambda_N'|}\mathfrak{N}_{\Lambda_N'}^{(\ell,\neg R)}-\rho_1|<\tfrac{\delta}{2}\}\mathbb{1}\{|\tfrac{1}{|\Lambda_N'|}\mathfrak{N}_{\Lambda_N'}^{(\ell,R)}-\rho_2|<\tfrac{\delta}{2}\}\Big|\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R}=\psi\Big].
\end{aligned}
\tag{4.41}
$$

Our next task is to write the expectation in the second line of (4.41) as an expectation with respect to $\Xi^{(\circ,\omega_{\mathrm{P}})}_{N,R}$, which is not able to express interaction between any two subboxes $z+W$ and $z'+W$ with distinct $z.z'\in2R\mathbb{Z}^d$. For this, we need to split all the interaction into the sum of the interactions in all the subboxes $z+W_R$ with $z\in2R\mathbb{Z}^d$, and we need to upper estimate the interaction between any two of such subboxes. This will be based on the following estimate of the interaction in terms of an integral of the square of particle numbers. Recall the notation from (1.10) and from (2.18)–(2.20) and (4.26) and that $F_{W,W}=\frac{1}{2}F^{(\mathcal{LL})}_{W,W}+F^{(\mathcal{LS})}_{W,W}+\frac{1}{2}F^{(\mathcal{SS})}_{W,W}$. For two (not necessarily disjoint) subboxes $W,\widetilde{W}$, we write $F_{W,\widetilde{W}}((\omega_W,\varpi_W),(\omega_{\widetilde{W}},\varpi_{\widetilde{W}}))$ for all the interaction of loops and shreds between W and $\widetilde{W}$ (i.e., the loops with all their particles in W respectively in $\widetilde{W}$).

Lemma 4.4 (Estimating interaction)**.** *Assume that the interaction potential v satisfies Assumption (V). Then, for any $M\in(0,\infty)$, with $\mathfrak{r}_M^{(v)}=M+\sup\operatorname{supp}(v)$, for any configuration $\omega\in\mathcal{L}^{(M)}$, and for any $R\in(1,\infty)$ and any $z,z'\in 2R\mathbb{Z}^d$,*

$$\begin{aligned}F_{W_z,W_{z'}}\big((\omega_{W_z},\varpi_{W_z}),(\omega_{W_{z'}},\varpi_{W_{z'}})\big)\le\beta\|v\|_\infty\Big(\int_{\mathcal{B}_{\mathfrak{r}_M^{(v)}}(W_z\cap W_{z'})}\mathrm{d}x\,\widetilde{\mathfrak{N}}^{(\ell)}_{\mathcal{B}_{\mathfrak{r}_M^{(v)}}(x)}(\omega_{W_z},\varpi_{W_z})^2\Big)^{1/2}&\\ \times\Big(\int_{\mathcal{B}_{\mathfrak{r}_M^{(v)}}(W_z\cap W_{z'})}\mathrm{d}x\,\widetilde{\mathfrak{N}}^{(\ell)}_{\mathcal{B}_{\mathfrak{r}_M^{(v)}}(x)}(\omega_{W_{z'}},\varpi_{W_{z'}})^2\Big)^{1/2},&\end{aligned}\tag{4.42}$$

where $\mathcal{B}_r(A)$ denotes the r-ball around a compact set A.

Proof. We write $\mathfrak{r}$ instead of $\mathfrak{r}_M^{(v)}$. The main observation is that any leg of ω in W (i.e., any leg that starts from a particle in W) stays in the M-neighbourhood of W and has therefore interaction only in the $\mathfrak{r}$-neighbourhood of W. Hence, any $\omega\in\mathcal{L}^{(M)}$ has the property that, for any two legs g,g' of a loop or shred in W_z and of a loop or shred in $W_{z'}$ (meaning that the initial site lies in W_z respectively in $W_{z'}$), respectively, that have a non-vanishing interaction with each other, there is a site $x\in\mathcal{B}_{\mathfrak{r}}(W_z\cap W_{z'})$ such that $|g(0)-x|\le\mathfrak{r}$ and $|g'(0)-x|\le\mathfrak{r}$. Denote by $\mathfrak{L}(\omega_W,\varpi_W)=\{f_{x,i}\colon x\in\zeta\cap W, i\in[\ell(f_x)]\}\cup\{g_i\colon i\in\Gamma_W(g),g\in\varpi_W\}$ the set of all the legs of loops or of shreds of $\omega_W=\sum_{x\in\zeta}\delta_{(x,f_x)}$ or of ϖ_W. (Recall that all particles of all the loops of ω_W are in W, according to (4.26).) Then, for any $z,z'\in 2R\mathbb{Z}^d$,

$$\begin{aligned}&F_{W_z,W_{z'}}\big((\omega_{W_z},\varpi_{W_z}),(\omega_{W_{z'}},\varpi_{W_{z'}})\big)\\&\quad=\sum_{g\in\mathfrak{L}(\omega_{W_z},\varpi_{W_z})}\sum_{g'\in\mathfrak{L}(\omega_{W_{z'}},\varpi_{W_{z'}})}V(g,g')\\&\quad\le\beta\|v\|_\infty\sum_{g,g'}\int_{\mathcal{B}_{\mathfrak{r}}(W_z\cap W_{z'})}\mathrm{d}x\,\mathbb{1}\{|g(0)-x|\le\mathfrak{r}\}\mathbb{1}\{|g'(0)-x|\le\mathfrak{r}\}\\&\quad\le\beta\|v\|_\infty\int_{\mathcal{B}_{\mathfrak{r}}(W_z\cap W_{z'})}\mathrm{d}x\,\widetilde{\mathfrak{N}}^{(\ell)}_{\mathcal{B}_{\mathfrak{r}}(x)}(\omega_{W_z},\varpi_{W_z})\widetilde{\mathfrak{N}}^{(\ell)}_{\mathcal{B}_{\mathfrak{r}}(x)}(\omega_{W_{z'}},\varpi_{W_{z'}})\\&\quad\le\beta\|v\|_\infty\Big(\int_{\mathcal{B}_{\mathfrak{r}}(W_z\cap W_{z'})}\mathrm{d}x\,\widetilde{\mathfrak{N}}^{(\ell)}_{\mathcal{B}_{\mathfrak{r}}(x)}(\omega_{W_z},\varpi_{W_z})^2\Big)^{1/2}\Big(\int_{\mathcal{B}_{\mathfrak{r}}(W_z\cap W_{z'})}\mathrm{d}x\,\widetilde{\mathfrak{N}}^{(\ell)}_{\mathcal{B}_{\mathfrak{r}}(x)}(\omega_{W_{z'}},\varpi_{W_{z'}})^2\Big)^{1/2}.\end{aligned}$$

□

In words, this estimates the amount of cross-interaction between two neighbouring subboxes against the product of the integrals over x in a neighbourhood of the intersection of the subboxes over the square of particle numbers in a certain box around x, and the amount of interaction within one of such subboxes in terms of the integral of the squares of the particle numbers in a neighbourhood of the subbox.

Let us proceed with the proof of Lemma 4.3, restarting from (4.41) and showing what Lemma 4.4 gives us for the interaction. We will, as usual, always write $\omega=\sum_{x\in\zeta}\delta_{(x,f_x)}$ for any $\omega\in\mathcal{L}$. We write $\mathcal{L}_W^{(M,\le L)}=\{\omega\in\mathcal{L}_W^{(M)}\colon\ell(f_x)\le L\,\forall x\in\zeta\}$. We pick now an arbitrary $\mathfrak{r}\in(0,\infty)$. Without loss of generality, we assume that the boxes $z+W_{\mathfrak{r}}$ with $z\in W\cap2\mathfrak{r}\mathbb{Z}^d$ cover W, i.e., $W=\bigcup_{z\in W\cap2\mathfrak{r}\mathbb{Z}^d}z+W_{\mathfrak{r}}$. Recall from Remark 2.11 that $\widetilde{\mathfrak{N}}^{(\ell)}_V$ counts particles only within V for any $V\subset W$; hence for $(\omega,\varpi)\in\mathcal{L}_W\times\mathcal{S}_W$, we have that $\widetilde{\mathfrak{N}}^{(\ell)}_W(\omega,\varpi)=\sum_{z\in W\cap2\mathfrak{r}\mathbb{Z}^d}\widetilde{\mathfrak{N}}^{(\ell)}_{z+W_{\mathfrak{r}}}(\omega,\varpi)$.

For $\omega \in A^{(\mathcal{L})}_{W;M,L,K}$ we easily see that $\widetilde{\mathfrak{N}}^{(\ell)}_{\mathcal{B}_{\mathfrak{r}^{(v)}_M}(x)}(\omega) \leq C_{M,L,K}$ for every $x \in \mathcal{B}_{\mathfrak{r}}(W)$ with some $C_{M,L,K}$ that depends only on M, L, and K. Analogously, for $\varpi \in A^{(\mathcal{S})}_{W;M,\mathfrak{r},T}$ we easily see that $\widetilde{\mathfrak{N}}^{(\ell)}_{\mathcal{B}_{\mathfrak{r}^{(v)}_M}(x)}(\varpi) \leq C_M T$ for every $x \in \mathcal{B}_{\mathfrak{r}}(W)$ with some $C_M \in (0,\infty)$ that depends only on M.

On the event $A_{W;\Theta}$ defined in (4.36), since R is much larger than M, for each $z \in Z^{(\circ)}_{N,R}$, the number of $z' \neq z$ such that there is some interaction between W_z and $W_{z'}$ is not larger than 3^d. Furthermore, the volume of $\mathcal{B}_{\mathfrak{r}}(\partial W_z)$ is not larger than CMR^{d-1}. Hence, we can estimate, with the help of Lemma 4.4,

$$\begin{aligned}\Phi_{\Lambda'_N,\Lambda'_N}(\omega) &= \sum_{z,z' \in Z^{(\circ)}_{N,R}} F_{W_z,W_{z'}}\big((\omega_{W_z},\varpi_{W_z}),(\omega_{W_{z'}},\varpi_{W_{z'}})\big)\\ &\leq \sum_{z \in Z^{(\circ)}_{N,R}} \Big(F_{W_z,W_z}(\omega_{W_z},\varpi_{W_z}) + C_{M,L,K,T}R^{d-1}\Big) \leq \frac{|\Lambda_N|}{|W|}\langle F_{W,W},\Xi^{(\circ,\omega)}_{N,R}\rangle + \frac{C_{M,L,K,T}}{R}|\Lambda_N|,\end{aligned} \tag{4.43}$$

with some $C_{M,L,K,T} \in (0,\infty)$ that does not depend on R nor on ω nor on N. (We choose the event $A_{W;\Theta}$ much smaller than what is needed for (4.43); this will come in handy in our further developments in Section 6.5.)

So far, we have argued that we may estimate, for any sufficiently large N,

$$\begin{aligned}&\mathrm{e}^{-\Phi_{\Lambda'_N,\Lambda'_N}}\mathbb{1}\{|\tfrac{1}{|\Lambda'_N|}\mathfrak{N}^{(\ell,\neg R)}_{\Lambda'_N} - \rho_1| \leq \tfrac{\delta}{2}\}\mathbb{1}\{|\tfrac{1}{|\Lambda'_N|}\mathfrak{N}^{(\ell,R)}_{\Lambda'_N} - \rho_2| \leq \tfrac{\delta}{2}\}\\ &\qquad \geq \mathrm{e}^{-|\Lambda_N|\frac{1}{|W|}\langle F_{W,W},\Xi^{(\circ)}_{N,R}\rangle}\mathrm{e}^{-\frac{C_{M,L,K,T}}{R}|\Lambda_N|}\prod_{z \in Z^{(\circ)}_{N,R}}\mathbb{1}_{A_{W_z;\Theta}}(\omega_{W_z},\varpi_{W_z})\\ &\qquad\quad \times \mathbb{1}\Big\{\frac{1}{N_R}\sum_{z \in Z^{(\circ)}_{N,R}}\frac{1}{|W|}\mathfrak{N}^{(\ell,\mathcal{L})}_{W_z} \in \mathcal{B}_{\delta/2}(\rho_1)\Big\}\mathbb{1}\Big\{\frac{1}{N_R}\sum_{z \in Z^{(\circ)}_{N,R}}\frac{1}{|W|}\mathfrak{N}^{(\ell,\mathcal{S})}_{W_z} \in \mathcal{B}_{\delta/2}(\rho_2)\Big\}.\end{aligned} \tag{4.44}$$

Under $\mathtt{Q}^{(\Lambda'_N,\mathrm{par})}_M[\,\cdot\,|\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R} = \psi]$, the indicators $\mathbb{1}_{A_{W_z;\Theta}}(\omega_{W_z},\varpi_{W_z})$ are independent over $z \in Z^{(\circ)}_{N,R}$ and their product has the probability

$$\big[\Pi^{(\mathcal{L})}_W(\mathtt{Q})(A^{(\mathcal{L})}_{W;M,L,K})\big]^{|Z^\circ_{N,R}|}\mathrm{e}^{|Z^\circ_{N,R}|\langle\log p_{W;M,\mathfrak{r},T},\psi\rangle},$$

where $p_{W;M,\mathfrak{r},T}$ is defined in (4.38). Using the notation for the conditional measure $\mathtt{Q}^{(\Lambda'_N,\mathrm{par})}_{W;\Theta}[\,\cdot\,|\partial\Xi^{(\circ,\omega,\mathcal{S})}_{N,R} = \psi]$ given just before the lemma, we obtain the assertion of the lemma, by substituting (4.44) into (4.41). $\square$

This lower bound is analogous to the upper bound in (4.25) with $Z_{N,R}$ replaced by $Z^\circ_{N,R}$ and the corresponding amendments and with $\overline{\mathcal{B}}_{2\delta}(\rho_i)$ replaced by $\mathcal{B}_{\delta/3}(\rho_i)$ and with additional auxiliary terms whose exponential rates turn out to be small when we later make $R, M, T, L \to \infty$. Note that we have now particle boundary condition in the (slightly smaller) box Λ'_N. Therefore, all the $\theta_z(\omega_{W_z})$ with $z \in Z^\circ_{N,R}$ have now the same distribution; in particular the $\theta_z(\omega_{W_z})$ are i.i.d.

In Lemma 5.3 below, we are going to take $\psi \in \mathcal{M}_1(\mathcal{T}_W)$ in a particular way and derive a suitable large-deviation lower bound for the term in the second line in Section 5.2. In Corollary 6.5 in Section 6.1 we explain abstractly how to find the large-N exponential rate of the expectation in the last two lines, and in Section 6.5 we proceed with making $N \to \infty$ in (4.39). In Section 9.2 we put everything together and finish the proof of the lower bound in Theorem 2.5.

5. The empirical measure of boundary-shred configurations

In this section we provide crucial information about the distribution of the empirical measure of the boundary-shred configuration, $\partial\Xi_{N,R}^{(\omega_{\rm P},\mathcal{S})}$, defined in (4.23), for the reference marked PPP $\omega_{\rm P}$ under $\mathtt{Q}^{(\Lambda_N,{\rm bc})}$. The main result of this section will be a lower bound for the large-N deviations in Lemma 5.3. This will be crucial for handling the term in the second line of (4.39). Finally, this term should and will turn out to be negligible in the limit $R\to\infty$ since it is "only" a boundary term after all, but the way to proving this will be long. A step towards this is done in Lemma 5.5, where the lower bound of Lemma 5.3 is further lower bounded for particular configurations. More work in Section 7.5 will show that a general configuration can be approximated with such particular ones. In contrast, for proving the upper bound, we will just drop the influence coming from $\partial\Xi_{N,R}^{(\omega_{\rm P},\mathcal{S})}$, i.e., we estimate probabilities involving this just by one, which is of course much easier.

In Lemma 5.3 and its proof we make a connection between long loops and interlacements; we need to construct large-loop configurations with a given empirical measure ψ of boundary-shred configurations. By far not every ψ admits the existence of loop-configurations in arbitrarily large boxes Λ_N that is consistent with ψ, the criterion for this property is rather subtle and has a global character. In Lemma 5.3 we first circumvent this problem by coining the term "admissible" for ψ's having this extension property and prove the lemma under this assumption. But in Lemma 5.4 we show that the requirement of ergodicity of shift-invariant random interlacement configurations implies admissibility; this is the reason that in Section 7.5 we need to approximate arbitrary loop-interlacement configurations with ergodic ones.

As a preparation, in Section 5.1 we look at the W-shreds emanating from a single loop and describe, given the boundary-shred configuration, the distribution of the resulting shred configuration via the strong Markov property of the Brownian motion. This is used in Section 5.2 in Lemma 5.3 for deriving the crucial LDP lower bound for the probability of the event $\{\partial\Xi_{N,R}^{(\omega_{\rm P},\mathcal{S})}\approx\psi\}$ for any admissible ψ. In Section 5.3 we derive further estimates for this lower bound for particular shred distributions. Furthermore, as an application and also for independent interest and further use later, we show that the expected number of shreds in a large box is negligible with respect to the volume of the box for any shift-invariant interlacement configuration measure with bounded entropy and bounded expected particle density; a property that is very suggesting, but surprisingly difficult to prove.

In the entire section, assume that the situation and the notation described in Section 1.4 is given. That is, the box Λ_N with volume $\sim N/\rho$ is decomposed into $N_R\sim|\Lambda_N|/|W_R|$ regular boxes $W_z=z+W$ of diameter $2R$, where R may depend on N and converges to a given $R\in\mathbb{N}$.

5.1. **The shreds of one loop.** In this section, we are examining the joint distribution of all the W-shreds of a given Brownian loop in a large box Λ containing the box W. Using the strong Markov property, we will collect all the pieces between entries and exits to and from W and drop the remaining pieces. That is, we will consider one W-shred $g\in\mathcal{C}_k$ as in (4.8), ignoring the last particle before $g(0)$, which is in $W^{\rm c}$. Recall that $\mathbb{P}_x^{(\Lambda,{\rm bc})}$ denotes the distribution of a Brownian motion $B=(B_t)_{t\in[0,\infty)}$ in Λ with boundary condition 'bc' starting from x. We assume that 'bc' is one of the three boundary conditions 'per', '0' or 'par'.

Define

$$p_x^{(W,\Lambda,{\rm bc})}(l,y)=\mathbb{P}_x^{(\Lambda,{\rm bc})}(\tau=l,B_{\tau\beta}\in{\rm d}y)/{\rm d}y,\qquad(x,l,y)\in\Lambda\times\mathbb{N}\times\Lambda,\tag{5.1}$$

where

$$\tau=\tau_W=\inf\{l\in\mathbb{N}\colon B_{l\beta}\notin W^{(B_0)}\}\tag{5.2}$$

is the first of the times in $\beta\mathbb{N}$ at which the motion changes the box, where we write $W^{(x)}=W_{z_x}$ with $x\in W_{z_x}$ for that subbox that contains x. Then p_x is a sub-probability density (with respect to the counting measure on $\mathbb{N}$ times the Lebesgue measure) on $\mathbb{N}\times(W^{(x)})^{\rm c}$ (for periodic boundary condition even a probability density). Furthermore, for a loop $B \in \mathcal{C}^{(\circlearrowleft)}$, we define $\tau_1 = \tau$ and $\tau_{j+1} = \inf\{l > \tau_j : l \in \mathbb{N}, B_{l\beta} \notin W^{(B_{\tau_j})}\}$ (the $(j+1)$-st time that the motion B changes the box at some time in $\beta\mathbb{Z}$) for $j \in \mathbb{N}$. Since B is a cycle, there is a unique smallest $\mathfrak{m}_R = \mathfrak{m}_R(B) \in \mathbb{N}$ such that $B_{\tau_{\mathfrak{m}_R+1}\beta} = B_{\tau_1\beta}$. The number $\mathfrak{m}_R(B)$ is the number of times that B changes the subbox at times in $\beta\mathbb{N}$.

In order to describe the pieces of the loop between the changes between the subboxes (i.e., the shreds), we introduce the following probability measure on $\mathcal{C}_l$: by

$$q_{x,y}^{(l,W,\Lambda,\mathrm{bc})}(\mathrm{d}f) = \frac{\mathbb{P}_x^{(l,\Lambda,\mathrm{bc})}(\tau = l, B \in \mathrm{d}f, B_{\tau\beta} \in \mathrm{d}y)}{p_x^{(W,\Lambda,\mathrm{bc})}(l,y)\,\mathrm{d}y} = \mathbb{P}_x^{(l,\Lambda,\mathrm{bc})}\big((B_s)_{s\in[0,l\beta]} \in \mathrm{d}f \mid \tau = l, B_{l\beta} = y\big) \tag{5.3}$$

we denote the distribution of a Brownian W-shred with l legs from $x \in W$ to $y \in W^{\mathrm{c}}$.

We also need the restricted versions of p_x and $q_{x,y}$ for shreds whose legs have spreads $< M$ for some $M \in (0,\infty)$. For this, we denote

$$\mathcal{C}_1^{(M)} = \{f \in \mathcal{C}_1 : \|f\|_{\mathrm{sp}} < M\}, \qquad \text{where } \|f\|_{\mathrm{sp}} = \max_{t\in[0,\beta]} |f(t) - f(0)|. \tag{5.4}$$

Denote

$$p_x^{(W,M)}(l,y) = \mathbb{P}_x\big(\tau = l, \forall j \in [l] : B_j \in \mathcal{C}_1^{(M)}, B_{\tau\beta} \in \mathrm{d}y\big)/\mathrm{d}y, \qquad (x,l,y) \in W \times \mathbb{N} \times W^{\mathrm{c}}, \tag{5.5}$$

and

$$\begin{aligned} q_{x,y}^{(l,W,M)}(\mathrm{d}f) &= \frac{\mathbb{P}_x(\tau = l, \forall j \in [l] : B_j \in \mathcal{C}_1^{(M)}, B \in \mathrm{d}f, B(\tau\beta) \in \mathrm{d}y)}{p_x^{(W,M)}(l,y)\,\mathrm{d}y} \\ &= \mathbb{P}_x\big(B \in \mathrm{d}f \mid \tau = l, B(l\beta) = y, \forall j \in [l] : B_j \in \mathcal{C}_1^{(M)}\big). \end{aligned} \tag{5.6}$$

Note that B does not leave the M-neighbourhood of the subbox in which it started. Hence, if this box is in the interior of Λ, then it satisfies particle-boundary condition in Λ.

We are going to identify the distribution of the lengths of the time lags between subsequent exit times from subboxes and the location of the Brownian cycle at these times.

For $k \in \mathbb{N}$ we consider the (non-normalised) measure

$$\widehat{\mathbb{P}}^{(k,\Lambda,\mathrm{bc})}(\cdot) = \int_\Lambda \mathrm{d}x\, \mu_{x,x}^{(k,\Lambda,\mathrm{bc})}(\{\mathfrak{m}_R \geq 1\} \cap \cdot). \tag{5.7}$$

Recall the decomposition of Λ_N into the subboxes $z + W$ with $z \in Z_{N,R}$ and the smaller box Λ'_N, which is the union of the $z + W$ over $z \in Z^\circ_{N,R}$, the (discrete) interior of $Z_{N,R}$, as introduced before Lemma 4.3.

Lemma 5.1 (Distribution of boundary shred configuration of one loop)**.** *Fix $R \in (0,\infty)$ and $W = W_R = [-R,R]^d$ and $M \in (0,R)$. Assume that Λ is a large centred box.*

(1) *Fix $m \in \mathbb{N}$ and $l_1, l_2, \dots, l_m \in \mathbb{N}$ such that $\sum_{j=1}^m l_j = k$. Fix $w_1, \dots, w_m \in W$ and $z_1, \dots, z_m \in Z_{N,R}$ such that $z_j \neq z_{j-1}$ for any j, where $z_0 = z_m$. Then*

$$\begin{aligned} &\widehat{\mathbb{P}}^{(k,\Lambda,\mathrm{bc})}\Big(\mathfrak{m}(B) = m, \forall j = 1, \dots, m : \tau_j - \tau_{j-1} = l_j, B_{l_j\beta} \in \mathrm{d}(z_j + w_j)\Big) \\ &\qquad = \bigotimes_{j=1}^m p_{z_{j-1}+w_{j-1}}^{(W,\Lambda,\mathrm{bc})}(l_i, z_j + w_j)\,\mathrm{d}w_j, \qquad z_0 = z_m, w_0 = w_m. \end{aligned} \tag{5.8}$$

Furthermore, under conditioning on the event

$$A_m\big((l_j,w_j)_{j\in[m]}\big) = \{\mathfrak{m}_R(B) = m, \forall j = 1, \dots, m : \tau_{j+1} - \tau_j = l_j, B_{l_j\beta} = z_j + w_j\}, \tag{5.9}$$

the shreds $(B_{s+\tau_j\beta})_{s\in[0,l_j\beta]}$ are independent over $j \in \{1, \dots, m\}$ and have distribution $q_{z_{j-1}+w_{j-1},z_j+w_j}^{(l_j,W,\Lambda,\mathrm{bc})}$.

(2) *For* $\mathrm{bc} = \mathrm{par}$ *and* $\widehat{\mathbb{P}}^{(k,\Lambda,\mathrm{par})}$ *restricted to all the legs of* B *lying in* $\mathcal{C}_1^{(M)}$, *we can replace* $p^{(W,\Lambda,\mathrm{par})}$ *by* $p^{(W,M)}$ *and* $q^{(l_j,W,\Lambda,\mathrm{par})}$ *by* $q^{(l_j,W,M)}$ *in these two statements.*

(3) *For any* $z \in Z^\circ_{N,R}$ *such that all legs start start in* W_z *have spread* $< M$ *and for any* j *satisfying* $z_{j-1} = z$, *we may replace* $p^{(W,\Lambda,\mathrm{per})}_{z_{j-1}+w_{j-1}}$ *by* $p^{(W,M)}_{z_{j-1}+w_{j-1}}$ *and* $q^{(l_j,W,\Lambda,\mathrm{per})}_{z_{j-1}+w_{j-1},z_j+w_j}$ *by* $q^{(l_j,W,M)}_{z_{j-1}+w_{j-1},z_j+w_j}$ *in these statements.*

Proof. (1) For any $m \in \mathbb{N}$ and $\widetilde{l}_1, l_2, \dots, l_m \in \mathbb{N}$ such that $\widetilde{l}_1 + \sum_{j=2}^{m-1} l_j \leq k$, and $z_1, \dots, z_m \in Z_{N,R}$ satisfying $z_j \neq z_{j-1}$ for all j and $w_1, \dots, w_m \in W$, we have, with $\tau_0 = 0$,

$$
\begin{aligned}
&\widehat{\mathbb{P}}^{(k,\Lambda,\mathrm{bc})}\big(\tau_1 = \widetilde{l}_1, B_{\widetilde{l}_1\beta} \in \mathrm{d}(z_1+w_1), \forall j = 1, \dots, m-1\colon \tau_j - \tau_{j-1} = l_j, B_{l_j\beta} \in \mathrm{d}\big(z_j + w_j\big)\big) \\
&\qquad = \int_\Lambda \mathrm{d}x_0\, p^{(W,\Lambda,\mathrm{bc})}_{x_0}\big(\widetilde{l}_1, \mathrm{d}(z_1+w_1)\big)\Big[\prod_{j=2}^m p^{(W,\Lambda,\mathrm{bc})}_{z_{j-1}+w_{j-1}}\big(l_j, \mathrm{d}(z_j+w_j)\big)\Big] \\
&\qquad\qquad \mathbb{P}^{(\Lambda,\mathrm{bc})}_{z_m+w_m}\Big(\tau > k - \widetilde{l}_1 - \sum_{j=2}^m l_j, B_{(k-\widetilde{l}_1-\sum_{j=2}^m l_j)\beta} \in \mathrm{d}x_0\Big) \\
&\qquad = \prod_{j=1}^m p^{(W,\Lambda,\mathrm{bc})}_{z_{j-1}+w_{j-1}}\big(l_j, z_j + w_j\big)\,\mathrm{d}w_j, \qquad l_1 = k - \sum_{j=2}^m l_j, z_0 = z_m, w_0 = w_m,
\end{aligned}
\tag{5.10}
$$

where we used the Markov property in the second step to concatenate the last piece of the Brownian cycle from the last change of a subbox at time $\tau_m\beta$ until the end of the time interval, $k\beta$.

The additional assertion stems also directly from the strong Markov property, applied successively at the times $\tau_j\beta$.

(2) follows from the definition of $p^{(W,M)}$, and (3) from the fact that, by $M < R$, the legs in such subboxes to not reach the boundary of Λ but their particles stay in the subbox, and hence periodic boundary condition in Λ can be replaced by particle boundary condition in W. $\square$

5.2. **Large-deviation lower bound for the boundary-shred configurations.** In this section we show how to lower bound the term in the second line on the right-hand side of (4.39). That is, we now use Lemma 5.1 to derive a large-deviations lower bound for the distribution of the boundary configuration $\partial\Xi_{N,R}^{(\omega_{\mathrm{P}},\mathcal{S})}$ defined in (4.23) of all the W-shreds, i.e., their lengths and initial and starting sites, for shreds coming from the R-crossing part of marked Poisson process ω_{P}.

The distribution of $\partial\Xi_{N,R}^{(\mathcal{S})}$ is not easy to describe. As it concerns large-deviation upper bounds, we will in Section 9.1 be satisfied with estimating the probability of $\partial\Xi_{N,R}^{(\mathcal{S})}$ being in a compact set simply against one. However, for the lower bound we have to work. Because of our preparations in Lemma 4.3, we need the lower bound only for particle boundary conditions. We will be consent with a large-deviation lower bound for the probability of the event $\{\partial\Xi_{N,R}^{(\mathcal{S})} = \psi\}$ for some given $\psi \in \mathcal{M}_1(\mathcal{T}_W)$ (possibly N-dependent) with certain properties. For a general ψ, it is not *a priori* clear whether or not this event has positive probability at all under any loop configuration measure in a large box Λ_N, i.e., whether or not it is possible to find a loop configuration in Λ_N such that the empirical measure of its boundary-shred triples is close to the given ψ. For the time being we are therefore working under the following assumption on ψ.

Definition 5.2 (Admissible boundary-shred configurations)**.** *Fix* $R > 0$ *and* $W = W_R = [-R, R]^d$. *We call* $\psi \in \mathcal{M}_1(\mathcal{T}_W)$ admissible *(more precisely,* W*-admissible) if, for any open neighbourhood* $\mathcal{B}$ *of* ψ, *for any sufficiently large* N, *there is a loop-configuration* $\omega \in \mathcal{L}_{\Lambda_N}$ *such that* $\partial\Xi_{N,R}^{(\omega,\mathcal{S})} \in \mathcal{B}$.

Actually, we will later need to apply the lower bound only to ψ's of the form $\partial\Pi_W^{(\mathcal{S})}(P)$ with $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L} \times \mathcal{S})$. An important ingredient will then be Lemma 5.4, which says that such a ψ is W-admissible if P is W-ergodic.

Recall the particle-number operator $\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})}$ from (4.16) and observe that the expected shred particle number of a configuration $\xi\in\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W)$ may be written, if $\psi=\partial\Pi_W^{(\mathcal{S})}(\xi)$,

$$\langle\mathfrak{N}_W^{(\ell,\mathcal{S})},\xi\rangle=\langle\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})},\psi\rangle=\int\psi(\mathrm{d}\mu)\sum_{i\in I}l_i,\qquad \mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}.$$

By $\mathcal{B}_\delta(x)$ we always denote the open δ-ball around x for any x in a metric space. If $x=\psi\in\mathcal{M}_1(\mathcal{T}_W)$, then we use the Prokhorov metric defined in (4.5), which induces the weak topology.

The following lemma will later be applied for Λ_N' and $Z_{N,R}^\circ$ instead of Λ_N and $Z_{N,R}$. Recall that $\mathtt{Q}_M^{(\Lambda_N,\mathrm{par})}$ is the PPP with intensity measure given in (4.31), i.e., with all leg spreads $<M$. Analogously, $\mu_{x,x}^{(k,M)}$ is the canonical Brownian loop measure starting at x with k particles and all leg spreads $<M$. We recall from Section 5.1 that $\mathfrak{m}_R(B)$ is the number of times that the loop B changes the subbox of radius R. By C_M we denote some constant $\in(0,\infty)$ that depends only on M (and possibly on d and β).

Lemma 5.3 (Lower LDP bound for boundary shred configurations)**.** *Assume the situation and the notation described in Section 1.4. That is, Λ_N is a centred box with volume $|\Lambda_N|\sim N/\rho$ for some $\rho\in(0,\infty)$, and is decomposed into the subboxes $z+W_R=z+[-R,R]^d$ for some $R\in(0,\infty)$ with $z\in 2R\mathbb{Z}^d\cap\Lambda_N$. We consider particle boundary condition and indicate it by 'par'. Fix a large parameter $M\in(0,R/2)$ and fix $\xi\in\mathcal{M}_1(\mathcal{L}_W^{(M)}\times\mathcal{S}_W^{(M)})$ such that $\langle\mathfrak{N}_{\partial W}^{(\ell,\mathcal{S})},\psi\rangle<\infty$, where $\psi=\partial\Pi_W^{(\mathcal{S})}(\xi)\in\mathcal{M}_1(\mathcal{T}_W)$, and assume that ψ is W-admissible. Then, for any sufficiently large R, there is $\psi_N\in\mathcal{M}_1(\mathcal{T}_W)$ that converges to ψ as $N\to\infty$ such that*

$$\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathtt{Q}_M^{(\Lambda_N,\mathrm{par})}\Big(\partial\Xi_{N,R}^{(\mathcal{S})}=\psi_N\Big)\geq-\frac{C_M}{R^{d/2}}+\frac{1}{|W|}\langle\psi,\mathfrak{p}_{W,M}\rangle,\tag{5.11}$$

where we recall (5.5) and introduce

$$\mathfrak{p}_{W,M}(\mu)=\sum_{i\in I}\log p_{x_i}^{(W,M)}(l_i,y_i),\qquad \mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}.\tag{5.12}$$

This lower bound is no surprise, since $\langle\psi,\mathfrak{p}_{W,M}\rangle$ is nothing but the rate of the product of all the probability densities for the shreds, given they start from some $x_i\in W$, having length l_i and terminating at $y_i\in W^{\mathrm{c}}$. The W-admissibility of ψ guarantees that there is at least one way to arrange the shred-triple configurations in all the subboxes (each configuration μ appears in $\approx\frac{|\Lambda_N|}{|W|}\psi(\mu)$ subboxes) in such a way that a global loop configuration arises. A similar upper bound is generally suggesting, but here one would have to upper bound the number of such arrangements, which appears a very difficult task. Since $\langle\psi,\mathfrak{p}_{W,M}\rangle$ should be roughly of W-surface order and therefore negligible in the limit $R\to\infty$, we decided to control only the lower bound so explicitly. Lemma 5.5 in Section 5.3 is devoted to deriving a lower bound for the last term on the right-hand side of (5.11).

Proof of Lemma 5.3. Note that the admissibility of ψ guarantees the existence of some $\psi_N\to\psi$ such that $\{\partial\Xi_{N,R}^{(\mathcal{S})}=\psi_N\}$ is not empty for all sufficiently large N. Picking M large enough (not depending on N), we also have that this event has a positive probability under $\mathtt{Q}_M^{(\Lambda_N,\mathrm{par})}$, and we assume that N is so large. We are going to carefully decompose this event and to identify a lower bound for its probability.

We are considering only the shred-configuration part of $\Xi_{N,R}$ under the PPP in Λ_N, that is, we neglect all loops that have particles only in one of the subboxes. In other words, we can assume that the intensity measure is equal to $\sum_{k\in\mathbb{N}}\frac{1}{k}\int_{\Lambda_N}\mathrm{d}x\,\mu_{x,x}^{(k,M,\Lambda_N,\mathrm{par})}(\{\mathfrak{m}_R\geq 1\}\cap\cdot)$, where $\mu_{x,x}^{(k,M,\Lambda_N,\mathrm{par})}$ is the Brownian loop measure for k-particle loops starting at x and having all leg spreads $<M$ with particle-boundary condition in Λ_N.

From now on we drop the superscript $\mathcal{S}$ at $\Xi_{N,R}$ and write $\Xi^{(B)}_{N,R}$ for the empirical shred configuration measure coming from just one single path $B\in\mathcal{C}$, then $\Xi_{N,R}=\sum_{k\in\mathbb{N}}\sum_{i=1}^{r_k}\Xi^{(B^{(i,k)})}_{N,R}$ if $B^{(i,k)}\in\mathcal{C}_k$ is the i-th Brownian loop of length k, and r_k is the number of loops of length k in the Brownian loop soup. We are going to use that the PPP is realised by a bunch of r_k independent Brownian loops $B^{(k)}_1,\dots,B^{(k)}_{r_k}$ with $k\in\mathbb{N}$ and normalised distribution, where the r_k are picked as Poisson-distributed random variables with parameter $\frac{1}{k}\int_\Lambda \mathrm{d}x\,\mu^{(k,M,\Lambda_N,\mathrm{par})}_{x,x}(\mathfrak{m}_R\geq 1)$. This gives the formula

$$
\begin{aligned}
\mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big(\partial\Xi_{N,R}=\psi_N\big)&=\mathrm{e}^{-\sum_k\frac{1}{k}\int_\Lambda\mathrm{d}x\,\mu^{(k,M,\Lambda_N,\mathrm{par})}_{x,x}(\mathfrak{m}_R\geq 1)}\\
&\sum_{(r_k)_{k\in\mathbb{N}}\in\mathbb{N}_0^{\mathbb{N}}}\Big(\prod_k\frac{k^{-r_k}}{r_k!}\Big)\Big(\bigotimes_k\big(\widehat{\mathbb{P}}^{(k,\Lambda_N,\mathrm{par})}_M\big)^{\otimes r_k}\Big)\Big(\sum_{k\in\mathbb{N}}\sum_{i=1}^{r_k}\partial\Xi^{(B^{(i,k)})}_{N,R}=\psi_N\Big),
\end{aligned}
\tag{5.13}
$$

where $\widehat{\mathbb{P}}^{(k,\Lambda_N,\mathrm{par})}$ is defined in (5.7) (note that the Poisson parameter $\frac{1}{k}\int_{\Lambda_N}\mathrm{d}x\,\mu^{(k,M,\Lambda_N,\mathrm{par})}_{x,x}(\mathfrak{m}_R\geq 1)$ is, up to the factor of $\frac{1}{k}$, equal to its total mass) and $\widehat{\mathbb{P}}^{(k,\Lambda_N,\mathrm{par})}_M$ is its restriction to $\bigcap_{z\in Z_{N,R}}\{\varpi^{(\mathcal{S})}_{W_z}\in\mathcal{S}^{(M)}_{W_z}\}$. The sum on $(r_k)_k$ may be restricted to those that satisfy $|\frac{1}{|\Lambda_N|}\sum_k kr_k-\rho_2|\leq C\delta$ for some C (where $\rho_2=\frac{1}{|W|}\langle\psi,\mathfrak{N}^{(\ell)}_{\partial W}\rangle$), but this restriction is implicit in the event $\{\sum_{k\in\mathbb{N}}\sum_{i=1}^{r_k}\partial\Xi^{(B^{(i,k)})}_{N,R}\in U_\delta\}$.

It is easy to prove that

$$
\lim_{N\to\infty}\frac{1}{|\Lambda_N|}\sum_{k\in\mathbb{N}}\frac{1}{k}\int_{\Lambda_N}\mathrm{d}x\,\mu^{(k,M,\Lambda_N,\mathrm{par})}_{x,x}(\mathfrak{m}_R\geq 1)=\sum_{k\in\mathbb{N}}\frac{1}{k}\mu^{(k,M)}_{0,0}(\mathfrak{m}_R\geq 1).
$$

Observe that

$$
\sum_{k\in\mathbb{N}}\frac{1}{k}\mu^{(k,M)}_{0,0}(\mathfrak{m}_R\geq 1)\leq\frac{C_M}{R^{d/2}},\qquad M,R\in(1,\infty),
\tag{5.14}
$$

for some $C_M\in(0,\infty)$, only depending on M (and on d and β). Indeed, from the definition of $\mathfrak{m}_R$ in Section 5.1 and the one of $\mu^{(k,M)}_{0,0}$ in (1.4) we see that the event $\{\mathfrak{m}_R\geq 1\}$ is empty if $k\leq R/M$, since the spreads $|B_{i\beta}-B_{(i-1)\beta}|$ are $\leq M$ under $\mu^{(k,M)}_{0,0}$ and hence only the sum on $k>R/M$ remains, and the total mass of $\mu^{(k,M)}_{0,0}$ is $\leq Ck^{-d/2}$; we leave the details to the reader. This explains the first term on the right-hand side of (5.11).

Since $R>2M$, every step from a box $z+W$ with size $\leq M$ can terminate only in that box or its $\boxplus$-neighbouring boxes. By $\boxplus$-neighbours of the origin we mean the members of the set $\{-1,0,1\}^d\setminus\{0\}$; analogously we talk of $\boxplus$-neighbouring boxes in the sense of an $2R$-upscaling of this notion.

Each $B^{(i,k)}$ contributes to $\partial\Xi_{N,R}$ a vector of triples $(x^{(i,k)}_j,l^{(i,k)}_j,y^{(i,k)}_j)\in W\times\mathbb{N}\times W^{\mathrm{c}}$, $j\in[m^{(i,k)}]$, with a distribution given by the product measure of the $p^{(k,M,\Lambda,\mathrm{par})}_{x_j}(l^{(i,k)}_j,\mathrm{d}y_j)$, where the positive integers $l^{(i,k)}_j$ sum up over j to k, and $x_j=z^{(i,k)}_{j-1}+w^{(i,k)}_{j-1}$ and $y_j=z^{(i,k)}_j+w^{(i,k)}_j=z^{(i,k)}_{j-1}+\widetilde{w}^{(i,k)}_j$, and $\widetilde{w}^{(i,k)}_j=\mathrm{e}^{(i,k)}_j+w^{(i,k)}_j\in W^{\mathrm{c}}$ and $\mathrm{e}^{(i,k)}_j=z^{(i,k)}_j-z^{(i,k)}_{j-1}$. We have to sum over all $m^{(i,k)}\in\mathbb{N}$ (since $\mathfrak{m}_R\geq 1$) and all $z^{(i,k)}_1,\dots,z^{(i,k)}_{m^{(i,k)}}\in Z_{N,R}$ and integrate over all $w^{(i,k)}_1,\dots,w^{(i,k)}_{m^{(i,k)}}\in W$ such that the event $\{\partial\Xi_{N,R}=\psi_N\}$ is met. Each $(z^{(i,k)}_0,z^{(i,k)}_1,\dots,z^{(i,k)}_{m^{(i,k)}})$ is a cycle, i.e., $z^{(i,k)}_0=z^{(i,k)}_{m^{(i,k)}}$. We may and shall assume that all the $w^{(i,k)}_j$ are distinct. For any j such that $z^{(i,k)}_{j-1}\in Z_{N,R}$, we know that $\mathrm{e}^{(i,k)}_j$ is a $\boxplus$-neighbour of the origin; more precisely $z^{(i,k)}_j$ and $z^{(i,k)}_{j-1}$ are $\boxplus$-neighbours. Each $w^{(i,k)}_j$ has a distance $\leq M$ to the boundary of W^{c}.

Without loss of generality (see [B07, Example 8.1.6(i)]), we may and shall restrict to a $\psi\in\mathcal{M}_1(\mathcal{T}_W)$ (the limit point of the ψ_N) that consists of finitely many atoms only, i.e., it is of the form $\psi=\sum_{\alpha\in I}s_\alpha\delta_{\mu_\alpha}$ with I a finite set and $\sum_\alpha s_\alpha=1$ and $s_\alpha>0$ for all $\alpha\in I$, and $\mu_\alpha=\sum_{\gamma\in J_\alpha}\delta_{(x^{(\alpha)}_\gamma,l^{(\alpha)}_\gamma,y^{(\alpha)}_\gamma)}\in\mathcal{T}_W$ a simple point measure. We may assume that all the $x^{(\alpha)}_\gamma$ with $\alpha\in I$ and $\gamma\in J_\alpha$ are distinct; the same for the $y^{(\alpha)}_\gamma$.

Let us identify $\partial\Xi_{N,R}^{(B^{(i,k)})}$ for one single loop $B^{(i,k)}$. For $z\in Z_{N,R}$, we see that $\theta_z(\Pi_{W_z}(\delta_{B^{(i,k)}}))=\sum_{j\in[m^{(i,k)}]\colon z_{j-1}^{(i,k)}=z}\delta_{(l_j^{(i,k)},w_{j-1}^{(i,k)},\widetilde{w}_j^{(i,k)})}$ is equal to the shred configuration that $B^{(i,k)}$ induces in the subbox W_z. Hence, writing $\mathbf{w}=(w_j^{(i,k)})_{k,i,j}$ and $\mathbf{z}=(z_j^{(i,k)})_{k,i,j}$ and $\mathbf{l}=(l_j^{(i,k)})_{k,i,j}$, the superposition

$$C_z(\mathbf{z},\mathbf{w},\mathbf{l})=\sum_{k\in\mathbb{N}}\sum_{i\in[r_k]}\sum_{j\in[m^{(i,k)}]\colon z_{j-1}^{(i,k)}=z}\delta_{(l_j^{(i,k)},z+w_{j-1}^{(i,k)},z+\widetilde{w}_j^{(i,k)})}\in\mathcal{T}_{W_z}$$

is the configuration that all the loops $B^{(i,k)}$ induce in the subbox W_z. Then

$$\partial\Xi_{N,R}=\sum_{k,i}\partial\Xi_{N,R}^{(B^{(i,k)})}=\frac{1}{\#Z_{N,R}}\sum_{z\in Z_{N,R}}\delta_{\theta_z(C_z(\mathbf{z},\mathbf{w},\mathbf{l}))}\in\mathcal{M}_1(\mathcal{T}_W).$$

We call every such collection

$$(\mu_z)_{z\in Z_{N,R}}=(C_z(\mathbf{z},\mathbf{w},\mathbf{l}))_{z\in Z_{N,R}}\in\bigtimes_{z\in Z_{N,R}}\mathcal{T}_{W_z}\tag{5.15}$$

admissible. In other words, a collection $(\mu_z)_{z\in Z_{N,R}}\in\bigtimes_{z\in Z_{N,R}}\mathcal{T}_{W_z}$ of configurations is called admissible if the arrangement of all these μ_z connects up precisely at all the boundaries of all the W_z and produces a bunch of continuous loops inside Λ_N. With this notion, we can write the probability term on the right-hand side of (5.13) as follows.

$$\begin{aligned}\Big(\bigotimes_k\big(\widehat{\mathbb{P}}_M^{(k,\Lambda,\mathrm{par})}\big)^{\otimes r_k}\Big)\Big(\sum_{k\in\mathbb{N}}\sum_{i=1}^{r_k}\partial\Xi_{N,R}^{(B^{(i,k)})}=\psi_N\Big)=\int_{\bigtimes_{z\in Z_{N,R}}\mathcal{T}_{W_z}}&\mathbb{1}\{(\mu_z)_{z\in Z_{N,R}}\text{ admissible}\}\\ \mathbb{1}\Big\{\frac{1}{N_R}\sum_{z\in Z_{N,R}}\delta_{\theta_z(\mu_z)}=\psi_N\Big\}\Big(\bigotimes_k\big(\widehat{\mathbb{P}}_M^{(k,\Lambda,\mathrm{par})}\big)^{\otimes r_k}\Big)&\big(\mathrm{d}((\mu_z)_{z\in Z_{N,R}})\big).\end{aligned}\tag{5.16}$$

Strictly speaking, the indicator on admissibility is superfluous, since the probability measure is concentrated on collections of μ_z that admit only global configurations of continuous paths. But we are going to calculate now the probability by means of a product measure over $z\in Z_{N,R}$, which ignores admissibility. Only under admissibility, it gives a proper lower bound. For a upper bound, we have to upper estimate the number of admissible collections $(\mu_z)_z$; the remainder is the same. For the lower bound, we rely here on the assumption of admissibility of ψ_N, which guarantees the existence of at least one admissible $(\mu_z)_z$.

Let us express what $\Delta=\psi_N$ means, where $\Delta=\frac{1}{N_R}\sum_{z\in Z_{N,R}}\delta_{\theta_z(\mu_z)}$. Since $\psi_N\to\psi$, and ψ is a simple point measure at the μ_α's, this means that, for any small $\delta\in(0,1)$ and all sufficiently large N and all $\alpha\in I$, we have $\Delta(B_\delta(\mu_\alpha))=\psi_N(B_\delta(\mu_\alpha))$, and the latter is equal to some $s_\alpha^{(N)}$ that converges to s_α as $N\to\infty$. We pick δ so small that all the δ-balls around the $x_\gamma^{(\alpha)}$'s and the δ-balls around the $y_\gamma^{(\alpha)}$'s are pairwise disjoint. Now observe that, for any α,

$$\begin{aligned}N_R\Delta(B_\delta(\mu_\alpha))&=\#\{z\colon\theta_z(\mu_z)\in\mathcal{B}_\delta(\mu_\alpha)\}\\&=\#\{z\colon\forall\gamma\in J_\alpha\exists(k,i,j)\colon z_{j-1}^{(i,k)}=z,w_{j-1}^{(i,k)}\in\mathcal{B}_\delta(x_\gamma^{(\alpha)}),l_j^{(i,k)}=l_\gamma^{(\alpha)},\widetilde{w}_j^{(i,k)}\in\mathcal{B}_\delta(y_\gamma^{(\alpha)})\}.\end{aligned}$$

Note that, for any $\alpha\in I$ and $\gamma\in J_\alpha$ and all (k,i), the index j in the description of the set is unique. We denote the set of all $(\mathbf{z},\mathbf{l},\mathbf{w})$ having the property on the right-hand side by A.

Recall the notation for certain densities $p_x(l,\cdot)$ from (5.1) and (5.5) (with various super-indices). Now using Lemma 5.1 and the independence of all the $B^{(i,k)}$, we can write (conceiving $p_x(l,\cdot)$ as a measure instead of a density)

$$\Big(\bigotimes_k (\widehat{\mathbb{P}}_M^{(k,\Lambda,\mathrm{par})})^{\otimes r_k}\Big)\Big(\frac{1}{N_R}\sum_{z\in Z_{N,R}}\delta_{\mu_z}=\psi_N\,\Big|\,(\mu_z)_{z\in Z_{N,R}}\text{ admissible}\Big)$$
$$=\sum_{\mathbf{z},\mathbf{l}}\int \mathrm{d}\mathbf{w}\,\mathbb{1}_A(\mathbf{z},\mathbf{l},\mathbf{w})\prod_{z\in Z_{N,R}}\prod_{k\in\mathbb{N}}\prod_{i\in[r_k]}\prod_{j\in[m^{(i,k)}]:\,z=z_{j-1}^{(i,k)}} p_{z+w_{j-1}^{(i,k)}}^{(W,M,\Lambda_N,\mathrm{par})}(l_j^{(i,k)},\mathrm{d}(z+\widetilde{w}_j^{(i,k)})). \tag{5.17}$$

We may replace the p-term by $p_{w_{j-1}^{(i,k)}}^{(W,M)}(l_j^{(i,k)},\mathrm{d}\widetilde{w}_j^{(i,k)})$, since every leg that starts from W_z has spread $\leq M$ and hence this term is invariant under shift by z. For these z, we obtain a lower bound by estimating

$$p_{w_{j-1}^{(i,k)}}^{(W,M)}(l_j^{(i,k)},\mathrm{d}\widetilde{w}_j^{(i,k)}))=p_{w_{j-1}^{(i,k)}}^{(W,M)}(l_\gamma^{(\alpha)},\mathrm{d}(\mathrm{e}_j^{(i,k)}+w_j^{(i,k)}))\geq \inf_{x\in\mathcal{B}_\delta(x_\gamma^{(\alpha)})} p_x^{(W,M)}(l_\gamma^{(\alpha)},\mathrm{d}(\mathrm{e}_j^{(i,k)}+w_j^{(i,k)})), \tag{5.18}$$

if (k,i,j) is such that $w_{j-1}^{(i,k)}\in\mathcal{B}_\delta(x_\gamma^{(\alpha)}), l_j^{(i,k)}=l_\gamma^{(\alpha)}, w_j^{(i,k)}\in\mathcal{B}_\delta(y_\gamma^{(\alpha)})$.

We further have

$$\begin{aligned}\text{l.h.s. of (5.17)} &= \sum_{\mathbf{z},\mathbf{l}}\int \mathrm{d}\mathbf{w}\,\mathbb{1}_A(\mathbf{z},\mathbf{l},\mathbf{w})\prod_{k\in\mathbb{N}}\prod_{i\in[r_k]}\prod_{j\in[m^{(i,k)}]} p_{w_{j-1}^{(i,k)}}^{(W,M)}(l_j^{(i,k)},\mathrm{d}\widetilde{w}_j^{(i,k)})\\ &\geq \prod_{\alpha\in I}\Big(\prod_{\gamma\in J_\alpha}\inf_{x\in\mathcal{B}_\delta(x_\gamma^{(\alpha)})} p_x^{(W,M)}(l_\gamma^{(\alpha)},\mathcal{B}_\delta(y_\gamma^{(\alpha)}))\Big)^{s_\alpha^{(N)}N_R}\\ &= \mathrm{e}^{-N_R\mathcal{O}(\delta)}\exp\Big(N_R\int\psi(\mathrm{d}\mu)\,\log\mathbb{P}_{\inf}^{(\partial W,M)}(\mathcal{B}_\delta(\mu))\Big),\end{aligned} \tag{5.19}$$

where $\mathbb{P}_{\inf}^{(\partial W,M)}$ is defined by

$$\mathbb{P}_{\inf}^{(\partial W,M)}(\mathcal{B}_\delta(\mu))=\prod_{i\in I}\inf_{x\in\mathcal{B}_\delta(x_i)} p_x^{(W,M)}(l_i,\mathcal{B}_\delta(y_i)),\qquad \mathcal{B}_\delta(\mu)=\bigtimes_{i\in I}\big[\mathcal{B}_\delta(x_i)\times\{l_i\}\times\mathcal{B}_\delta(y_i)\big], \tag{5.20}$$

where $\mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}$ and $\mathcal{B}_\delta(z)$ is the ball with radius δ around $z\in\mathbb{R}^d$.

Note that this lower bound does not depend on $(r_k)_k$. For the lower bound, one can pick one choice of $(r_k)_{k\in\mathbb{N}}$ and lower estimate the sum over $(r_k)_k$ against this single summand. This implies, for some $C\in(0,\infty)$,

$$\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathsf{Q}_M^{(\Lambda_N,\mathrm{par})}\big(\partial\Xi_{N,R}^{(\mathcal{S})}=\psi_N\big)\geq -\frac{C_M}{R^{d/2}}-\frac{C\delta}{|W|}+\frac{1}{|W|}\int\psi(\mathrm{d}\mu)\,\log\mathbb{P}_{\inf}^{(\partial W,M)}(\mathcal{B}_\delta(\mu)).$$

Since the left-hand side does not depend on δ, we can make $\delta\downarrow 0$ on the right-hand side and obtain the assertion in (5.11). □

We now give an existence assertion for admissible boundary-shred configurations ψ in a setting that will be relevant in Section 9 when we finish the proof of the lower bound in Theorem 2.5. We keep using the notation that we reminded on at the beginning of Section 5. We call a measure $P\in\mathcal{M}_1(\mathcal{L}\times\mathcal{S})$ *ergodic with respect to* W_R or simply W_R-*ergodic*, if it is W_R-shift invariant (i.e., $\theta_x(P)=P$ for $x\in 2R\mathbb{Z}^d$) and ergodic under shifts by vectors in $2R\mathbb{Z}^d$.

Lemma 5.4 (W_R-admissible ψ's)**.** *Fix $R\in(0,\infty)$ and put $W=W_R=[-R,R]^d$. Pick $P\in\mathcal{M}_1(\mathcal{L}\times\mathcal{S})$ and assume that P is W-ergodic. Then $\psi=\partial\Pi_W^{(\mathcal{S})}(P)$ is W-admissible.*

Proof. Put $\xi=\Pi_W(P)\in\mathcal{M}_1(\mathcal{L}_W)\times\mathcal{S}_W)$. Extending the definition (4.20) to loops and interlacements, we conceive $\Xi_{N,R}=\Xi_{N,R}^{(\omega,\varpi)}$ as the empirical measure coming from loops and interlacements in Λ_N (and $\omega\in\mathcal{L}$ is the loop configuration and $\varpi\in\mathcal{S}$ the interlacement configuration). Indeed, it is an empirical measure (discrete mixture) of W-shifted copies of $\Pi_W(\omega,\varpi)$, which have distribution ξ under

$P(\mathrm{d}(\omega,\varpi))$. Here we use the shift-invariance of P, more precisely only the invariance under shifts by vectors $\in 2R\mathbb{Z}^d$.

According to the ergodic theorem for P, for any open neighbourhood $U(\xi)\subset\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W)$ of ξ,

$$\lim_{N\to\infty} P(\Xi_{N,R}\in U(\xi))=1.$$

In particular, the event $\{(\omega,\varpi)\in\mathcal{L}\times\mathcal{S}\colon \Xi_{N,R}^{(\omega,\varpi)}\in U(\xi)\}$ is non-empty for all sufficiently large N. This implies that, for any N, there is a configuration $(\omega_N,\varpi_N)\in\mathcal{L}_{\Lambda_N}\times\mathcal{S}_{\Lambda_N}$ (namely $\omega_N=\Pi^{(\mathcal{L})}_{\Lambda_N}(\omega+\varpi)$ and $\varpi_N=\Pi^{(\mathcal{S})}_{\Lambda_N}(\omega+\varpi)$) such that $\Xi_{N,R}^{(\omega_N,\varpi_N)}$ (which is equal to $\Xi_{N,R}^{(\omega,\varpi)}$) lies in $\mathcal{U}(\xi)$. From (ω_N,ϖ_N), we can construct a loop configuration $\widetilde\omega_N\in\mathcal{L}_{\Lambda_N}$ as follows. We keep the loop configuration ω_N unchanged and modify all the shreds in ϖ_N by concatenating all Λ_N-shreds to a loop by removing the last leg (the only one that leaves Λ_N) and adding a piece from its termination site in Λ_N to its initial site in Λ_N that runs within Λ_N. It is easily seen that the above modification can be done in such a way that the distance between $\Xi_{N,R}^{(\omega_N,\varpi_N)}$ and $\Xi_{\widetilde N,R}^{(\widetilde\omega_N)}$ vanishes as $N\to\infty$, so $\Xi_{\widetilde N,R}^{(\widetilde\omega_N)}$ lies also in $U(\xi)$ for all large N.

Now let $\mathcal{B}(\psi)$ be some open neighbourhood of ψ, and pick the neighbourhood $\mathcal{B}(\xi)$ in such a way that $\{\partial\Pi^{(\mathcal{S})}_W(\xi')\colon \xi'\in\mathcal{B}(\xi)\}\subset\mathcal{B}(\psi)$. Therefore, for any large N, $\partial\Xi_{\widetilde N,R}^{(\widetilde\omega_N,\mathcal{S})}$ lies in $\mathcal{B}(\psi)$. Hence ψ is admissible. □

5.3. **Estimating the LDP lower bound.** In this section, we give in Lemma 5.5 a further lower bound for the LDP lower bound of Lemma 5.3, which shows that it is negligible for all the (approximations of) ψ's that we will later apply it to. Furthermore, we use this in order to prove that the expected shred number in a large box under a shift-invariant interlacement-configuration distribution is negligible with respect to the volume of the box, see Corollary 5.8, which needs the preparation in Lemma 5.7.

The idea in the proof of Lemma 5.5 is that we will be able to control $\langle\psi,\mathfrak{p}_{W,M}\rangle$ (see (5.12)) only for ψ's that are concentrated on shred configurations in W_R with legs that have a bounded spread, whose shreds have a bounded spread-out behaviour on longer scales, and that produce a bounded number of particles in any subbox of a given size. More precisely, with large parameters $M\in(1,\infty)$ and $\vartheta,S\in\mathbb{N}$, introduce the set of configurations of shreds whose legs have a spread $<M$ and which satisfy an upper bound for subpieces of length S,

$$\begin{aligned}\mathcal{S}^{(M,\vartheta,S)}=\Big\{\sum_{g\in\Gamma}\delta_g\in\mathcal{S}\colon &\forall g\in\Gamma,\forall i\in[\ell(g)],\|g_i\|_{\mathrm{sp}}<M,\\ &\forall g\in\Gamma,\forall j\in[\lfloor\tfrac{1}{S}\ell(g)\rfloor], g(((jS)\wedge\ell(g))\beta)-g((j-1)S\beta)|\le\vartheta\sqrt{S}\Big\},\end{aligned}\tag{5.21}$$

where we remind that $g\in\mathcal{C}_{\ell(g)}$ is a shred of length $\ell(g)$. Write $\mathcal{S}_W^{(M,\vartheta,S)}=\Pi_W^{(\mathcal{S})}(\mathcal{S}^{(M,\vartheta,S)})$ and $A^{(\mathcal{S})}_{W;\Theta}=A^{(\mathcal{S})}_{W;M,\mathfrak{r},T}\cap\mathcal{S}_W^{(M,\vartheta,S)}$, where $\Theta=(M,\mathfrak{r},T,\vartheta,S)$, then we will assume that ψ is concentrated on $A^{(\mathcal{S})}_{W;\Theta}$.

Lemma 5.5 (Lower estimates for expectations of $\mathfrak{p}_{W,M}$)**.** *For any $R,M\in(1,\infty)$, fix a shred configuration measure $\xi\in\mathcal{M}_1(\mathcal{S}^{(M)}_{W_R})$.*

(1) *There is a C_M (only depending on M, d and β) such that, for any $R,M\in(1,\infty)$ and any shred configuration measure $\xi\in\mathcal{M}_1(\mathcal{S}^{(M)}_{W_R})$,*

$$-\langle\partial\Pi^{(\mathcal{S})}_{W_R}(\xi),\mathfrak{p}_{W,M}\rangle\le C_M\langle\xi,\mathfrak{N}^{(\ell,\mathcal{S})}_{W_R}\rangle.\tag{5.22}$$

(2) *There are constants $C_{M,\mathfrak{r},T}$ (only depending on $M,\mathfrak{r},T$ and d, β) and C (depending only on d and β) such that, for any $R,M,\mathfrak{r},T,\vartheta\in(0,\infty)$ and $S\in\mathbb{N}$ and for any $\xi\in\mathcal{M}_1(\mathcal{S}_{W_R})$ that is*

concentrated on $A^{(\mathcal{S})}_{W_R;\Theta}$, *where* $\Theta=(M,\mathfrak{r},T,\vartheta,S)$,

$$-\langle\partial\Pi^{(\mathcal{S})}_{W_R}(\xi),\mathfrak{p}_{W,M}\rangle\leq\Big[\frac{C_{M,\mathfrak{r},T}}{R}+\frac{C\vartheta^2}{S}\Big]\langle\xi,\mathfrak{N}^{(\ell,\mathcal{S})}_{W_R}\rangle. \tag{5.23}$$

Proof. Let us first derive the lower bound in (5.24) for $p_x^{(W,M)}(l,i)$, also using that $|x-y|\leq lM$ and that $y\in W_{R+M}\setminus W_R$ and $x\in W_R\setminus W_{R-M}$ for any (x,l,y) in the support of any $\mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}$ in the support of $\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)$.

Recall that $\tau\beta\in\beta\mathbb{N}$ is the first time $\in\beta\mathbb{N}$ at which the shred leaves W. The steps of the underlying random walk $(B_{k\beta})_{k\in\mathbb{N}_0}$ are Gaussian, conditioned on being $<M$, and have therefore a density that is bounded from below (by some M-depending constant). Therefore, for any $(x,l,y)\in(W_R\setminus W_{R-M})\times\mathbb{N}\times(W_{R+M}\setminus W_R)$, using the Markov property at time $\beta(l-1)$, writing g_M for the Gaussian density restricted to $B_M(0)$,

$$\begin{aligned}p_x^{(W,M)}(l,y)&\geq\int_{W\cap B_M(y)}\mathbb{P}_x^{(M)}(\tau>l-1,B_{\beta(l-1)}\in\mathrm{d}w)g_M(w-y)\\&\geq C_M\mathbb{P}_x^{(M)}(\tau>l-1,B_{\beta(l-1)}\in W\cap B_M(y)).\end{aligned}$$

Now we distinguish the cases where $l\gg R^2$ (where the long stay of the walk in W is a late-escape deviation), $\varepsilon|x-y|^2\leq l\leq CR^2$ (where the walk has so much time to arrive in $W\cap B_M(y)$ that the CLT can be applied) and $\frac{|x-y|}{M}\leq l\ll|x-y|^2$ (where the sprint to $W\cap B_M(y)$ is a large or moderate deviation, since it makes very long steps).

Indeed, if $l\geq CR^2$ for some $C\in(0,\infty)$, then we can estimate $\mathbb{P}_x^{(M)}(\tau>l-1,B_{\beta(l-1)}\in W\cap B_M(y))$ from below against $\mathrm{e}^{-cl/R^2}C_MR^{1-d}$ since the probability for a long path to stay in a given box decays exponentially in the length of the path divided by the square of the radius of the box, according to Brownian scaling; and since the endpoint of such a path has a positive density in $W_R\setminus W_{R-M}$, which has volume $\geq C_MR^{d-1}$. (This asymptotics may be made much more precise with the help of the principal eigenvalue of the Laplace operator in W_R, which is $\sim C/R^2$, and the corresponding eigenfunction, but this would be cumbersome.)

Second, if $\varepsilon|x-y|^2\leq l\leq CR^2$ for some small $\varepsilon\in(0,1)$, then, on the event $\{\tau>l-1\}$, the path $(B_{\beta k})_{k\in\{0,1,\dots,l-1\}}$ behaves like a Brownian motion with diffusive constant $\asymp l$ and arrives in $B_M(y)$ with probability $\geq\mathrm{e}^{-c|x-y|^2/l}C_M$, according to Donsker's invariance principle, where the factor C_M expresses the volume of $B_M(y)$. (Another explanation of the term $\mathrm{e}^{-c|x-y|^2/l}$ is that the Brownian motion path can be divided into l steps of length $\approx|x-y|/l$ each.)

Furthermore, if we consider the case $\frac{|x-y|}{M}\leq l\leq\varepsilon|x-y|^2$, then we estimate $\mathbb{P}_x^{(M)}(\tau>l-1,B_{\beta(l-1)}\in W\cap B_M(y))\geq\mathrm{e}^{-c|x-y|^2/l}$, according to a large-deviation principle (if $l\asymp|x-y|$), respectively a moderate-deviation principle (if $|x-y|\ll l\leq\varepsilon|x-y|^2$). Indeed, the walk has to make $\asymp l$ steps of sizes $\asymp\frac{|x-y|}{l}$, and the probability for making such large steps is approximately Gaussian.

Summarizing the second and third case, we obtain

$$\log p_x^{(W,M)}(l,y)\geq\begin{cases}-c\frac{l}{R^2}+\log(C_MR^{1-d}), & \text{if } l\geq CR^2,\\-c\frac{|x-y|^2}{l}-C_M, & \text{if } \frac{|x-y|}{M}\leq l\leq CR^2.\end{cases} \tag{5.24}$$

For an arbitrary $\xi\in\mathcal{M}_1(\mathcal{S}^{(M)}_{W_R})$, abbreviate $\psi=\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)$. Then the left-hand side of (5.22) or (5.23) is an integral with respect to $\psi(\mathrm{d}\mu)$, where $\mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}$. Hence, it can be estimated from above against

$$\int\psi(\mathrm{d}\mu)\sum_{i\in I}\Big[\big(c\tfrac{l_i}{R^2}-\log(C_MR^{1-d})\big)\mathbb{1}\{l\geq CR^2\}+\big(c\tfrac{|x_i-y_i|^2}{l_i}+C_M\big)\mathbb{1}\{\tfrac{|x_i-y_i|}{M}\leq l_i\leq CR^2\}\Big]. \tag{5.25}$$

Estimating $c\frac{l_i}{R^2}\mathbb{1}\{l \geq CR^2\} \leq c\frac{l_i}{R^2}$ and using in the next term that $\mathbb{1}\{l \geq CR^2\} \leq \frac{l}{CR^2}$, the first part can be estimated, for any large R, against

$$\frac{C_M}{R}\int \psi(\mathrm{d}\mu)\sum_{i\in I} l_i = \frac{C_M}{R}\langle \xi, \mathfrak{N}^{(\ell,\mathcal{S})}_{W_R}\rangle.$$

Furthermore, we can simply estimate the second part against $C_M l_i$ and obtain the bound in (5.22).

Now assume that ξ is concentrated on $A^{(\mathcal{S})}_{W;\Theta}$. We recall that $\sum_{i\in I}\frac{|x_i-y_i|^2}{l_i}$ is indeed integrated with respect to ξ, under which $|f_i(jS\beta) - f_i((j+1)S\beta)| \leq \vartheta\sqrt{S}$ holds for any $j \in \mathbb{Z}$ and for any $i \in I$, where we recall that we wrote $f_i \in \mathcal{C}_{l_i}$ for the i-th shred (with $f_i(0) = x_i$ and $f_i(\ell(f_i)\beta) = y_i$), such that we can estimate $|x_i - y_i| \leq \sum_{j=1}^{\lceil l_i/S\rceil} |f_i(jS\beta) - f_i((j-1)S\beta)| \leq \frac{\vartheta l_i}{\sqrt{S}}$. Therefore, the second part is upper bounded against

$$\int \xi(\mathrm{d}\varpi)\sum_{i\in I}\Big[c\frac{|x_i-y_i|^2}{l_i} + C_M\Big] \leq \int \xi(\mathrm{d}\varpi)\sum_{i\in I}\Big[C\frac{\vartheta^2 l_i}{S} + C_M\Big] = \frac{C\vartheta^2}{S}\langle \xi, \mathfrak{N}^{(\ell,\mathcal{S})}_{W_R}\rangle + C_M\int \xi(\mathrm{d}\varpi)\,|I|.$$

Recall that ξ is concentrated on $A^{(\mathcal{S})}_{W_R;\Theta}$, which in particular means that in any of the boxes $z+W_{\mathfrak{r}}$ there are no more than T particles in the shreds and each leg has spread $< M$ with ξ-probability one. This implies that, with ξ-probability one, there are no more than of surface order (i.e., $O(R^{d-1})$) of shreds, and the proportionality factor depends on $M, \mathfrak{r}$ and T. This shows that $C_M\int \xi(\mathrm{d}\varpi)\,|I| \leq C_{M,\mathfrak{r},T}R^{d-1}$ for some constant that depends only on M, $\mathfrak{r}$, and T (and on d and on β). Finally, we absorb the C_M from above in $C_{M,\mathfrak{r},T}$, which finishes the proof of (5.23). □

Remark 5.6 (Simpler estimates for Brownian bridges) It is clear that, under shred-configuration measures ξ whose shreds are independent Brownian bridges (for example $\Pi^{(\mathcal{S})}_W(\mathtt{Q})$), there are much simpler upper bounds for $-\langle \xi, \mathfrak{p}_{W_R,M}\rangle$, but the main point in Lemma 5.5 is that we need to control expectations under pretty arbitrary ξs and therefore need to put the condition that *all* of them lie in $\mathcal{S}^{(M,\vartheta,S)}_W$ under ξ. Indeed, if all the shreds under ξ are Brownian bridges, then the only term in the above proof that needs and offers a simpler argument is $\int \xi(\mathrm{d}\varpi)\sum_{i\in I}\frac{|x_i-y_i|^2}{l_i}$, which is easily estimated by estimating $\frac{|x_i-y_i|^2}{l_i} \leq M\sum_{j=1}^{l_i/S}|f_i(jS\beta) - f_i((j+1)S\beta)|$, which gives, by the Gaussian nature of $f_i(jS\beta) - f_i((j+1)S\beta)$,

$$\int \xi(\mathrm{d}\varpi)\sum_{i\in I}\frac{|x_i-y_i|^2}{l_i} \leq MC\sqrt{S}\int \xi(\mathrm{d}\varpi)\sum_{i\in I}\frac{l_i}{S} \leq \frac{C_M}{\sqrt{S}}\langle \xi, \mathfrak{N}^{(\ell,\mathcal{S})}_{W_R}\rangle.$$

◊

Let us append now some useful estimate for the expected number of interlacement-shreds of shift-invariant interlacement configuration measures in a large box. Together with Lemma 5.5, it will lead to a flexible and useful bound in Corollary 5.8 below.

Lemma 5.7 (Estimating the expected number of interlacement-shreds)**.** *There is a $C \in (0,\infty)$ such that, for any $P \in \mathcal{M}^{(\mathrm{s})}_1(\mathcal{S})$, and for any $R \in \mathbb{N}$,*

$$\int \partial\Pi^{(\mathcal{S})}_{W_R}(P)(\mathrm{d}\mu)\,|\mu| \leq \frac{C|W_R|}{R}\Big(J_{W_R}(\Pi_{W_R}(P)) + \langle P, \mathfrak{N}^{(\ell)}_U\rangle - \frac{1}{|W_R|}\langle \partial\Pi^{(\mathcal{S})}_{W_R}(P), \mathfrak{p}_{W_R}\rangle\Big), \tag{5.26}$$

where we recall the definition of $\mathfrak{p}$ from (5.12) (with $M = \infty$) and (5.5).

Additionally, if $P \in \mathcal{M}^{(\mathrm{s})}_1(\mathcal{S}^{(M)})$ for some $M \in (1,\infty)$, then (5.26) holds with $\mathfrak{p}_{W_R}$ replaced by $\mathfrak{p}_{W_R,M}$.

Proof. We write W instead of W_R. We claim that there is a constant $C \in (0,\infty)$ such that

$$\int \partial\Pi_W^{(\mathcal{S})}(P)(\mathrm{d}\mu)\,|\mu| \leq \frac{C}{R}\int_{\mathcal{S}} P(\mathrm{d}\varpi)\sum_g \mathbb{1}\{g(0)\in W\}|g(\beta)-g(0)|, \qquad R\in\mathbb{N}, \tag{5.27}$$

where $\sum_g$ is the sum on all legs in the entire interlacement configuration (there are no loops). In other words, the expected number of W-shreds under $\Pi_W(P)$ is not larger than C/R times the expected sum of all jump lengths of all the legs with starting sites in W. Indeed, our argument for this is as follows. Fix one of the $2d$ hyperplanes that define the boundaries of W, say $H=\{x\in\mathbb{R}^d\colon x_1=R\}$. Consider all the W-shreds that make a final jump over this hyperplane (i.e., with a starting site with first index $\in[-R,R)$ and a target site with first index in (R,∞)). Put $I_x=[x,x+1)\times[-R,R]^{d-1}$ and call $M(x,y)$ the number of jumps of any leg in the configuration from I_x into I_y. For $z\in\mathbb{Z}$, consider $M_z=\sum_{x=z-R}^{z-1}\sum_{y=z}^{\infty}M(x,y)$, then the the number of shreds with final jump over H is not larger than M_R. By shift-invariance of P, the expectation of M_z does not depend on z. Hence, the expected number of W-shreds that make a final jump over H is not larger than C/R times the expectation of

$$\sum_{z=-R}^{R} M_z = \sum_{x,y,z\in\mathbb{Z}\colon -R\leq x\leq z\leq y,z\leq R} M(x,y) \leq \sum_{x=-R}^{R}\sum_{y\geq x}|y-x|M(x,y) \leq \sum_g \mathbb{1}\{g(0)\in W\}|g(\beta)-g(0)|,$$

where $\sum_g$ is the sum over all legs $\in\mathcal{C}_1$ in the entire configuration. Note that the expectation of M_z (i.e., the one of M_R) is not smaller than the left-hand side of (5.27). Summing over all the $2d$ hyperplanes, our argument implies (5.27).

Now we estimate the expectation on the right-hand side of (5.27) by using (2.7). We write the sum over the legs as $\sum_{i,k}\mathbb{1}\{f_i((k-1)\beta)\in W\}|f_i(k\beta)-f_i((k-1)\beta)|$, where $\varpi=\sum_{i\in I}\delta_{f_i}$ is the shred configuration in W. We abbreviate the kernel $(\Pi_W(P))_{\mathcal{T}_W\to\mathcal{S}_W}(\mu,\cdot)$ by $P(\mu,\cdot)$ and see (by first considering a cut-off version of $\sum_{i,k}|f_i(k\beta)-f_i((k-1)\beta)|$) that

$$\begin{aligned}
&\int \Pi_W^{(\mathcal{S})}(P)(\mathrm{d}\varpi)\sum_{i,k}\mathbb{1}\{f_i((k-1)\beta)\in W\}|f_i(k\beta)-f_i((k-1)\beta)|\\
&\quad=\int_{\mathcal{T}_W}\partial\Pi_W^{(\mathcal{S})}(P)(\mathrm{d}\mu)\int_{\mathcal{S}_W}P(\mu,\mathrm{d}\varpi)\sum_{i,k}|f_i(k\beta)-f_i((k-1)\beta)|\\
&\quad\leq\int\partial\Pi_W^{(\mathcal{S})}(P)(\mathrm{d}\mu)\Big[H_{\mathcal{S}_W}(P(\mu,\cdot)\mid \mathtt{K}_W(\mu,\cdot))+\log\int\mathtt{K}_W(\mu,\mathrm{d}\varpi)\Big(\prod_{i,k}\mathrm{e}^{|f_i(k\beta)-f_i((k-1)\beta)|}\Big)\Big]\\
&\quad\leq|W|J_W(\Pi_W(P))+\int\partial\Pi_W^{(\mathcal{S})}(P)(\mathrm{d}\mu)\sum_i\log\int q_{x_i,y_i}^{(W,l_i)}(\mathrm{d}f)\prod_{k=1}^{l_i}\mathrm{e}^{|f(k\beta)-f((k-1)\beta)|},
\end{aligned}\tag{5.28}$$

where we recall the definition of $\mathtt{K}_W$ in (2.5), and we wrote $\mu=\sum_i\delta_{(x_i,l_i,y_i)}$, as usual. Recalling the definition of $p_x^{(W)}(l,y)$ from (5.5) and (5.6) (both with $M=\infty$) and writing g_β for the Gaussian density with variance 2β, we further estimate

$$\begin{aligned}
\int q_{x_i,y_i}^{(W,l_i)}(\mathrm{d}f)\prod_{k=1}^{l_i}\mathrm{e}^{|f(k\beta)-f((k-1)\beta)|}&=\frac{1}{p_{x_i}^{(W)}(l_i,y_i)}\int_{W^{l_i-1}}\mathrm{d}w_1\dots\mathrm{d}w_{l_i-1}\prod_{k=1}^{l_i-1}\big[\mathrm{e}^{|w_k-w_{k-1}|}g_\beta(w_k-w_{k-1})\big]\\
&\leq\frac{1}{p_{x_i}^{(W)}(l_i,y_i)}C^{l_i-1}C',
\end{aligned}$$

with $C=\log\mathbb{E}_0[\mathrm{e}^{|B_\beta|}]$ and $C'=\sup_{x\in\mathbb{R}^d}[\mathrm{e}^{|x|}g_\beta(x)]$, and we wrote $w_0=x_i$ and $w_{l_i}=y_i$. We estimate the C-terms against C^{l_i} (using our convention that C is a generic constant) and see that the last term on the right-hand side of (5.28) is not larger than $\int\partial\Pi_W^{(\mathcal{S})}(P)(\mathrm{d}\mu)\,\sum_i[Cl_i-\log p_{x_i}^{(W)}(l_i,y_i)]=C\langle P,\mathfrak{N}_W^{(\ell,\mathcal{S})}\rangle-$

$\int \partial\Pi_W^{(\mathcal{S})}(P)(\mathrm{d}\mu)\,\sum_{i\in I}\log p_{x_i}^{(W)}(l_i,y_i)$. Now use the shift-invariance of P to see that $\frac{1}{|W_R|}\langle P,\mathfrak{N}_W^{(\ell,\mathcal{S})}\rangle = \langle P,\mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle$, which finishes the proof (recalling (5.12) for the second term). □

As a consequence, here is an upper bound for the expected number of interlacement-shreds in a large box W_R that vanishes as $R\to\infty$ for every shift-invariant measure P that has finite particle density and finite entropy $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$ (thanks to (2.16)).

Corollary 5.8 (Estimating the expected number of interlacement-shreds)**.** *For any $M\in(1,\infty)$, there is a $C_M\in(1,\infty)$ such that, for any $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{S}^{(M)})$ and any sufficiently large R,*

$$\frac{1}{|W_R|}\int\partial\Pi_{W_R}^{(\mathcal{S})}(P)(\mathrm{d}\mu)\,|\mu|\leq\frac{1}{R}\Big(J_{W_R}(\Pi_{W_R}(P))+C_M\langle P,\mathfrak{N}_U^{(\mathcal{S},\ell)}\rangle\Big).$$

Proof. In (5.26) for $M<\infty$, use (5.22) with $\xi=\Pi_{W_R}(P)$ for estimating the last term and use the shift-invariance of P. □

6. Large deviation analysis: making $N\to\infty$

In this section, we derive the necessary LDP-framework for the thermodynamic limit as $N\to\infty$ for the empirical measure $\Xi_{N,R}^{(\omega_{\mathrm{P}})}$ defined at the beginning of Section 4.4, where ω_{P} is the marked PPP under $\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}$. In Section 6.1 we give first some technical preparation and then the basic large deviation principle that we need (see Corollary 6.5) and in Section 6.2 some crucial properties of the rate function, J_W, which will be important for the proofs of the main results, Theorems 2.5 and (2.3). In order to apply later the upper bound of that LDP, we need to derive some crucial assertions about compactness, which we do in Section 6.3. In Section 6.4 we apply it to derive an upper bound for the limit superior as $N\to\infty$ of $\frac{1}{|\Lambda_N|}$ times the logarithm of the right-hand side of (4.25); in Section 6.5 we give a lower bound for the limit inferior of $\frac{1}{|\Lambda_N|}$ times the logarithm of the right-hand side of (4.39).

As always, we assume that the situation of Section 1.4 is given, i.e., the regular decomposition of the centred box Λ_N into subboxes of radius R, which possibly depends on N and converges to a given $R\in\mathbb{N}$. Recall the notion of a relative entropy from (2.6).

6.1. The conditional large-deviation principle.

The empirical measure $\Xi_{N,R}^{(\omega_{\mathrm{P}})}$ defined in (4.20) is a mixture over $z\in Z_{N,R}$ of the $\mathcal{L}_W\times\mathcal{S}_W$-valued random variable pairs $(\theta_z(\omega_{W_z}^{(\mathcal{L})}),\theta_z(\omega_{W_z}^{(\mathcal{S})}))$ defined in (4.26), and we consider the distribution under the PPP $\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}$. The first member of these pairs comes from those loops that are entirely contained in some of these subboxes, while the second one, the shred part, comes from those loops that are R-crossing. If these pairs would be i.i.d. over z and their distribution would not depend on N, then Sanov's theorem (see [DZ10, Section]) would immediately give the validity of the LDP with an explicit rate function. But the shred-parts are far from being i.i.d., and the distribution of the pairs depends on N (namely via the boundary condition).

Hence, we need to take care of these two obstacles. The second one is only technical: because of the boundary condition (periodic or particle) in Λ_N, the distribution in the subboxes depend slightly on N and on z. We will handle this problem with the help of the concept of exponential equivalence (which says that two exponentially equivalent sequences satisfy the same LDP, if any, see [DZ10, Section 4.2.2]) and will be using the following technical lemma, whose proof is straight-forward and is left to the reader.

Lemma 6.1 (Perturbed empirical measures are exponentially equivalent)**.** *On a Polish space $\mathcal{X}$, assume that an i.i.d. sequence $(X_z)_{z\in[N]}$ is given and another sequence $(X_z^{(N)})_{z\in[N]}$ that satisfies*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{z=1}^{N}\log\mathbb{E}\big[\mathrm{e}^{f(X_z^{(N)})}\big]=\log\mathbb{E}[\mathrm{e}^{f(X_1)}],\qquad f\in\mathcal{C}_{\mathrm{b}}(\mathcal{X}).\tag{6.1}$$

Then the empirical measures $\frac{1}{N}\sum_{z=1}^N \delta_{X_z^{(N)}}$ *and* $\frac{1}{N}\sum_{z=1}^N \delta_{X_z}$ *are exponentially equivalent.*

In order to apply this, we need to prove that the distributions of the pairs satisfy the criterion in (6.1). Recall the Brownian kernel $\mathtt{K}_W$ defined in (2.5) and introduce also the version $\mathtt{K}_W^{(\Lambda_N,\mathrm{bc})}(\mu,\cdot)\in\mathcal{M}_1(\mathcal{S}_W)$, where $q^{(l_i,W)}$ is replaced by $q^{(l_i,W,\Lambda_N,\mathrm{bc})}$; see (5.3). This measure is concentrated on the set of those collections $(f_i)_i$ of W-shreds such that f_i is a length-l_i shred with $f_i(0)=x_i$ and $f_i(l_i\beta)=y_i$ for any $i\in I$.

For any $\xi\in\mathcal{M}_1(\mathcal{S}_W)$ and any $W\Subset\mathbb{R}^d$, we denote by $\xi_{\mathcal{T}_W\to\mathcal{S}_W}$ the kernel from $\mathcal{T}_W$ into $\mathcal{S}_W$ that is defined by letting $\xi_{\mathcal{T}_W\to\mathcal{S}_W}(\mu,\cdot)$ be a regular conditional version of ξ given $\partial\Pi_W^{(\mathcal{S})}(\xi)=\mu$ for $\mu\in\mathcal{T}_W$. Then ξ is equal to $\partial\Pi_W^{(\mathcal{S})}(\xi)\otimes\xi_{\mathcal{T}_W\to\mathcal{S}_W}$.

Lemma 6.2 (The shred-configuration kernel)**.** *We consider either periodic or particle boundary condition.*

(1) *For any $N\in\mathbb{N}$ and any $z\in Z_{N,R}$, the map $\mathcal{T}_{W_z}\times\mathcal{G}_{W_z}^{(\mathcal{S})}\ni(\mu,\mathrm{d}\xi)\mapsto\mathtt{K}_{W_z}^{(\Lambda_N,\mathrm{bc})}(\mu,\mathrm{d}\xi)$ is equal to the kernel $\Pi_{W_z}^{(\mathcal{S})}(\mathtt{Q}^{(\Lambda_N,\mathrm{bc})})_{\mathcal{T}_{W_z}\to\mathcal{S}_{W_z}}$, where we recall that $W_z=z+W$.*
(2) *The empirical measure of the kernels from (1) converges to the kernel $\mathtt{K}_W$ in the following sense. Assume that $\frac{1}{\#Z_{N,R}}\sum_{z\in Z_{N,R}}\delta_{\mu_z}\to\delta_\mu$ in $\mathcal{M}_1(\mathcal{T}_W)$ as $N\to\infty$, then*
$$\lim_{N\to\infty}\frac{1}{\#Z_{N,R}}\sum_{z\in Z_{N,R}}\log\mathbb{E}_{\mu_z}\big[\mathrm{e}^{f(\theta_z(\Pi_{W_z}^{(\mathcal{S})}))}\big]=\log\mathtt{K}_W\Big(\mu,\mathrm{e}^{f(\Pi_W^{(\mathcal{S})})}\Big),\qquad f\in\mathcal{C}_\mathrm{b}(\mathcal{S}_W),\tag{6.2}$$
where $\mathbb{E}_{\mu_z}[\cdot]=\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}[\cdot\mid\theta_z(\partial\Pi_{W_z}^{(\mathcal{S})})=\mu_z]$.
(3) *Both under $\mathtt{Q}^{(\Lambda_N,\mathrm{par})}$ and $\mathtt{Q}^{(\Lambda_N,\mathrm{per})}$, under the assumption that $\frac{1}{\#Z_{N,R}}\sum_{z\in Z_{N,R}}\delta_{\mu_z}\to\delta_\mu$ in $\mathcal{M}_1(\mathcal{T}_W)$ as $N\to\infty$, the pairs $(\theta_z(\omega_{W_z}^{(\mathcal{L})}),\theta_z(\omega_{W_z}^{(\mathcal{S})}))$, $z\in Z_{N,R}$ satisfy (6.1).*

Proof. (1) Lemma 5.1(1) implies that $\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}$, conditional on the family $(\mu_z)_{z\in Z_{N,R}}=(\theta_z(\partial\Pi_{W_z}^{(\mathcal{S})}(\omega_\mathrm{P})))_{z\in Z_{N,R}}$ of the boundary shred configurations, the shred-configuration family $(\theta_z(\Pi_{W_z}^{(\mathcal{S})}(\omega_\mathrm{P})))_{z\in Z_{N,R}}$ has the distribution $\bigotimes_{z\in Z_{N,R}}\mathtt{K}_{W_z}^{(\Lambda_N,\mathrm{bc})}(\mu_z,\cdot)$. By superposition of all the independent loops of the loop soup, it implies also that, for any z, the conditional distribution of the family of shred configurations in W_z given the boundary shred configuration μ_z is given by the product measure of the $q_{x_i,y_i}^{(l_i,W_z,\Lambda_N,\mathrm{bc})}$.

(2) For 'bc' equal to particle boundary condition, according to Lemma 5.1(2), the boundary condition has no effect, since all particles involved in the definition of q lie in W anyway, i.e., $q^{(W,\Lambda_N,\mathrm{bc})}=q^{(W)}$. Hence, by Assertion (1), (6.2) is trivially true without taking the limit. However, for periodic boundary condition, one needs to prove that $\mathtt{K}_W^{(\Lambda_N,\mathrm{bc})}(\mu,\cdot)$ is close to $\mathtt{K}_W(\mu,\cdot)$, i.e., $q_{x_i,y_i}^{(l_i,W,\Lambda_N,\mathrm{bc})}$ is close to $q_{x_i,y_i}^{(l_i,W)}$ in the sense of (6.2). We leave this to the reader, see the hints in the proof of (3).

(3) Observe that, since the empirical measures of the μ_z are given and converge, the loop part and the shred part are independent, and the necessary assertion for the shred part is in (2). The proof for the corresponding loop part is similar, hence we restrict ourselves to giving just some hints and leave the details to the reader.

Note that we can estimate, for some $C\in(0,\infty)$, only depending on R (and on β and on d), for any sufficiently large N,
$$p_x^{(W,\Lambda_N,\mathrm{per})}(l,y)\le p_x^{(W)}(l,y)\big[1+\mathrm{e}^{-CN^{2/d}/l}\big],\qquad x\in W,l\in\mathbb{N},y\in W^\mathrm{c}.\tag{6.3}$$
The reason for this is the following. According to [BR97, Theorem 6.3.8], there are, for any $K\in(0,\infty)$, constants C_1,C_2,C_3 such that the solution $g_t^{(\Lambda,\mathrm{per})}(x,y)$ to the heat equation with the Laplace operator with periodic boundary condition in the box Λ satisfies
$$0\le g_t^{(\Lambda,\mathrm{per})}(x,y)-g_t(x,y)\le C_1 g_{t/C_2}(x,y)\,\mathrm{e}^{-C_3\mathrm{dist}(y,\partial\Lambda)^2/t},\qquad x,y\in\Lambda,t\in(0,K],\tag{6.4}$$

where g_t denotes the free fundamental solution, i.e., the standard Gaussian kernel with variance $2t$. We leave the remainder of the details to the reader. □

Let us turn to the first of the two obstacles. This is more serious. Indeed, the first members of the pairs are independent over z and (ignoring the small dependence on z and N) also identically distributed. But the second ones are not at all, since neighbouring subboxes have a strong dependence, and this dependence has very long correlations. Nevertheless, there is a way out of that, and this is a conditioning on the boundary configurations. The idea is that, under this conditioning, the configurations inside the boxes are independent with a distribution that depends only on the boundary configuration (by the virtue of the Markov property of the Brownian motion). More precisely, we condition on the boundary empirical measure of $\Xi_{N,R}$, namely

$$\partial\Xi_{N,R}^{(\mathcal{S})} = \partial\Pi_W^{(\mathcal{S})}(\Xi_{N,R}) = \frac{1}{\#Z_{N,R}} \sum_{z\in Z_{N,R}} \delta_{\partial\Pi_W^{(\mathcal{S})}(\theta_z \varpi_{W_z}^{(\mathcal{S})})} \in \mathcal{M}_1(\mathcal{T}_W) \tag{6.5}$$

and obtain that $\Xi_{N,R}$ is indeed an empirical measure of independent random variables, and we also have sufficient information about their distributions.

In order to catch the large deviation behaviour of the conditional distribution of $\Xi_{N,R}$, we need to employ an LDP of the following type. The following LDP is taken from [PRV13, Thm. 17]; it goes back to an unpublished manuscript by C. Léonard; see [ADPZ11, Thm. 1] for a conditional version of this LDP.

Lemma 6.3 (Sanov-type LDP for type-dependent independent random variables). *Let $\mathcal{X},\mathcal{Y}$ be two Polish spaces and pick $\rho \in \mathcal{M}_1(\mathcal{X})$ and let $(x_i^{(n)})_{i\in\mathbb{N}}$ be a sequence in $\mathcal{X}$ such that $\frac{1}{n}\sum_{i=1}^n \delta_{x_i^{(n)}}$ converges weakly towards ρ as $n\to\infty$. Let $\zeta\colon \mathcal{X}\times\sigma(\mathcal{Y})\to[0,1]$ be a continuous Markov kernel from $\mathcal{X}$ to $\mathcal{Y}$, where $\sigma(\mathcal{Y})$ denotes the Borel sigma field on $\mathcal{Y}$. Furthermore, let $(Y_i^{(n)})_{i\in[n]}$ have the distribution $\bigotimes_{i\in[n]}\zeta(x_i^{(n)},\cdot)$. Then the empirical pair measure $\frac{1}{n}\sum_{i=1}^n \delta_{(x_i^{(n)},Y_i^{(n)})}$ satisfies an LDP with rate function*

$$\mathcal{M}_1(\mathcal{X}\times\mathcal{Y}) \ni q \mapsto \begin{cases} H_{\mathcal{X}\times\mathcal{Y}}(q\,|\,\rho\otimes\zeta) & \text{if } \pi_1(q)=\rho,\\ +\infty & \text{otherwise,}\end{cases} \tag{6.6}$$

where π_1 is the canonical projection $\mathcal{X}\times\mathcal{Y}\to\mathcal{X}$, and $\pi_1(q)$ is the corresponding image measure.

We recall that $\rho\otimes\zeta$ is defined by $\rho\otimes\zeta(\mathrm{d}(x,y)) = \rho(\mathrm{d}x)\zeta(x,\mathrm{d}y)$.

Based on Lemma 6.3, we can give a conditional LDP for $\Xi_{N,R}$ for a fixed box $W=W_R=[-R,R]^d$. We need to take care of a consistency property (or marginal property) that $\Xi_{N,R}$ asymptotically has: the rate function is equal to ∞ in any configuration measures $\xi\in\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W)$ that cannot be extended to the larger box 3^kW for some $k\in\mathbb{N}$ such that the extension is W-shift invariant (meaning that projection of the extension to any subbox $W_z=z+W$ of 3^kW, $z\in 2R\mathbb{Z}^d$, is equal to ξ after shifting back to W. (This property is analogous to the marginal property of the rate function for the empirical measures $\frac{1}{N}\sum_{k=1}^N \delta_{(M_{k-1},M_k)}$ of a Markov chain $(M_n)_{n\in\mathbb{N}}$, whose two marginal measures are equal in the limit $N\to\infty$, and therefore the rate function is equal to ∞ if this property is not satisfied; see [DZ10, Section 3.1.3].) Here is a precise definition. We introduce the set $\mathcal{R}_{3^k,W}$ of rectangles $\subset 3^kW$ of the form $\bigcup_{z\in 2RZ} z+W$ for some $Z\subset\mathbb{Z}^d$. Then we introduce the set of W-shift invariant configurations in 3^kW,

$$\begin{aligned}\widetilde{\mathcal{M}}_1^{(k)} = \Big\{\xi^{(k)}\in\mathcal{M}_1(\mathcal{L}_{3^kW}\times\mathcal{S}_{3^kW})\colon \forall\widetilde{W}\in\mathcal{R}_{3^k,W}\forall z\in 2R\mathbb{Z}^d\colon\\ z+\widetilde{W}\in\mathcal{R}_{3^k,W}\Longrightarrow \theta_z(\Pi_{z+\widetilde{W}}(\xi^{(k)})) = \Pi_{\widetilde{W}}(\xi^{(k)})\Big\}\end{aligned} \tag{6.7}$$

and the set

$$\mathcal{M}_1^{(k)}(\mathcal{L}_W \times \mathcal{S}_W) = \big\{\xi \in \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W) \colon \exists \xi^{(k)} \in \widetilde{\mathcal{M}}_1^{(k)} \colon \Pi_{3^k W \to W}(\xi^{(k)}) = \xi\big\} = \Pi_{3^k W \to W}(\widetilde{\mathcal{M}}_1^{(k)}), \tag{6.8}$$

where we remind that $\Pi_{3^k W\to W} \colon \mathcal{M}_1(\mathcal{L}_{3^k W} \times \mathcal{S}_{3^k W}) \to \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$ is the well-known projection operator from (4.18), here with domain $\mathcal{M}_1(\mathcal{L}_{3^k W} \times \mathcal{S}_{3^k W})$.

Lemma 6.4 (LDP for $\Xi_{N,R}$, conditional on $\partial\Pi_W^{(\mathcal{S})}(\Xi_{N,R})$)**.** *Fix a window $W = [-R,R]^d$ and any $\psi \in \mathcal{M}_1(\mathcal{T}_W)$. Then, under $\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}$ conditioned on $\{\partial\Pi_W^{(\mathcal{S})}(\Xi_{N,R}) = \psi_N\}$ for some $\psi_N \to \psi$, the sequence $(\Xi_{N,R})_{N\in\mathbb{N}}$ satisfies an LDP on $\mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$ with speed $|\Lambda_N|$ with rate function*

$$J_W^{(\psi)}(\xi) = \begin{cases} \frac{1}{|W|} H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi \,\big|\, \Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\psi \otimes \mathtt{K}_W]\big) & \text{if } \partial\Pi_W^{(\mathcal{S})}(\xi) = \psi \text{ and } \xi \in \bigcap_{k\in\mathbb{N}} \mathcal{M}_1^{(k)}(\mathcal{L}_W \times \mathcal{S}_W), \\ +\infty & \text{otherwise.} \end{cases} \tag{6.9}$$

Proof. First we show that the rate function is ∞ outside $\mathcal{M}_1^{(k)}(\mathcal{L}_W \times \mathcal{S}_W)$, for any $k \in \mathbb{N}$. Clearly, $\mathcal{M}_1^{(k)}(\mathcal{L}_W \times \mathcal{S}_W)$ is closed, since all the projection operators appearing in the definition of $\mathcal{M}_1^{(k)}(\mathcal{L}_W \times \mathcal{S}_W)$ are continuous.

Consider the measure

$$\Xi_{N,R}^{(k)}(\omega) = \frac{1}{\# Z_{N,R}} \sum_{z \in Z_{N,R}} \delta_{\theta_z(\Pi_{z+3^k W}(\omega))} \in \mathcal{M}_1(\mathcal{L}_{3^k W} \times \mathcal{S}_{3^k W}), \qquad \omega \in \mathcal{L} \times \mathcal{S}. \tag{6.10}$$

This is the empirical measure of the loop/shred configuration in the boxes $z + 3^k W$ with $z \in Z_{N,R} = \{z \in 2R\mathbb{Z}^d \colon z + W \subset \Lambda_N\}$ (these boxes overlap each other significantly by disjoint unions of shifted copies of W), shifted to the origin by z. (It is of no importance for this proof if we replace $Z_{N,R}$ by $\{z \in 2R\mathbb{Z}^d \colon z + 3^k W \subset \Lambda_N\}$ or not, since we only rely on the limit as $N \to \infty$.)

Up to boundary effects, its $\Pi_{3^k W\to W}$-projection is equal to $\Xi_{N,R}(\omega)$, by which we mean that

$$\sup_\omega \mathrm{d_{TV}}\big(\Pi_{3^k W \to W}(\Xi_{N,R}^{(k)}(\omega)), \Xi_{N,R}(\omega)\big) \to 0, \qquad N \to \infty, \tag{6.11}$$

where $\mathrm{d_{TV}}$ denotes the total-variation distance on $\mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$. Similarly,

$$\sup_\omega \mathrm{d_{TV}}\big(\theta_z(\Pi_{z+\widetilde{W}}(\Xi_{N,R}^{(k)}(\omega))), \Pi_{\widetilde{W}}(\Xi_{N,R}^{(k)}(\omega))\big) \to 0, \qquad N \to \infty, z \in \mathbb{Z}^d, \widetilde{W}, z + \widetilde{W} \in \mathcal{R}_{3^k, W}. \tag{6.12}$$

This means that the distance between $\Xi_{N,R}^{(k)}$ and $\widetilde{\mathcal{M}}_1^{(k)}$ vanishes as $N \to \infty$, uniformly in ω. By continuity of $\Pi_{3^k W\to W}$, also the distance between $\Pi_{3^k W\to W}(\Xi_{N,R}^{(k)})$ and $\Pi_{3^k W\to W}(\widetilde{\mathcal{M}}_1^{(k)})$ vanishes as $N \to \infty$. Combining with (6.11), we see that the distance between $\Xi_{N,W}$ and $\Pi_{3^k W\to W}(\widetilde{\mathcal{M}}_1^{(k)})$ vanishes as $N \to \infty$, uniformly in ω.

As a consequence, for any open set $\mathcal{U} \subset \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$ that entirely lies in the complement of $\mathcal{M}_1^{(k)}(\mathcal{L}_W \times \mathcal{S}_W) = \Pi_{3^k W\to W}(\widetilde{\mathcal{M}}_1^{(k)})$ and has a positive distance to it, the probability of $\{\Xi_{N,R} \in \mathcal{U}\}$ under $\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}$ is zero for all large N, since the event is empty. Indeed, the distance of $\mathcal{U}$ to $\Pi_{3^k W\to W}(\widetilde{\mathcal{M}}_1^{(k)})$ is positive, but $\Xi_{N,R}$ comes arbitrarily close to it. This implies that the rate function is equal to $+\infty$ outside $\mathcal{M}_1^{(k)}(\mathcal{L}_W \times \mathcal{S}_W)$ for any $k \in \mathbb{N}$.

Now we use Lemma 6.3 to prove the LDP and to identify the rate function. We conceive the state space $\mathcal{L}_W \times \mathcal{S}_W$ as the product $\mathcal{T}_W \times (\mathcal{L}_W \times \mathcal{S}_W)$, which is meant as the set of those pairs $(\mu, (\omega, \varpi))$ that are compatible in the sense that $\partial\Pi_W^{(\mathcal{S})}(\omega, \varpi) = \mu$. That is, the $\mathcal{L}$-part of ξ does not depend on μ, and μ is the configuration of all the triples of initial sites, lengths and terminal sites of the shreds of ϖ. Hence, in Lemma 6.3, $\mathcal{X} = \mathcal{T}_W$ and $\mathcal{Y} = \mathcal{L}_W \times \mathcal{S}_W$, and the projection π_1 there is called $\partial\Pi_W^{(\mathcal{S})}$ in our situation.

Now consider $\omega \in \mathcal{L}_{\Lambda_N}$, distributed under $\Pi_W^{(\mathcal{L})}(\mathtt{Q}^{(\Lambda_N,\mathrm{bc})})$. Conditionally given $(\theta_z(\partial\Pi_{W_z}^{(\mathcal{S})}(\omega)))_{z\in Z_{N,R}}$, the family of pairs $(\theta_z(\omega_{W_z}^{(\mathcal{L})}),\theta_z(\omega_{W_z}^{(\mathcal{S})}))_{z\in Z_{N,R}}$ has the distribution $\bigotimes_{z\in Z_{N,R}}\zeta_z(\partial\Pi_W^{(\mathcal{S})}(\omega),\cdot)$, where $\zeta_z(\mu,\cdot)$ is, for general μ, the product measure of the distribution $\Pi_W^{(\mathcal{L})}(\mathtt{Q}^{(\Lambda_N,\mathrm{bc})})$ (not depending on μ) and $\mathbb{P}_\mu^{(W_z,\Lambda_N,\mathrm{bc})}$ (see Lemma 6.2(3)). We are under the assumption that the empirical measure of the $\theta_z(\partial\Pi_{W_z}^{(\mathcal{S})}(\omega))$ (namely $\partial\Xi_{N,R}^{(\omega)}$) converges towards ψ. If $\zeta_z(\partial\Pi_W^{(\mathcal{S})}(\omega),\cdot)$ would not depend on z nor on N (it does through our boundary condition in $\Pi_W^{(\mathcal{L})}(\mathtt{Q}^{(\Lambda_N,\mathrm{bc})})$), i.e., if it would be just equal to $\zeta(\mu,\mathrm{d}(\omega,\varpi))=\Pi_W^{(\mathcal{L})}(\mathtt{Q})(\mathrm{d}\omega)\mathtt{K}_W(\mu,\mathrm{d}\varpi)$ (see (2.5)), then Lemma 6.3 would be directly applicable, and we would obtain that the empirical measure of these pairs (and this is $\Xi_{N,R}^{(\omega)}$) satisfies an LDP with the rate function in (6.9). Indeed, the reference measure in (6.9) is equal to $\psi\otimes\zeta(\mathrm{d}\mu,\mathrm{d}(\omega,\varpi))=\psi(\mathrm{d}\mu)\Pi_W^{(\mathcal{L})}(\mathtt{Q})(\mathrm{d}\omega)\mathtt{K}_W(\mu,\mathrm{d}\varpi)=\Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes[\psi\otimes\mathtt{K}_W](\mathrm{d}(\omega,\varpi))$. Now apply Lemmas 6.2 and 6.3 to finish the proof. □

Hence, it is natural to restrict to the set of ξ's and ψ's to those that satisfy $\partial\Pi_W^{(\mathcal{S})}(\xi)=\psi$ and to use only the rate function

$$J_W(\xi)=J_W^{(\partial\Pi_W^{(\mathcal{S})}(\xi))}(\xi)=\frac{1}{|W|}H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi\,\big|\,\Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes[\partial\Pi_W^{(\mathcal{S})}(\xi)\otimes\mathtt{K}_W]\big),\qquad \xi\in\bigcap_{k\in\mathbb{N}}\mathcal{M}_1^{(k)}(\mathcal{L}_W\times\mathcal{S}_W). \tag{6.13}$$

Note that $\Pi_W(P)$ lies automatically in $\bigcap_{k\in\mathbb{N}}\mathcal{M}_1^{(k)}(\mathcal{L}_W\times\mathcal{S}_W)$ for $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$, hence for considering $J_W(\Pi_W(P))$ later we will not need to take extra care of the restriction.

Let us formulate Lemma 6.4 in a way in which we will actually apply it later. For the lower bound, because of our approach in Lemmas 4.3 and (5.5), we need to condition the integration to the event $\{\Xi_{N,R}^{(\circ)}(A_{W;\Theta})=1\}$, where we collect all the cutting parameters in the tuple $\Theta=(M,L,K,\mathfrak{r},T,S,\vartheta)$ and summarize the definitions in (4.34), (4.35) and (4.36) and (5.21) by putting $A_{W;\Theta}=A_{W;M,\mathfrak{r},K,L,T}^{(\mathcal{L})}\times(A_{W;M,\mathfrak{r},T}^{(\mathcal{S})}\cap\mathcal{S}^{(M,\vartheta,S)})$. By $\mathtt{Q}_{W;\Theta}^{(\Lambda_N,\mathrm{par})}$ we denote probability with respect to the PPP with intensity measure $\nu_{\Lambda_N,W;\Theta}^{(\mathrm{par})}$, which is defined as the restriction of $\nu_{\Lambda_N}^{(\mathrm{par})}$ defined in (1.6) to the event $\bigcap_{z\in Z_{N,R}}\{(\omega_{W_z},\varpi_{W_z})\in A_{W_z;\Theta}\}$.

Corollary 6.5 (Upper and lower LDP bounds for $\Xi_{N,R}$)**.** *Fix $R\in\mathbb{N}$ and $W=W_R=[-R,R]^d$ and a function $F\colon\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W)\to\mathbb{R}$ and a measurable set $B\subset\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W)$.*

(1) *If F is lower semi-continuous and bounded from below and B closed and $B'\subset\mathcal{M}_1(\mathcal{T}_W)$ compact, then*

$$\limsup_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}\Big[\mathrm{e}^{-|\Lambda_N|F(\Xi_{N,R})}\mathbb{1}\{\Xi_{N,R}\in B\}\{\partial\Xi_{N,R}^{(\mathcal{S})}\in B'\}\Big]\\ \le-\inf_{\xi\in B\cap\bigcap_{k\in\mathbb{N}}\mathcal{M}_1^{(k)}(\mathcal{L}_W\times\mathcal{S}_W)\colon\partial\Pi_W^{(\mathcal{S})}(\xi)\in B'}\big(F(\xi)+J_W(\xi)\big).$$

(2) *Fix all the parameters $M,L,K,\mathfrak{r},T,\vartheta,S\in(0,\infty]$ and put $\Theta=(M,L,K,\mathfrak{r},T,\vartheta,S)$. Then, for any W-admissible $\psi\in\mathcal{M}_1(\mathcal{T}_W)$ and for any sequence $\psi_N\to\psi$ such that $\mathtt{Q}_{W;\Theta}^{(\Lambda_N,\mathrm{bc})}(\partial\Pi_W^{(\mathcal{S})}(\Xi_{N,R})=\psi_N)>0$ for any N, if F is upper semi-continuous and B is subset of and open in $A_{W;\Theta}$,*

$$\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathtt{Q}_{W;\Theta}^{(\Lambda_N,\mathrm{bc})}\Big[\mathrm{e}^{-|\Lambda_N|F(\Xi_{N,R})}\mathbb{1}\{\Xi_{N,R}\in B\}\,\Big|\,\partial\Pi_W^{(\mathcal{S})}(\Xi_{N,R})=\psi_N\Big]\\ \ge-\inf_{\xi\in B\colon\partial\Pi_W^{(\mathcal{S})}(\xi)=\psi}\big(F(\xi)+J_{W;\Theta}(\xi)\big),$$

where

$$J_{W;\Theta}(\xi)=\frac{1}{|W|}H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi\,\big|\,\Pi_W^{(\mathcal{L})}(\mathtt{Q}_M)_{L,K}\otimes[\partial\Pi_W^{(\mathcal{S})}(\xi)\otimes\mathtt{K}_W^{(M,\mathfrak{r},T,\vartheta,S)}]\big),\qquad\xi\in\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W),\tag{6.14}$$

where $\Pi_W^{(\mathcal{L})}(\mathtt{Q}_M)_{L,K}$ *is the conditional version of* $\Pi_W^{(\mathcal{L})}(\mathtt{Q}_M)$ *given* $A^{(\mathcal{L})}_{W;M,L,K}$*, and* $\mathtt{K}_W^{(M,\mathfrak{r},T,\vartheta,S)}(\mu,\cdot)$ *is the conditional version of* $\mathtt{K}_W(\mu,\cdot)$ *given* $A^{(\mathcal{S})}_{W;M,\mathfrak{r},T}\cap\mathcal{S}_W^{(M,\vartheta,S)}$*. (In particular,* $J_{W;\Theta}(\xi)=\infty$ *if* $\xi(A_{W;\Theta})<1$*.)*

The upper bound also motivates the following Section 6.3: we will need good compactness criteria on $\mathcal{M}_1(\mathcal{T}_W)$. The lower bound will later be applied for $\psi=\partial\Pi^{(\mathcal{S})}_{W_R}(P)$ with an ergodic $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$, evoking Lemma 5.4.

Proof. (1) follows, by the virtue of the arguments in the proof of Varadhan's lemma (see [DZ10, Section 4.3]) from Lemma 6.4 by dropping all probability terms that concern $\partial\Pi_W^{(\mathcal{S})}(\Xi_{N,R})$. However, for the application of Varadhan's lemma, we need to ensure that the rate function J_W has compact level sets, and this is not true since its reference measure has no autonomous $\mathcal{M}_1(\mathcal{T}_W)$-part. But, by the proof of Lemma 6.8(1), the intersection of the level set with $\{\xi\colon\partial\Pi_W^{(\mathcal{S})}(\xi)\in B'\}$ is compact, which suffices for the upper-bound part of Varadhan's lemma.

To prove (2), again by the virtue of the arguments in the proof of Varadhan's lemma, we only need to observe that the proof of Lemma 6.4 applies also for the restricted and conditional version $\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}_{W;\Theta}$ instead of $\mathtt{Q}^{(\Lambda_N,\mathrm{bc})}$ and that $J_{W;\Theta}(\xi)=\infty$ if $\partial\Pi_W^{(\mathcal{S})}(\xi)\neq\psi$. □

6.2. **Properties of** J_W**.** We are going to collect some crucial properties of J_W defined in (6.13), in particular of $J_{W_R}(\Pi_{W_R}(P))$ for shift-invariant distributions P of loops and interlacements. These properties will be essential for all the future developments. Our main result here is Proposition 6.6; it has four assertions.

In the first two assertions, we show that the J-value in a box mW (with $m\in\mathbb{N}$) of a given configuration measure ξ is not smaller than the J-value in W of a kind of convex combination of parts of ξ. This will be used in the proof of Theorem 2.3 when we show that the limit inferior of $J_{W_R}(\Pi_{W_R}(P))$ as $R\to\infty$ is large enough for any $P\in\mathcal{M}_1(\mathcal{L}\times\mathcal{S})$. The third assertion is a technical help here in form of a certain lower-semicontinuity property. The fourth assertion is in a sense dual to the first one, as it solves the question how to construct from a configuration measure ξ in a box W a configuration measure $\xi^{(k)}$ in the box 3^kW (in fact, an extension of ξ) such that its J-values do not significantly increase if k gets large. (This will be crucial in the proof of the upper bound in Theorem 2.5, when we have to make a connection from the variational problem in W_R to the $\mathbb{R}^d$-version of it, $\chi(\rho_1,\rho_2)$.) Here we construct rather comfortable extensions $\xi^{(k)}$ of ξ which lie in the nice set $\widetilde{\mathcal{M}}_1^{(k)}$ of W-shift invariant measures defined in (6.7); in particular they are extensions of each other.

The main body of our arguments relies on LDP-arguments instead of the explicit entropy formula in (6.13). One of the biggest difficulties comes from the fact that extensions of point measures on shreds are much more complicated than extensions of point measures of loops, and also the latter is not at all clear in our setting, since our W-loops have all their particles in W: an extension to a larger box $\widetilde{W}$ should also produce loops that have particles both in W and $\widetilde{W}$. Our LDP-arguments guarantee also that.

For $m\in\mathbb{N}$, denote by $Z_{m,R}$ the set of those $y\in 2R\mathbb{Z}^d$ such that $W_{mR}=mW_R$ is equal to the pairwise disjoint (up to boundaries) union of all the $y+W_R$ with $y\in Z_{m,R}$. Furthermore, consider the map

$$f_{m,R}\colon\mathcal{M}_1(\mathcal{L}_{W_{mR}}\times\mathcal{S}_{W_{mR}})\to\mathcal{M}_1(\mathcal{L}_{W_R}\times\mathcal{S}_{W_R}),\qquad f_m(\xi)=\frac{1}{\#Z_{m,R}}\sum_{z\in Z_{m,R}}\theta_z(\Pi_{W_{mR}\to z+W_R}(\xi)).\tag{6.15}$$

Recall the notion of admissibility from Definition 5.2 and the definitions of $\mathcal{L}^{(M)}$ and $\mathcal{S}^{(M)}$ from (4.30) and (4.32), respectively. Recall the definition of the set $\widetilde{\mathcal{M}}_1^{(k)}$ from (6.7).

Proposition 6.6 (Estimates for J_W)**.** *Fix $M, R \in (1,\infty)$ and any odd $m \in \mathbb{N}$. Recall $\mathfrak{p}_{W_R,M}$ defined in (5.5). The constants C and $C_{M,\mathfrak{r},T}$ stem from Lemma 5.5, while C_M appears first in Lemma 5.3. Then the following hold.*

(1) *For any $\xi \in \mathcal{M}_1(\mathcal{L}^{(M)}_{W_{mR}} \times \mathcal{S}^{(M)}_{W_{mR}})$ such that $\psi = \partial\Pi^{(\mathcal{S})}_{W_{mR}}(\xi)$ is W_{mR}-admissible,*

$$J_{W_R;M}(f_{m,R}(\xi)) \leq J_{W_{mR};M}(\xi) + \frac{C_M}{(mR)^{d/2}} - \frac{1}{|W_{mR}|}\langle \psi, \mathfrak{p}_{W_{mR},M}\rangle, \tag{6.16}$$

and, for any $x \in \mathbb{R}^d$ such that $x + W_R \subset W_{mR}$,

$$J_{W_R;M}(\theta_x(\Pi_{x+W_R}(\xi))) \leq J_{W_{mR};M}(\xi) + \frac{C_M}{(mR)^{d/2}} - \frac{1}{|W_{mR}|}\langle \psi, \mathfrak{p}_{W_{mR},M}\rangle. \tag{6.17}$$

where $J_{W;M}$ is defined in (6.14) with all other parameters K, L, T, ϑ equal to ∞.

(2) *For any $P \in \mathcal{M}^{(\mathrm{s})}_1(\mathcal{L}^{(M)} \times \mathcal{S}^{(M)})$ such that P is W_{mR}-ergodic and $\Pi^{(\mathcal{S})}_{W_R}(P)$ is concentrated on $A^{(\mathcal{S})}_{W_R;M,\mathfrak{r},T} \cap \mathcal{S}^{(\vartheta,S)}_{W_R}$,*

$$J_{W_R;M}(\Pi_{W_R}(P)) \leq J_{W_{mR};M}(\Pi_{W_{mR}}(P)) + \frac{C_M}{(mR)^{d/2}} + \Big[\frac{C_{M,\mathfrak{r},T}}{mR} + \frac{C\vartheta^2}{S}\Big]\langle P, \mathfrak{N}^{(\ell)}_U\rangle. \tag{6.18}$$

(3) *For any $P, P_1, P_2, \dots \in \mathcal{M}^{(\mathrm{s})}_1(\mathcal{L} \times \mathcal{S})$ such that P_n converges weakly to P, we have $J_{W_R}(\Pi_{W_R}(P)) \leq \liminf_{n\to\infty} J_{W_{R_n}}(\Pi_{W_{R_n}}(P_n))$ for any sequence $R_n \downarrow R$.*

(4) *For any $\xi \in \mathcal{M}_1(\mathcal{L}^{(M)}_{W_R} \times \mathcal{S}^{(M)}_{W_R})$ such that $\psi = \partial\Pi^{(\mathcal{S})}_{W_R}(\xi)$ is W_R-admissible, and for any $k \in \mathbb{N}$,*

$$J_{W_R;M}(\xi) \geq \inf\big\{J_{3^kW_R;M}(\xi^{(k)}) \colon \xi^{(k)} \in \widetilde{\mathcal{M}}^{(k)}_1, \Pi_{3^kW_R\to W_R}(\xi^{(k)}) = \xi\big\} - \frac{C_M}{R^{d/2}} + \frac{1}{|W_R|}\langle \psi, \mathfrak{p}_{W_R,M}\rangle. \tag{6.19}$$

Proof. Recall that $\mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M$ denotes the PPP with intensity measure given in (4.31). All asymptotic assertions as $N \to \infty$ in the following are meant under this measure.

(1) Note that $f_{m,R}(\Xi_{N,mR}) = \Xi_{N,R}$ for any large $N \in \mathbb{N}$ that is an odd multiple of mR. Hence, for any δ-neighbourhood $\mathcal{B}_\delta$ of ξ, there is an δ'-neighbourhood $\mathcal{B}'_{\delta'}$ of $f_{m,R}(\xi)$ such that

$$\mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big(\Xi_{N,mR} \in \mathcal{B}_\delta\big) \leq \mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big(\Xi_{N,R} \in \mathcal{B}'_{\delta'}\big). \tag{6.20}$$

We take $\mathcal{B}_\delta$ such that $\widetilde{U} = \partial\Pi^{(\mathcal{S})}_{W_R}(\overline{\mathcal{B}}'_{\delta'}) = \{\partial\Pi^{(\mathcal{S})}_{W_R}(\xi') \in \mathcal{M}_1(\mathcal{T}_W) \colon \xi' \in \overline{\mathcal{B}}'_{\delta'}\}$ is compact. Then, according to Lemma 6.4 (or Corollary 6.5(1)), the right-hand side can be further estimated as follows.

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big(\Xi_{N,R} \in \mathcal{B}'_{\delta'}\big) \leq -\inf_{\psi\in\widetilde{U}}\inf_{\xi'\in\overline{\mathcal{B}}'_{\delta'}} J^{(\psi)}_{W_R;M}(\xi') = -\inf_{\xi'\in\overline{\mathcal{B}}'_{\delta'}} J_{W_R;M}(\xi'),$$

where the last step derives from the fact that $J^{(\psi)}_{W_R;M}(\xi') = \infty$ if $\psi \neq \partial\Pi^{(\mathcal{S})}_{W_R}(\xi')$. Because of the lower semicontinuity of $J_{W_R;M}$, we can estimate $-\inf_{\xi'\in\overline{\mathcal{B}}'_{\delta'}} J_{W_R;M}(\xi') \leq -J_{W_R;M}(f_{m,R}(\xi)) + C_\delta$ for some C_δ that vanishes as $\delta \downarrow 0$; here we use that $\overline{\mathcal{B}}'_{\delta'}$ is an δ'-neighbourhood of $f_{m,R}(\xi)$, and δ' vanishes as $\delta \downarrow 0$.

The left-hand side of (6.20) is lower estimated using a similar strategy to what we will present in Section 6.5. Indeed, we can lower bound

$$\mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big(\Xi_{N,mR} \in \mathcal{B}_\delta\big) \geq \mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big(\Xi_{N,mR} \in \mathcal{B}'_{\delta'} \,\big|\, \partial\Xi^{(\mathcal{S})}_{N,mR} = \psi_N\big)\mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big(\partial\Xi_{N,mR} = \psi_N\big),$$

where $\psi_N \in \mathcal{M}_1(\mathcal{T}_{W_R})$ are picked such that they converge towards ψ as $N \to \infty$. Now Lemma 5.3 and Corollary 6.5(2) give that

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathtt{Q}_M^{(\Lambda_N,\mathrm{par})}\big(\Xi_{N,mR} \in \mathcal{B}_\delta\big) \geq - \inf_{\widetilde{\xi}\in\mathcal{B}'_{\delta'}\colon \partial\Pi^{(\mathcal{S})}_{W_{mR}}(\widetilde{\xi})=\psi} J_{W_{mR};M}(\widetilde{\xi}) - \frac{C_M}{(mR)^{d/2}} + \frac{1}{|W_{mR}|}\langle \psi, \mathfrak{p}_{W_{mR},M}\rangle,$$

recalling that $\psi = \partial\Pi^{(\mathcal{S})}_{W_{mR}}(f_{m,R}(\xi))$ is W_{mR}-admissible. Clearly, the infimum on the right-hand side is not larger than $J_{W_{mR};M}(\xi)$. This yields

$$-J_{W_R;M}(f_{m,R}(\xi)) + C_\delta \geq -J_{W_{mR};M}(\xi) - \frac{C_M}{(mR)^{d/2}} + \frac{1}{|W_{mR}|}\langle \psi, \mathfrak{p}_{W_{mR},M}\rangle. \tag{6.21}$$

Making $\delta \downarrow 0$, this implies (6.16).

For proving (6.17), fix $x \in \mathbb{R}^d$ such that $x + W_R \subset W_{mR}$ and an odd $m \in \mathbb{N} \setminus \{1\}$ and define

$$f^{(x)}\colon \mathcal{M}_1(\mathcal{L}_{W_{mR}} \times \mathcal{S}_{W_{mR}}) \to \mathcal{M}_1(\mathcal{L}_{W_R} \times \mathcal{S}_{W_R}), \qquad f^{(x)}(\xi^{(m)}) = \theta_x\big(\Pi_{x+W_R}(\xi^{(m)})\big). \tag{6.22}$$

Then $f^{(x)}(\Xi_{N,mR}(\omega)) = \Xi^{(x)}_{N,mR}(\omega) = \frac{1}{\# Z_{N,mR}} \sum_{z\in Z_{N,mR}} \delta_{\theta_{z+x}(\Pi_{z+x+W_R}(\omega))}$. We proceed now as above with $f_{m,R}$ replaced by $f^{(x)}$. Indeed, for any δ-neighbourhood $\mathcal{B}_\delta$ of $\Pi_{W_{mR}}(P)$, there is an δ'-neighbourhood $\mathcal{B}'_{\delta'}$ of $f^{(x)}(\Pi_{W_{mR}}(P)) = \theta_x(\Pi_{x+W_R}(P))$ such that

$$\mathtt{Q}_M^{(\Lambda_N,\mathrm{par})}\big(\Xi_{N,mR} \in \mathcal{B}_\delta\big) \leq \mathtt{Q}_M^{(\Lambda_N,\mathrm{par})}\big(\Xi^{(x)}_{N,mR} \in \mathcal{B}'_{\delta'}\big). \tag{6.23}$$

For further estimating the right-hand side, observe that Lemma 6.4 applies to $\Xi^{(x)}_{N,mR}$ when conditioned on $\{\partial\Xi^{(x)}_{N,mR} = \psi_N\}$ for some $\psi_N \to \psi \in \mathcal{M}_1(\mathcal{T}_{W_R})$, where we define $\partial\Xi^{(x)}_{N,mR} = \frac{1}{\# Z_{N,mR}} \sum_{z\in Z_{N,mR}} \delta_{\theta_{z+x}(\partial\Pi^{(\mathcal{S})}_{z+x+W_R}(\omega))}$, since both $\mathtt{Q}_M$ and the kernel $\mathtt{K}_W$ are shift-invariant and can be equally applied in $x + W_R$ as in W_R. Therefore, as above, we arrive at (6.17).

(2) We apply (1) to $\xi = \Pi_{W_{mR}}(P)$ and note that $f_{m,R}(\Pi_{W_{mR}}(P)) = \Pi_{W_R}(P)$ by stationarity of P. Note that $\partial\Pi^{(\mathcal{S})}_{W_{mR}}(P)$ is W_{mR}-admissible by Lemma 5.4. Use Lemma 5.5 for mR instead of R in order to estimate the right-hand side of (6.16).

(3) Fix $R \in (0,\infty)$ and a sequence $(R_n)_{n\in\mathbb{N}}$ that tends to R from above. Using (2.7), we see that, for any bounded measurable function $f\colon \mathcal{L}_{W_{R_n}} \times \mathcal{S}_{W_{R_n}} \to \mathbb{R}$,

$$|W_{R_n}| J_{W_{R_n}}(\Pi_{W_{R_n}}(P)) \geq \langle \Pi_{W_{R_n}}(P), f\rangle - \log\big[\Pi^{(\mathcal{L})}_{W_{R_n}}(\mathtt{Q}) \otimes [\partial\Pi^{(\mathcal{S})}_{W_{R_n}}(P) \otimes \mathtt{K}_{W_{R_n}}\big](\mathrm{e}^f).$$

We use this for a function f that depends only on $\mathcal{L}_{W_R} \times \mathcal{S}_{W_R}$, i.e., for an f of the form $f = g \circ \Pi_{W_R}$ with some bounded measurable $g\colon \mathcal{L}_{W_R} \times \mathcal{S}_{W_R} \to \mathbb{R}$. Using that $R \leq R_n$ and hence $\Pi_{W_R} \circ \Pi_{W_{R_n}} = \Pi_{W_R}$, we see that

$$\langle \Pi_{W_{R_n}}(P), f\rangle = \langle P, f \circ \Pi_{W_{R_n}}\rangle = \langle P, g \circ \Pi_{W_R}\rangle = \langle \Pi_{W_R}(P), g\rangle.$$

On the other hand, for such an f, the integral of e^f with respect to $\Pi^{(\mathcal{L})}_{W_{R_n}}(\mathtt{Q})$ is equal to the integral of e^g with respect to $\Pi^{(\mathcal{L})}_{W_R}(\mathtt{Q})$, and the integral of e^g with respect to $\partial\Pi^{(\mathcal{S})}_{W_n}(P_n) \otimes \mathtt{K}_{W_{R_n}}$ is asymptotic to its integral with respect to $\partial\Pi^{(\mathcal{S})}_{W_R}(P) \otimes \mathtt{K}_{W_R}$ as $n \to \infty$, since the P_n-probability that a W_{R_n}-shred is not a W_R-shred vanishes. Picking g such that the supremum over all bounded measurable functions $\mathcal{L}_{W_R} \times \mathcal{S}_{W_R} \to \mathbb{R}$ is approached in the formula (2.7) for $|W_R| J_{W_R}(\Pi_{W_R}(P))$, we arrive at the assertion.

(4) Recall the empirical process $\Xi^{(k)}_{N,R}$ defined in (6.10) and recall from (6.11) that $\Xi_{N,R}$ and $\Pi_W(\Xi^{(k)}_{N,R})$ have a vanishing total-variation distance as $N \to \infty$, where we abbreviate here and in

the rest of the proof $\Pi_W = \Pi_{3^kW\to W}$ and $W = W_R$. We also note that

$$\Xi^{(k)}_{N,R} = \frac{1}{\#(3^kW\cap\mathbb{Z}^d)}\sum_{y\in 3^kW\cap\mathbb{Z}^d}\Xi^{(k,y)}_{N,R},\qquad \text{where } \Xi^{(k,y)}_{N,R} = \frac{1}{\#Z_{N,3^kR}}\sum_{z\in Z_{N,3^kR}}\delta_{\theta_{y+z}(\Pi_{y+z+3^kW}(\omega))},\tag{6.24}$$

and note, for any $y\in 3^kW\cap\mathbb{Z}^d$, that $(\Xi^{(k,y)}_{N,R})_{N\in\mathbb{N}}$ can be shown to be exponentially equivalent to $(\Xi^{(k,0)}_{N,R})_{N\in\mathbb{N}} = (\Xi_{N,3^kR})_{N\in\mathbb{N}}$, since they are only a fixed shift of each other, which corresponds to replacing Λ_N by $\Lambda_N + y$. In particular, every $(\Xi^{(k,y)}_{N,R})_{N\in\mathbb{N}}$ satisfies, according to (a variant of) Corollary 6.5, the LDP upper bound with rate function $J_{3^kW;M}$.

We introduce the projection $\pi_y\colon \mathcal{M}_1(\mathcal{L}_{3^kW}\times\mathcal{S}_{3^kW})^{3^kW\cap\mathbb{Z}^d}$ on the y-th factor. We first show that $((\Xi^{(k,y)}_{N,R})_y)_{N\in\mathbb{N}}$ satisfies the LDP upper bound with rate function $(\xi_y)_y\mapsto\max_y J_{3^kW}(\xi_y)$. Indeed, for any closed set U in $\mathcal{M}_1(\mathcal{L}_{3^kW}\times\mathcal{S}_{3^kW})^{3^kW\cap\mathbb{Z}^d}$, we can easily estimate, for any y,

$$\mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big((\Xi^{(k,y)}_{N,R})_y\in U\big)\le\mathsf{Q}^{(\Lambda_N,\mathrm{par})}_M\big(\Xi^{(k,y)}_{N,R}\in\pi_y(U)\big)\le\exp\Big\{-(1+o(1))|\Lambda_N|\inf_{\pi_y(U)}J_{3^kW,M}\Big\}.$$

We abbreviate $G((\xi_y)_{y\in 3^kW\cap\mathbb{Z}^d}) = \frac{1}{\#(3^kW\cap\mathbb{Z}^d)}\sum_y\xi_y$, then $\Xi^{(k)}_{N,R} = G((\Xi^{(k,y)}_{N,R})_y)$ satisfies, according to the (upper-bound half of the) contraction principle from the theory of large deviations (see [DZ10, Section 4.2.1]) the upper-bound half of the LDP with rate function $\mathcal{M}_1(\mathcal{L}_{3^kW}\times\mathcal{S}_{3^kW})\ni\xi^{(k)}\mapsto\inf\{\max_y J_{3^kW,M}\circ\pi_y((\xi_y)_y)\colon G((\xi_y)_y)=\xi^{(k)}\}$. Since $\Xi^{(k)}_{N,R}$ has a vanishing distance to the set $\widetilde{\mathcal{M}}^{(k)}_1$, the rate function is $=\infty$ in points $\xi^{(k)}\in(\widetilde{\mathcal{M}}^{(k)}_1)^{\mathrm{c}}$. Note that this rate function is not smaller than $J_{3^kW,M}(\xi^{(k)})$, since, by convexity, $\max_y J_{3^kW,M}\circ\pi_y((\xi_{y'})_{y'})\ge G((J_{3^kW,M}(\xi_y)_y)\ge J_{3^kW,M}(G((\xi_y))_y) = J_{3^kW,M}(\xi^{(k)})$. Hence, $\Xi^{(k)}_{N,R}$ satisfies the upper-bound half of the LDP with rate function $\xi^{(k)}\mapsto J_{3^kW,M}(\xi^{(k)})$ if $\xi^{(k)}\in\widetilde{\mathcal{M}}^{(k)}_1$ and $\mapsto\infty$ otherwise. Now recall that $\Xi_{N,R}$ (and therefore $\Pi_W(\Xi^{(k)}_{N,R})$, since they have vanishing total-variation distance) satisfies the LDP lower bound with rate function $\xi\mapsto J_{W,M}(\xi)+\frac{C_M}{R^{d/2}}-\frac{1}{|W|}\langle\psi,\mathfrak{p}_{W,M}\rangle$ (for a proof, adapt the arguments of the proof of (1)). Again using the contraction principle for the function Π_W, we see that (6.19) is satisfied. □

6.3. **Compactnesses.** In Section 6.4 we will apply the LDP of Corollary 6.5(1) to the right-hand side of (4.25). For this, it is necessary to generate compactness in the integration area beforehand. Furthermore, we need to introduce a condition that makes the two particle densities that appear in (4.25) continuous functions of $\Xi_{N,R}$, since it is crucial for our purposes to keep control on the condensate density and the total density of particles. The purpose of this section is to introduce this condition and to show how it can be incorporated in (4.25).

Part of our solution to this problem is to prove the compactness of the sub-level sets of J_W, which will be also used in the proof of Theorem 2.3. But we will also need to explicitly control the expected particle number of the boundary-shred configuration $\partial\Pi^{(\mathcal{S})}_W(\xi)$, since this is not controlled in the sub-level sets.

More explicitly, we will apply the (upper bound of the) LDP on the event $\{\Xi^{(\mathcal{L})}_{N,R}\in K^{(\mathcal{L})}\}\cap\{\partial\Xi^{(\mathcal{S})}_{N,R}\in K^{(\mathcal{S})}\}$, where $K^{(\mathcal{L})}$ and $K^{(\mathcal{S})}$ are sets that need to satisfy three crucial properties:

(1) they are closed, and $K^{(\mathcal{S})}$ induces compactness for the $\mathcal{M}_1(\mathcal{T}_W)$-part on the sub-level sets of J_W,
(2) the probability of $\Xi^{(\mathcal{L})}_{N,R}$, respectively $\partial\Xi^{(\mathcal{S})}_{N,R}$, being in their complement is extremely small on the exponential scale,
(3) on these sets, the number of particles is a continuous function of $\Xi^{(\mathcal{L})}_{N,R}$ respectively $\partial\Xi^{(\mathcal{S})}_{N,R}$.

The problem with (3) is that, *a priori*, the maps $\xi\mapsto\langle\xi,\mathfrak{N}^{(\ell,\mathcal{L})}_W\rangle$, respectively $\psi\mapsto\langle\psi,\mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W}\rangle$, are not continuous in the weak topology since neither $\mathfrak{N}^{(\ell,\mathcal{L})}_W$ nor $\mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W}$ are bounded. Therefore, we make use of Assumption (V), in particular (2.23), which gives an efficient bound on having an extraordinarily high number of particles in the subboxes. Hence, (2) will be handled in terms of the transformed probability measure $\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{bc})}$ defined in (1.13).

6.3.1. *Loop part.* Following Remark 2.11, we decided to work (instead with the number $\mathfrak{N}_{W_R}^{(\ell,\mathcal{L})}$ of particles in all loops in W_R; see (2.15)) with the number $\widetilde{\mathfrak{N}}_{y+U}^{(\ell,\mathcal{L})}(\omega)$ of particles in $y+U$ (recall that $U=[-\frac{1}{2},\frac{1}{2}]^d=W_{1/2}$) coming from any of the loops of $\omega\in\mathcal{L}_W$ starting in W. Note that $\sum_{y\in Z_{R,1/2}}\widetilde{\mathfrak{N}}_{y+U}^{(\ell,\mathcal{L})}(\omega)=\widetilde{\mathfrak{N}}_{W}^{(\ell,\mathcal{L})}(\omega)=\mathfrak{N}_{W}^{(\ell,\mathcal{L})}(\omega)$, where $Z_{R,1/2}$ (in a small abuse of notation) is the set of those $y\in\frac{1}{2}\mathbb{Z}^d$ such that $y+U\subset W_R$ (which implies $|Z_{R,1/2}|\leq|W_R|$). Then we consider the set

$$K_R^{(\mathcal{L})}(L)=\Big\{\xi\in\mathcal{M}_1(\mathcal{L}_{W_R})\colon\sum_{y\in Z_{R,1/2}}\big\langle\xi,\big(\widetilde{\mathfrak{N}}_{y+U}^{(\ell,\mathcal{L})}\big)^{5/4}\big\rangle\leq L|W_R|\Big\},\qquad L\in(0,\infty). \tag{6.25}$$

In words, under any $\xi\in K_R^{(\mathcal{L})}(L)$, the number of particles in any shift of U in W_R is not extremely unevenly distributed, uniformly over the subbox and over ξ.

The set $K_R^{(\mathcal{L})}(L)$ turns out to possess the mentioned three properties:

Lemma 6.7 (Restriction of the loop-part)**.** *Suppose that Assumption (V) holds, then the following assertions hold.*

(1) *For any $L\in(0,\infty)$ and for any $R\in\mathbb{N}$, the set $K_R^{(\mathcal{L})}(L)$ is closed in the weak topology.*
(2)
$$\lim_{L\to\infty}\sup_{R\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{|\Lambda_N|}\log\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{par})}\Big(\Xi_{N,R}^{(\mathcal{L})}\in\big(K_R^{(\mathcal{L})}(L)\big)^{\mathrm{c}}\Big)=-\infty. \tag{6.26}$$
(3) *For any $L\in(0,\infty)$ and for any $R\in\mathbb{N}$, the map $\xi\mapsto\langle\xi,\widetilde{\mathfrak{N}}_W^{(\ell,\mathcal{L})}\rangle$ is continuous on the set $K_R^{(\mathcal{L})}(L)$.*

Proof. We drop the super-indices '$\mathcal{L}$', and we abbreviate $W=W_R$.

(1) Since the map $\xi\mapsto\langle\xi,\widetilde{\mathfrak{N}}_U^{(\ell)}\mathbb{1}\{\widetilde{\mathfrak{N}}_U^{(\ell)}>L_k\}\rangle$ is lower semi-continuous for any $k\in\mathbb{N}$, $K_R(L)$ is closed.

(2) Now we show (6.26). We adapt and extend the proof of Lemma 3.2 and will also employ Lemma 3.1 (which uses (2.23) from Assumption (V)). As always, we assume that Λ_N is equal to the (up to boundaries) disjoint union of the boxes $W_z=z+W$ with $z\in Z_{N,R}$ (and R may depend on N). We use that $\#Z_{N,R}\sim|\Lambda_N|/|W|$ as $N\to\infty$.

First note that, for any $y\in Z_{R,1/2}$, in the limit as $N\to\infty$,

$$\big\langle\Xi_{N,R}(\omega_{\mathrm{P}}),\big(\widetilde{\mathfrak{N}}_{y+U}^{(\ell)}\big)^{5/4}\big\rangle\sim\frac{|W|}{|\Lambda_N|}\sum_{z\in Z_{N,R}}\big(\widetilde{\mathfrak{N}}_{z+y+U}^{(\ell)}(\omega_{\mathrm{P}})\big)^{5/4}.$$

Hence, for any $L\in(0,\infty)$, recalling the notation in (4.26),

$$\begin{aligned}\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{par})}\big(\Xi_{N,R}\in(K_R(L))^{\mathrm{c}}\big)&\leq\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{par})}\Big(\sum_{z\in Z_{N,R}}\sum_{y\in Z_{R,1/2}}\big(\widetilde{\mathfrak{N}}_{z+y+U}^{(\ell)}(\omega_{W_z})\big)^{5/4}>L|\Lambda_N|\Big)\\&\leq\mathrm{e}^{-L|\Lambda_N|}\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{par})}\Big(\mathrm{e}^{\sum_{z\in Z_{N,R}}\sum_{y\in Z_{R,1/2}}\left(\widetilde{\mathfrak{N}}_{z+y+U}^{(\ell)}(\omega_{W_z})\right)^{5/4}}\Big)\\&\leq\frac{\mathrm{e}^{-L|\Lambda_N|}}{\widehat{Z}_N^{(\mathrm{bc})}(\Lambda_N)}\prod_{z\in Z_{N,R}}\mathsf{Q}^{(\Lambda_N,\mathrm{par})}\Big(\mathrm{e}^{\sum_{y\in Z_{R,1/2}}\left(\widetilde{\mathfrak{N}}_{z+y+U}^{(\ell)}(\omega_{W_z})\right)^{5/4}}\mathrm{e}^{-\sum_{y\in Z_{R,1/2}}\Phi_{z+y+U,z+y+U}(\omega_{W_z})}\Big),\end{aligned} \tag{6.27}$$

where we used that, under particle boundary condition, the configurations in the subboxes W_z are i.i.d. over z, and we estimated the total interaction $\Phi_{\Lambda_n,\Lambda_N}$ in Λ_N from below against the sum of the interactions within the $z+y+U$ (i.e., the interactions between any pair of different legs that start in $z+y+U$).

We apply Lemma 3.1 for any z and y to the expectation of $\mathrm{e}^{-\Phi_{z+y+U,z+y+U}}$ over all the legs starting in $z+y+U$, conditioned on their starting sites, which are the particles. Hence, there is some $C\in(0,\infty)$ such that this conditional expectation is upper bounded by $\mathrm{e}^{-C(\widetilde{\mathfrak{N}}^{(\ell)}_{z+y+U})^{3/2}\mathbb{1}\{\widetilde{\mathfrak{N}}^{(\ell)}_{z+y+U}\geq C\}}$. Abbreviate $f(m)=m^{5/4}-Cm^{3/2}\mathbb{1}\{m\geq C\}$. Hence, when carrying out the expectation over all the (independent!) legs in the configuration in each subbox $z+y+U$, we obtain from (6.27) the following upper bound:

$$\widehat{\mathsf{Q}}^{(\Lambda_N,\mathrm{par})}\big(\Xi_{N,R}\in(K_R(L))^{\mathrm{c}}\big)\leq\frac{\mathrm{e}^{-L|\Lambda_N|}}{\widehat{Z}_N^{(\mathrm{bc})}(\Lambda_N)}\prod_{z\in Z_{N,R}}\mathsf{Q}^{(\Lambda_N,\mathrm{par})}\Big[\mathrm{e}^{\sum_{y\in Z_{R,1/2}}f(\widetilde{\mathfrak{N}}^{(\ell)}_{z+y+U}(\omega_{W_z}))}\Big]. \tag{6.28}$$

Since f is bounded from above (here is the point where we need the decomposition into the sum over y of the 5/4-moments of the particle numbers in the unit boxes in the definition of $K_R^{(\mathcal{L})}(L)$ instead of directly bounding only the 5/4-moment of the particle number in W), and since the number of summands (z,y) does not depend on R and is $\sim|\Lambda_N|$, and noting that the normalization constant $\widehat{Z}_N^{(\mathrm{bc})}(\Lambda_N)$ has a positive exponential rate, this shows that (6.26) holds.

(3) First note that the map $\xi\mapsto\langle\xi,\widetilde{\mathfrak{N}}^{(\ell)}_W\rangle$ is lower semicontinuous on $K_R^{(\mathcal{L})}(L)$, since $\widetilde{\mathfrak{N}}^{(\ell)}_W$ can be monotonously approached from below with nonnegative bounded continuous functions. Furthermore, the upper semicontinuity can be easily be derived, since on $K_R^{(\mathcal{L})}(L)$ there is a uniform upper bound for a higher moment of $\widetilde{\mathfrak{N}}^{(\ell)}_W$. More precisely, for any sequence $(\xi_n)_n$ in $K_R^{(\mathcal{L})}(L)$, tending weakly towards some $\xi\in K_R(L)$, we use that $\widetilde{\mathfrak{N}}^{(\ell)}_W=\sum_{y\in Z_{R,1/2}}\widetilde{\mathfrak{N}}^{(\ell)}_{y+U}$ and estimate, for any $n\in\mathbb{N}$ and any $T\in(0,\infty)$,

$$\begin{aligned}\big|\langle\xi_n-\xi,\widetilde{\mathfrak{N}}^{(\ell)}_W\rangle\big|&\leq\sum_{y\in Z_{R,1/2}}\Big(\big|\langle\xi_n-\xi,\widetilde{\mathfrak{N}}^{(\ell)}_{y+U}\mathbb{1}\{\widetilde{\mathfrak{N}}^{(\ell)}_{y+U}\leq T\}\rangle\big|+T^{-1/4}\langle\xi_n+\xi,(\widetilde{\mathfrak{N}}^{(\ell)}_{y+U})^{5/4}\rangle\Big)\\&\leq\sum_{y\in Z_{R,1/2}}\big|\langle\xi_n-\xi,\widetilde{\mathfrak{N}}^{(\ell)}_{y+U}\mathbb{1}\{\widetilde{\mathfrak{N}}^{(\ell)}_{y+U}\leq T\}\rangle\big|+T^{-1/4}L|W|.\end{aligned}$$

The latter term can be made arbitrarily small by picking T large, and then the former sum can be made arbitrarily small by picking n large. □

6.3.2. *Compactness for shred part.* Analogously to the loop part, we present now a strategy to compactify the boundary-shred-part in the variational formula in (4.25) such that the number of particles in the shreds, i.e., the map $\psi\mapsto\langle\psi,\mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W_R}\rangle$, becomes continuous, where we recall from (4.16) that $\mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W_R}(\mu)=\sum_{i\in I}l_i$ denotes the number of particles in the configuration $\mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}$. We also denote by $\mathfrak{N}^{(\ell,\mathcal{S})}_{y+U}(\varpi)$ the number of particles in $y+U$ from all the shreds in $\varpi\in\mathcal{S}_W$ (compare to (2.15)).

Then we consider the set

$$K_R^{(\mathcal{S})}(L)=\Big\{\xi\in\mathcal{M}_1(\mathcal{S}_{W_R})\colon\sum_{y\in Z_{R,1/2}}\big\langle\xi,\big(\mathfrak{N}^{(\ell,\mathcal{S})}_{y+U}\big)^{5/4}\big\rangle\leq L|W_R|\Big\},\qquad L\in(0,\infty). \tag{6.29}$$

The set $K_R^{(\mathcal{S})}$ turns out to possess the mentioned three properties:

Lemma 6.8 (Compactification of the shred-part). *Suppose that Assumption (V) holds. Then the following holds.*

(1) *For any $R,L\in\mathbb{N}$, the set $K_R^{(\mathcal{S})}(L)$ is closed in the weak topology. Furthermore, the set*

$$\big\{\xi\in\mathcal{M}_1(\mathcal{L}_{W_R}\times\mathcal{S}_{W_R})\colon\Pi^{(\mathcal{S})}_{W_R}(\xi)\in K_R^{(\mathcal{S})}(L),\langle\partial\Pi^{(\mathcal{S})}_{W_R}(\xi),\mathfrak{N}^{(\mathcal{S})}_{\partial W_R}\rangle=\rho_2,J_{W_R}(\xi)\leq C\big\} \tag{6.30}$$

is compact for any $\rho_2,C\in(0,\infty)$.

(2)

$$\lim_{L\to\infty}\sup_{R\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{|\Lambda_N|}\log\widehat{\mathtt{Q}}^{(\Lambda_N,\mathrm{par})}\Big(\Xi^{(\mathcal{S})}_{N,R}\in\big(K^{(\mathcal{S})}_R(L)\big)^{\mathrm{c}}\Big)=-\infty. \tag{6.31}$$

(3) *For any $R,L\in\mathbb{N}$, the map $\psi\mapsto\int\psi(\mathrm{d}\mu)\sum_{i\in I}l_i$ is continuous on $K^{(\mathcal{S})}_R(L)$.*

Proof. Much of this proof is analogous to the one of Lemma 6.7, and we leave the details to the reader. Indeed, the closedness of $K^{(\mathcal{S})}_R(L)$ is analogous, the proof of (2) and the one of (3) as well. However, the compactness of the set in (6.30) needs to be shown. Here we can restrict to showing the relative compactness of the set of ξ such that $J_{W_R}(\xi)\le C$ and $\langle\partial\Pi^{(\mathcal{S})}_{W_R}(\xi),\mathfrak{N}^{(\mathcal{S})}_{\partial W_R}\rangle\le\rho_2$, since the continuity of the map $\xi\mapsto\langle\partial\Pi^{(\mathcal{S})}_{W_R}(\xi),\mathfrak{N}^{(\mathcal{S})}_{\partial W_R}\rangle$ follows from assertion (3).

First recall from the end of Section 4.2 that the set K_{ρ_2} of all $\psi\in\mathcal{M}_1(\mathcal{T}_{W_R})$ such that $\langle\psi,\mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W_R}\rangle\le\rho_2$ is relatively compact. Hence, we only need to show that its intersection with the sub-level set of J_{W_R} (more precisely, the set of ξ such that $\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)$ is in that set and $J_{W_R}(\xi)\le C$) is relative compact in $\mathcal{M}_1(\mathcal{L}_{W_R}\times\mathcal{S}_{W_R})$.

Pick a sequence $(\xi_n)_{n\in\mathbb{N}}$ in that set. Using the standard Lemma 8.2 below we see that its loop-part, i.e., the sequence $(H_{\mathcal{L}_W}(\Pi^{(\mathcal{L})}_{W_R}(\xi_n)\mid\Pi^{(\mathcal{L})}_{W_R}(\mathtt{Q})))_{n\in\mathbb{N}}$, is bounded. Therefore, as it is proved in the large-deviations literature as part of the proof of Sanov's theorem, $(\Pi^{(\mathcal{L})}_{W_R}(\xi_n))_{n\in\mathbb{N}}$ is tight. Therefore, we only need to show that the shred-part $(\Pi^{(\mathcal{S})}_{W_R}(\xi_n))_{n\in\mathbb{N}}$ is also tight. Then $(\Pi_{W_R}(\xi_n))_{n\in\mathbb{N}}$ is also tight, since, for any compact sets $K^{(\mathcal{L})}\subset\mathcal{L}_W$ and $K^{(\mathcal{S})}\subset\mathcal{S}_W$, also $K^{(\mathcal{L})}\times K^{(\mathcal{S})}$ is compact, and its complement is contained in $[(K^{(\mathcal{L})})^{\mathrm{c}}\times\mathcal{S}_W]\cup[\mathcal{L}_W\times(K^{(\mathcal{S})})^{\mathrm{c}}]$.

Let us show tightness of the sequence $(\Pi^{(\mathcal{S})}_{W_R}(\xi_n))_{n\in\mathbb{N}}$. Again from (2.16) and Lemma 8.2 we see that it lies in the set

$$A_R=\big\{\xi\in\mathcal{M}_1(\mathcal{S}_{W_R})\colon\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)\in K_{\rho_2},H_{\mathcal{S}_{W_R}}(\xi\mid\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)\otimes\mathtt{K}_{W_R})\le C_R\big\}, \tag{6.32}$$

for some $C_R\in\mathbb{R}$.

We show now that this set is compact. The following is an adaptation of the standard proof of compactness of level sets of entropies, as it can be found in the literature on Sanov's theorem (see [DZ10, Section 6.2]). Indeed, let a small $\delta\in(0,1)$ be given. Since K_{ρ_2} is compact, there is a compact set $L_1\subset\mathcal{T}_{W_R}$ such that $\psi(L_1^{\mathrm{c}})\le\delta$ for any $\psi\in K_R$. Note that there is an $\alpha\in\mathbb{N}$ such that, for any $\mu=\sum_i\delta_{(x_i,y_i,l_i)}\in L_1$, we have $|y_i|\le\alpha$, $l_i\le\alpha$ and $|\mu|\le\alpha$. Additionally, let a small $\varepsilon\in(0,1)$ be given. Then, for any $\mu\in\mathcal{T}_{W_R}$, there is a compact set $L_2(\mu)\subset\mathcal{S}_{W_R}$ such that $\mathtt{K}_{W_R}(\mu,L_2(\mu)^{\mathrm{c}})\le\varepsilon$. Then the set $L=\bigcup_{\mu\in L_1}\{\mu\}\times L_2(\mu)$ is compact.

We denote a regular version of the conditional distribution of ξ given $\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)$ by $\xi(\mu,\cdot)$ (in earlier notation $\xi_{\mathcal{T}_{W_R}\to\mathcal{S}_{W_R}}(\mu,\cdot)$). Using the characterisation of the entropy from (2.7), we see by taking $f=\mathbb{1}_{L_2(\mu)^{\mathrm{c}}}\log M$ for any large M that, for any $\xi\in\mathcal{M}_1(\mathcal{S}_{W_R})$,

$$H_{\mathcal{S}_{W_R}}(\xi(\mu,\cdot)\mid\mathtt{K}_{W_R}(\mu,\cdot))\ge\xi(\mu,L_2(\mu)^{\mathrm{c}})\log M-\log\big(1-\mathtt{K}_{W_R}(\mu,L_2(\mu)^{\mathrm{c}})+M\mathtt{K}_{W_R}(\mu,L_2(\mu)^{\mathrm{c}})\big),$$

and hence

$$\xi(\mu,L_2(\mu)^{\mathrm{c}})\le\frac{1}{\log M}\Big[H_{\mathcal{S}_{W_R}}(\xi(\mu,\cdot)\mid\mathtt{K}_{W_R}(\mu,\cdot))+\log(1+(M-1)\varepsilon)\Big].$$

Now we assume that $\xi\in A_R$ and integrate over $\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)(\mathrm{d}\mu)$ to obtain

$$\begin{aligned}\xi\Big(\bigcup_{\mu\in\mathcal{T}_{W_R}}\{\mu\}\times L_2(\mu)^{\mathrm{c}}\Big)&=\int\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)(\mathrm{d}\mu)\,\xi(\mu,L_2(\mu)^{\mathrm{c}})\\&\le\frac{1}{\log M}H_{\mathcal{S}_{W_R}}(\xi\mid\partial\Pi^{(\mathcal{S})}_{W_R}(\xi)\otimes\mathtt{K}_{W_R})+\frac{\log(1+(M-1)\varepsilon)}{\log M}\\&\le\frac{c|W_R|+\log(1+(M-1)\varepsilon)}{\log M}.\end{aligned}$$

Now picking $M = 1 + \frac{1}{\varepsilon}$ and making ε small, we can press the right-hand side under the given small number $\delta \in (0,1)$. Now we want to argue that $\xi(L^{\mathrm{c}}) \leq 2\delta$ for any $\xi \in A_R$.

Indeed, split L^{c} into the union of $\{\mu\} \times L_2(\mu)^{\mathrm{c}}$ with $\mu \in L_1$ and the union of $\{\mu\} \times L_2(\mu)$ with $\mu \in L_1^{\mathrm{c}}$ to see that the ξ-measure of the complement of the former is below δ by the choice of L_1 and the one of the latter is below δ by the preceding argument. This finishes the proof of tightness of A_R and therefore the proof of the entire lemma. $\square$

6.4. **Upper large deviations bound.** In this section we derive an upper bound for the large-N exponential rate of $\widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N, \rho_1, \rho_2)$ defined in (4.24), on the base of the upper bound of Lemma 4.2 in (4.25). Recall that we fixed $\rho_1, \rho_2 \in (0,\infty)$ with $\rho = \rho_1 + \rho_2$. Furthermore, as always, we fix $R \in \mathbb{N}$ and decompose the box Λ_N into the subboxes $z + W$ with $z \in Z_{N,R}$, where R might depend on N and converges then to R. Furthermore, we fix a small $\delta \in (0,1)$.

We want to apply Corollary 6.5(1) to the right-hand side of (4.25). For this, we insert the indicator on $\{\Xi^{(\mathcal{L})}_{N,R} \in K^{(\mathcal{L})}_R(L)\} \cap \{\Xi^{(\mathcal{S})}_{N,R} \in K^{(\mathcal{S})}_R(L)\}$, plus the one on its complement, and we note that the second part (the one with the indicator on the complement) is negligible on the exponential scale in the limit $N \to \infty$ by Lemmas 6.7 and 6.8, if L is taken large enough (not depending on R, according to assertion (2) there). On the first event, the map $\xi \mapsto \langle \xi, \frac{1}{|W|} F_{W,W} \rangle$ is lower semicontinuous, and the two maps $\xi \mapsto \langle \xi, \mathfrak{N}^{(\ell,\mathcal{L})}_W \rangle$ and $\psi \mapsto \langle \psi, \mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W} \rangle$ are continuous; hence the two indicators on the right-hand side of (4.25) are on closed sets, and the $\mathcal{M}_1(\mathcal{T}_W)$-part is even continuous. Applying Corollary 6.5(1), we arrive at

$$
\begin{aligned}
\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} & \log \widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N, \rho_1, \rho_2) \\
& \leq - \inf \Big\{ \frac{1}{|W|} \langle \xi, F_{W,W} \rangle + J_W(\xi) \colon \xi \in \bigcap_{k \in \mathbb{N}} \mathcal{M}_1^{(k)}(\mathcal{L}_W \times \mathcal{S}_W), \\
& \qquad \Pi^{(\mathcal{L})}_W(\xi) \in K^{(\mathcal{L})}_R(L), \Pi^{(\mathcal{S})}_W(\xi) \in K^{(\mathcal{S})}_R(L), \\
& \qquad \langle \xi, \tfrac{1}{|W|} \mathfrak{N}^{(\ell,\mathcal{L})}_W \rangle \in \overline{\mathcal{B}}_{2\delta}(\rho_1), \langle \xi, \tfrac{1}{|W|} \mathfrak{N}^{(\ell,\mathcal{S})}_W \rangle \in \overline{\mathcal{B}}_{2\delta}(\rho_2) \Big\}.
\end{aligned}
\tag{6.33}
$$

In Section 8 we show how to handle the limit as $R \to \infty$ for the crucial entropy term $J_W(\xi)$ if $\xi = \Pi_W(P)$ for some $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L} \times \mathcal{S})$. The finish of the proof of the upper bound in Theorem 2.5 will be carried out in Section 9.1.

6.5. **Lower large deviations bound.** Now we apply the large deviations assertions of Corollary 6.5(2) to our preliminary lower bound in Lemma 4.3 in (4.39) to obtain a lower bound for the limiting free energy. We keep $R \in \mathbb{N}$ fixed and put $W = [-R,R]^d$. Furthermore, we fix also all the auxiliary parameters $\mathfrak{r}, M, K, L, T \in (0,\infty)$ and recall the events in (4.34), (4.35) and (4.36). We also recall that Λ'_N is the 'interior' of Λ_N, i.e., the union of the W_z with $z \in Z^{\circ}_{N,R}$. Recall also other notation from Section 4.6, in particular that we write $\|f\|_{\mathrm{sp}} = \max_{t \in [0,\beta]} |f(t) - f(0)|$ for the spread of a leg $f \in \mathcal{C}_1$. We pick any admissible $\psi^* \in \mathcal{M}_1(\mathcal{T}_W)$ (see Definition 5.2). We assume that $\langle \psi^*, \frac{1}{|W|} \mathfrak{N}^{(\ell,\mathcal{S})}_{\partial W} \rangle \in \mathcal{B}_{\delta/3}(\rho_2)$. For some small $\delta \in (0,1)$, we insert the indicator on the event $\{\partial \Xi^{(\circ,\omega_{\mathrm{P}},\mathcal{S})}_{N,R} \in \mathcal{B}_\delta(\psi^*)\}$ into the expectation on the right-hand side of (4.39), where $\mathcal{B}_\delta(\psi^*)$ is a δ-neighbourhood of ψ^*. Recall the kernel $\mathtt{K}_W$ from (2.5). We are going to use $C \in (0,\infty)$ as a generic constant that depends only on d, β, ρ_1, ρ_2 or v and may change its value from appearance to appearance.

In view of an application of Corollary 6.5(2) we switched already in (4.39) in Section 4.6 to the integration with respect the conditional version given an event, namely $\bigcap_{z \in Z^{\circ}_{N,R}} \{(\omega_{W_z}, \varpi_{W_z}) \in A_{W_z;M,L,K,T}\}$, on which the particle numbers in loops and in shreds, respectively, and the interaction are continuous functionals of $\Xi_{N,R}$. This is necessary in order that the event

$$
B = \Big\{ \xi \in \mathcal{M}_1(A_{W;\mathfrak{r},M,K,L,T}) \colon \langle \xi, \tfrac{1}{|W|} \mathfrak{N}^{(\ell,\mathcal{L})}_W \rangle \in \mathcal{B}_{\delta/3}(\rho_1) \text{ and } \langle \xi, \tfrac{1}{|W|} \mathfrak{N}^{(\ell,\mathcal{S})}_W \rangle \in \mathcal{B}_{\delta/3}(\rho_2) \Big\}
$$

is open in $\mathcal{M}_1(A_{W;\mathfrak{r},M,K,L,T})$ and the map $\xi \mapsto \langle F_{W,W}, \xi\rangle$ is continuous. Indeed the functionals $\mathfrak{N}_W^{(\ell,\mathcal{L})}$, $\mathfrak{N}_W^{(\ell,\mathcal{S})}$ and $F_{W,W}$ need to be bounded and continuous on our restriction, since we are working in the weak topology. Note that their continuity is not a problem, since (see Section 4.1) convergence of loop-shred configurations is defined leg-wise, and V defined in (1.10) is continuous. Indeed, on $A_{W;\mathfrak{r},M,L,K,T}$, the three functions $\mathfrak{N}_W^{(\ell,\mathcal{L})}$, $\mathfrak{N}_W^{(\ell,\mathcal{S})}$ and $F_{W,W}$ are bounded. This is clear for the first two, and for $F_{W,W}$ this follows from Lemma 4.4 with $z = z'$. Hence, we are in the setting of our crucial LDP-tool, Corollary 6.5(2).

Now taking the limit $N \to \infty$ in (4.39) gives, according to Lemma 5.3 and Corollary 6.5(2),

$$
\begin{aligned}
\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N, \rho_1, \rho_2) \geq & -\frac{C_{M,L,K,T}}{R} - \frac{C_M}{R^{d/2}} + \langle \psi^*, \mathfrak{p}_{W_R,M}\rangle \\
& - \inf\Big\{ \frac{1}{|W|}\langle \xi, F_{W,W}\rangle + J_{W;\mathfrak{r},M,L,K,T}(\xi) - \langle \log p_{W;\mathfrak{r},M,L,K,T}, \psi^*\rangle \colon \\
& \qquad \xi \in \mathcal{M}_1(A_{W;\mathfrak{r},M,L,K,T}), \partial\Pi_W^{(\mathcal{S})}(\xi) = \psi^*, \langle \xi, \tfrac{1}{|W|}\mathfrak{N}_W^{(\ell,\mathcal{L})}\rangle \in \mathcal{B}_{\delta/3}(\rho_1)\Big\},
\end{aligned}
\tag{6.34}
$$

where $J_{W;\mathfrak{r},M,L,K,T}$ is defined in (6.14) with $\vartheta = \infty$. Note that, for $\xi \in \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$ that is concentrated on $A_{W;\mathfrak{r},M,L,K,T}$ with $\partial\Pi_W^{(\mathcal{S})}(\xi) = \psi^*$, we have

$$
\begin{aligned}
J_{W;\mathfrak{r},M,L,K,T}(\xi) - \langle \log p_{W;\mathfrak{r},M,L,K,T}, \psi^*\rangle &= \int_{A_{W;\mathfrak{r},M,L,K,T}} \xi(\mathrm{d}(\omega,\varpi)) \log \frac{\mathrm{d}\xi}{\mathrm{d}[\Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\psi^* \otimes \mathtt{K}_W]]}(\omega,\varpi) \\
&= J_W(\xi),
\end{aligned}
$$

since the integral on $A^{\mathrm{c}}_{W;\mathfrak{r},M,L,K,T}$ does not contribute. We take ψ^* of this form.

In order to apply Lemma 5.5 to the third term on the right-hand side of (6.34), we additionally require that ξ is concentrated on $\mathcal{L}_W \times \mathcal{S}_W^{(M,\vartheta,S)}$ where $\mathcal{S}^{(M,\vartheta,S)}$ is defined in (5.21). We further confine the infimum to those ξ's of the form $\xi = \Pi_W(P)$ with an ergodic measure $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$. Then $\psi^* = \partial\Pi_W^{(\mathcal{S})}(P)$ is admissible according to Lemma 5.4. This implies that, for any $R, M, L, K, T, \vartheta, S, \delta, \delta' \in (0,\infty)$,

$$
\begin{aligned}
\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N, \rho_1, \rho_2) \geq & -\frac{C_{M,L,K,T}}{R} - \frac{C_M}{R^{d/2}} - \Big[\frac{C_{M,\vartheta,S}}{R} + \frac{C\vartheta^2}{S}\Big](\rho_2 + \delta') \\
& - \inf\Big\{ \frac{1}{|W|}\langle \Pi_W(P), F_{W,W}\rangle + J_W(\Pi_W(P)) \colon P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S}) \text{ ergodic}, \\
& \quad \Pi_W(P)(A_{W;\mathfrak{r},M,L,K,T}) = 1 = \Pi_W^{(\mathcal{S})}(P)(\mathcal{S}_W^{(M,\vartheta,S)}), \langle P, \mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle \in \mathcal{B}_{\delta/3}(\rho_1), \langle P, \mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle \in \mathcal{B}_{\delta'}(\rho_2)\Big\}.
\end{aligned}
\tag{6.35}
$$

In Lemma 7.6 we will show how to get rid of the two restrictions of P being ergodic and $\Pi_W(P)$ being concentrated on $A_{W;\mathfrak{r},M,L,K,T}$. In Section 9.2, we put this all together, make $R \to \infty$ and afterwards $M, T \to \infty$ and then $L, K \to \infty$ and finish the proof of the lower bound in Theorem 2.5.

7. Analysis of the variational formula

This section is devoted to proofs of a number of results that evolve around the variational formula $\chi(\rho_1, \rho_2)$ in (2.24). In Section 7.1 we give the proof of Lemma 2.12 from Section 2.4, the regularity properties of $\chi(\rho_1, \rho_2)$. The other sections give a number of technical preparations for some crucial points in the proofs of Theorems 2.3 and 2.5. That is, in Section 7.2 we explicitly decompose the entropies for loop point processes and for interlacement point processes according to the various parts of these point processes. In Section 7.3 we use this to prove Lemma 2.13, i.e., that every loop/interlacement configuration can be approximated with a pure loop-configuration. In Section 7.4 we show that entropies of loop-shred configuration measures that are concentrated on uniformly bounded configurations approximate the entropy of arbitrary loop-shred configurations if the bound diverges. Using this and a similar technique as in Section 7.3, in Section 7.5 we carry out an ergodic approximation in our main variational formula, i.e., we prove that the minimizers can be approximated by ergodic loop/interlacements configurations that are additionally uniformly bounded in various senses.

7.1. **Proof of Lemma 2.12.** In this section, we prove Lemma 2.12. As a preparation, we give a crucial lower bound on the expected energy F_U defined in (2.22), which will be important for obtaining compactness properties. By $\widetilde{\mathfrak{N}}^{(\ell)}_W(\omega,\varpi)$ we denote the number of particles in W in the loop/interlacement configuration (ω,ϖ), regardless where the loop starts in which is the particle is contained (if any). In Remark 2.11, we argued that $\widetilde{\mathfrak{N}}^{(\ell)}_W$ has the same expectation under any $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ as $\mathfrak{N}^{(\ell)}_W$ defined in and below (2.15).

Lemma 7.1 (Lower bound for expected interaction)**.** *Suppose that v satisfies Assumption (V). Then, for any ergodic $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$,*

$$\langle P,F_U\rangle\geq\frac{C\beta}{2}\big\langle P,\big(\widetilde{\mathfrak{N}}^{(\ell)}_U\big)^2\big\rangle-\frac{5C\beta}{4},\tag{7.1}$$

where C is introduced in (2.23). Moreover, the maps $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})\ni P\mapsto\langle P,\widetilde{\mathfrak{N}}^{(\ell,\mathcal{L})}_U\rangle$ and $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})\ni P\mapsto\langle P,\widetilde{\mathfrak{N}}^{(\ell,\mathcal{S})}_U\rangle$ are continuous on the set $\{P\colon\langle P,(\widetilde{\mathfrak{N}}^{(\ell)}_U)^2\rangle\leq L\}$ for any $L\in(0,\infty)$.

Proof. According to Remark 2.11, we can replace $\langle P,F_U\rangle$ by the expectation of $\widetilde{F}_U(\omega,\varpi)$, the amount of interaction in the loop/interlacement configuration between any leg that starts in U (regardless, whether it is in a loop or in an interlacement) and any other leg. Define $\widetilde{F}_W$ in the same way with U replaced by $W=[-R,R]^d$. Define $\mathfrak{L}(\omega,\varpi)$ as the set of all legs of ω or of ϖ. Then, by shift-invariance, for $R\in\mathbb{N}$, we can estimate

$$\begin{aligned}\langle P,F_U\rangle&=\frac{1}{|W|}\langle P,\widetilde{F}_W\rangle\geq\frac{1}{|W|}\int_0^\beta\mathrm{d}s\int P(\mathrm{d}(\omega,\varpi))\sum_{g,g'\in\mathfrak{L}(\omega,\varpi)\colon g(s),g'(s)\in W}\mathbb{1}\{g\neq g'\}v(g(s)-g'(s))\\&\geq\frac{C}{|W|}\int_0^\beta\mathrm{d}s\int P(\mathrm{d}(\omega,\varpi))\Big(\frac{\widetilde{\mathfrak{N}}_W(s)^2}{|W|}-\widetilde{\mathfrak{N}}_W(s)\Big),\end{aligned}$$

where we denote $\widetilde{\mathfrak{N}}_W(s)=\#\{g\in\mathfrak{L}(\omega,\varpi)\colon g(s)\in W\}$, and we used the assumption (2.23) of super-stability in Assumption (V). Jensen's inequality gives

$$\langle P,F_U\rangle\geq\frac{C\beta}{|W|}\int\mathrm{d}P\Big(\frac{\mathfrak{G}_W^2}{|W|}-\mathfrak{G}_W\Big)=C\beta\int\mathrm{d}P\Big(\Big(\frac{\mathfrak{G}_W}{|W|}\Big)^2-\frac{\mathfrak{G}_W}{|W|}\Big),$$

where $\mathfrak{G}_W=\frac{1}{\beta}\int_0^\beta\mathrm{d}s\,\widetilde{\mathfrak{N}}_W(s)$ denotes $\frac{1}{\beta}$ times the total local time in W of all the legs in the loop/interlacements configuration. We estimate $x^2-x\geq\frac12x^2$ for $x\geq2$ and $x^2-x\geq-\frac14$ for $x\leq2$; hence

$$\langle P,F_U\rangle\geq-\frac{C\beta}{4}+\frac{C\beta}{2}\int\mathrm{d}P\Big(\frac{\mathfrak{G}_W}{|W|}\Big)^2\mathbb{1}\{\mathfrak{G}_W\geq2|W|\}.$$

Denote by $\widetilde{\mathfrak{N}}^{(\ell,M)}_{\widetilde{W}}$ the number of particles in the entire configuration that lie in $\widetilde{W}$ (regardless where its loop starts, if it is contained in a loop) and have spread $\leq M$. If $R>M$, then $\mathfrak{G}_W\geq\widetilde{\mathfrak{N}}^{(\ell,M)}_{W_{R-M}}$, since all legs that start in W_{R-M} and have spread $\leq M$ are entirely contained in $W=W_R$. Using this, we estimate

$$\langle P,F_U\rangle\geq-\frac{C\beta}{4}+\frac{C\beta}{2}\int\mathrm{d}P\Big(\frac{\widetilde{\mathfrak{N}}^{(\ell,M)}_{W_{R-M}}}{|W|}\Big)^2\mathbb{1}\{\widetilde{\mathfrak{N}}^{(\ell,M)}_{W_{R-M}}\geq2|W|\}.$$

Making $R \to \infty$ (recalling that $W = W_R$) and using the spatial ergodic theorem, and making afterwards $M \to \infty$, we see that

$$\langle P, F_U\rangle \geq -\frac{C\beta}{4} + \frac{C\beta}{2}\int \mathrm{d}P\,\big(\widetilde{\mathfrak{N}}_U^{(\ell)}\big)^2 1\!\!1\{\widetilde{\mathfrak{N}}_U^{(\ell)} \geq 2\} \geq \frac{C\beta}{2}\big\langle P, \big(\widetilde{\mathfrak{N}}_U^{(\ell)}\big)^2\big\rangle - \frac{5C\beta}{4}.$$

More precisely, by ergodicity, $\widetilde{\mathfrak{N}}_{W_{R-M}}^{(\ell,M)}/|W|$ converges P-almost surely to $\widetilde{\mathfrak{N}}_U^{(\ell,M)}$ as $R \to \infty$, and we can use Lebesgue's theorem after lower estimating the integrand against $(\widetilde{\mathfrak{N}}_{W_{R-M}}^{(\ell,M)}/|W|)^2 \wedge K 1\!\!1\{\widetilde{\mathfrak{N}}_{W_{R-M}}^{(\ell,M)} \geq 2|W|\}$ for some K. Afterwards, we make $K \to \infty$ and $M \to \infty$, using the monotone convergence theorem.

Now about the continuities. Recall that we are working with the topology on $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L} \times \mathcal{S})$ that is defined by the weak topologies induced by all the projections $\Pi_W^{(\mathcal{L})}$ and $\Pi_W^{(\mathcal{S})}$ with $W \Subset \mathbb{R}^d$. Since the counting functionals are nonnegative, the lower semi-continuity is clear (for example via Fatou's lemma). As it concerns the shred-part, the upper semi-continuity follows from the estimate, for any $K \in (0, \infty)$,

$$\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{S})} \leq \widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{S})}\, 1\!\!1\{\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{S})} \leq K\} + \frac{1}{K}\Big(\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{S})}\Big)^2,$$

since the P-integral of the last term vanishes uniformly in P in that set as $K \to \infty$, and the first one is a bounded and continuous function of $\Pi_U^{(\mathcal{S})}$. For the loop-part, we need to distinguish particles in loops that are entirely contained in U and in the other loops. The first ones are functionals of $\Pi_U^{(\mathcal{L})}$, the latter ones are functionals of $\Pi_U^{(\mathcal{S})}$. More precisely, writing $f_x \subset W$ if $f_{x,k}(0) \in W$ for all $k \in [\ell(f_x)]$, and using the notation $N_U(f) = \sum_{k\in[\ell(f)]} 1\!\!1_U(f_k(0))$ for $f \in \mathcal{C}^{(\circlearrowleft)}$, we have

$$\begin{aligned}\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{L})}(\omega) \leq &\Big(\sum_{x\in\zeta\cap U\colon f_x\subset U} N_U(f_x)\Big) 1\!\!1\Big\{\sum_{x\in\zeta\cap U\colon f_x\subset U} N_U(f_x) \leq K\Big\} + \frac{1}{K}\Big(\sum_{x\in\zeta\cap U\colon f_x\subset U} N_U(f_x)\Big)^2\\ &+ \Big(\sum_{x\in\zeta\colon f_x\not\subset U} N_U(f_x)\Big) 1\!\!1\Big\{\sum_{x\in\zeta\colon f_x\not\subset U} N_U(f_x) \leq K\Big\} + \frac{1}{K}\Big(\sum_{x\in\zeta\colon f_x\not\subset U} N_U(f_x)\Big)^2.\end{aligned}$$

The first term on the right is a bounded and continuous function of $\Pi_U^{(\mathcal{L})}(\omega)$, the third is a bounded and continuous function of $\Pi_U^{(\mathcal{S})}(\omega)$, and each of the two other terms are not bigger than $\frac{1}{K}\widetilde{\mathfrak{N}}_U^{(\ell)}(\omega,\varpi)^2$, whose integral with respect to P from that set is $\leq \frac{L}{K}$. This shows the upper semi-continuity of $P \mapsto \langle P, \widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{L})}\rangle$ on the set of $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L} \times \mathcal{S})$ satisfying $\langle P, (\widetilde{\mathfrak{N}}_U^{(\ell)})^2\rangle \leq L$. □

Now we proceed with the proof of Lemma 2.12.

Proof of (1): The compactness of K follows from Theorem 2.3, noting that, for any $P \in K$ and any $R \in \mathbb{N}$,

$$\langle \partial\Pi_{W_R}^{(\mathcal{S})}(P), (\mathfrak{N}_{\partial W_R}^{(\ell)})^2\rangle = \langle \Pi_{W_R}^{(\mathcal{S})}(P), (\mathfrak{N}_{W_R}^{(\ell)})^2\rangle \leq |W_R| \sum_{y\in Z_{R,1/2}} \langle \Pi_{y+U}^{(\mathcal{S})}(P), (\mathfrak{N}_{y+U}^{(\ell)})^2\rangle \leq |W_R|^2 L,$$

using that the distribution of $\mathfrak{N}_{\partial W_R}^{(\ell)}$ (under $\partial\Pi_{W_R}^{(\mathcal{S})}(P)$) and of $\mathfrak{N}_{W_R}^{(\ell)}$ (under $\Pi_{W_R}^{(\mathcal{S})}(P)$) are identical. Hence, K is contained in a set of the form (2.17) and is therefore compact, according to Theorem 2.3.

Furthermore, the maps $P \mapsto \langle P, \mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle$ and $P \mapsto \langle P, \mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle$ are continuous on K, which can be proved similarly to Lemma 7.1.

From Lemma 7.1 it directly follows that the value of the variational formula on the right-hand side of (2.24) with restriction to ergodic Ps does not depend on the L in the definition of the set K, if the variational problem in (2.24) is restricted to K and L is large enough. According to Lemma 7.6, for every $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ with finite entropy and energy and particle density there is an ergodic measure $\in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ with hardly differing values of all these quantities, therefore the statement is also true for (2.24) without restrictions.

Proof of (2): Proving the convexities of χ and $\widetilde{\chi}$ are elementary exercises, which we leave to the reader; they rely on the convexity of the maps $P \mapsto \langle P, F_U\rangle$ and $P \mapsto \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$. Continuity of $\overline{\chi}$ and of $\chi(\rho_1,\cdot)$ and $\chi(\cdot,\rho_2)$ in $(0,\infty)$ for fixed ρ_1, respectively ρ_2, follows from convexity. The limiting statements as $\rho_1 \to \infty$ and $\rho_2 \to \infty$ follow directly from Lemma 7.1 after using Jensen's inequality.

Proof of (3): For $\rho_1 = 0 = \rho_2$, only the empty measure (the shift-invariant distribution of loops and interlacements that has none of them) is admissible in the variational problem. When examining its value in $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(\cdot)$, we first see that the interlacement part is zero, since also the reference measure $\partial\Pi_W^{(\mathcal{S})}$ is empty. Furthermore, the $\mathtt{Q}$-part is equal to the entropy $H(0 \mid (q_k)_{k\in\mathbb{N}})$ (see (2.6)) of the zero sequence $m = (m_k)_{k\in\mathbb{N}} = (0)_k = 0$ with respect to $(q_k)_k = (\frac{1}{k}4\pi\beta k)^{-d/2})_{k\in\mathbb{N}}$, and this entropy is equal to $q = \sum_{k\in\mathbb{N}} q_k = (4\pi\beta)^{-d/2}\zeta(1+\frac{d}{2})$ with ζ the Riemann zeta function (recall (1.12)). That is, $\chi(0,0) = q$. For proving the second assertion, we need to upper bound $\frac{1}{\rho_1}(\chi(\rho_1,0) - \chi(0,0))$ and make $\rho_1 \downarrow 0$. We get an upper bound for $\chi(\rho_1,0)$ by inserting the marked Poisson point process $\mathtt{Q}^{(m)}$, defined as the reference process $\mathtt{Q}$, but with $(q_k)_{k\in\mathbb{N}}$ replaced by $m = (m_k)_{k\in\mathbb{N}} = \rho_1\delta_1 = (\rho_1, 0, 0, \dots)$, which is admissible in the variational formula. Since the interaction potential v is bounded, the expected interaction of $\mathtt{Q}^{(m)}$ is not larger than $C\rho_1^2$ for some $C \in (0,\infty)$ and all sufficiently small ρ_1. Its relative entropy density with respect to $\mathtt{Q}$ is equal to $H(m \mid (q_k)_k) = \sum_{k\in\mathbb{N}} q_k - \rho_1 + \rho_1 \log \frac{\rho_1}{q_1}$. Hence,

$$\frac{1}{\rho_1}(\chi(\rho_1,0) - \chi(0,0)) \le \frac{1}{\rho_1}\Big(C\rho_1^2 - \rho_1 + \rho_1 \log\frac{\rho_1}{q_1}\Big) \le \log\frac{\rho_1}{q_1},$$

which tends to $-\infty$ as $\rho_1 \downarrow 0$. By noting that $\chi(\rho_1,0) \ge \chi^{(v=0)}(\rho_1,0)$, which tends to $\chi(0,0)$ as $\rho_1 \downarrow 0$, we also see that $\chi(\rho_1,0)$ is continuous in $\rho_1 = 0$.

Proof of (4) and (5): The two assertions (4) and (5) rely on the same standard arguments, which are based on the assertions in (1) and the lower semi-continuity of the two maps $P \mapsto \langle P, F_U\rangle$ and $P \mapsto \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$, the latter according to Theorem 2.3. We leave the details to the reader.

7.2. **Preparation: entropy decomposition.** Let us have a general look at the entropy terms that appear in the definition (6.13) of $J_W(\xi)$ with respect to the reference measures, and decompose them into its pieces, first for loops. The next two lemmas will be jointly used in later sections for loop-interlacement distributions $\xi \in \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$ according to the formula

$$\begin{aligned} |W| J_W(\xi) &= H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi \,\big|\, \Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_W^{(\mathcal{S})}(\xi) \otimes \mathtt{K}_W]\big) \\ &= H_{\mathcal{L}_W}\big(\Pi_W^{(\mathcal{L})}(\xi) \mid \Pi_W^{(\mathcal{L})}(\mathtt{Q})\big) + \int_{\mathcal{L}_W} \Pi_W^{(\mathcal{L})}(\xi)(\mathrm{d}\omega)\, H_{\mathcal{S}_W}\big(\xi_\omega^{(\mathcal{S})} \,\big|\, \partial\Pi_W^{(\mathcal{S})}(\xi) \otimes \mathtt{K}_W\big), \end{aligned} \tag{7.2}$$

writing $\xi_\omega^{(\mathcal{S})}$ for the kernel $\xi_{\mathcal{L}_W \to \mathcal{S}_W}(\omega,\cdot)$.

The entropy of a random loop configuration with respect to the Poisson loop reference measure (the Brownian loop soup) in a set W is decomposed into the following partial entropies according to the random mechanisms:

(1) the expected number of k-length loops in W as a sequence in $k \in \mathbb{N}$,
(2) the number of k-length loops in W for any k,
(3) the locations of the starting sites of all these loops, and finally
(4) the loops themselves.

We write $\omega = \sum_{k\in\mathbb{N}} \sum_{x\in\zeta_k} \delta_{(x,f_x)}$ for loop configurations, with $\zeta_k = \{x \in \zeta \colon \ell(f_x) = k\}$. Furthermore, $\mathfrak{N}_W(\omega)$ denotes the number of loops in W and $\mathfrak{N}_W^{(\delta_k)}(\omega)$ the number of loops of length k in W. Let $m_{W,k}(\xi) = \frac{1}{|W|}\int \xi(\mathrm{d}\omega)\, \mathfrak{N}_W^{(\delta_k)}(\omega)$ denote the normalized expected number of loops of length k in W

under $\xi \in \mathcal{M}_1(\mathcal{L}_W)$. The Poisson parameter for $\Pi^{(\mathcal{L})}_{W,k}(\mathtt{Q})$, the projection of the PPP $\mathtt{Q}$ on the set of k-length loops with all particles in W, is equal to

$$q_{W,k} = \frac{1}{k}\int_W \mathrm{d}x\, \mu^{(W,k,\mathrm{par})}_{x,x}(\mathcal{C}^{(\circlearrowleft)}_k) = \frac{1}{k}\int_W \mathrm{d}x\, \mu^{(k)}_{x,x}(\tau_W > k), \tag{7.3}$$

where we recall that $\mu^{(W,k,\mathrm{par})}_{x,y}$ is the (not normalized) Brownian bridge measure from x to y with all the particles in W, and we recall from (5.2) that $\beta\tau_W$ denotes the first time $\in \beta\mathbb{N}$ of an exit from W. We write $\mathcal{L}^{(k)}_W$ for the set of k-length loop configurations in W. Introduce the projection $\Pi_k\colon \mathcal{L} \to \mathcal{L}^{(k)}$ defined by $\Pi_k(\omega) = \sum_{x\in\zeta} \mathbb{1}\{\ell(f_x) = k\}\delta_{(x,f_x)} = \sum_{x\in\zeta_k}\delta_{(x,f_x)}$. By $\overline{\nu} = \nu/\nu(\mathcal{X})$ we define the normalised version of a positive and finite measure ν on a Polish space $\mathcal{X}$.

Lemma 7.2 (Entropy of loop distributions)**.** *For any box $W \Subset \mathbb{R}^d$ and any $\xi \in \mathcal{M}_1(\mathcal{L}_W)$,*

$$\begin{aligned}
&H_{\mathcal{L}_W}\big(\xi \mid \Pi^{(\mathcal{L})}_W(\mathtt{Q})\big)\\
&\quad= |W|\, H\big((m_k)_k \mid \tfrac{1}{|W|}(q_k)_k\big) + H_{\mathcal{L}_W}\Big(\xi \,\Big|\, \textstyle\bigotimes_{k\in\mathbb{N}} \Pi_k(\xi)\Big)\\
&\qquad+ \sum_{k\in\mathbb{N}} H_{\mathbb{N}_0}(\Pi_k(\xi)\circ \mathfrak{N}_W^{-1} \mid \mathrm{Poi}_{m_k|W|})\\
&\qquad+ \sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}} \Pi_k(\xi)(\mathfrak{N}_W = n)\, H\big(\Pi^{(n)}_k(\xi)\circ(\zeta_k\cap W)^{-1} \mid \overline{\mathrm{Leb}}^{\otimes n}_W\big)\\
&\qquad+ \sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}_0} \int \Pi^{(n)}_k(\xi)(\zeta_k\cap W\in\mathrm{d}A)\, \Big[H\Big(\Pi^{(A)}_k(\xi)\,\Big|\, \bigotimes_{x\in A}\Pi^{(A)}_{k,x}(\xi)\Big) + \sum_{x\in A} H\big(\Pi^{(A)}_{k,x}(\xi)\mid \overline{\mu}^{(W,k,\mathrm{par})}_{x,x}\big)\Big],
\end{aligned} \tag{7.4}$$

where we abbreviate $m_k = m_{W,k}(\xi)$ and $q_k = m_{W,k}(\Pi^{(\mathcal{L})}_W(\mathtt{Q}))$ and denote by $\Pi_k, \Pi^{(n)}_k, \Pi^{(A)}_k, \Pi^{(A)}_{k,x}\colon \mathcal{L}_W \to \mathcal{L}^{(k)}_W$ the projections on the set of loops of length k, respectively on the set of configurations in W with precisely n loops of length k and no loops else, respectively on the set of configurations of loops of length k with set of initial sites equal to $A\subset W$.

Proof. We write $\Pi^{(\mathcal{L})}_{W,k} = \Pi^{(\mathcal{L})}_W\circ\Pi_k = \Pi_k\circ\Pi^{(\mathcal{L})}_W$. In the following we drop the super-index $\mathcal{L}$. Note that $\Pi_W(\mathtt{Q})$ is equal to the convolution of the measures $\Pi_{W,k}(\mathtt{Q})$ over $k\in\mathbb{N}$. Then one sees elementarily that

$$H_{\mathcal{L}_W}(\xi \mid \Pi_W(\mathtt{Q})) = \sum_{k\in\mathbb{N}} H_{\mathcal{L}^{(k)}_W}(\Pi_k(\xi)\mid \Pi_{W,k}(\mathtt{Q})) + H_{\mathcal{L}_W}\Big(\xi\,\Big|\, \bigotimes_{k\in\mathbb{N}}\Pi_k(\xi)\Big). \tag{7.5}$$

Let us further decompose $H_{\mathcal{L}^{(k)}_W}(\Pi_k(\xi) \mid \Pi_{W,k}(\mathtt{Q}))$. We write a configuration in $\mathcal{L}^{(k)}_W$ as $\omega_k = \sum_{x\in\zeta_k\cap W}\delta_{(x,f_x)}$. Note that $|\zeta_k\cap W|$ has the distribution $\mathrm{Poi}_{q_{W,k}}$ under $\Pi_{W,k}(\mathtt{Q})$, where we recall (1.4). By the properties of a Poisson point process, $\Pi_{W,k}(\mathtt{Q})$ can be decomposed as

$$\Pi_{W,k}(\mathtt{Q})(\mathrm{d}\omega_k) = \mathrm{Poi}_{q_{W,k}}(n)\,\overline{\mathrm{Leb}}^{\otimes n}_W(\mathrm{d}A)\bigotimes_{x\in A}\overline{\mu}^{(k,W)}_{x,x}(\mathrm{d}f_x), \qquad n = \mathfrak{N}_W(\omega_k) = |A|, A = \zeta_k\cap W. \tag{7.6}$$

We now write $\Pi_k(\xi)$ in an analogous manner. We abbreviate $\Pi^{(n)}_k(\xi)$ for the conditional distribution under $\Pi_k(\xi)$ given that $|\zeta_k\cap W| = n$ and $\Pi^{(A)}_k(\xi)$ for the conditional distribution under $\Pi_k(\xi)$ given that $\zeta_k\cap W = A$, and $\Pi^{(A)}_{k,x}(\xi)$ for the distribution of any f_x under $\Pi^{(A)}_k(\xi)$. Write $m_k = \frac{1}{|W|}\Pi_k(\xi)(\mathfrak{N}^{(\ell)}_W)$ for the normalised expected particle number in the configuration under $\Pi_k(\xi)$ in W. Then we can write, with the same notation as above,

$$\begin{aligned}
\Pi_k(\xi)(\mathrm{d}\omega_k) &= \mathrm{Poi}_{m_k|W|}(n)\frac{\mathrm{d}\Pi_k(\xi)\circ\mathfrak{N}_W^{-1}}{\mathrm{d}\mathrm{Poi}_{m_k|W|}}(n)\,\overline{\mathrm{Leb}}^{\otimes n}_W(\mathrm{d}A)\,\frac{\mathrm{d}\Pi^{(n)}_k(\xi)\circ(\zeta_k\cap W)^{-1}}{\mathrm{d}\overline{\mathrm{Leb}}^{\otimes n}_W}(A)\\
&\quad\times \frac{\mathrm{d}\Pi^{(A)}_k(\xi)}{\mathrm{d}\bigotimes_{x\in A}\Pi^{(A)}_{k,x}(\xi)}\big((f_x)_{x\in A}\big)\bigotimes_{x\in A}\frac{\mathrm{d}\Pi^{(A)}_{k,x}(\xi)}{\mathrm{d}\overline{\mu}^{(W,k,\mathrm{par})}_{x,x}}(f_x)\bigotimes_{x\in A}\overline{\mu}^{(W,k,\mathrm{par})}_{x,x}(\mathrm{d}f_x).
\end{aligned}$$

Hence, we see that the density is given by

$$\frac{\mathrm{d}\Pi_k(\xi)}{\mathrm{d}\Pi_{W,k}(\mathtt{Q})}(\omega_k) = \frac{\mathrm{dPoi}_{m_k|W|}(n)}{\mathrm{dPoi}_{q_{W,k}}(n)} \frac{\mathrm{d}\Pi_k(\xi)\circ\mathfrak{N}_W^{-1}}{\mathrm{dPoi}_{m_k|W|}}(n) \frac{\mathrm{d}\Pi_k^{(n)}(\xi)\circ(\zeta_k\cap W)^{-1}}{\mathrm{d}\overline{\mathrm{Leb}}_W^{\otimes n}}(A)$$
$$\times \frac{\mathrm{d}\Pi_k^{(A)}(\xi)}{\mathrm{d}\bigotimes_{x\in A}\Pi_{k,x}^{(A)}(\xi)}\big((f_x)_{x\in A}\big)\prod_{x\in A}\frac{\mathrm{d}\Pi_{k,x}^{(A)}(\xi)}{\mathrm{d}\overline{\mu}_{x,x}^{(k,W)}}(f_x).$$

Now take the logarithm and integrate with respect to $\Pi_k(\xi)(\mathrm{d}\omega_k)$ and use (7.5), to obtain the assertion in (7.4). $\square$

Now we do the same for the shreds. We write ξ_W for the kernel $\xi_{\mathcal{T}_W\to\mathcal{S}_W}$. We also recall the notation

$$q_{x,y}^{(n,W)}(\mathrm{d}f) = \mathbb{P}_x(B\in\mathrm{d}f \mid \tau_W = n, B_{\tau\beta} = y), \qquad n\in\mathbb{N}, x\in W, y\in W^{\mathrm{c}}, \tag{7.7}$$

for the distribution of an n-leg W-shred from x to y (which appears in the definition (2.5) of $\mathtt{K}_W$).

Lemma 7.3 (Entropy of shred distributions)**.** *For any box $W \Subset \mathbb{R}^d$ and any $\xi\in\mathcal{M}_1(\mathcal{S}_W)$,*

$$\begin{aligned} H_{\mathcal{S}_W}\big(\xi\,\big|\,\partial\Pi_W^{(\mathcal{S})}(\xi)\otimes\mathtt{K}_W\big) &= \int_{\mathcal{T}_W}\partial\Pi_W^{(\mathcal{S})}(\xi)(\mathrm{d}\mu)\,H_{\mathcal{S}_W}\Big(\xi(\mu,\cdot)\,\Big|\,\bigotimes_{i\in I}\Pi_i(\xi(\mu,\cdot))\Big)\\ &\quad+\int_{\mathcal{T}_W}\partial\Pi_W^{(\mathcal{S})}(\xi)(\mathrm{d}\mu)\sum_{i\in I}H_{\mathcal{C}_{l_i}}\big(\Pi_i(\xi(\mu,\cdot))\,\big|\,q_{x_i,y_i}^{(l_i,W)}\big),\end{aligned} \tag{7.8}$$

where we wrote $\mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}$ and $\Pi_i\colon\mathcal{C}^I\to\mathcal{C}$ for the canonical projection on the i-th factor and $\xi(\mu,\cdot)$ for the kernel $(\xi)_{\mathcal{T}_W\to\mathcal{S}_W}(\mu,\cdot)$.

Proof. We drop all super-indices $\mathcal{S}$. Then we see that

$$\begin{aligned} H_{\mathcal{S}_W}\big(\xi\,\big|\,\partial\Pi_W(\xi)\otimes\mathtt{K}_W\big) &= \int_{\mathcal{T}_W}\partial\Pi_W(\xi)(\mathrm{d}\mu)\,H_{\mathcal{S}_W}\big(\xi(\mu,\cdot)\,\big|\,\mathtt{K}_W(\mu,\cdot)\big)\\ &= \int_{\mathcal{T}_W}\partial\Pi_W(\xi)(\mathrm{d}\mu)\,\Big[H_{\mathcal{S}_W}\Big(\xi(\mu,\cdot)\,\Big|\,\bigotimes_{i\in I}\Pi_i(\xi(\mu,\cdot))\Big)+\sum_{i\in I}H_{\mathcal{C}_{l_i}}\big(\Pi_i(\xi(\mu,\cdot))\,\big|\,q_{x_i,y_i}^{(l_i,W)}\big)\Big].\end{aligned}$$

$\square$

7.3. **Proof of Lemma 2.13.** In this section, we prove Lemma 2.13, which says that $\chi(\rho_1+\rho_2,0)\le\chi(\rho_1,\rho_2)$ for any $\rho_1,\rho_2\in[0,\infty)$. For this, we will replace an arbitrary loop-interlacements configuration by a loop configuration without changing much the number of particles per volume and without increasing the energy nor the entropy terms. We do this by rewiring the ends of all the shreds at the boundary of a large box $W=W_R$ in such a way that all the shreds of interlacements are turned into the shreds of one long loop just outside that box, i.e., in $W_{R+M}\setminus W_R$. The new configuration has only loops in W_{R+M} and no shreds. Our manipulation does not change anything inside W_R, which is the main part, and the changes in $W_{R+M}\setminus W_R$ have only a vanishing influence in the limit as $R\to\infty$, as it concerns the three main quantities: entropy, expected interaction energy per volume and particle densities. Finally, we put independent copies of the new configuration into the boxes $z+W_{R+M}$, $z\in 2(R+M)\mathbb{Z}^d$, and mix the configuration in $\mathbb{R}^d$ over $x\in W_{R+M}$ and control the entropy of this configuration.

We need some more notation. We need some large parameters $R,M,T\in(1,\infty)$. As usual, we write $W=[-R,R]^d$. Denote by $\widetilde{\Pi}_{W;M}\colon\mathcal{L}_W\times\mathcal{S}_W\to\mathcal{L}_W^{(M)}\times\mathcal{S}_W^{(M)}$ the natural projection on loops, respectively shreds, whose legs $f\in\mathcal{C}_1$ have throughout spread $\|f\|_{\mathrm{sp}}=\max_{t\in[0,\beta]}|f(t)-f(0)|<M$. We denote by $\widetilde{J}_{W;M}$ the normalized entropy function as in (6.13), but with the reference measure $\Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes[\partial\Pi_W^{(\mathcal{S})}(P)\otimes\mathtt{K}_W]$ replaced by its image measure under $\widetilde{\Pi}_{W;M}$. Note that $\widetilde{J}_{W;M}(\xi\circ\widetilde{\Pi}_{W;M}^{-1})\le$

$J_W(\xi)$ for any $\xi \in \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$, by monotonicity of the relative entropy in the reference σ-field, see [G11, Proposition 15.5(c)].

Furthermore, we write $\widetilde{W} = W_{R+M} = [-R-M, R+M]^d$ and denote by $A_{\widetilde{W};M,T} = \{(\omega,\varpi) \in \mathcal{L}^{(M)}_{\widetilde{W}} \times \mathcal{S}^{(M)}_{\widetilde{W}} : \widetilde{\mathfrak{N}}^{(\ell)}_{z+U}(\omega,\varpi) \leq T\, \forall z \in \widetilde{W} \cap \mathbb{Z}^d\}$, where we recall that $U = [-\frac12, \frac12]^d$ is the unit box and $\widetilde{\mathfrak{N}}^{(\ell)}_{z+U}$ denotes the number of particles in the loop/shred configuration in $z+U$, regardless where the loop starts; see also Remark 2.11. (This event is very similar to $A_{\widetilde{W};\frac12,M,K,L,T}$ defined in (4.36).) Furthermore, let $\widetilde{\Pi}_{\widetilde{W};M,T} : \mathcal{L}_{\widetilde{W}} \times \mathcal{S}_{\widetilde{W}} \to A_{\widetilde{W};M,T}$ be some projection onto $A_{\widetilde{W};M,T}$.

Let us explain the structure of our proof. We may assume that $\rho_1, \rho_2 \in (0,\infty)$. Pick $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ with $\langle P, \mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle = \rho_1$ and $\langle P, \mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle = \rho_2$. Furthermore, fix $\varepsilon \in (0,1)$. Pick M and T so large that $\langle \Pi_U(P), \widetilde{\mathfrak{N}}_U^{(\ell)} \circ \widetilde{\Pi}_{U;M,T}\rangle \geq \rho_1 + \rho_2 - \varepsilon$, where $\widetilde{\mathfrak{N}}_U^{(\ell)} = \widetilde{\mathfrak{N}}_U^{(\mathcal{L},\ell)} + \widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{S})}$ is the number of all particles in loops and in shreds in U.

We are going to prove that, for any sufficiently large M and T, and for any sufficiently large $R \in (1,\infty)$, there exists some ergodic $\widetilde{P} \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L})$ such that $|\langle \Pi_U(\widetilde{P}), \widetilde{\mathfrak{N}}_U^{(\mathcal{L},\ell)} \circ \widetilde{\Pi}_{U;M,T}\rangle - \langle \Pi_U(P), \widetilde{\mathfrak{N}}_U^{(\ell)} \circ \widetilde{\Pi}_{U;M,T}\rangle| \leq \varepsilon$ and $\langle \frac{1}{|\widetilde{W}|} F_{\widetilde{W},\widetilde{W}} \circ \widetilde{\Pi}_{\widetilde{W};M,T}, \Pi_{\widetilde{W}}(\widetilde{P})\rangle \leq \langle \frac{1}{|W|} F_{W,W} \circ \widetilde{\Pi}_{W;M,T}, \Pi_W(P)\rangle + \varepsilon$ and $J_{\widetilde{W}}(\Pi_{\widetilde{W}}(\widetilde{P})) \leq \widetilde{J}_{W;M}(\Pi_W(P) \circ \widetilde{\Pi}^{-1}_{W;M}) + \varepsilon$.

Indeed, this implies for any ε and all sufficiently large M and T and for any sufficiently large R that

$$
\begin{aligned}
\inf\Big\{&\Big\langle \frac{1}{|W_{R+M}|} F_{W_{R+M},W_{R+M}} \circ \widetilde{\Pi}_{W_{R+M};M,T}, \Pi_{W_{R+M}}(P)\Big\rangle + J_{W_{R+M}}(\Pi_{W_{R+M}}(P)) \colon \\
&\qquad\qquad P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}) \text{ ergodic}, \langle \Pi_U(P), \mathfrak{N}_U^{(\mathcal{L},\ell)} \circ \widetilde{\Pi}_{U;M,T}\rangle \in \mathcal{B}_\varepsilon(\rho_1+\rho_2)\Big\} \\
&\leq 2\varepsilon + \inf\Big\{\Big\langle \frac{1}{|W|} F_{W,W} \circ \widetilde{\Pi}_{W;M,T}, \Pi_W(P)\Big\rangle + \widetilde{J}_{W;M}(\Pi_W(P) \circ \widetilde{\Pi}^{-1}_{W;M}) \colon \\
&\qquad\qquad P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S}), \langle P, \mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle = \rho_1, \langle P, \mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle = \rho_2\Big\}.
\end{aligned}
$$

The right-hand side is not larger than $3\varepsilon + \chi(\rho_1,\rho_2)$, as we show now. Indeed, clearly $F_{W,W} \circ \widetilde{\Pi}_{W;M,T} \leq F_{W,W}$. Furthermore, splitting all the contributions to $F_{W,W}$, one sees that, for any $\omega \in \mathcal{L}$ and $\varpi \in \mathcal{S}$,

$$
\begin{aligned}
F_{W,W}(\Pi_W(\omega,\varpi)) \leq \sum_{z \in W\cap\mathbb{Z}^d} \Big[&\frac12 F^{(\mathcal{L}\mathcal{L})}_{z+U,z+U} + F^{(\mathcal{L}\mathcal{S})}_{z+U,z+U} + \frac12 F^{(\mathcal{S}\mathcal{S})}_{z+U,z+U} \\
&+ F^{(\mathcal{L}\mathcal{L})}_{z+U,z+U^{\mathrm{c}}} + F^{(\mathcal{L}\mathcal{S})}_{z+U,z+U^{\mathrm{c}}} + F^{(\mathcal{L}\mathcal{S})}_{z+U^{\mathrm{c}},z+U} + F^{(\mathcal{S}\mathcal{S})}_{z+U,z+U^{\mathrm{c}}}\Big](\omega,\varpi).
\end{aligned}
\tag{7.9}
$$

Now integrate both sides with respect to P and use its shift-invariance, to see that $\langle \frac{1}{|W|} F_{W,W} \circ \widetilde{\Pi}_{W;M,T}, P\rangle \leq \langle F_U, P\rangle$, where we recall (2.22) for the definition of F_U. Furthermore, $\widetilde{J}_{W;M}(\Pi_W(P) \circ \widetilde{\Pi}^{-1}_{W;M}) \leq \varepsilon + J_W(\Pi_W(P)) \leq \varepsilon + \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$ by (2.16).

The left-hand side converges, as $R, M, T \to \infty$ and $\varepsilon \downarrow 0$ to $\chi(\rho_1+\rho_2, 0)$, as we show now. Indeed, for any $\varepsilon \in (0,1)$ and any ergodic $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L})$ with $\langle P, \widetilde{\mathfrak{N}}_U^{(\mathcal{L},\ell)}\rangle = \rho_1 + \rho_2$ and finite values of $\mathrm{h}^{(\mathcal{L})}(P \mid \mathtt{Q})$ and $\langle F_U, P\rangle$, pick M and T so large that P is amenable to the formula on the left and such that $\langle F_U \circ \widetilde{\Pi}_{U;M,T}, P\rangle \geq \langle F_U, P\rangle - \varepsilon$. Making $R \to \infty$ and using Theorem 2.3, we see that $J_{W_{R+M}}(\Pi_{W_{R+M}}(P))$ converges towards $\mathrm{h}^{(\mathcal{L})}(P \mid \mathtt{Q})$. Furthermore, making $R \to \infty$ we see that $\langle \frac{1}{|W_{R+M}|} F_{W_{R+M},W_{R+M}} \circ \widetilde{\Pi}_{W_{R+M};M,T}, \Pi_{W_{R+M}}(P)\rangle$ converges towards $\langle F_U \circ \widetilde{\Pi}_{U;M,T}, P\rangle$. Since $\chi(\rho_1+\rho_2, 0)$ is the infimum over all $P \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{L})$ (not only the ergodic ones), we have arrived at the desired estimate, $\chi(\rho_1+\rho_2,0) \leq \chi(\rho_1,\rho_2)$.

Here is our construction of $\widetilde{P}$. Recall

$$
W = W_R = [-R,R]^d \qquad \text{and} \qquad \widetilde{W} = W_{R+M} = [-R-M, R+M]^d \qquad \text{and} \qquad W' = \widetilde{W} \setminus W.
$$

Consider $\xi_R = \Pi_W(P)\circ\widetilde{\Pi}_{W;M}^{-1} \in \mathcal{M}_1(\mathcal{L}_W^{(M)}\times\mathcal{S}_W^{(M)})$. Furthermore, assume that M is also so large that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\nu\circ\widetilde{\Pi}_{W;M}^{-1}}(\omega,\varpi)\le(1+\varepsilon)^{\widetilde{\mathfrak{N}}_W^{(\ell)}(\omega,\varpi)},\qquad R>1,(\omega,\varpi)\in\mathcal{L}_W\times\mathcal{S}_W, \tag{7.10}$$

where $\nu=\Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes[\partial\Pi_W^{(\mathcal{S})}(\xi_R)\otimes\mathtt{K}_W]$ (this is possible since all the leg spreads are i.i.d. under ν). We are going to construct from ξ_R some loop-configuration measure $\xi_R^{(M)}\in\mathcal{M}_1(\mathcal{L}_{\widetilde{W}})$ as follows. First, we add an empty loop configuration $\delta_{\underline{0}}$ in $W'=\widetilde{W}\setminus W$ (i.e., no loop starts in W'). Furthermore, we extend all the initial and terminal sites of all the W-shreds of ξ_R by appending independent Brownian bridge pieces with particles only within W' such that a single large loop in $\widetilde{W}$ arises. Hence, $\Pi_W(\xi_R^{(M)})=\Pi_W(\xi_R)=\xi_R$, and $\xi_R^{(M)}$ has no $\widetilde{W}$-shreds. Then we decompose $\mathbb{R}^d$ regularly into the boxes $z+\widetilde{W}$ with $z\in 2(R+M)\mathbb{Z}^d$ and put into each such subbox one independent copy of a loop-configuration with distribution $\theta_z(\xi_R^{(M)})$ (where θ_z is the shift-operator such that $\theta_z(z)=0$). Call the distribution of this configuration $\widetilde{P}_{R,M}=\bigotimes_{z\in2(R+M)\mathbb{Z}^d}\theta_z(\xi_R^{(M)})$ and define $P_{R,M}=\frac{1}{|W_{R+M}|}\int_{W_{R+M}}\mathrm{d}x\,\widetilde{P}_{R,M}\circ\theta_x^{-1}$. Then $P_{R,M}\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L})$ is obviously stationary and contains no interlacement. The proof of [G11, Theorem 14.12] applies and shows that $P_{R,M}$ is ergodic.

We need to show the following:

$$\lim_{R\to\infty}\langle P_{R,M},\widetilde{\mathfrak{N}}_U^{(\mathcal{L},\ell)}\circ\widetilde{\Pi}_{U;M,T}\rangle-\langle P,[\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{L})}+\widetilde{\mathfrak{N}}_U^{(\ell,\mathcal{S})}]\circ\widetilde{\Pi}_{U;M,T}\rangle = 0, \tag{7.11}$$

$$\limsup_{R\to\infty}\Big[\Big\langle\frac{1}{|\widetilde{W}|}F_{\widetilde{W},\widetilde{W}}\circ\widetilde{\Pi}_{\widetilde{W};M,T},P_{R,M}\Big\rangle-\Big\langle\frac{1}{|W|}F_{W,W}\circ\widetilde{\Pi}_{W;M,T},P\Big\rangle\Big] \le 0, \tag{7.12}$$

$$\limsup_{R\to\infty}[J_{\widetilde{W}}(\Pi_{\widetilde{W}}(P_{R,M}))-\widetilde{J}_{W;M}(\Pi_W(P)\circ\widetilde{\Pi}_{W;M}^{-1})] \le \varepsilon. \tag{7.13}$$

Then $\widetilde{P}=P_{R,M}$ with sufficiently large R is suitable, and the assertion follows.

Properties (7.11) and (7.12) are easy to show and do not need details of the construction:

Proof of (7.11)*:* Since $|\widetilde{W}|/|W|\to1$, it suffices to show that $\lim_{R\to\infty}\frac{1}{|W|}\langle\xi_R^{(M)},\widetilde{\mathfrak{N}}_{\widetilde{W}}^{(\ell)}\circ\Pi_{\widetilde{W};M,T}\rangle=\lim_{R\to\infty}\frac{1}{|W|}\langle\xi_R,[\widetilde{\mathfrak{N}}_W^{(\mathcal{L},\ell)}+\widetilde{\mathfrak{N}}_W^{(\mathcal{L},\ell)}]\circ\Pi_{W;M,T}\rangle$. Recall that $\xi_R^{(M)}$ has its particles from (1) the loops of ξ_R in W, (2) the shreds of ξ_R in W, and (3) the added shreds in W'. Therefore, it suffices to show that the expected amount of the particles of (3) is negligible in the limit $R\to\infty$. But this is clear since $\widetilde{\mathfrak{N}}_{W'}^{(\ell,\mathcal{S})}\circ\Pi_{W';M,T}\le|W'|T\le CMT|W|/R$ for some constant C that depends only on d.

Proof of (7.12)*:* Again, we only have to show that the expected $F_{W',\widetilde{W}}\circ\Pi_{\widetilde{W};M,T}$-interactions of the added Brownian shreds in W' with itself and with any other part of the configuration in $\widetilde{W}$ are $o(|W|)$ as $R\to\infty$. But this is again clear, since $F_{W',\widetilde{W}}\circ\Pi_{\widetilde{W};M,T}\le C_{M,T}|W|/R$ for some constant $C_{M,T}$ that depends only on M and T (and possibly on β, d and v); see Lemma 4.4.

Now let us show that property (7.13) is satisfied.

Let us first make a connection between $J_{\widetilde{W}}(\Pi_{\widetilde{W}}(P_{R,M}))$ and $J_{\widetilde{W}}(\Pi_{\widetilde{W}}(\widetilde{P}_{R,M}))$ (the latter is $J_{\widetilde{W}}(\xi_R^{(M)})$). By convexity of $J_{\widetilde{W}}$ and by (6.17) in Proposition 6.6(1) with $m=3$ (it is clear that $\widetilde{P}_{R,M}$ is $\widetilde{W}$-ergodic and hence $3\widetilde{W}$-ergodic), we have, for all R sufficiently large,

$$\begin{aligned}J_{\widetilde{W}}(\Pi_{\widetilde{W}}(P_{R,M}))&\le\frac{1}{|\widetilde{W}|}\int_{\widetilde{W}}\mathrm{d}x\,J_{\widetilde{W}}(\Pi_{\widetilde{W}}(\widetilde{P}_{R,M}\circ\theta_x^{-1}))=\frac{1}{|\widetilde{W}|}\int_{\widetilde{W}}\mathrm{d}x\,J_{\widetilde{W}}(\theta_x(\Pi_{x+\widetilde{W}}(\widetilde{P}_{R,M})))\\&\le J_{W_{3(R+M)}}(\Pi_{W_{3(R+M)}}(\widetilde{P}_{R,M}))+\varepsilon,\end{aligned} \tag{7.14}$$

where we used that, by construction, $\widetilde{P}_{R,M}$ has no shred-part and therefore has $\psi=0$ for $\psi=\partial\Pi_{W_{3(R+M)}}^{(\mathcal{S})}(\widetilde{P}_{R,M})$.

Since, by construction, $\Pi_{W_{3(R+M)}}(\widetilde{P}_{R,M})$ and also the reference measure in the entropy $J_{W_{3(R+M)}}$, decomposes into the product of 3^d shifted copies of $\Pi_{\widetilde{W}}(\widetilde{P}_{R,M})$ (recall that it has no loops in $z+W'$ for all the $z\in 2(R+M)\mathbb{Z}^d$ and therefore an empty boundary-shred part at the boundaries of all these subboxes $z+\widetilde{W}$), the right-hand side is not larger than $J_{\widetilde{W}}(\Pi_{\widetilde{W}}(\widetilde{P}_{R,M}))+\varepsilon$. Hence, it will be sufficient to prove (7.13) with $J_{\widetilde{W}}(\Pi_{\widetilde{W}}(\widetilde{P}_{R,M}))=J_{\widetilde{W}}(\xi_R^{(M)})$ instead of $J_{\widetilde{W}}(\Pi_{\widetilde{W}}(P_{R,M}))$. More explicitly, we need to show that $\limsup_{R\to\infty}[J_{\widetilde{W}}(\xi_R^{(M)})-\widetilde{J}_{W;M}(\xi_R)]\leq 0$. Recall that the reference measure for the entropy of $\xi_R^{(M)}$ is $\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q})$, while the one for ξ_R is equal to $[\Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes[\partial\Pi_W^{(\mathcal{S})}(\xi_R)\otimes\mathtt{K}_W]]\circ\widetilde{\Pi}_{W;M}^{-1}$.

However, proving property (7.13) needs the details of the construction of $\xi_R^{(M)}$ from ξ_R, which we are going to describe now. Fix a configuration $(\omega,\varpi)\in\mathcal{L}_W^{(M)}\times\mathcal{S}_W^{(M)}$. Write $\varpi=\sum_{i\in I}\delta_{f_i}$ and $\mu=\partial\Pi_W^{(\mathcal{S})}(\varpi)=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}$ with $l_i=\ell(f_i)$ and $f_i(0)=x_i$ and $f_i(\beta l_i)=y_i$. Then $x_i\in W$ and $y_i\in W'$ for any $i\in I$, since each leg of a loop in ϖ has spread $\leq M$ (we consider ϖ under the distribution ξ_R, which has this property almost surely). With some bijection $\sigma\colon I\to I$ that has only one cycle, we sample, conditionally given μ, a collection of random lengths $(L_i)_{i\in I}$ with distribution $\bigotimes_{i\in I}p^{(W')}_{y_i,x_{\sigma(i)}}$, where we define

$$p^{(W')}_{y,x}(n)=\mathbb{P}_y(\tau_{W'}=n\mid B_{\tau_{W'}\beta}=x),\qquad y\in W',x\in(W')^{\rm c},n\in\mathbb{N},\tag{7.15}$$

(compare to (5.5)). Then we append, for $i\in I$, an L_i-length Brownian bridge B_i with particles only in W' at y_i terminating at $x_{\sigma(i)}$. In other words, each B_i has distribution $p^{(W')}_{y_i,x_{\sigma(i)}}\otimes q^{(\cdot,W')}_{y_i,x_{\sigma(i)}}$, where

$$q^{(n,W')}_{y,x}({\rm d}f)=\mathbb{P}_y\big((B_s)_{s\in[0,n\beta]}\in{\rm d}f\mid\tau_{W'}=n,B_{n\beta}=x\big),\qquad y\in W',x\in(W')^{\rm c},n\in\mathbb{N},f\in\mathcal{C}_n,\tag{7.16}$$

(compare to (5.3), $\tau_{W'}$ was defined in (5.2)), and $(B_i)_{i\in I}$ is independent.

In this way, all shreds of ξ_R are turned into a single loop $f\in\mathcal{C}^{(\circlearrowleft)}$ in $\widetilde{W}$ with length $\ell(f)=n=\sum_i(l_i+L_i)$. By default, we take the starting site as $x_{i^*}\in W$, where i^* is uniformly distributed over I. This loop runs subsequently through the functions f_{i^*}, g_{i^*}, $f_{\sigma(i^*)}$, $g_{\sigma(i^*)}$, $f_{\sigma^2(i^*)}$, $g_{\sigma^2(i^*)}$ and so on, more precisely, it is the concatenation of these functions in this order. The change from one f_i to some g_i happens at the stopping times where the next particle $\in W'$ appears, and the changes from g_j to $f_{\sigma(j)}$ happen at the stopping times when the next particle $\in W$ appears. Finally, we define $\xi_R^{(M)}$ as the distribution of $\omega+\delta_f\in\mathcal{L}_{\widetilde{W}}$ under $\delta_{\underline{0}}\otimes\xi_R({\rm d}(\omega,\varpi))\otimes\bigotimes_{i\in I}[p^{(W')}_{y_i,x_{\sigma(i)}}\otimes q^{(\cdot,W')}_{y_i,x_{\sigma(i)}}]$.

Let us first identify the reference measure $\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q})$ explicitly. Recall from Section 4.1 that $\mathcal{L}_{\widetilde{W}}=\mathcal{L}_{W\cup W'}$ consists of the three parts $\mathcal{L}_W$, $\mathcal{L}_{W'}$ and the set $\mathcal{L}_{W,W'}$ of loops in $\widetilde{W}=W\cup W'$ that have particles both in W and in W', but nowhere else. The reference measure $\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q})$ is equal to the convolution of $\Pi_W^{(\mathcal{L})}(\mathtt{Q})$, $\Pi_{W'}^{(\mathcal{L})}(\mathtt{Q})$ and $\Pi_{W,W'}^{(\mathcal{L})}(\mathtt{Q})$. Observe that $\Pi_{W,W'}^{(\mathcal{L})}(\mathtt{Q})({\rm d}\delta_f)$, having just one loop with starting site at x_{i^*} with length n, can be decomposed, according to the Markov property. Therefore, using also (7.3) and

$$q_{W,W',k}=\frac{1}{k}\int_W{\rm d}x\,\mu_{x,x}^{(k,W\cup W',{\rm par})}(\mathfrak{m}_{W,W'}\geq 1),\tag{7.17}$$

where we recall that $\mathfrak{m}_{W,W'}$ is the number of times that the Brownian loop changes between W and W', and abbreviating

$$A_\sigma({\rm d}\mu)=\bigotimes_{i\in I}\Big[\mathbb{P}_{x_i}(\tau_W=l_i,B(\tau_W\beta)\in{\rm d}y_i)\mathbb{P}_{y_{\sigma^{-1}(i)}}(B(\tau_{W'}\beta)\in{\rm d}x_i)\Big],\tag{7.18}$$

we can write

$$\begin{aligned}\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q})\big({\rm d}(\omega+\underline{0}+\delta_f)\big)&=\Pi_W^{(\mathcal{L})}(\mathtt{Q})({\rm d}\omega)\,{\rm e}^{-\sum_{k\in\mathbb{N}}q_{W',k}}\,{\rm e}^{-\sum_{k\in\mathbb{N}}[q_{W,W',k}+q_{W',W,k}]}q_{W,W',n}B_n\\&\otimes A_\sigma({\rm d}\mu)\otimes\bigotimes_{i\in I}\Big(q^{(l_i,W)}_{x_i,y_i}({\rm d}f_i)\otimes q^{(L_i,W')}_{y_i,x_{\sigma(i)}}({\rm d}g_i)\Big),\qquad\text{with }B_n=\frac{1}{|I|}\sum_{i\in I}\frac{1}{\mu^{(n,W\cup W',{\rm par})}_{x_i,x_i}(\mathfrak{m}_{W,W'}\geq 1)}.\end{aligned}\tag{7.19}$$

Indeed, the exponential terms in the first line are the probability that no W'-loop exists and that one n-loop in $W \cup W'$ with particles both in W and in W' exists and no loop else; the product of the measures over i is the probability distribution of the W-shreds and the W'-shreds of the loop f (because of the cyclic structure and the uniform choice of the starting site, one does not see here the starting site explicitly); and $B(n)$ is the normalization for a length-n loop in $W \cup W'$ that has particles both in W and in W' and starts from some point that is uniformly distributed over $\{x_i : i \in I\}$.

Accordingly, also $\xi_R^{(M)}$ splits into these three parts, observing also the dependences on ω and δ_f and recalling that we added no loop that starts in W':

$$\begin{aligned}\xi_R^{(M)}\big(\mathrm{d}(\omega + \underline{0} + \delta_f)\big) &= \xi_R\big(\mathrm{d}(\omega, \varpi)\big)\ \mathrm{e}^{-\sum_{k\in\mathbb{N}}[q_{W',k}+q_{W',W,k}]} \bigotimes_{i\in I}\Big[p_{y_i,x_{\sigma(i)}}^{(W')}(L_i)\, q_{y_i,x_{\sigma(i)}}^{(L_i,W')}(\mathrm{d}g_i)\Big]\\ &= \Pi_W^{(\mathcal{L})}(\xi_R)(\mathrm{d}\omega)\, \mathrm{e}^{-\sum_{k\in\mathbb{N}}[q_{W',k}+q_{W',W,k}]}\, (\xi_R)_{\mathcal{L}_W\to\mathcal{S}_W}(\omega, \mathrm{d}\varpi) \otimes \bigotimes_{i\in I}\Big[p_{y_i,x_{\sigma(i)}}^{(W')}(L_i)\, q_{y_i,x_{\sigma(i)}}^{(L_i,W')}(\mathrm{d}g_i)\Big],\end{aligned} \tag{7.20}$$

where we recall that $\varpi = \sum_{i\in I}\delta_{f_i}$ and $\mu = \partial\Pi_W^{(\mathcal{S})}(\varpi) = \sum_{i\in I}\delta_{(x_i,l_i,y_i)}$. Then, writing $\partial\Pi_W^{(\mathcal{L},\mathcal{T})}$ for the projection on $\mathcal{L}_W \times \mathcal{T}_W$,

$$\begin{aligned} &H_{\mathcal{L}_{\widetilde W}}(\xi_R^{(M)} \mid \Pi_{\widetilde W}^{(\mathcal{L})}(\mathtt{Q})) = H_{\mathcal{L}_W}(\Pi_W^{(\mathcal{L})}(\xi_R) \mid \Pi_W^{(\mathcal{L})}(\mathtt{Q}))\\ &\qquad + \int \partial\Pi_W^{(\mathcal{L},\mathcal{T})}(\xi_R)(\mathrm{d}(\omega,\mu)) H_{\mathcal{S}_W}\big((\xi_R)_{(\mathcal{L}_W\times\mathcal{T}_W)\to\mathcal{S}_W}(\omega,\mu,\cdot)\,\big|\,\mathtt{K}_W(\mu,\cdot)\big)\\ &\qquad + \sum_{k\in\mathbb{N}} q_{W,W',k} - \int \xi_R^{(M)}(\mathrm{d}\varpi)\, \log\big(q_{W,W',n}B(n)\big) + \int \Pi_W^{(\mathcal{L})}(\xi_R)(\mathrm{d}\omega)\, H_{\mathcal{T}_W}\big((\xi_R)_{\mathcal{L}_W\to\mathcal{T}_W}(\omega,\cdot)\,\big|\,A_\sigma\big)\\ &\quad = H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi_R\,\big|\,\Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_W^{(\mathcal{S})}(\xi_R)\otimes \mathtt{K}_W]\big)\\ &\qquad + \sum_{k\in\mathbb{N}} q_{W,W',k} - \int \xi_R^{(M)})(\mathrm{d}\varpi)\, \log\big(q_{W,W',n}B(n)\big) + \int \Pi_W^{(\mathcal{L})}(\xi_R)(\mathrm{d}\omega)\, H_{\mathcal{T}_W}\big((\xi_R)_{\mathcal{L}_W\to\mathcal{T}_W}(\omega,\cdot)\,\big|\,A_\sigma\big), \end{aligned} \tag{7.21}$$

as one sees from (7.2), together with a comparison with Lemma 7.3. Now divide this by $|W_{R+M}|$ and make $R\to\infty$ and observe that $|W_{R+M}|/|W_R| \to 1$. For the entropy of ξ_R, we need to switch from the reference measure $\Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_W^{(\mathcal{S})}(\xi_R)\otimes\mathtt{K}_W]$ to its image measure under $\widetilde\Pi_{W;M}$. This gives for the first term on the right-hand side of (7.21), using (7.10),

$$\begin{aligned} &\frac{1}{|W|} H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi_R\,\big|\,\Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_W^{(\mathcal{S})}(\xi_R)\otimes \mathtt{K}_W]\big)\\ &\qquad = \widetilde J_{W;M}(\xi_R) + \frac{1}{|W|}\int \xi_R(\mathrm{d}(\omega,\varpi))\, \log\frac{\mathrm{d}\Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_W^{(\mathcal{S})}(\xi_R)\otimes \mathtt{K}_W]}{\mathrm{d}\Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_W^{(\mathcal{S})}(\xi_R)\otimes \mathtt{K}_W]\circ\widetilde\Pi_{W;M}^{-1}}(\omega,\varpi)\\ &\qquad \le \widetilde J_{W;M}(\xi_R) + \frac{1}{|W|}\int \Pi_W(P)(\mathrm{d}(\omega,\varpi))\, \widetilde{\mathfrak{N}}_W^{(\ell)}(\omega,\varpi)\log(1+\varepsilon) \le \widetilde J_{W;M}(\xi_R) + C\varepsilon, \end{aligned}$$

where C depends only on $\rho_1 + \rho_2$.

We have to show that all other terms on the right-hand side of (7.21) vanish (i.e., have a vanishing upper bound) with a prefactor of $1/|W|$ as $R\to\infty$. We leave the proof of $\lim_{R\to\infty}\frac{1}{|W_R|}\sum_{k\in\mathbb{N}} q_{W,W',k} = 0$ to the reader.

Let us turn to the next term. Recall that $n = \sum_{i\in I}(l_i + L_i)$ is random. A lower bound for B_n is easy to get by estimating the μ-term against the free Gaussian kernel with parameter βn, i.e., $\mu_{x_i,x_i}^{(n,W\cup W',\mathrm{par})}(\mathfrak{m}_{W,W'} \ge 1) \le (4\pi\beta n)^{-d/2}$, such that it is clear that $\liminf_{R\to\infty}\frac{1}{|W_R|}\int \xi_R^{(M)}(\mathrm{d}\varpi)\,\log B_n \ge 0$. Concerning the other part of that term, we claim that we may estimate $q_{W,W',n} \ge \mathrm{e}^{-nC/R^2}$ for any large R and n. Indeed, lower bound the event $\{\mathfrak{m}_{W,W'} \ge 1\} \cap \{B_0 = x_i = B_{n\beta}\} \cap \{B_{k\beta} \in W \cup W'\ \forall k \in [n]\}$ by the event that the Brownian motion on the time interval $[0,\beta n]$, starting from and terminating at x_i, runs entirely within $W \cup W'$ and visits both W and W'. Then, for large n, the probability of this event has a logarithmic asymptotics in n with a parameter equal to the principal Dirichlet eigenvalue of the Laplace operator in $W \cup W'$. This eigenvalue is, as $R \to \infty$, asymptotic to some constant times $1/(R+M)^2 \sim 1/R^2$. This implies that

$$\begin{aligned}
-\frac{1}{|W|}\int \xi_R^{(M)}(\mathrm{d}\varpi)\,\log q_{W,W',n} &\leq \frac{C}{R^2}\,\frac{1}{|W|}\int \xi_R^{(M)}(\mathrm{d}\varpi_R)\sum_{i\in I}(l_i+L_i)\\
&\leq \frac{C}{R^2}\,\frac{1}{|W|}\Big(\langle \Pi_W^{(\mathcal{S})}(\xi_R), \mathfrak{N}_W^{(\ell)}\rangle + \int \partial\Pi_W^{(\mathcal{S})}(\xi_R)(\mathrm{d}\mu)\sum_{i\in I}\sum_{m\in\mathbb{N}} m p_{y_i,x_{\sigma(i)}}^{(W')}(m)\Big)\\
&\leq \frac{C}{R^2} + \frac{C}{R^2}\frac{1}{|W|}\int \partial\Pi_W^{(\mathcal{S})}(\xi_R)(\mathrm{d}\mu)\sum_{i\in I}|x_{\sigma(i)}-y_i|,
\end{aligned}$$

using the following Lemma 7.4. We now estimate simply $|x_{\sigma(i)} - y_i| \leq CR$ and $|I| \leq \sum_{i\in I} l_i = \mathfrak{N}_W^{(\ell,\mathcal{S})}(\varpi)$, to see that the last summand is of order $O(1/R)$ and that the right-hand side vanishes as $R \to \infty$.

Lemma 7.4 (Expected length of Brownian bridge in W')**.** *For any $M \in (0,\infty)$, there is a constant $C_M \in (0,\infty)$ (depending on d, β and M) such that, for any $R \in \mathbb{N}$,*

$$\sum_{n\in\mathbb{N}} n p_{y,x}^{(W')}(n) \leq C_M|x-y|, \qquad y \in W', x \in (W')^{\mathrm{c}}. \tag{7.22}$$

Proof. For the proof, it suffices to replace W' by $[-R-M, R+M]^{d-1} \times [0,M]$ (which we will still write as W' in this proof), since $W_{R+M} \setminus W_R$ is a finite union of such sets, modulo shifts and orthogonal transformations. We write $x = (\underline{x}, x^{(d)})$ with $\underline{x} \in [-R-M, R+M]^{d-1}$ and $x^{(d)} \in [0,M]$. The assertion needs to be proved only for large R and large $|\underline{x} - \underline{y}|$. The intuitive reason for (7.22) is that $\tau_{W'}$ has exponential tails as $R \to \infty$ under $\mathbb{P}_y(\cdot \mid B(\tau_{W'}\beta) = x)$, uniformly in $x \in W_R$ and $y \in W'$, since the d-th component of the Brownian bridge lies in $[0,M]$. This implies that the conditioned Gaussian random walk $(B_{n\beta})_{n\in\mathbb{N}}$ is forced to assume a drift into the direction $w(\underline{x} - \underline{y})$ for some $w \in (0,\infty)$ that asymptotically depends only on M.

Let us come to the details. The vector $(\underline{B}_{n\beta})_{n\in\mathbb{N}}$ of the first $d-1$ components and the d-th component $(B_{n\beta}^{(d)})_{n\in\mathbb{N}}$ of the Gaussian random walk are independent, and the stopping time $\tau_{W'}$ depends only on the latter; indeed, we write it as the exit time $\tau_{[0,M]}$ from $[0,M]$ for the d-th component. Let us denote the one-step distribution density of the Gaussian walk by g and its $(d-1)$-dimensional version by $\underline{g}$ and with d-th component version $g^{(d)}$ (these are nothing but the usual Gaussian densities with variance parameter 2β in the respective dimensions). We now integrate over $B_{n\beta}$ to get, in the limit as $R \to \infty$, uniformly in large n and in $y \in W'$ and $x \in (W')^{\mathrm{c}}$ for $|\underline{x} - \underline{y}| \to \infty$,

$$\begin{aligned}
\frac{\mathbb{P}_y(B_{\tau_{W'}\beta} \in \mathrm{d}x)}{\mathrm{d}x} p_{y,x}^{(W')}(n+1) &= \int_{W'} \mathbb{P}_y(\tau_{W'} > n, B_{n\beta} \in \mathrm{d}z) g(z-x)\\
&\sim \mathbb{E}_{\underline{y}}\big[\underline{g}(\underline{B}_{n\beta} - \underline{x})\big]\,\mathbb{E}_{y^{(d)}}\big[\mathbb{1}\{\tau_{[0,M]} > n\} g^{(d)}(B_{n\beta}^{(d)} - x^{(d)})\big]\\
&= \mathrm{e}^{-|\underline{x}-\underline{y}|^2/4\beta(n+1)}\,\mathrm{e}^{-nC_M}\,\mathrm{e}^{o(n)},
\end{aligned}$$

where the first term on the right-hand side comes out of a quadratic extension in the exponent, and the second from subadditivity (the probability of $\{\tau_{[0,M]} > n\}$ decays exponentially in n, while the $g^{(d)}$-expectation is bounded in n). Hence, the term $p_{y,x}^{(W')}(n+1)$ is strictly maximal for $n \sim |\underline{x} - \underline{y}|/\sqrt{4\beta C_M}$. Hence, the sum of $(n+1)p_{y,x}^{(W')}(n+1)$ over n comes mainly from the summands around that value of n, and hence it is asymptotic to $|\underline{x} - \underline{y}|/\sqrt{4\beta C_M}$. This implies the assertion in (7.22).

The right-hand side of the last display is equal to $\mathrm{e}^{-|\underline{y}-\underline{x}|(\sqrt{C_M/\beta}+o(1))}$ for this n. Since the $p_{y,x}^{(W')}(n+1)$ do not decay exponentially fast for $n/|\underline{x}-\underline{y}|$ in a neighbourhood of $\sqrt{4\beta C_M}$, but away from it and sum up to one on n, we also see that $\mathbb{P}_y(B_{\tau_{W'\beta}}\in \mathrm{d}x)/\mathrm{d}x=\mathrm{e}^{-|\underline{y}-\underline{x}|(\sqrt{C_M/\beta}+o(1))}$ as $|\underline{y}-\underline{x}|\to\infty$.

□

Now let us turn to the last term on the right-hand side of (7.21), where we recall the definition of A_σ in (7.18). We are going to write the measure $(\xi_R)_{\mathcal{L}_W\to\mathcal{T}_W}(\omega,\cdot)=\frac{\Pi_W^{(\mathcal{L},\mathcal{T})}(\xi_R)(\mathrm{d}\omega,\cdot)}{\Pi_W^{(\mathcal{L})}(\xi_R)(\mathrm{d}\omega)}$ with the help of projections of the measure $\widetilde{\xi}=\widetilde{\Pi}_{\widetilde{W};M}(\Pi_{\widetilde{W}}(P))$. This measure has a finite entropy with respect to the reference measure in the definition of $J_{\widetilde{W}}$, hence, it has a density $\widetilde{g}\colon\mathcal{L}_{\widetilde{W}}\times\mathcal{S}_{\widetilde{W}}\to[0,\infty)$ with respect to the reference measure $\widetilde{\nu}=\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q})\otimes[\partial\Pi_{\widetilde{W}}^{(\mathcal{S})}(P)\otimes\mathtt{K}_{\widetilde{W}}]$. We conceive the projections $\pi_1=\Pi_W^{(\mathcal{L},\mathcal{T})}\colon\mathcal{L}_{\widetilde{W}}\times\mathcal{S}_{\widetilde{W}}\to\mathcal{L}_W\times\mathcal{T}_W$ and $\pi_2=\Pi_W^{(\mathcal{L})}\colon\mathcal{L}_{\widetilde{W}}\times\mathcal{S}_{\widetilde{W}}\to\mathcal{L}_W$ as projections from configurations in $\widetilde{W}$ to loop/boundary-shred-configurations in W, respectively to loop configurations in W. The factorisation lemma from measure theory implies that there are measurable functions $g_1\colon\mathcal{L}_W\times\mathcal{T}_W\to[0,\infty)$ and $g_2\colon\mathcal{L}_W\to[0,\infty)$ such that $\widetilde{g}=g_2\circ\pi_2=g_1\circ\pi_1$ and $g_1=g_2\circ\pi_3$, where π_3 is the projection from $\mathcal{L}_W\times\mathcal{T}_W$ on $\mathcal{L}_W$, that is, $g_1(\omega,\mu)=g_2(\omega)$. Therefore, they are densities of the image measures of $\widetilde{\xi}$ with respect to the image measures of $\widetilde{\nu}$ under π_1, respectively under π_2, that is,

$$g_1(\omega,\mu)=\frac{\mathrm{d}\pi_1(\widetilde{\xi})}{\mathrm{d}\pi_1(\widetilde{\nu})}(\omega,\mu)\qquad\text{and}\qquad g_2(\omega)=\frac{\mathrm{d}\pi_2(\xi_R)}{\mathrm{d}\pi_2(\widetilde{\nu})}(\omega),\qquad\omega\in\mathcal{L}_W,\mu\in\mathcal{T}_W.$$

Recall from (8.5) that $(\xi_R)_{\mathcal{L}_W\to\mathcal{T}_W}(\omega,\mathrm{d}\mu)=\frac{\Pi_W^{(\mathcal{L},\mathcal{T})}(\xi_R)(\mathrm{d}\omega,\mathrm{d}\mu)}{\Pi_W^{(\mathcal{L})}(\xi_R)(\mathrm{d}\omega)}$ and note that

$$\begin{aligned}\Pi_W^{(\mathcal{L},\mathcal{T})}(\xi_R)&=\Pi_W^{(\mathcal{L},\mathcal{T})}(\widetilde{\Pi}_{W;M}(\Pi_W(P)))=\Pi_W^{(\mathcal{L},\mathcal{T})}(\widetilde{\Pi}_{W;M}(\Pi_{\widetilde{W}\to W}(\Pi_{\widetilde{W}}(P))))\\&=\Pi_W^{(\mathcal{L},\mathcal{T})}(\Pi_{\widetilde{W}\to W}(\widetilde{\Pi}_{\widetilde{W};M}(\Pi_{\widetilde{W}}(P)))=\pi_1(\widetilde{\xi}),\end{aligned}$$

since $\widetilde{\Pi}_{W;M}\circ\Pi_{\widetilde{W}\to W}=\Pi_{\widetilde{W}\to W}\circ\widetilde{\Pi}_{\widetilde{W};M}$ (by $\Pi_{\widetilde{W}\to W}$ we mean Π_W with domain $\mathcal{L}_{\widetilde{W}}\times\mathcal{S}_{\widetilde{W}}$). Analogously, we see that $\Pi_W^{(\mathcal{L})}(\xi_R)=\pi_2(\widetilde{\xi})$. Hence,

$$\frac{(\xi_R)_{\mathcal{L}_W\to\mathcal{T}_W}(\omega,\mathrm{d}\mu)}{A_\sigma(\mathrm{d}\mu)}=\frac{\pi_1(\widetilde{\xi})(\mathrm{d}\omega,\mathrm{d}\mu)}{\pi_2(\widetilde{\xi})(\mathrm{d}\omega)A_\sigma(\mathrm{d}\mu)}=\frac{g_1(\omega,\mu)}{g_2(\omega)}\,\frac{\mathrm{d}\pi_1(\widetilde{\nu})}{\mathrm{d}[\pi_2(\widetilde{\nu})\otimes A_\sigma]}(\omega,\mu)=\frac{\mathrm{d}\pi_1(\widetilde{\nu})}{\mathrm{d}[\pi_2(\widetilde{\nu})\otimes A_\sigma]}(\omega,\mu),$$

where the last step derives from $g_1=g_2\circ\pi_3$.

Note that $\pi_2(\widetilde{\nu})=\Pi_W^{(\mathcal{L})}(\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q}))=\Pi_W^{(\mathcal{L})}(\mathtt{Q})$, since the remainder part of $\widetilde{\nu}$ does not create loops in W. On the other hand, $\pi_1(\widetilde{\nu})=\Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes\partial\Pi_W^{(\mathcal{S})}(\Pi_{\widetilde{W}\setminus W}^{(\mathcal{L},\widetilde{W})}(\mathtt{Q})\otimes[\partial\Pi_{\widetilde{W}}^{(\mathcal{S})}(P)\otimes\mathtt{K}_{\widetilde{W}}])$, where we denote by $\Pi_{\widetilde{W}\setminus W}^{(\mathcal{L},\widetilde{W})}$ the projection on all the loops that start in $\widetilde{W}\setminus W$ and have all the particles in $\widetilde{W}$. Therefore,

$$\frac{\mathrm{d}\pi_1(\widetilde{\nu})}{\mathrm{d}[\pi_2(\widetilde{\nu})\otimes A_\sigma]}(\omega,\mu)=\frac{\mathrm{d}\partial\Pi_W^{(\mathcal{S})}(\Pi_{\widetilde{W}\setminus W}^{(\mathcal{L},\widetilde{W})}(\mathtt{Q})\otimes[\partial\Pi_{\widetilde{W}}^{(\mathcal{S})}(P)\otimes\mathtt{K}_{\widetilde{W}}])}{\mathrm{d}A_\sigma}(\mu).$$

The measure in the numerator is equal to $A_{\sigma'}$ for some random bijective $\sigma'\colon I\to I$ whose distribution depends on $\partial\Pi_{\widetilde{W}}^{(\mathcal{S})}(P)$. Hence, the quotient is of the form $\prod_{i\in I}\mathbb{P}_{(\sigma')^{-1}(i)}(B(\tau_{W'\beta})\in\mathrm{d}x_i)/\mathbb{P}_{\sigma^{-1}(i)}(B(\tau_{W'\beta})\in\mathrm{d}x_i)$, which is bounded from above by $C_M^{|\mu|}$ for some constant C_M, depending only on M, d and β. We have to integrate its logarithm with respect to ξ_R, which gives $\log C_M$ times the expectation of $|\mu|$ under $\partial\Pi_W^{(\mathcal{S})}(\xi_R)=\partial\Pi_W^{(\mathcal{S})}(P)$, i.e., the expected number of W-shreds under P. This is $o(|W|)$, according to Corollary 5.8. This shows that the last term on the right-hand side of (7.21) is $o(|W|)$ and hence negligible.

This finishes the proof of Lemma 2.13.

7.4. **Preparation: entropy of projection image measures.** Fix W and recall the events $A_{W;\mathfrak{r},M,L,K,T}$ defined in (4.34)–(4.36) and the restricted shred-configuration set $\mathcal{S}_W^{(M,\vartheta,S)}$ defined in (5.21) and the next line. In this section, we show that the entropy $J_W(\xi)$ is well approximated with the entropy $J_W(\widetilde{\xi})$ of a measure $\widetilde{\xi}$ that approximates ξ and is concentrated on the above restricted set of configurations, if the parameters are picked large. More precisely, we estimate the entropy $J_W(\xi\circ\Pi^{-1}_{W;\mathfrak{r},M,L,K,T,\vartheta,S})$ of some projection $\Pi_{W;\mathfrak{r},M,L,K,T,\vartheta,S}$ of an arbitrary measure ξ on $A_{W;\mathfrak{r},M,L,K,T}\cap\mathcal{S}_W^{(M,\vartheta,S)}$ in terms of $J_W(\xi)$. We want to show that the projected entropy is not much larger than the original entropy after taking $W\uparrow\mathbb{R}^d$ and M,L,K and $T,\vartheta\to\infty$, uniformly in $S\in\mathbb{N}$. This will be used in the proof of the ergodic approximation in Lemma 7.6, which will be used in several further proofs.

Lemma 7.5 (Entropy of projection image)**.** *Fix a box $W\Subset\mathbb{R}^d$ and $\mathfrak{r},M,L,K,T,\vartheta\in(0,\infty)$ and $S\in\mathbb{N}$. Then there is a projection $\Pi=\Pi_{W;\mathfrak{r},M,L,K,T,\vartheta,S}\colon\mathcal{L}_W\times\mathcal{S}_W\to A_{W;\mathfrak{r},M,L,K;T}\cap(\mathcal{L}_W\times\mathcal{S}_W^{(M,\vartheta,S)})$ and there are $\varepsilon^{(\mathcal{L})}_{M,W,L}$ and $\varepsilon^{(\mathcal{S})}_{M,\vartheta,S,T}\in(0,1)$ satisfying, for any S,*

$$\lim_{L\to\infty}\lim_{M\to\infty}\limsup_{W\uparrow\mathbb{R}^d}\varepsilon^{(\mathcal{L})}_{M,W,L}=0\qquad\textit{and}\qquad\lim_{M,\vartheta,T\to\infty}\varepsilon^{(\mathcal{S})}_{M,\vartheta,S,T}=0,$$

such that, for any $\xi\in\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W)$ with finite expected total particle number $\langle\xi,\mathfrak{N}_W^{(\ell)}\rangle$, and for any $S\in\mathbb{N}$,

$$J_W(\xi\circ\Pi^{-1})\le J_W(\xi)+\varepsilon^{(\mathcal{L})}_{M,W,L}+\Big(\frac{1}{K}+\varepsilon^{(\mathcal{S})}_{M,\vartheta,S,T}\Big)\frac{1}{|W|}\langle\xi,\mathfrak{N}_W^{(\ell)}\rangle. \tag{7.23}$$

Proof. For the ease of notation, we write $\xi^{(\mathcal{L})}$ instead of $\Pi_W^{(\mathcal{L})}(\xi)$ and $\xi^{(\mathcal{S})}$ instead of $\Pi_W^{(\mathcal{S})}(\xi)$ and $\partial\xi^{(\mathcal{S})}$ instead of $\partial\Pi_W^{(\mathcal{S})}(\xi)$. Furthermore, we write $\mathtt{Q}_W$ instead of $\Pi_W^{(\mathcal{L})}(\mathtt{Q})$. Furthermore, we drop the indices W, $\mathfrak{r},M,L,K$, and T,ϑ,S and write $\Pi^{(\mathcal{L})}$ and $\Pi^{(\mathcal{S})}$ for the respective projections on $A^{(\mathcal{L})}$ and $A^{(\mathcal{S})}\cap\mathcal{S}_W^{(M,\vartheta,S)}$ (to be defined in this proof). Then we put $\Pi=\Pi^{(\mathcal{L})}\otimes\Pi^{(\mathcal{S})}$. We are going to use the decomposition (7.2) of the entropies into a loop part and a shred part.

Note that, on any measurable space Ω,

$$H(P\circ f^{-1}\mid Q\circ f^{-1})\le H(P\mid Q),\qquad P,Q\in\mathcal{M}_1(\Omega),f\colon\Omega\to\Omega\text{ mb.},\tag{7.24}$$

since the left-hand side is equal to the relative entropy of P with respect to Q on the σ-field generated by f, and relative entropies are increasing in the σ-field [G11, Prop. 15.5(c)].

Therefore, we have

$$\begin{aligned}
&H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi\circ\Pi^{-1}\mid\mathtt{Q}_W\otimes[\partial(\xi\circ\Pi^{-1})\otimes\mathtt{K}_W]\big)\\
&\qquad=H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi\circ\Pi^{-1}\mid[\mathtt{Q}_W\otimes[\partial\xi\otimes\mathtt{K}_W]]\circ\Pi^{-1}\big)+\int\mathrm{d}\xi\circ\Pi^{-1}\,\log\frac{\mathrm{d}[\mathtt{Q}_W\otimes[\partial\xi\otimes\mathtt{K}_W]]\circ\Pi^{-1}}{\mathrm{d}\mathtt{Q}_W\otimes[\partial(\xi\circ\Pi^{-1})\otimes\mathtt{K}_W]}\\
&\qquad\le H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi\mid\mathtt{Q}_W\otimes[\partial\xi\otimes\mathtt{K}_W]\big)+\int_{\mathcal{L}_W}\mathrm{d}\xi^{(\mathcal{L})}\,\log\frac{\mathrm{d}[\mathtt{Q}_W\circ(\Pi^{(\mathcal{L})})^{-1}]}{\mathrm{d}\mathtt{Q}_W}\circ\Pi^{(\mathcal{L})}\\
&\qquad\quad+\int_{\mathcal{S}_W}\mathrm{d}\xi^{(\mathcal{S})}\,\log\frac{\mathrm{d}[\partial\xi\otimes\mathtt{K}_W]\circ(\Pi^{(\mathcal{S})})^{-1}}{\mathrm{d}[\partial\xi\otimes\mathtt{K}_W]}\circ\Pi^{(\mathcal{S})}-\int_{\mathcal{S}_W}\mathrm{d}\xi^{(\mathcal{S})}\,\log\frac{\mathrm{d}[\partial(\xi^{(\mathcal{S})}\circ(\Pi^{(\mathcal{S})})^{-1})\otimes\mathtt{K}_W]}{\mathrm{d}\partial\xi^{(\mathcal{S})}\otimes\mathtt{K}_W}\circ\Pi^{(\mathcal{S})}.
\end{aligned}\tag{7.25}$$

Note that, in the last term on the right-hand side of (7.25), we can drop the $\mathtt{K}_W$-terms and see that this is a relative entropy, which is nonnegative; hence we can drop it with "$\le$".

Now we handle the second term on the right-hand side of (7.25). We use the projection $\Pi = \Pi^{(\mathcal{L})}_{W;\mathfrak{r},M,L,K}\colon \mathcal{L}_W \to A^{(\mathcal{L})}_{W;\mathfrak{r},M,L,K}$ defined, for all $\omega = \sum_{x\in\zeta}\delta_{(x,f_x)}$, by

$$\Pi(\omega) = \Pi^{(\mathcal{L})}_{W;\mathfrak{r},M,L,K}(\omega) = \sum_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\sum_{l\in[L]}\Big[\mathbb{1}\{|\zeta_z^{(M,l)}|\leq K\}\sum_{x\in\zeta_z^{(M,l)}}\delta_{(x,f_x)}$$
$$+\mathbb{1}\{|\zeta_z^{(M,l)}|>K\}\frac{1}{\binom{|\zeta_z^{(M,l)}|}{K}}\sum_{B\subset\zeta_z^{(M,l)}\colon |B|=K}\sum_{x\in B}\delta_{(x,f_x)}\Big],$$

where $\zeta_z^{(M,l)} = \{x\in\zeta\cap(z+W_\mathfrak{r})\colon \ell(f_x)=l, f_x\in\mathcal{C}^{(M,\circlearrowleft)}\}$, where $\mathcal{C}^{(M,\circlearrowleft)}$ denotes the set of all loops all of whose legs have a spread $\leq M$. In other words, Π first removes all loops that are longer than L or have a leg with spread $>M$ and then removes in each subbox $z+W_\mathfrak{r}$ and for any $l\in[L]$, if necessary, a uniformly distributed choice of loops such that no more than K loops of length $=l$ survive. This algorithm can be conceived as being carried out independently in each of these subboxes.

Recall that $\mathcal{L}_W^{(M)}$ denotes the set of loop configurations in $\mathcal{L}_W$ (i.e., with all particles in W) that have only legs in $\mathcal{C}^{(M)}$, i.e., with all the spreads $\leq M$. With an additional super-index l we restrict to configurations with all loop lengths $=l$, and the super-index $\leq L$ means the union over $l\in[L]=\{1,\dots,L\}$. With the index $W;\mathfrak{r},W'$ for $W'\subset W$ we mean those configurations with particles in W that have initial site in W'. All this notation also applies to measures Q.

For ω in the image of Π, the pre-image $\Pi^{-1}(\{\omega\})$ is equal to the set of configurations of the form $\sum_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}(\omega_z+\omega'_z+\sum_{l\in[L]}\omega''_{z,l})$, where $\omega_z=\sum_{x\in\zeta\cap(z+W_\mathfrak{r})}\delta_{(x,f_x)}$ (the projection of ω to that subbox), and $\omega'_z\in\mathcal{L}_{W;\mathfrak{r},z+W_\mathfrak{r}}\setminus\mathcal{L}^{(M,\leq L)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ and $\omega''_{z,l}=0$ (empty configuration) if $|\zeta_z^{(M,l)}|<K$ and $\omega''_{z,l}\in\mathcal{L}^{(M,l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ otherwise. Note that all the configurations ω'_z and $\omega''_{z,l}$ are independent over z and l under Q_W. For fixed z and l, we split $\mathsf{Q}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ into the PPP $\mathsf{Q}^{(M,l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ on $\mathcal{L}^{(M,l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ and $\mathcal{L}^{(\infty,l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}\setminus\mathcal{L}^{(M,l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$. The first one has an intensity measure with total mass

$$q_z^{(M,l)} = \frac{1}{l}\int_{z+W_\mathfrak{r}}\mathrm{d}x\,\mu_{x,x}^{(M,l,W,\mathrm{par})}(\mathcal{C}^{(\circlearrowleft)}),$$

(recall (1.6)), where $\mu_{x,x}^{(M,l,W,\mathrm{par})}$ is the restriction of $\mu_{x,x}^{(l,W,\mathrm{par})}$ to loops configurations with loops in the set $\mathcal{C}^{(M,\circlearrowleft)}=\{f\in\mathcal{C}^{(\circlearrowleft)}\colon\forall i\colon f_{x,i}\in\mathcal{C}_M\}$. These two PPPs are independent. Therefore, the density f of $\mathsf{Q}_W\circ\Pi^{-1}$ with respect to Q_W is the product

$$f(\omega) = \prod_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\Big[\mathrm{e}^{\sum_{l=L+1}^\infty q_z^{(l)}}\prod_{l\in[L]}f_{z,l}(\omega_z^{(l)})\Big] = \mathrm{e}^{\nu^{(W,\mathrm{par})}(\mathcal{C}^{(\circlearrowleft)}_{>L})}\prod_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\prod_{l\in[L]}f_{z,l}(\omega_z^{(l)}),\tag{7.26}$$

with a density $f_{z,l}$ of $\mathsf{Q}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}\circ\Pi^{-1}_{z+W_\mathfrak{r}}$ with respect to $\mathsf{Q}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$, where $\mathsf{Q}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ denotes the PPP on the set $\mathcal{L}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ of loop configurations with starting sites in $z+W_\mathfrak{r}$ and all the particles contained in W. The q-term expresses that, for $l>L$, the pre-image of the empty configuration is equal to $\mathcal{L}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ and the empty configuration has probability $\mathrm{e}^{-q_z^{(l)}}$ under $\mathsf{Q}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$. Recall that $\nu^{(W,\mathrm{par})}$ denotes the intensity measure of the PPP on W, and we wrote $\mathcal{C}^{(\circlearrowleft)}_{>L}$ for the set of loops with lengths $>L$.

Hence, the only difference between the distribution of a given $\omega_z^{(l)}\in\mathcal{L}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}\setminus\mathcal{L}^{(M,l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ under $\mathsf{Q}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}\circ\Pi^{-1}$ and under $\mathsf{Q}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}$ is the change of the Poisson parameter from $q_z^{(M,l)}$ to $q_z^{(l)}=q_z^{(\infty,l)}$. Therefore, for ω_z with $|\zeta_z^{(M,l)}|<K$, we have

$$f_{z,l}(\omega_z^{(l)}) = \frac{\mathrm{d}\mathsf{Q}^{(M,l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}}{\mathrm{d}\mathsf{Q}^{(l)}_{W;\mathfrak{r},z+W_\mathfrak{r}}}(\omega_z^{(l)}) = \mathrm{e}^{q_z^{(l)}-q_z^{(M,l)}}\Big(\frac{q_z^{(M,l)}}{q_z^{(l)}}\Big)^{|\zeta_{z,l}|},$$

where $\zeta_{z,l}$ is the set of initial sites of the loops in $z+W_\mathfrak{r}$ with length l.

For $\omega_z^{(l)}$ with $|\zeta_z^{(M,l)}| = K$, in the pre-image there may be additional loops in $\mathcal{L}_{W;\mathfrak{r},z+W_\mathfrak{r}}^{(M,l)}$ with uniformly and independently distributed locations, such that an additional Poisson density comes in, but also a combinatorial term that assigns the m points to the additional loops and the other K to the existing loop:

$$\begin{aligned} f_{z,l}(\omega_z^{(l)}) &= \frac{\mathrm{d}\mathsf{Q}_{W;\mathfrak{r},z+W_\mathfrak{r}}^{(M,l)}}{\mathrm{d}\mathsf{Q}_{W;\mathfrak{r},z+W_\mathfrak{r}}^{(l)}}(\omega_z^{(l)}) \sum_{m\in\mathbb{N}} \frac{\mathrm{Poi}_{q_z^{(M,l)}}(K+m)}{\mathrm{Poi}_{q_z^{(M,l)}}(K)\mathrm{Poi}_{q_z^{(M,l)}}(m)} \mathsf{Q}_{W;\mathfrak{r},z+W_\mathfrak{r}}(\{\omega_z'\colon |\zeta_{z,l}| = m\}) \\ &= \mathrm{e}^{q_z^{(l)} - q_z^{(M,l)}} \Big(\frac{q_z^{(M,l)}}{q_z^{(l)}}\Big)^{|\zeta_{z,l}|} \mathrm{e}^{q_z^{(M,l)}} = \mathrm{e}^{q_z^{(l)}} \Big(\frac{q_z^{(M,l)}}{q_z^{(l)}}\Big)^{|\zeta_{z,l}|}. \end{aligned}$$

Recall that we picked $\mathfrak{r}$ such that W is the disjoint (up to boundaries) union of the subboxes $z+W_\mathfrak{r}$ with $z \in W \cap 2\mathfrak{r}\mathbb{Z}^d$. Therefore $\sum_z \sum_{l\in\mathbb{N}} q_z^{(l)} = \nu^{(W,\mathrm{par})}(\mathcal{C}^{(\circlearrowleft)})$ and $\sum_z \sum_{l\in\mathbb{N}} q_z^{(M,l)} = \nu^{(W,\mathrm{par})}(\mathcal{C}^{(M,\circlearrowleft)})$, the total mass of the reference measure of the set of all loop configurations with all spreads of the legs $\leq M$. Therefore, we can summarize

$$\begin{aligned} f(\omega) &= \mathrm{e}^{\nu^{(W,\mathrm{par})}(\mathcal{C}_{>L}^{(\circlearrowleft)})} \mathrm{e}^{\nu^{(W,\mathrm{par})}(\mathcal{C}_{\leq L}^{(\circlearrowleft)}) - \nu^{(W,\mathrm{par})}(\mathcal{C}_{\leq L}^{(M,\circlearrowleft)})} \prod_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d} \prod_{l\in[L]} \Big[\Big(\frac{q_z^{(M,l)}}{q_z^{(l)}}\Big)^{|\zeta_{z,l}|} \mathrm{e}^{q_z^{(M,l)} 1\!\!1\{|\zeta_{z,l}|=K\}}\Big] \\ &\leq \mathrm{e}^{\varepsilon_{M,W,L}^{(\mathcal{L})}|W|} \exp\Big\{\sum_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d} \sum_{l\in[L]} q_z^{(M,l)} 1\!\!1\{|\zeta_{z,l}| = K\}\Big\} \\ &\leq \mathrm{e}^{\varepsilon_{M,W,L}^{(\mathcal{L})}|W|} \exp\Big\{\frac{1}{K} \sum_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d} \sum_{l\in[L]} q_z^{(M,l)} |\zeta_{z,l}|\Big\} \\ &\leq \mathrm{e}^{\varepsilon_{M,W,L}^{(\mathcal{L})}|W|} \mathrm{e}^{\frac{|\zeta|}{K}}, \end{aligned}$$

where we put $\varepsilon_{M,W,L}^{(\mathcal{L})} = \frac{1}{|W|}[\nu^{(W,\mathrm{par})}(\mathcal{C}_{>L}^{(\circlearrowleft)}) + \nu^{(\mathrm{par})}(W \times [\mathcal{C}_{\leq L}^{(\circlearrowleft)} \setminus \mathcal{C}_{\leq L}^{(M,\circlearrowleft)}])]$.

Hence, the second term on the right-hand side of (7.25) can be estimated as follows.

$$\begin{aligned} \int \mathrm{d}\xi^{(\mathcal{L})} \log\Big(\frac{\mathrm{d}(\mathsf{Q}^{(\mathcal{L})} \circ (\Pi^{(\mathcal{L})})^{-1})}{\mathrm{d}\mathsf{Q}^{(\mathcal{L})}} \circ \Pi^{(\mathcal{L})}\Big) &= \int \mathrm{d}\xi^{(\mathcal{L})} \log f(\Pi^{(\mathcal{L})}(\omega)) \leq |W|\varepsilon_{M,W,L}^{(\mathcal{L})} + \frac{1}{K}\int \mathrm{d}\xi^{(\mathcal{L})} |\zeta| \\ &\leq |W|\varepsilon_{M,W,L}^{(\mathcal{L})} + \frac{1}{K}\langle \xi, \mathfrak{N}_W^{(\ell)}\rangle. \end{aligned}$$

We leave it to the reader to prove that $\lim_{L\to\infty} \lim_{M\to\infty} \limsup_{W\uparrow\mathbb{R}^d} \varepsilon_{M,W,L}^{(\mathcal{L})} = 0$.

Now we turn to estimating the third term on the right-hand side of (7.25). We define $\Pi^{(\mathcal{S})}\colon \mathcal{S}_W \to A_{W;\mathfrak{r},M,T}^{(\mathcal{S})} \cap \mathcal{S}_W^{(M,\vartheta,S)}$ by describing in words the removal procedure as follows. From a given shred configuration $\varpi = \sum_{g\in\Gamma} \delta_g$, first all shreds are removed that do not lie in

$$\mathcal{C}^{(M,\vartheta,S)} = \bigcup_{k\in\mathbb{N}} \big\{f \in \mathcal{C}_k\colon f_i \in \mathcal{C}^{(M)}, |f(i\beta) - f((i+S)\beta)| \leq \vartheta\sqrt{S}\ \forall i\big\}.$$

Then, with a fixed enumeration $(z_1, z_2, \dots, z_N)$ of $W \cap 2\mathfrak{r}\mathbb{Z}^d$, we successively for $i = 1, \dots, N$ remove shreds from the configuration such that the particle number in $z_i + W_\mathfrak{r}$ is not larger than T. We write $\mathcal{S}_W^{\neg(M,\vartheta,S)}$ for the set of all configurations of W-shreds that do not lie in $\mathcal{C}^{(M,\vartheta,S)}$, i.e., the set of $\sum_{g\in\Gamma}\delta_g$ with $g \in \mathcal{C} \setminus \mathcal{C}^{(M,\vartheta,S)}$ for all $g \in \Gamma$.

For simplicity, we write Π instead of $\Pi^{(\mathcal{S})}$. For a given ϖ in the image of Π, the pre-image $\Pi^{-1}(\{\varpi\})$ is contained in the set of all W-shred configurations $\varpi + \varpi' + \varpi''$ with $\varpi' \in \mathcal{S}_W^{\neg(M,\vartheta,S)}$ and $\varpi'' \in \mathcal{S}_W^{(M,\vartheta,S)}$ being either the empty configuration or satisfying $\widetilde{\mathfrak{N}}_{z+W_\mathfrak{r}}(\varpi'' + \varpi) > T$ for some $z \in W \cap 2\mathfrak{r}\mathbb{Z}^d$. Denote $\mathcal{S}_{z,\varpi}^{(M,\vartheta,S)} = \{\varpi'' \in \mathcal{S}_W^{(M,\vartheta,S)}\colon \widetilde{\mathfrak{N}}_{z+W_\mathfrak{r}}(\varpi'') > T - \widetilde{\mathfrak{N}}_{z+W_\mathfrak{r}}(\varpi)\}$, then

$$\Pi^{-1}(\{\varpi\}) \subset \{\varpi\} \cup \Big(\{\varpi\} + (\mathcal{S}_W^{\neg(M,\vartheta,S)} \setminus \{0\}) + \bigcup_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d} \mathcal{S}_{z,\varpi}^{(M,\vartheta,S)}\Big).$$

Then, writing $\partial\xi$ instead of $\partial\Pi_W^{(\mathcal{S})}(\xi)$ and μ for $\partial\varpi$ and $\mathtt{K}$ instead of $\mathtt{K}_W$, we have, using the product structure of $\mathtt{K}$,

$$(\partial\xi\otimes\mathtt{K})\circ\Pi^{-1}(\mathrm{d}\varpi)\leq\partial\xi(\mathrm{d}\mu)\mathtt{K}(\mu,\mathrm{d}\varpi)\Big[1+\int_{\mathcal{T}_W}\frac{\xi(\mathrm{d}\mu+\mathrm{d}\mu')}{\partial\xi(\mathrm{d}\mu)}\mathbb{1}\{|\mu'|\geq 1\}\int_{(\mathcal{S}_W^{\neg(M,\vartheta,S)}\setminus\{0\})+\bigcup_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\mathcal{S}_{z,\varpi}^{(M,\vartheta,S)}}\mathtt{K}(\mu',\mathrm{d}\varpi')\Big].$$

Hence, the third term on the right-hand side of (7.25) can be estimated as follows (using $\log(1+x)\leq x$):

$$\int_{\mathcal{S}_W}\mathrm{d}\xi^{(\mathcal{S})}\,\log\frac{\mathrm{d}[\partial\xi\otimes\mathtt{K}_W]\circ(\Pi^{(\mathcal{S})})^{-1}}{\mathrm{d}[\partial\xi\otimes\mathtt{K}_W]}\circ\Pi^{(\mathcal{S})}\leq\int_{\mathcal{S}_W}\xi(\mathrm{d}\varpi)\int_{\mathcal{T}_W}\frac{\partial\xi(\mathrm{d}\mu+\mathrm{d}\mu')}{\partial\xi(\mathrm{d}\mu)}\mathbb{1}\{|\mu'|\geq 1\}\mathtt{K}\Big(\mu',(\mathcal{S}_W^{\neg(M,\vartheta,S)}\setminus\{0\})+\bigcup_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\mathcal{S}_{z,\Pi(\varpi)}^{(M,\vartheta,S)}\Big).\tag{7.27}$$

Since $\mathcal{S}_W^{\neg(M,\vartheta,S)}$ and $\bigcup_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\mathcal{S}_{z,\Pi(\varpi)}^{(M,\vartheta,S)}$ are disjoint sets of W-shreds, its elements can be written as the sum over configurations of these two sets. Depending on these two, μ' can be split into the sum of $\widetilde{\mu}=\sum_{i\in\widetilde{I}}\delta_{(x_i,l_i,y_i)}$ and $\mu'\setminus\widetilde{\mu}=\sum_{i\in I'}\delta_{(x_i,l_i,y_i)}$, and the last term in the last display decomposes into the according product of $\mathtt{K}(\widetilde{\mu},\mathcal{S}_W^{\neg(M,\vartheta,S)}\setminus\{0\})$ and $\mathtt{K}(\mu'\setminus\widetilde{\mu},\bigcup_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\mathcal{S}_{z,\Pi(\varpi)}^{(M,\vartheta,S)})$. We estimate the second term as follows.

$$\begin{aligned}\mathtt{K}\Big(\mu'\setminus\widetilde{\mu},\bigcup_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\mathcal{S}_{z,\Pi(\varpi)}^{(M,\vartheta,S)}(\Pi(\varpi))\Big)&\leq\mathtt{K}\Big(\mu'\setminus\widetilde{\mu},\bigcup_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\{\varpi'\colon\widetilde{\mathfrak{N}}_{z+W_{\mathfrak{r}}}(\varpi')>T-\widetilde{\mathfrak{N}}_{z+W_{\mathfrak{r}}}(\Pi(\varpi))\}\Big)\\&\leq\frac{1}{T}\sum_{z\in W\cap 2\mathfrak{r}\mathbb{Z}^d}\int_{\mathcal{S}_W}\mathtt{K}(\mu'\setminus\widetilde{\mu},\mathrm{d}\varpi')\,\widetilde{\mathfrak{N}}_{z+W_{\mathfrak{r}}}(\varpi'+\Pi(\varpi))\\&\leq\frac{1}{T}\Big[\mathfrak{N}_W^{(\ell,\mathcal{S})}(\varpi)+\sum_{i\in I'\setminus\widetilde{I}}l_i\Big],\end{aligned}$$

where in the last step we used that $\Pi(\varpi)\leq\varpi$ and that W is the (disjoint, up to boundaries) union of the boxes $z+W_{\mathfrak{r}}$, $z\in W+2\mathfrak{r}\mathbb{Z}^d$.

For estimating the other, we use that the probability of the set of single W-shreds of length k, namely $\mathcal{C}_k^{(M,\vartheta,S)}$ under $\mathtt{K}(\delta_{(x,l,y)},\cdot)$ for any (x,l,y) has a probability $\geq(1-\varepsilon_M)^k(1-\varepsilon_{\vartheta,S})^{k/S}$ for some ε_M and $\varepsilon_{\vartheta,S}$ that vanish if $M\to\infty$ respectively $\vartheta\to\infty$ (both do not depend on R). The reason is that all spreads of the legs and all the differences $g(iS\beta)-g((i-1)S\beta)$ are independent. Since these W-shreds are independent under $\mathtt{K}(\widetilde{\mu},\cdot)$, we can estimate $\mathtt{K}(\widetilde{\mu},\mathcal{S}_W^{\neg(M,\vartheta,S)}\setminus\{0\})\leq 1-\prod_{i\in\widetilde{I}}(1-\varepsilon_M)^{l_i}(1-\varepsilon_{\vartheta,S})^{l_i/S}\leq C(\varepsilon_M+\varepsilon_{\vartheta,S})\sum_{i\in\widetilde{I}}l_i$.

We estimate now the product against the maximum of these two upper bounds (also depending on whether $\widetilde{\mu}=0$ or $\mu'\setminus\widetilde{\mu}=0$) and the other one against one, and for simplicity the entire term against the sum of the two upper bounds. Substituting this above, we see that the right-hand side of (7.27) is not larger than

$$\int_{\mathcal{S}_W}\xi(\mathrm{d}\varpi)\int_{\mathcal{T}_W}\frac{\partial\xi(\mathrm{d}\mu+\mathrm{d}\mu')}{\partial\xi(\mathrm{d}\mu)}\Big[C(\varepsilon_M+\varepsilon_{\vartheta,S})\sum_{i\in\widetilde{I}}l_i+\frac{1}{T}\mathfrak{N}_W^{(\ell,\mathcal{S})}(\varpi)+\frac{1}{T}\sum_{i\in I'}l_i\Big]\leq\varepsilon_{M,\vartheta,S,T}^{(\mathcal{S})}\langle\xi,\mathfrak{N}_W^{(\ell,\mathcal{S})}\rangle,$$

with an obvious choice of $\varepsilon_{M,\vartheta,S,T}^{(\mathcal{S})}$ such that $\lim_{M,\vartheta,T\to\infty}\varepsilon_{M,\vartheta,S,T}^{(\mathcal{S})}=0$ for any S. □

7.5. **Ergodic approximation.** In the main result of this section, Lemma 7.6, we prove a technical, but crucial approximation that will be used in several important other proofs. The main message is that an arbitrary shift-invariant loop-interlacement distribution can be replaced by an ergodic measure that has only loops and interlacements with additional nice properties, without changing the crucial properties (particle densities in loops and in interlacements, interaction energy, entropy) much. More precisely, in the crucial variational formula $\chi(\rho_1,\rho_2)$ in (2.24), we show that the infimum can be restricted, with only a small error, to the set of ergodic random configurations such that their projection on $W=[-R,R]^d$ is concentrated on the event $A_{W;\mathfrak{r},M,L,K,T}$ defined in (4.34), (4.35) and (4.36) and on $\mathcal{S}_W^{(M,\vartheta,S)}$ (defined in (5.21)) if M,L,K,T,ϑ are picked large enough, uniformly in $S\in\mathbb{N}$. Here we will use Lemma 7.5 to handle the entropy part of the variational formula.

Lemma 7.6 is crucial for applying several preliminary results that rely on such special properties of the random configurations, since these imply for example that the error term in the large-deviations lower bound in Lemma 5.5 can be controlled and that the interaction energy per unit volume is bounded. More explicitly, Lemma 7.6 will be used in the proof of the lower bound of Theorem 2.5 in Section 9.2 (it is what we need in order to go on from (6.35)) and in the proof of Theorem 2.3 in Section 8 and was already used in the proof of Lemma 2.12 in Section 7.1.

We follow a similar ansatz as in the proof of Lemma 2.13. Indeed, for a given loop/shred configuration in a given large box, we do not change anything in the configuration in the box, but we rewire all the ends of the shreds (the loop shreds and the interlacement shreds) in such a way that they are now connected with certain regularly placed sites at the boundary of a slightly larger box. Doing this independently in all these slightly larger subboxes (which decompose $\mathbb{R}^d$), they connect up with the new shreds in the neighbouring subbox. This leads to a global configuration whose crucial three quantities (entropy, interaction energy and particle densities) are hardly changed and which has much better independence properties, in particular, is ergodic.

Lemma 7.6 (Ergodic approximation)**.** *Fix $\rho_1,\rho_2\in(0,\infty)$ and fix $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ with finite $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$ and finite $\langle F_U,P\rangle$ and $\langle P,\mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle=\rho_1$, $\langle P,\mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle=\rho_2$. Furthermore, fix $\varepsilon\in(0,1)$.*

Then, for any sufficiently large ϑ,M,L,K,T,R (in this order), there is an ergodic $P'\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ such that $\Pi_{W_{R+M}}(P')(A_{W_{R+M};\mathfrak{r},M,L,K,T})=1=\Pi^{(\mathcal{S})}_{W_{R+M}}(P')(S^{(M,\vartheta,S)}_{W_{R+M}})$, $\langle P',\mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle\in\mathcal{B}_\varepsilon(\rho_1)$, $\langle P',\mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle\in\mathcal{B}_\varepsilon(\rho_2)$ and $\frac{1}{|W_{R+M}|}\langle F_{W_{R+M},W_{R+M}},\Pi_{W_{R+M}}(P')\rangle\leq\varepsilon+\langle F_U,P\rangle$ and $J_{W_{R+M}}(\Pi_{W_{R+M}}(P'))\leq\varepsilon+J_{W_R}(\Pi_{W_R}(P))$.

Proof. Our proof is in the spirit of the one of Lemma 2.13 in Section 7.3. Again we abbreviate

$$W=W_R=[-R,R]^d\qquad\text{and}\qquad\widetilde W=W_{R+M}=[-R-M,R+M]^d\qquad\text{and put}\qquad W'=\widetilde W\setminus W.$$

As usual, $\Pi_{\widetilde W}$ is the well-known projection on $\widetilde W$. Denote by $\widetilde\Pi_{\widetilde W}=\Pi_{\widetilde W;\mathfrak{r},M,L,K,T}$ the projection onto $A_{\widetilde W;\mathfrak{r},M,L,K,T}$ introduced in Lemma 7.5 under the name Π (with $\widetilde W$ instead of W). We pick M,K,L,ϑ,T so large that the particle densities of the $\widetilde\Pi$-projection of P are close enough to the ones of P and the error terms in Lemma 7.5 are small enough, more precisely, $|\langle\mathfrak{N}_U^{(\ell,\mathcal{L})}\circ\widetilde\Pi_U,\Pi_U(P)\rangle-\rho_1|\leq\varepsilon$ and $|\langle\mathfrak{N}_U^{(\ell,\mathcal{S})}\circ\widetilde\Pi_U,\Pi_U(P)\rangle-\rho_2|\leq\varepsilon$ and that $\lim_{R'\to\infty}\varepsilon^{(\mathcal{L})}_{M,[-R',R']^d,L}+(\frac1K+\varepsilon^{(\mathcal{S})}_{M,\vartheta,S,T})(\rho_1+\rho_2+1)\leq\varepsilon$.

Then, for any sufficiently large R we will construct an ergodic $\widetilde P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ such that

$$\frac{1}{|\widetilde W|}\langle F_{\widetilde W,\widetilde W}\circ\widetilde\Pi_{\widetilde W},\Pi_{\widetilde W}(\widetilde P)\rangle-\frac{1}{|W|}\langle F_{W,W}\circ\widetilde\Pi_W,\Pi_W(P)\rangle\ \leq\ \varepsilon,\tag{7.28}$$

$$\frac{1}{|\widetilde W|}\langle[\mathfrak{N}_{\widetilde W}^{(\ell,\mathcal{L})}+\mathfrak{N}_{\widetilde W}^{(\ell,\mathcal{S})}]\circ\widetilde\Pi_{\widetilde W},\Pi_{\widetilde W}(\widetilde P)\rangle-\frac{1}{|W|}\langle[\mathfrak{N}_W^{(\ell,\mathcal{L})}+\mathfrak{N}_W^{(\ell,\mathcal{S})}]\circ\widetilde\Pi_W,\Pi_W(P)\rangle|\ \leq\ \varepsilon,\tag{7.29}$$

$$J_{\widetilde W}(\Pi_{\widetilde W}(\widetilde P))-J_W(\Pi_W(P)\circ\widetilde\Pi_W^{-1})\ \leq\ \varepsilon.\tag{7.30}$$

This will end the proof of this lemma with $P'=\widetilde P\circ\widetilde\Pi_{\widetilde W}^{-1}$, since $\frac{1}{|W|}\langle F_{W,W}\circ\widetilde\Pi_W,\Pi_W(P)\rangle\leq\langle F_U,P\rangle$ and, if R is large enough, $J_{\widetilde W}(\Pi_{\widetilde W}(\widetilde P)\circ\widetilde\Pi_{\widetilde W}^{-1})\leq\varepsilon+J_{\widetilde W}(\Pi_{\widetilde W}(\widetilde P))$ (see Lemma 7.5 and our choice of $M,K,L;T$ above), and $J_W(\Pi_W(P)\circ\widetilde\Pi_W^{-1})\leq J_W(\Pi_W(P))+\varepsilon$ again by Lemma 7.5.

Let us now construct $\widetilde{P}$. This will be partially similar to the proof of Lemma 2.13. Again, we extend the W-projection of P to a random configuration in $\widetilde{W}$, but here we do it in such a way that all W-shreds are extended to the boundary of $\widetilde{W}$ in such a way that their composition in all the boxes $z+\widetilde{W}$ with $z\in 2(R+M)\mathbb{Z}^d$ concatenates to interlacements; this time we are not creating new loops. We do this for all shreds, also the loop shreds, since we will later show that the expected number of particles in these loops is negligible in the limit $W\uparrow\mathbb{R}^d$; hence it will not change much in the expected particle numbers if we turn all loops that have particles outside W into interlacements.

Decompose $\mathbb{R}^d$ regularly into boxes $\widetilde{W}_z=z+\widetilde{W}$ with $z\in 2(R+M)\mathbb{Z}^d$ and assume that R is much larger than M. Put $\xi_R=\Pi_W(P)\circ\widetilde{\Pi}_W^{-1}$ and $\psi_R=\partial\Pi_W^{(\mathcal{S})}(\xi_R)$. We will construct a loop/shred configuration $\xi_R^{(M)}\in\mathcal{M}_1(\mathcal{L}_{\widetilde{W}}\times\mathcal{S}_{\widetilde{W}})$ that extends ξ_R by appending independent Brownian W'-shreds in such a way that no new loop (or hardly any) is created in $\widetilde{W}$ and all W-shreds are connected with $\partial\widetilde{W}$ in a certain regular way. We will do this independently over $z\in 2(R+M)\mathbb{Z}^d$, such that all these added shreds in neighbouring subboxes automatically connect up such that they are concatenated to interlacements in $\mathbb{R}^d$.

For this, pick some deterministic index set Φ with $|\Phi|\asymp|W|/R\asymp R^{d-1}$ and a collection of points $(X_a)_{a\in\Phi}$ in $W_{R+M}^\circ\setminus W_{R+M-1}$ that are more than one away from the corners of W_{R+M}, i.e., from $(R+M)\{-1,1\}^d$. Then each X_a is closest to precisely one of the $2d$ neighbouring subboxes $z+W_{R+M}$ with $z\in 2(R+M)\{-1,1\}^d$ (i.e., there is precisely one $z\in 2(R+M)\{-1,1\}^d$ such that $X_a\in\mathcal{B}_1(\widetilde{W}\cap(z+\widetilde{W}))$, where $\mathcal{B}_1$ denotes the 1-neighbourhood). Assume that, for any $z\in 2(R+M)\{-1,1\}^d$, the number of a such that X_a is closest to $z+W$ is the same. Put $Y_a^{(z)}=X_a$ for these z. Furthermore, put $X_a^{(z)}=z+X_a$ for $z\in 2(R+M)\mathbb{Z}^d$, and define $Y_a^{(z')}$ as above for any neighbour z' of z and $X_a^{(z)}$. We conceive, for any $z\in 2(R+M)Z^d$, the pair $(X_a^{(z)},Y_a^{(z)})\in(z+\widetilde{W},(z+\widetilde{W})^{\rm c})$ as the entry-exit sites of a $(z+\widetilde{W})$-shred. They lie perfectly opposite to each other close to the boundary of $z+\widetilde{W}$. We put $X_a^{(0)}=X_a$ and $Y_a=Y_a^{(0)}$.

Now let $\mu=\sum_{i\in I}\delta_{(x_i,l_i,y_i)}\in\mathcal{T}_W$ be given such that $y_i\in\widetilde{W}$ for all $i\in I$. We are considering μ under $\partial\Pi_W^{(\mathcal{S})}(\xi_R)$, i.e., we can see μ as the configuration of all the starting/terminal sites and lengths of all the W-shreds in the configuration that is sampled under ξ_R. Note that ξ_R has, by definition as a random configuration in the event $A_{W;\mathfrak{r},M,L,K;T}$, with probability one, only a bounded number of particles in any subbox of radius $\mathfrak{r}$ around sites in $2\mathfrak{r}\mathbb{Z}^d$ and hence not more than $\asymp R^{d-1}$ particles close to ∂W. Therefore, we may assume that $|I|\le O(R^{d-1})$.

We pick an injective map $I\ni i\mapsto a_i\in\Phi$ and connect X_{a_i} with x_i and y_i with Y_{a_i} by means of an independent Brownian W'-shred, for any $i\in I$. For all other a's, i.e., for $a\in\Phi\setminus I_\Phi$ with $I_\Phi=\{a_i\colon i\in I\}$, we connect X_a with Y_a with an additional independent Brownian W'-shred. The distribution of the resulting configuration is called $\xi_R^{(M)}$, i.e., the extension of ξ_R to a loop/shred configuration in $\widetilde{W}$ without loops in W'.

Having constructed $\xi_R^{(M)}$ in this way, we take independent measures $\xi_R^{(M,z)}\in\mathcal{M}_1(\mathcal{L}_{\widetilde{W}_z}\times\mathcal{S}_{\widetilde{W}_z})$ with $z\in 2(R+M)\mathbb{Z}^d$ such that their shift $\theta_z(\xi_R^{(M,z)})$ has the same distribution as $\xi_R^{(M)}$. Then we put

$$\widetilde{P}_{R,M}=\bigotimes_{z\in 2(R+M)\mathbb{Z}^d}\xi_R^{(M,z)}\in\mathcal{M}_1(\mathcal{L}_W\times\mathcal{S}_W),\tag{7.31}$$

$$P_{R,M}=\frac{1}{|\widetilde{W}|}\int_{\widetilde{W}}{\rm d}x\,\theta_x(\widetilde{P}_{R,M})\in\mathcal{M}_1^{\rm (s)}(\mathcal{L}_{\widetilde{W}}\times\mathcal{S}_{\widetilde{W}}),\tag{7.32}$$

the latter which is obviously stationary. The proof of [G11, Theorem 14.12] applies and shows that $P_{R,M}$ is ergodic. We will show that $\widetilde{P}=P_{R,M}$ satisfies (7.28)–(7.30) for R large enough, which ends the proof of the lemma.

Note that $P_{R,M}$ has its interlacements only from our construction from the $|\Phi|$-many connections of $X_a^{(z)}$ with $Y_a^{(z)}$ (partially via x_i and y_i), which perfectly connect each other in the sites $Y_a^{(z)} = X_a^{(z')}$, if $z+\widetilde W$ and $z'+\widetilde W$ are neighbouring subboxes. All the loops of $P_{R,M}$ are the $(z+W)$-loops for some $z\in 2(R+M)\mathbb{Z}^d$.

The proofs of (7.28) and (7.29) are more or less identical to the proofs of (7.11) and (7.12) in the proof of Lemma 2.13; we leave the details to the reader; let us only mention that here also the following Lemma 7.7 is used, which says that our decision to turn all W-loops that have particles also in W' into interlacements does not noticeably change the particle densities: For shift-invariant loop-distributions, the expected number of particles in loops that have particles in W, but not all their particles, is negligible:

Lemma 7.7. *For any $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L})$ with $\langle P,\mathfrak{N}_U^{(\ell)}\rangle\in(0,\infty)$, we have*

$$\lim_{R\to\infty}\frac{1}{|W_R|}\langle \Pi^{(\mathcal{S})}_{W_R}(P),\mathfrak{N}^{(\ell)}_{W_R}\rangle = 0.$$

Proof. Note that $\Pi^{(\mathcal{S})}_{W_R}(P)=\Pi_{W_R}(P)-\Pi^{(\mathcal{L})}_{W_R}(P)$. By shift-invariance, for $R\in\mathbb{N}$,

$$\frac{1}{|W_R|}\langle \Pi_{W_R}(P),\mathfrak{N}^{(\ell)}_{W_R}\rangle=\frac{1}{|W_R|}\sum_{z\in W_R\cap\mathbb{Z}^d}\langle \Pi_{W_R}(P),\mathfrak{N}^{(\ell)}_{z+U}\rangle=\langle P,\mathfrak{N}^{(\ell)}_U\rangle.$$

Furthermore, the normalized integral with respect to the second is asymptotically not smaller than $\langle P,\mathfrak{N}^{(\ell)}_U\rangle$. Indeed, denoting by $\mathfrak{N}^{(\ell,\leq M)}_{\widetilde W}$ the number of particles in all the loops f that start in $\widetilde W\Subset\mathbb{R}^d$ and have spread $\max_{t\in[0,\beta\ell(f)]}|f(t)-f(0)|\leq M$, we see that

$$\frac{1}{|W_R|}\langle \Pi^{(\mathcal{L})}_{W_R}(P),\mathfrak{N}^{(\ell)}_{W_R}\rangle\geq\frac{1}{|W_R|}\sum_{z\in W_{R-M}\cap\mathbb{Z}^d}\langle \Pi_{W_R}(P),\mathfrak{N}^{(\ell,\leq M)}_{z+U}\rangle=\frac{|W_{R-M}|}{|W_R|}\langle P,\mathfrak{N}^{(\ell,\leq M)}_U\rangle\to\langle P,\mathfrak{N}^{(\ell,\leq M)}_U\rangle$$

as $R\to\infty$, and the last term converges to $\langle P,\mathfrak{N}^{(\ell)}_U\rangle$ as $M\to\infty$, according to the monotonous convergence theorem. □

Proof of (7.30)*:* Analogously to the argument around (7.14), we can argue also here that it suffices to show that $J_{\widetilde W}(\xi_R^{(M)})\leq J_W(\xi_R)+\varepsilon$. Indeed, the estimate in (7.14) is analogously valid here as well, with the difference that the second error term on the right-hand side of (6.17) is not trivial as in (7.14) (since $\psi=0$ there), but needs an argument, since $\psi=\partial\Pi^{(\mathcal{S})}_{3\widetilde W}(\widetilde P_{R,M})$ is non-trivial: For large R we have

$$J_{\widetilde W}(\Pi_{\widetilde W}(P_{R,M}))\leq J_{3\widetilde W}(\Pi_{3\widetilde W}(\widetilde P_{R,M}))+\frac{\varepsilon}{2}-\langle\psi,\mathfrak{p}_{3\widetilde W,M}\rangle. \tag{7.33}$$

In order to show that the last term is negligible, we point out that all the $3\widetilde W$-shreds f_i' of $\Pi_{3\widetilde W}(\widetilde P_{R,M})$ satisfy the condition $|f_i'(jS\beta)-f_i'((j-1)S\beta)/\leq\vartheta\sqrt{S}$ either almost surely for those j such that $f_i'([(j-1)S\beta,jS\beta])$ lies in $z+W$ for some $z\in 2(R+M)\mathbb{Z}^d$ (since they come from ξ_R or an independent copy of it, which is concentrated on $A^{(\mathcal{S})}_{W_R;\Theta}$) or in expectation, since these pieces are independent Brownian bridges by construction (see the argument in Remark 5.6). Therefore, making S large enough, we may assume that $J_{\widetilde W}(\Pi_{\widetilde W}(P_{R,M}))\leq J_{3\widetilde W}(\Pi_{3\widetilde W}(\widetilde P_{R,M}))+\varepsilon$.

In order to see that $J_{3\widetilde W}(\Pi_{3\widetilde W}(\widetilde P_{R,M}))=J_{\widetilde W}(\Pi_{\widetilde W}(\widetilde P_{R,M}))=J_{\widetilde W}(\xi_R^{(M)})$, observe that the configurations in the 3^d subboxes $z+\widetilde W$ with $z\in 2(R+M)\{-1,0,1\}^d$ are independent copies from each other and that therefore the density of $\Pi_{3\widetilde W}(P)(\omega,\mu,\cdot)$ with respect to $\mathtt{K}_{3\widetilde W}(\mu,\cdot)$, conditional on the boundary-shred configuration μ, decomposes into a product of the corresponding $(z+\widetilde W)$-versions (since $\Pi_{3\widetilde W}(P)$ decomposes into i.i.d. parts by definition, and $\mathtt{K}_{3\widetilde W}$ decomposes by the Markov property), and therefore the entropy of $\Pi_{3\widetilde W}(P)$ decomposes into the sum over these entropies. Because of the prefactor $\frac{1}{|\widetilde{W}|}$ in the definition of $J_{\widetilde{W}}$, we obtain that the $J_{3\widetilde{W}}$-value equals the $J_{\widetilde{W}}$-value. Hence, for proving (7.30), we only need to show that $J_{\widetilde{W}}(\xi_R^{(M)}) \leq J_W(\xi_R) + \varepsilon$.

The distribution of $\xi_R^{(M)}$ can be written, using the Markov property, as follows. Let $(\omega, \varpi) \in \mathcal{L}_W \times \mathcal{S}_W$ with $\varpi = \sum_{i\in I} \delta_{f_i}$ and $\mu = \partial\Pi_W^{(\mathcal{S})}(\varpi) = \sum_{i\in I} \delta_{(x_i,l_i,y_i)} \in \mathcal{T}_W$. Recall that $|I| \leq |\Phi|$ and that we did not add any $\widetilde{W}$-loops. We can write, using the Markov property and the independence of all the Brownian W'-shreds,

$$
\begin{aligned}
\xi_R^{(M)}\big(\mathrm{d}(\omega', \varpi')\big) &= \xi_R\big(\mathrm{d}(\omega, \varpi)\big)\, \mathrm{e}^{-\sum_{k\in\mathbb{N}}[q_{W',k} + q_{W',W,k}]} \\
&\quad \otimes \bigotimes_{i\in I} \Big[p^{(W')}_{X_{a_i},x_i}(\mathrm{d}m_i) q^{(m_i,W')}_{X_{a_i},x_i}(\mathrm{d}g_i)\, p^{(W')}_{y_i,Y_{a_i}}(\mathrm{d}m_i') q^{(m_i',W')}_{y_i,Y_{a_i}}(\mathrm{d}g_i')\Big] \\
&\quad \otimes \bigotimes_{a\in\Phi\setminus I_\Phi} \Big[p^{(W')}_{X_a,Y_a}(\mathrm{d}n_a) q^{(n_a,W')}_{X_a,Y_a}(\mathrm{d}g_a')\Big],
\end{aligned}
\tag{7.34}
$$

where we remind on the definition of the p- and the q-terms in (7.15) and (7.16). We used the notation $\omega' = \omega$ (seen as an element of $\mathcal{L}_{\widetilde{W}}$ with only W-loops) and $\varpi' = \sum_{i\in I} \delta_{g_i'\diamond f_i\diamond g_i} + \sum_{a\in\Phi\setminus I_\Phi} \delta_{g_a'}$ and $\mu' = \sum_{i\in I} \delta_{(X_{a_i},L_{a_i},Y_{a_i})} + \sum_{a\in\Phi\setminus I_\Phi} \delta_{(X_a,n_a,Y_a)}$ with $L_{a_i} = m_i + \ell(f_i) + m_i'$.

Analogously, we write the distribution of the reference measure, using the above notation. Recall from the proof of Lemma 2.13 that the reference measure $\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q})$ is the convolution of $\Pi_W^{(\mathcal{L})}(\mathtt{Q})$, $\Pi_{W'}^{(\mathcal{L})}(\mathtt{Q})$ and $\Pi_{W,W'}^{(\mathcal{L})}(\mathtt{Q})$ (recall that the latter measure is the restriction of $\mathtt{Q}$ to the set of loops that have all particles in $W \cup W'$ and have at least one in W and one in W'). Hence, the loop part is similar to (7.19) without the n-loop, and we have

$$
\begin{aligned}
\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_{\widetilde{W}}^{(\mathcal{S})}(\xi_R^{(M)}) \otimes \mathtt{K}_{\widetilde{W}}]\big(\mathrm{d}(\omega', \varpi')\big) &= \Pi_W^{(\mathcal{L})}(\mathtt{Q})(\mathrm{d}\omega)\, \mathrm{e}^{-\sum_{k\in\mathbb{N}}[q_{W',k}+q_{W',W,k}]}\, \mathrm{e}^{-\sum_{k\in\mathbb{N}} q_{W,W',k}} \\
&\quad \otimes \partial\Pi_W^{(\mathcal{S})}(\xi_R)(\mathrm{d}\mu) \bigotimes_{i\in I} \Big[p^{(W')}_{X_{a_i},x_i}(\mathrm{d}m_i) q^{(m_i,W')}_{X_{a_i},x_i}(\mathrm{d}g_i) p^{(W')}_{y_i,Y_{a_i}}(\mathrm{d}m_i') q^{(m_i',W')}_{y_i,Y_{a_i}}(\mathrm{d}g_i')\Big] \\
&\quad \otimes \bigotimes_{a\in\Phi\setminus I_\Phi} \Big[p^{(W')}_{X_a,Y_a}(\mathrm{d}n_a) q^{(n_a,W')}_{X_a,Y_a}(\mathrm{d}g_a')\Big] \otimes \mathtt{K}_W(\mu, \mathrm{d}\varpi).
\end{aligned}
\tag{7.35}
$$

Summarizing,

$$
\frac{\mathrm{d}\xi_R^{(M)}}{\mathrm{d}\Pi_{\widetilde{W}}^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_{\widetilde{W}}^{(\mathcal{S})}(\xi_R^{(M)}) \otimes \mathtt{K}_{\widetilde{W}}]}(\omega', \varpi') = \frac{\mathrm{d}\xi_R}{\mathrm{d}\Pi_W^{(\mathcal{L})}(\mathtt{Q}) \otimes [\partial\Pi_W^{(\mathcal{S})}(\xi_R) \otimes \mathtt{K}_W]}(\omega, \varpi)\mathrm{e}^{\sum_{k\in\mathbb{N}} q_{W,W',k}}.
$$

Hence, we see that

$$
J_{\widetilde{W}}(\xi_R^{(M)}) = J_W(\xi_R) + \frac{1}{|W|} \sum_{k\in\mathbb{N}} q_{W,W',k},
$$

and as in the proof of Lemma 2.13, one sees that the last term is $\leq \varepsilon$ for R large enough. This ends the proof of (7.30) and hence the proof of this lemma. $\square$

8. Proof of Theorem 2.3: specific relative entropy density

In this section we establish a new kind of *specific relative entropy per volume* of ergodic loop/interlacement configurations, namely with respect to the product of $\Pi_W^{(\mathcal{L})}(\mathtt{Q})$ and the product of the boundary-configuration and the kernel $\mathtt{K}_W$ defined in (2.5). In other words, we prove Theorem 2.3. In particular, we identify the limit of the entropy term $J_W(\xi)$ defined in (2.13) (see also (6.13)) appearing in the variational formula in (6.33) and (6.35) in the limit $R \to \infty$, i.e., in the limit $W = W_R = [-R, R]^d \uparrow \mathbb{R}^d$. Recall that, in $d \geq 3$, $\mathtt{K}_W$ is a regular version of the distribution of $\Pi_W^{(\mathcal{S})}(\mathtt{R})$ given $\partial\Pi_W^{(\mathcal{S})}(\mathtt{R})$ for the Brownian β-spacing interlacement PPP, $\mathtt{R}$, of Section A, see Lemma A.4, but we are not going to use this fact.

Remark 8.1 The usual starting point of such a proof is to prove the superadditivity of the map $W \mapsto H_{\mathcal{L}_W\times\mathcal{S}_W}(\Pi_W(P) \mid \Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes[\partial\Pi_W^{(\mathcal{S})}(P)\otimes \mathtt{K}_W])$ on the set of all boxes $W \Subset \mathbb{R}^d$, but we did not succeed in doing this, as it concerns the interlacement part. Instead, we will be relying on Proposition 6.6, which needs ergodicity, combined with Lemma 7.6, which approximates arbitrary stationary P's by ergodic ones that have several useful additional properties. $\Diamond$

Proof of Theorem 2.3. Let P be in $\mathcal{M}_1^{(\rm s)}(\mathcal{L}\times\mathcal{S})$ with finite expected particle number in the unit box U.

(1) We first prove that the limit in (2.14) *exists and satisfies* (2.16). Pick an $R\in(0,\infty)$ and pick any sequence $(R_n)_{n\in\mathbb{N}}$ tending to ∞. We would like to apply Proposition 6.6(2) and (3) to get a lower bound for $\liminf_{n\to\infty} J_{W_{R_n}}(\Pi_{W_{R_n}}(P))$ that is close to $J_{W_R}(\Pi_{W_R}(P))$, but we can do this only for W_R-ergodic P's that are concentrated on $\mathcal{L}\times\mathcal{S}^{(M,\vartheta,S)}$ (see (5.21)) for some M,ϑ,S. Hence, we first need to replace $\Pi_{W_{R_n}}(P)$ by some measure like that, using Lemma 7.6. Indeed, with parameters $\mathfrak{r},M,L,K,T,\vartheta,S$, we apply Lemma 7.6 (see also the proof of Lemma 7.5) to see that, for any given small $\varepsilon\in(0,1)$ and any n there is an ergodic $P_n\in\mathcal{M}_1^{(\rm s)}(\mathcal{L}^{(M)}\times\mathcal{S}^{(M,\vartheta,S)})$ such that $\Pi^{(\mathcal{S})}_{W_{R_n}}(P_n)$ is concentrated on $A^{(\mathcal{S})}_{W_{R_n};M,\mathfrak{r},T}\cap\mathcal{S}^{(\vartheta,S)}_{W_{R_n}}$ and satisfies $|\langle P_n-P,\mathfrak{N}^{(\mathcal{S},\ell)}_{W_{R_n}}\rangle|\le\varepsilon|W_{R_n}|$ such that

$$J_{W_{R_n}}(\Pi_{W_{R_n}}(P))\ge J_{W_{R_n}}(\Pi_{W_{R_n}}(P_n))-\varepsilon,$$

if the parameters M,L,K,T,ϑ are picked large enough. Now we can apply Proposition 6.6(2). Indeed, we write $R_n=m_nR^{(m_n)}$ with $m_n\in\mathbb{N}$ and $m_n\to\infty$ and $R^{(m_n)}\downarrow R$ as $n\to\infty$ and get

$$J_{W_{R_n}}(\Pi_{W_{R_n}}(P_n))\ge J_{W_{R^{(m_n)}}}(\Pi_{W_{R^{(m_n)}}}(P_n))-\frac{C_M}{(m_nR^{(m_n)})^{d/2}}-\Big[\frac{C_{M,\mathfrak{r},T}}{m_nR^{(m_n)}}+\frac{C\vartheta^2}{S}\Big]\langle P_n,\mathfrak{N}_U^{(\mathcal{S},\ell)}\rangle.$$

We can pick $S=\vartheta^3$. Now we combine the last two displays and make $n\to\infty$ and use that $|\langle P_n-P,\mathfrak{N}^{(\mathcal{S},\ell)}_{W_{R_n}}\rangle|\le\varepsilon|W_{R_n}|$ and that both P and P_n are shift-invariant to get that

$$\liminf_{n\to\infty}J_{W_{R_n}}(\Pi_{W_{R_n}}(P))\ge\liminf_{n\to\infty}J_{W_{R^{(m_n)}}}(\Pi_{W_{R^{(m_n)}}}(P_n))-\frac{C}{\vartheta}\big(\varepsilon+\langle P,\mathfrak{N}_U^{(\mathcal{S},\ell)}\rangle\big)-\varepsilon. \tag{8.1}$$

We can make $M,K,L,T,\vartheta\to\infty$ to obtain

$$\begin{aligned}\liminf_{n\to\infty}J_{W_{R_n}}(\Pi_{W_{R_n}}(P))&\ge\liminf_{M,K,L,T,\vartheta\to\infty}\liminf_{n\to\infty}J_{W_{R^{(m_n)}}}(\Pi_{W_{R^{(m_n)}}}(P_n))-\varepsilon\\&\ge J_{W_R}(\Pi_{W_R}(P))-\varepsilon,\end{aligned}\tag{8.2}$$

where we used the following argument for the J-term. Indeed, we want to use Proposition 6.6(3). Note that, from the construction of P_n for W_{R_n} in the proof of Lemma 7.6, we see that $\Pi_{W_{R_n}}(P_n)=\Pi_{W_{R_n}}(P)\circ\widetilde{\Pi}^{-1}_{W_{R_n}}$, where $\widetilde{\Pi}_{W_{R_n}}$ is the projection onto $A_{W_{R_n};M,L,K,T}$ introduced in Lemma 7.5. Therefore, it is easy to see that P_n weakly converges (in our usual sense that all the projections $\Pi_{W_{R'}}(P_n)$ weakly converge, $R'\in\mathbb{N}$) towards P if all the parameters n,M,K,L,T,ϑ grow to ∞. Furthermore, $\Pi_{W_{R^{(m_n)}}}(P_n)=\Pi_{W_{R^{(m_n)}}}(P)\circ\widetilde{\Pi}^{-1}_{W_{R^{(m_n)}}}$. In order to see that, in this limit inferior the limit $P_n\Longrightarrow P$ and the limit as $R^{(m_n)}\downarrow R$ according to Proposition 6.6(3) can be combined such that the last step of (8.2) appears, use first (7.24) for $f=\widetilde{\Pi}_{W_{R^{(m_n)}}}$ and then prove that the density $\mathrm{d}\nu_n\circ\widetilde{\Pi}^{-1}_{W_{R^{(m_n)}}}/\mathrm{d}\nu_n$ tends to one in the limit as $n,M,K,L,T,\vartheta\to\infty$, where $\nu_n=\mathtt{Q}_{W_{R^{(m_n)}}}\otimes[\partial\Pi_{W_{R^{(m_n)}}}(P)\otimes\mathtt{K}_{W_{R^{(m_n)}}}]$ is the reference measure. We leave the proof of the latter to the reader.

Now (8.2) implies that the limit inferior is not smaller than the limit superior, i.e., the limit in (2.14) exists. Furthermore, from (8.2) we also see that (2.16) holds.

(2) Now we show that $P\mapsto \mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$ is lower semicontinuous. Recall that we are using on $\mathcal{M}_1(\mathcal{L}\times\mathcal{S})$ the weak topology induced by all the Π_W, i.e., P_n converges towards P if and only if $\Pi_W(P_n)$ weakly converges towards $\Pi_W(P)$ for any $W\Subset\mathbb{R}^d$. We write $\mathcal{L}_W\times\mathcal{S}_W$ as $[\mathcal{L}_W\times\mathcal{T}_W]\times\mathcal{S}_W$ and write $(\Pi_W(P))_{\mathcal{L}_W\times\mathcal{T}_W\to\mathcal{S}_W}(\omega,\mu,\cdot)$ for the corresponding kernel (by disintegration). Also the map

$P \mapsto (\Pi_W(P))_{\mathcal{L}_W\times\mathcal{T}_W\to\mathcal{S}_W}(\omega,\mu,\cdot)$ is continuous for any $\omega\in\mathcal{L}_W$ and $\mu\in\mathcal{T}_W$. It follows from the characterisation of the entropy in (2.7) and the fact that limits of lower semi-continuous functions are lower semi-continuous that relative entropies are lower semi-continuous in the first argument. With the help of writing $\partial\xi=\partial\Pi_W^{(\mathcal{S})}(P)$ and $\xi_\mu(\mathrm{d}(\omega,\varpi))=(\Pi_W(P))_{\mathcal{T}_W\to\mathcal{L}_W\times\mathcal{S}_W}(\mu,\mathrm{d}(\omega,\varpi))$ and

$$\mathcal{H}_W(P)=|W|J_W(\Pi_W(P))=\int_{\mathcal{T}_W}\partial\xi(\mathrm{d}\mu)\,H_{\mathcal{L}_W\times\mathcal{S}_W}\big(\xi_\mu\mid\Pi_W^{(\mathcal{L})}(\mathtt{Q})\otimes\mathtt{K}_W(\mu,\cdot)\big)\tag{8.3}$$

and Fatou's lemma, the assertion follows.

(3) Next, we show the convexity of the map $P\mapsto J_W(\Pi_W(P))$. This follows from the convexity of the map $(\mu,\nu)\mapsto H(\mu\mid\nu)$ for any relative entropy H, see [G11, Proposition 15.5(d)]. Since limit of convex functions are convex, the convexity of $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(\cdot)$ on $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ follows. The proof of concavity given in the proof of [G11, Proposition 15.14] applies also here, using (8.3). Indeed, if one replaces P by $sP_1+(1-s)P_2$ for $s\in(0,1)$ and $P_1,P_2\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$, then $\mathcal{H}_W(sP_1+(1-s)P_2)$ can be lower estimated (using the monotonicity of the logarithm) against $s\mathcal{H}_W(P_1)+(1-s)\mathcal{H}_W(P_2)+s\log s+(1-s)\log(1-s)$, and dividing by $|W|$ and letting $|W|\to\infty$ implies the concavity of $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(\cdot)$ on the set of all ergodic $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$.

(4) Now we prove the compactness of the restricted level set of $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(\cdot)$*, given in* (2.17). Assume that $c\in\mathbb{R}$ and a sequence of compact sets $K_R\subset\mathcal{T}_{W_R}$ with $R\in\mathbb{N}$ are given (we take the sequence $(R_N)_{N\in\mathbb{N}}$ as $\mathbb{N}$ without loss of generality). Because of lower semicontinuity of $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(\cdot)$ and of $P\mapsto\langle P,\mathfrak{N}_U^{(\ell)}\rangle$, the set in (2.17) is closed, hence we only need to prove its tightness. Pick a sequence $(P_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_1(\mathcal{L}\times\mathcal{S})$ in that set. Using (2.16), we see that, for any $R\in\mathbb{N}$, the projections $\Pi_{W_R}(P_n)$ lie for any n in the restricted level set

$$\big\{\xi\in\mathcal{M}_1(\mathcal{L}_{W_R}\times\mathcal{S}_{W_R})\colon\partial\Pi_{W_R}^{(\mathcal{S})}(\xi)\in K_R(L),J_{W_R}(\xi)\le c'\big\},\tag{8.4}$$

for some $c'\in\mathbb{R}$. According to Lemma 6.7(1), this set is compact. Therefore, $(\Pi_{W_R}(P_n))_{n\in\mathbb{N}}$ is tight. This finishes the proof, because of the following argument: We can assume that $(\Pi_{W_R}(P_n))_{n\in\mathbb{N}}$ converges towards some $\xi^{(W_R)}\in\mathcal{M}_1(\mathcal{L}_{W_R}\times\mathcal{S}_{W_R})$ as $n\to\infty$. Using a standard diagonal argument, we can assume that the n-subsequence works for all the $R\in\mathbb{N}$. Since $(\Pi_{W_R}(P_n))_{R\in\mathbb{N}}$ is consistent for any $n\in\mathbb{N}$, also $(\xi^{(W_R)})_{R\in\mathbb{N}}$ is consistent. Now Proposition 4.1 yields the existence of a $P\in\mathcal{M}_1(\mathcal{L}\times\mathcal{S})$ such that $\Pi_{W_R}(P)=\xi^{(W_R)}$ for any $R\in\mathbb{N}$. It is easy to see that P_n converges (along the chosen subsequence) towards P. □

Let us close this section with making some comments and remarks. First, let us state the following useful general inequality (which was already used in the proof of Theorem 2.3).

For a probability measure ν on the product of two Polish spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ with projections $\Pi_i\colon\mathcal{X}_1\times\mathcal{X}_2\to\mathcal{X}_i$, we write $\nu_{1\to2}$ for the Markov kernel from $\mathcal{X}_1$ into $\mathcal{M}_1(\mathcal{X}_2)$ that is defined by the disintegration formula

$$\nu\big(\mathrm{d}(x_1,x_2)\big)=\Pi_1(\nu)(\mathrm{d}x_1)\nu_{1\to2}^{(x_1)}(\mathrm{d}x_2).\tag{8.5}$$

In other words, $\nu_{1\to2}$ is a regular version of the conditional distribution of X_2 given X_1 if (X_1,X_2) is a random vector with distribution ν. Let us also recall the standard measure-theoretic notation $\mu\otimes K(\mathrm{d}(x_1,x_2))=\mu(\mathrm{d}x_1)\,K(x_1,\mathrm{d}x_2)$ and $\mu K(\mathrm{d}x_2)=\int_{\mathcal{X}_1}\mu(\mathrm{d}x_1)\,K(x_1,\mathrm{d}x_2)$ for $\mu\in\mathcal{M}_1(\mathcal{X}_1)$ and a kernel K from $\mathcal{X}_1$ into $\mathcal{M}_1(\mathcal{X}_2)$. By H_1,H_2 and H we denote the relative entropies on the spaces $\mathcal{M}_1(\mathcal{X}_1),\mathcal{M}_1(\mathcal{X}_2)$, and $\mathcal{M}_1(\mathcal{X}_1\times\mathcal{X}_2)$, respectively.

Lemma 8.2 (Entropy on products)**.** *For any two Polish spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ with projections $\Pi_i\colon\mathcal{X}_1\times\mathcal{X}_2\to\mathcal{X}_i$,*

$$H(\nu\mid\rho)\ge H_1(\Pi_1(\nu)\mid\Pi_1(\rho))+H_2(\Pi_1(\nu)\nu_{1\to2}\mid\Pi_1(\nu)\rho_{1\to2}),\qquad\nu,\rho\in\mathcal{M}_1(\mathcal{X}_1\times\mathcal{X}_2).\tag{8.6}$$

Proof. We may assume that ν has a density with respect to ρ. Then

$$\begin{aligned} H(\nu \mid \rho) &= \int_{\mathcal{X}_1} \Pi_1(\nu)(\mathrm{d}x_1) \int_{\mathcal{X}_2} \nu_{1\to 2}^{(x_1)}(\mathrm{d}x_2)\, \log\Big(\frac{\mathrm{d}(\Pi_1(\nu))}{\mathrm{d}(\Pi_1(\rho))}(x_1)\frac{\mathrm{d}(\nu_{1\to 2}^{(x_1)})}{\mathrm{d}(\rho_{1\to 2}^{(x_1)})}(x_2)\Big) \\ &= H_1(\Pi_1(\nu) \mid \Pi_1(\rho)) + \int_{\mathcal{X}_1} \Pi_1(\nu)(\mathrm{d}x_1)\, H_2\big(\nu_{1\to 2}^{(x_1)} \mid \rho_{1\to 2}^{(x_1)}\big) \\ &\geq H_1(\Pi_1(\nu) \mid \Pi_1(\rho)) + H_2\big(\Pi_1(\nu)\nu_{1\to 2} \mid \Pi_1(\nu)\rho_{1\to 2}\big), \end{aligned} \tag{8.7}$$

where in the last step we used Jensen's inequality for the convex function $(\nu,\rho)\mapsto H_2(\nu\mid\rho)$; see [G11, Proposition 15.5(d)]. □

Remark 8.3 ($\mathrm{h}\geq \mathrm{h}^{(\mathcal{L})}$) An application of (8.6) shows that $\mathrm{h}\geq\mathrm{h}^{(\mathcal{L})}$, where h was defined in (2.8), and we define $\mathrm{h}^{(\mathcal{L})}(P)$ as $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)$ for any $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L})$, where P is considered as a loop-interlacement configuration with empty interlacement part. Indeed, for any $W\Subset\mathbb{R}^d$, observe that $\Pi_W(\mathcal{L})$ is the direct sum of $\mathcal{L}_W=\Pi_W^{(\mathcal{L})}(\mathcal{L})$ and the set of all simple point measures that start in W, but have not all their particles in W. Using (8.6), we see that

$$H_{\Pi_W(\mathcal{L})}\big(\Pi_W(P)\,\big|\,\Pi_W(\mathtt{Q})\big)\geq H_{\mathcal{L}_W}\big(\Pi_W^{(\mathcal{L})}(P)\,\big|\,\Pi_W^{(\mathcal{L})}(\mathtt{Q})\big),$$

where we dropped on the right-hand side another entropy term. ◊

9. Finish of the proof of Theorem 2.5: making $R\to\infty$

After all the preparations, we finish the proof of Theorem 2.5 in this section: the proof of the upper bound is finished in Section 9.1, the one of the lower bound in Section 9.2.

9.1. **Upper bound.** With the material from Sections 8 and 6.4, we now finish the proof of the upper bound in Theorem 2.5, based on (6.33). Recall that $W=W_R=[-R,R]^d$ and that χ was defined in (2.24) and the set $\mathcal{M}_1^{(k)}(\mathcal{L}_W\times\mathcal{S}_W)$ in (6.8). Starting from (6.33), it is sufficient to prove the following.

$$\begin{aligned} \liminf_{R\to\infty}\inf\Big\{&\frac{1}{|W_R|}\langle\xi,F_{W_R,W_R}\rangle+J_{W_R}(\xi)\colon \xi\in\bigcap_{k\in\mathbb{N}}\mathcal{M}_1^{(k)}(\mathcal{L}_{W_R}\times\mathcal{S}_{W_R}),\\ &\Pi_{W_R}^{(\mathcal{L})}(\xi)\in K_R^{(\mathcal{L})}(L),\Pi_{W_R}^{(\mathcal{S})}(\xi)\in K_R^{(\mathcal{S})}(L),\langle\xi,\tfrac{1}{|W_R|}\mathfrak{N}_{W_R}^{(\ell,\mathcal{L})}\rangle=\rho_1,\langle\xi,\tfrac{1}{|W_R|}\mathfrak{N}_{W_R}^{(\ell,\mathcal{S})}\rangle=\rho_2\Big\}\\ \geq\inf\Big\{&\langle P,F_U\rangle+\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)\colon P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S}),\langle P,\mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle\in\overline{\mathcal{B}}_{2\delta}(\rho_1),\langle P,\mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle\in\overline{\mathcal{B}}_{2\delta}(\rho_2)\Big\}, \end{aligned} \tag{9.1}$$

where we recall the definition of $J_W(\xi)$ from (6.13). With the help of the continuity properties of χ from Lemma 2.12, the limit as $\delta\downarrow 0$ of the right-hand side is easily seen to coincide with $\chi(\rho_1,\rho_2)$, and the proof of (2.25) is then finished.

Recall the two sets $K_R^{(\mathcal{L})}(L)$ and $K_R^{(\mathcal{S})}(L)$ from Lemmas 6.7 and 6.8, respectively, and note that L does not depend on R and is picked large enough such that (6.33) is true.

Now we turn to the proof of (9.1). For the ease of notation, we now write W instead of W_R. Assume that $\xi^{(W)}$ is an approximate minimiser in the variational formula on the left-hand side of (9.1), that is, it is amenable to that formula and approaches the infimum up to some quantity that vanishes as $R\to\infty$. Put

$$\mathcal{H}=\liminf_{R\to\infty}J_W(\xi^{(W)})\qquad\text{and}\qquad\mathcal{F}=\liminf_{R\to\infty}\frac{1}{|W|}\langle\xi^{(W)},F_{W,W}\rangle. \tag{9.2}$$

Without loss of generality, $\mathcal{H}$ and $\mathcal{F}$ are finite. Let $\varepsilon\in(0,1)$ be given. We need to construct a $P\in\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$ that is amenable for the formula on the right-hand side of (9.1) (i.e., satisfies $\langle P,\mathfrak{N}_U^{(\ell,\mathcal{L})}\rangle\in\overline{\mathcal{B}}_{2\delta}(\rho_1)$, and $\langle P,\mathfrak{N}_U^{(\ell,\mathcal{S})}\rangle\in\overline{\mathcal{B}}_{2\delta}(\rho_2)$) and such that $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P)\leq\mathcal{H}+\varepsilon$ and $\langle P,F_U\rangle\leq\mathcal{F}+\varepsilon$, which will finish the proof. Let M be so large that the constant c_M introduced in Lemma 4.3 satisfies

$\log c_M \geq -\varepsilon/8$. Let R be so large that $C_M R^{-d/2} \leq \varepsilon/8$, where C_M is from (5.14). Furthermore, assume that R is also so large that

$$J_W(\xi^{(W)}) \leq \mathcal{H} + \varepsilon/8 \qquad \text{and} \qquad \frac{1}{|W|}\langle \xi^{(W)}, F_{W,W}\rangle \leq \mathcal{F} + \varepsilon/8. \tag{9.3}$$

In our first step, we are going to replace $\xi^{(W)}$ by loop-shred distributions in larger and larger boxes in such a way that the entropy and energy do not really increase and the expected particle densities in loops and in shreds does not really change. (In a later step, we will use Proposition 4.1 to further extend this measure to an element of $\mathcal{M}_1(\mathcal{L}\times\mathcal{S})$.) We want to use Proposition 6.6(4) for that and Lemma 5.5 for estimating the error term, but need first to replace $\xi^{(W)}$ by an approximation that guarantees that we are allowed to apply Proposition 6.6(4) and that the error term is small. For this, we need to employ Lemma 7.5 with fixed parameters $\mathfrak{r} \in (1,\infty)$ and $S \in \mathbb{N}$ and large parameters R, M, K, L, T, ϑ. Put $\Theta = (\mathfrak{r}, M, L, K, T, \vartheta, S)$. Pick the parameters in Θ so large that the error term in Lemma 7.5 is $\leq \varepsilon/4$ and pick $\xi^{(W,\Theta,0)} \in \mathcal{M}_1(\mathcal{L}_W \times \mathcal{S}_W)$ (namely, $\xi^{(W,\Theta,0)} = \xi^{(W)} \circ \Pi^{-1}$ with the projection Π of Lemma 7.5) that is concentrated on $A_{W;\Theta} = A_{W;\mathfrak{r},M,K,L,T} \cap (\mathcal{L}_W \times \mathcal{S}_W^{(M,\vartheta,S)})$ such that

$$J_{W;M}(\xi^{(W,\Theta,0)}) \leq J_W(\xi^{(W)}) + \frac{\varepsilon}{4} \qquad \text{and} \qquad \frac{1}{|W|}\langle F_{W,W}, \xi^{(W,\Theta,0)} - \xi^{(W)}\rangle \leq \frac{\varepsilon}{4}, \tag{9.4}$$

and $|\langle \mathfrak{N}_W^{(\mathcal{L},\ell)}, \xi^{(W,\Theta,0)} - \xi^{(W)}\rangle| \leq \frac{\varepsilon}{4}|W|$ and $|\langle \mathfrak{N}_W^{(\mathcal{S},\ell)}, \xi^{(W,\Theta,0)} - \xi^{(W)}\rangle| \leq \frac{\varepsilon}{4}|W|$ and $\Pi_W^{(\mathcal{L})}(\xi^{(W,\Theta,0)}) \in K_{3R}^{(\mathcal{L})}(L+\frac14)$ and $\Pi_W^{(\mathcal{S})}(\xi^{(W,\Theta,0)}) \in K_{3R}^{(\mathcal{S})}(L+\frac14)$. (The fact that all these integrals w.r.t. $\xi^{(W,\Theta,0)}$ can be made as close to the ones w.r.t. $\xi^{(W)}$ follows from monotonicity of Π in the cutting parameters M, K, L, ϑ.)

Now we apply Proposition 6.6(4) to $\xi^{(W,\Theta,0)}$ successively for $k=1$ (producing $\xi^{(W,\Theta,1)}$, then for $k=1$ and W replaced by $3W$ (producing $\xi^{(W,\Theta,2)}$ and so on. We pick the parameters R, S so large that the error term in Proposition 6.6(4) (using Lemma 5.5 for the last term) is $\leq \frac{\varepsilon}{8}3^{-k}$ for the k-th application of Proposition 6.6(4). In this way, we obtain a sequence of $\xi^{(W,\Theta,k)} \in \widetilde{\mathcal{M}}_1^{(k)} \subset \mathcal{M}_1(\mathcal{L}_{3^kW} \times \mathcal{S}_{3^kW})$ such that the approximation error in the infimum is $\leq \varepsilon/8$; more precisely,

$$J_{3^kW;M}(\xi^{(W,\Theta,k)}) \leq \frac{\varepsilon}{4} + J_{W;M}(\xi^{(W,\Theta,0)}) \qquad \text{and} \qquad \Pi_{3^kW\to 3^{k-1}W}(\xi^{(W,\Theta,k)}) = \xi^{(W,\Theta,k-1)}, \tag{9.5}$$

and every $\xi^{(W,\Theta,k)}$ is 3^kW-shift invariant. In particular, the family $(\xi^{(W,\Theta,k)})_{k\in\mathbb{N}}$ is consistent, and every projection of any $\xi^{(W,\Theta,k)}$ on a shift $z+W$ with $z \in 2R\mathbb{Z}^d$ is (after shifting back to the origin) equal to $\xi^{(W,\Theta,0)}$.

Note that $\xi^{(W,\Theta,k)}$ is concentrated on $A_{3^kW;\Theta}$, since its projections on any discrete shift of W in 3^kW is equal (after shifting to the origin) to $\xi^{(W,\Theta,0)}$. Therefore all shreds in it have all the properties of the shreds in the configurations in $A_{W;\Theta}$ (namely a bounded leg spread, a bound on the extension of all subparts, and an upper bound for the particle number in each of the boxes $W_\mathfrak{r}+z$). Therefore, the expected interaction $\frac{1}{|3^kW|}\langle F_{3^kW,3^kW}, \xi^{(W,\Theta,k)}\rangle$ is essentially equal to $\frac{1}{|W|}\langle F_{W,W}, \xi^{(W,\Theta,0)}\rangle$, up to the interaction between any two neighbouring boxes of radius R, which can be upper bounded by some constant (depending only on $M, L, K, T, \mathfrak{r}$ and d, β, v) times $1/R$, since it is a surface-order term. By making R large enough, this error term is below $\varepsilon/8$. Therefore,

$$\frac{1}{|3^kW|}\langle F_{3^kW,3^kW}, \xi^{(W,\Theta,k)}\rangle \leq \frac{\varepsilon}{4} + \frac{1}{|W|}\langle F_{W,W}, \xi^{(W,\Theta,0)}\rangle, \qquad k \in \mathbb{N}. \tag{9.6}$$

By the same argument, but without error terms, the normalized expected particle numbers for loops and shreds, $\frac{1}{|W|}\langle \mathfrak{N}_W^{(\ell,\mathcal{L})}, \xi^{(W,\Theta,0)}\rangle$ and $\frac{1}{|W|}\langle \mathfrak{N}_W^{(\ell,\mathcal{S})}, \xi^{(W,\Theta,0)}\rangle$, do not change when going from W to 3^kW and from $\xi^{(W,\Theta,0)}$ to $\xi^{(W,\Theta,k)}$. In particular, $\frac{1}{|3^kW|}\langle \mathfrak{N}_{3^kW}^{(\ell,\mathcal{L})}, \xi^{(W,\Theta,k)}\rangle \in \overline{B}_{2\delta}(\rho_1)$ and $\frac{1}{|3^kW|}\langle \mathfrak{N}_{3^kW}^{(\ell,\mathcal{S})}, \xi^{(W,\Theta,k)}\rangle \in \overline{B}_{2\delta}(\rho_2)$.

Now we use Proposition 4.1 to obtain from the consistent family $(\xi^{(W,\Theta,k)})_{k\in\mathbb{N}}$ some $\widetilde{P} \in \mathcal{M}_1(\mathcal{L}\times\mathcal{S})$ that extends all the $\xi^{(W,\Theta,k)}$s. By the W-shift invariance of all the $\xi^{(W,\Theta,k)}$s, P is invariant under any shift by a vector $\in 2R\mathbb{Z}^d$. Now we replace $\widetilde{P}$ by $P = \frac{1}{|W|}\int_W \mathrm{d}x\, \widetilde{P}\circ\theta_x^{-1}$, which is shift-invariant, i.e., lies in $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{L}\times\mathcal{S})$. We are going to show that $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P) \leq 2\varepsilon + J_W(\xi^{(W)})$ and $\langle F_U, P\rangle \leq 2\varepsilon + \frac{1}{|W|}\langle F_{W,W}, \xi^{(W)}\rangle$ and that both the loop and the interlacement particle densities of P are close to ρ_1, respectively to ρ_2. This will imply (9.1) and hence end the proof of the upper bound in Theorem 2.5.

It is relatively easy to see that the loop-particle density and the interlacement-particle densities lie in $\mathcal{B}_{3\delta}(\rho_1)$, respectively in $\mathcal{B}_{3\delta}(\rho_2)$. Indeed, use that $\frac{1}{|3^kW|}\langle \mathfrak{N}^{(\ell,\mathcal{L})}_{3^kW}, \xi^{(W,\Theta,k)}\rangle \sim \frac{1}{|3^kW|}\langle \mathfrak{N}^{(\ell,\mathcal{L})}_{3^kW}, P\rangle \to \langle \mathfrak{N}^{(\ell,\mathcal{L})}_{U}, P\rangle$ as $k\to\infty$ since $\langle \mathfrak{N}^{(\ell,\mathcal{L})}_{3^kW}, P\rangle$ lies between $\langle \mathfrak{N}^{(\ell,\mathcal{L})}_{3^kW-W}, P\rangle$ and $\langle \mathfrak{N}^{(\ell,\mathcal{L})}_{3^kW+W}, P\rangle$ and since, according to Lemma 7.7, the expected number of particles in box-shreds of loops is negligible with respect to the box volume. A similar argument applies to the expected particle density in interlacements; we leave the details to the reader.

Now we show that $J_{3^kW;M}(\Pi_{3^kW}(P)) \leq \mathcal{H} + 2\varepsilon$ for any $k\in\mathbb{N}$, which implies, via Theorem 2.3, that $\mathrm{h}^{(\mathcal{L},\mathcal{S})}(P) \leq \mathcal{H} + 2\varepsilon$. Indeed, using convexity of $J_{3^kW;M}$ and (6.17) in Proposition 6.6(1) with $m=3$, we see that, for any $k\in\mathbb{N}$,

$$
\begin{aligned}
J_{3^kW;M}(\Pi_{3^kW}(P)) &\leq \frac{1}{|W|}\int_W \mathrm{d}x\, J_{3^kW;M}(\theta_x(\Pi_{x+3^kW}(\widetilde{P}))\\
&\leq \varepsilon + J_{3^{k+1}W}(\Pi_{3^{k+1}W;M}(\widetilde{P})) = \varepsilon + J_{3^{k+1}W;M}(\xi^{(W,\Theta,k+1)})\\
&\leq \varepsilon + \frac{\varepsilon}{4} + J_{W;M}(\xi^{(W,\Theta,0)}) \leq \frac{3\varepsilon}{2} + J_W(\xi^{(W)}) \leq 2\varepsilon + \mathcal{H},
\end{aligned}
$$

where we used also (9.5) and (9.4) and one of the requirements for R.

It is only left to show that $\langle F_U, P\rangle \leq 2\varepsilon + \mathcal{F}$, where we recall the definition of F_U from (2.22). We first point out that, according to the spatial ergodic theorem,

$$
\langle F_U, P\rangle = \lim_{k\to\infty}\frac{1}{|3^kW|}\Big\langle \sum_{z\in\mathbb{Z}^d\colon z+U\subset 3^kW} F_{z+U}, P\Big\rangle, \tag{9.7}
$$

since F_U is bounded and has bounded reach almost surely on the support of P. This is seen as follows. Recall that every $\xi^{(W,\Theta,k)}$ is concentrated on the set $A_{3^kW;\Theta}$ of configurations; these have a globally bounded interaction between any box $z+U$ and its complement $(z+U)^{\mathrm{c}}$, where 'bounded' is meant in terms of spatial distance, number of interacting loops/shreds and in amount of interaction. More precisely, we may pick some $C\in(0,\infty)$ (only depending on the parameters in Θ and on the length of the support of the interaction potential v), such that $F_{z+U}(\omega,\varpi)$ is equal to the total interaction between $z+U$ and $z+W_C$ only, almost surely with respect to $\xi^{(W,\Theta,k)}$. And if $z+U\subset 3^kW$, then the integration of F_{z+U} with respect to P can be replaced by the one with respect to $\xi^{(W,\Theta,k+1)}$.

Recall from (2.21) that $F_{W,W}(\omega,\varpi) = \frac{1}{2}F^{(\mathcal{L}\mathcal{L})}_{W,W}(\omega) + F^{(\mathcal{L}\mathcal{S})}_{W,W}(\omega,\varpi) + \frac{1}{2}F^{(\mathcal{S}\mathcal{S})}_{W,W}(\varpi)$, whose definitions we recall from (2.18)–(2.20). Therefore, almost surely with respect to P, for any $x\in W$ and any $k\in\mathbb{N}$, we get from splitting the interaction between 3^kW and 3^kW into the interactions between $z+U$ and

$z+U^{\mathrm{c}}$ and sum on z,

$$\begin{aligned}
F_{3^kW,3^kW} &\geq \sum_{z\in\mathbb{Z}^d\colon z+U\subset 3^kW} \Big[\frac{1}{2}F_{z+U,z+U}+F_{z+U,W\cap(z+U^{\mathrm{c}})}\Big]\\
&\geq \sum_{z\in\mathbb{Z}^d\colon z+U\subset W_{3^k-C}} \Big[\frac{1}{2}F_{z+U,z+U}+F_{z+U,z+(W_C\setminus U)}\Big]\\
&= \sum_{z\in\mathbb{Z}^d\colon z+U\subset W_{3^k-C}} \Big(\tfrac{1}{2}F^{(\mathcal{L}\mathcal{L})}_{z+U,z+U}+F^{(\mathcal{L}\mathcal{S})}_{z+U,z+U}+\tfrac{1}{2}F^{(\mathcal{S}\mathcal{S})}_{z+U,z+U}\\
&\qquad +F^{(\mathcal{L}\mathcal{L})}_{z+U,z+(W_C\setminus U)}+F^{(\mathcal{L}\mathcal{S})}_{z+U,z+(W_C\setminus U)}+F^{(\mathcal{L}\mathcal{S})}_{z+(W_C\setminus U),z+U}+F^{(\mathcal{S}\mathcal{S})}_{z+U,z+(W_C\setminus U)}\Big)\\
&= \sum_{z\in\mathbb{Z}^d\colon z+U\subset W_{3^kR-C}} F_{z+U},
\end{aligned}$$

recalling the definition of F_U from (2.22). Certainly, the sum of F_{z+U} on $z\in\mathbb{Z}^d$ with $z+U\subset 3^kW$ but $z+U\not\subset W_{3^kR-C}$ is negligible in (9.7), and

$$\frac{1}{|3^kW|}\Big|\Big\langle F_{3^kW,3^kW}-\frac{1}{|W|}\int_W \mathrm{d}x\, F_{x+3^kW,x+3^kW},\xi^{(W,\Theta,k+1)}\Big\rangle\Big|\to 0,\qquad k\to\infty.$$

Therefore,

$$\begin{aligned}
\langle F_U,P\rangle &\leq \lim_{k\to\infty}\frac{1}{|3^kW|}\langle F_{3^kW,3^kW},P\rangle=\lim_{k\to\infty}\frac{1}{|3^kW|}\Big\langle\int_W \mathrm{d}x\, F_{x+3^kW,x+3^kW},\xi^{(W,\Theta,k+1)}\Big\rangle\\
&=\lim_{k\to\infty}\frac{1}{|3^kW|}\Big\langle F_{3^kW,3^kW},\xi^{(W,\Theta,k+1)}\Big\rangle.
\end{aligned}$$

Now using subsequently (9.6), (9.4) and (9.3), we see that $\langle F_U,P\rangle\leq\mathcal{F}+2\varepsilon$, which finishes the proof.

9.2. **Lower bound.** For finishing the proof of the lower bound in Theorem 2.5, we go back to (6.35). Applying Lemma 7.6 and (2.16), we see that, given any small $\varepsilon\in(0,1)$, we can make the parameters ϑ,M,L,K,T and then R (in this order) so large that the variational formula on the right-hand side of (6.35) is not larger than $\varepsilon+\chi(\rho_1,\rho_2)$, also using the continuity of $\chi(\rho_1,\rho_2)$ in (ρ_2,ρ_2); see Lemma 2.12. This gives, for any $\delta,\delta'\in(0,1)$,

$$\begin{aligned}
\liminf_{N\to\infty}\frac{1}{|\Lambda_N|}\log\widehat{Z}^{(\mathrm{bc})}_{N,R,\delta}(\Lambda_N,\rho_1,\rho_2)\geq -\frac{C_{M,L,K,T}}{R}-\frac{C_M}{R^{d/2}}-\Big[\frac{C_{M,\vartheta,S}}{R}+\frac{C\vartheta^2}{S}\Big](\rho_2+\delta')\\
-\varepsilon-\chi(\rho_1,\rho_2).
\end{aligned}\tag{9.8}$$

We make now first $S\to\infty$ and then $R\to\infty$, which makes the first three terms disappear. We have finished the proof of the lower bound in Theorem 2.5.

Appendix A. The Brownian interlacement PPP with spacing β

In this appendix we introduce, for $d\geq 3$, an interesting Poisson point process $\mathtt{R}=\mathtt{R}^{(u,\beta)}$ of random interlacements that has some connection to the specific relative entropy $\mathrm{h}^{(\mathcal{L},\mathcal{S})}$ defined in (2.14), or, more directly, some connection to the kernel $\mathtt{K}_W$ defined in (2.5). We survey its construction and some of its properties. Processes of this type were recently recognized as being instrumental in the description of the long loops in the Bose gas. In the present paper, such process does not appear in our description of the free energy, since our entropy $\mathrm{h}^{(\mathcal{L},\mathcal{S})}$ can be entirely characterised in terms of the kernel $\mathtt{K}_W$ defined in (2.5), but it admits enlightening interpretations. Because of Lemma A.4 below, $\mathrm{h}^{(\mathcal{L},\mathcal{S})}$ can be seen as a kind of specific relative entropy with respect to the reference PPP $\mathtt{Q}\otimes\mathtt{R}$ of loops and interlacements, but note the crucial appearance of the boundary-particle distribution $\partial\Pi^{(\mathcal{S})}_W(P)$ in (2.13). Nevertheless, the process $\mathtt{R}$ is at the heart of the limiting description of the Bose gas and its condensate, so we decided to spend some words on it, even though it is not involved in our analysis of the interacting Bose gas.

Here is an ad-hoc definition of this process: Let $d \geq 3$ and fix $\beta \in (0, \infty)$. The *Brownian interlacement Poisson point process with particle spacing* β, or for short *Brownian interlacement process*, $\mathtt{R} \in \mathcal{M}_1^{(\mathrm{s})}(\mathcal{S})$, is the distribution of the random interlacement point process $\varpi^{(\mathrm{B})} = \sum_{g \in \Gamma_{\mathrm{B}}} \delta_g$, which consists of independent two-sided Brownian motions $g \in \mathcal{C}_\infty$ with parameter set $\mathbb{R}$, identified as equal modulo time shift by β. The only free parameter of this process is the expected total amount of particles in the unit box $U = [-\frac{1}{2}, \frac{1}{2}]^d$, which we call $u \in (0, \infty)$. We sometimes write $\mathtt{R}^{(u,\beta)}$ or $\mathtt{R}^{(u)}$; if we take $u = 1$, then we drop this super-index.

The first construction of such a process was carried out in the seminal paper [Sz10] for simple random walk on $\mathbb{Z}^d$; later the Brownian version was constructed in [Sz13]. The version that we need in this paper is for Gaussian random walks instead, with interpolation by independent Brownian bridges. The Gaussian interlacement PPP has been already constructed in [AFY21], but without interpolation, and without stating the distributional properties that we will need; hence we are giving a short account on the construction and on the properties that we need. Alternatively, we could also refer to the Brownian version constructed in [Sz13] and furnish each interlacement $g\colon \mathbb{R} \to \mathbb{R}^d$ independently with a determination of where $\beta\mathbb{Z}$ is in the domain $\mathbb{R}$, which leads presumably to the same process that we are going to construct now, but we prefer to derive some distributional properties along our construction.

We are following the construction of the interlacement PPP for simple random walk on $\mathbb{Z}^d$ given in [Sz10, Section 1] and translate it to the Gaussian random walk in $\mathbb{R}^d$. Once we have constructed this discrete-time process, we scale it up in time by a factor of β and interpolate it between the times in $\beta\mathbb{Z}$ with independent Brownian bridges with time-interval $[0, \beta]$. See [LP18] for general references to the PPP theory.

Here is a slightly more explicit definition of the Gaussian interlacement PPP: It is the PPP with a σ-finite intensity measure Υ on the set of trajectories $\mathbb{Z} \to \mathbb{R}^d$ that is uniquely determined by the following property: For $W \Subset \mathbb{R}^d$, the restriction of Υ to the set of trajectories that visit W is the (not normalised) measure that starts with a site $\in W$ at time 0 that is 'distributed' according to the equilibrium measure of the Gaussian random walk on W, and has two independent trajectories: one into the negative time axis, conditioned on never entering W again, and one in the positive time axis, being just a Gaussian random walk in $\mathbb{Z}^d$. (By transience, it visits W only finitely often.) This W-dependent measure turns out to be consistent in W such that the described property fixes it uniquely.

Let us come to the details. By $\mathbb{P}_x$ we denote the probability measure under which a Gaussian random walk $(X_n)_{n \in \mathbb{N}_0}$ (i.e., with steps that are independent and Gaussian distributed with variance 2β) starts from $x \in \mathbb{R}^d$. Since we are in $d \geq 3$, this is a random variable with values in the set

$$\mathcal{T}_+ = \Big\{ \mathbf{x} = (x_n)_{n \in \mathbb{N}_0} \in (\mathbb{R}^d)^{\mathbb{N}_0} \colon \lim_{n \to \infty} x_n = \infty \Big\}.$$

By $\mathcal{T} = \{\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in (\mathbb{R}^d)^{\mathbb{Z}} \colon \lim_{n \to \pm\infty} x_n\}$ we denote the set of two-sided trajectories that visit any compact set only finitely often, and by $\mathcal{T}^*$ its quotient space with respect to the identification of functions if they are time-shifts of each other, i.e., $\mathcal{T}^* = \{\{\theta_n f \colon n \in \mathbb{Z}\} \colon f \in \mathcal{T}\}$, where θ_n is the time-shift by n. By $\pi^* \colon \mathcal{T} \to \mathcal{T}^*$ we denote the canonical projection. By $\mathcal{T}_W = \{(x_n)_{n \in \mathbb{Z}} \in \mathcal{T} \colon x_n \in W \text{ for some } n \in \mathbb{Z}\}$ we denote the set of trajectories that visit W, and by $\mathcal{T}_W^*$ its image under π^*.

As we now need some little theory of capacities and equilibrium measures for the Gaussian random walk, we restrict to special compact subsets, namely boxes. For any box $W \Subset \mathbb{R}^d$, we denote by e_W the density of the equilibrium measure of the Gaussian random walk, i.e.,

$$\mathrm{e}_W(x) = \mathbb{1}_W(x) \mathbb{P}_x(\widetilde{H}_W = \infty), \qquad x \in \mathbb{R}^d, \tag{A.1}$$

where $\widetilde{H}_W = \inf\{n \in \mathbb{N} \colon X_n \in W\}$ is the entry time into W. It is clear that $\mathrm{e}_W(x) > 0$ in the box W, but $= 0$ outside the box. By $\mathbb{P}_x^{(W)}$ we denote the conditional measure $\mathbb{P}_x(\cdot \mid \widetilde{H}_W < \infty)$ given that the path visits W. Now we introduce the path measure Q_W on $\mathcal{T}$ as follows:

$$Q_W\big((X_{-n})_{n\in\mathbb{N}_0} \in A, X_0 \in \mathrm{d}x, (X_n)_{n\in\mathbb{N}_0} \in B\big) = \mathbb{P}_x^{(W)}(A)\mathrm{e}_W(x)\mathbb{P}_x(B)\,\mathrm{d}x, \qquad x \in \mathbb{R}^d, A, B \subset \mathcal{T}_+ \text{ mb.} \tag{A.2}$$

We also need the last time $L_W = \sup\{n \in \mathbb{N}_0 \colon X_n \in W\}$ of a visit to W and the set of finite-length trajectories

$$\mathcal{T}_{W\to W} = \{\mathbf{x} = (x_n)_{n\in\{0,\dots,N\}} \in (\mathbb{R}^d)^{N+1} \colon N \in \mathbb{N}, x_0 \in W, x_N \in W\} \tag{A.3}$$

that start and terminate in W.

We can now formulate and prove the existence and uniqueness of the measure Υ that we will take as the intensity measure of our PPP.

Lemma A.1 (Existence, uniqueness and properties of Υ)**.** (1) *There is a unique σ-finite measure Υ on $\mathcal{T}^*$ such that*

$$\mathbb{1}_{\mathcal{T}^*_W}\,\Upsilon = \pi^* \circ Q_W, \qquad W \subset \mathbb{R}^d \text{ a box.} \tag{A.4}$$

(2) Υ *is invariant under time-reversal and spatial shifts.*

(3) *For any $W \Subset \mathbb{R}^d$ and any $\mathbf{x} = (x_n)_{n\in\{0,\dots,N\}} \in \mathcal{T}_{W\to W}$,*

$$\begin{aligned} Q_W\big((X_{-n})_{n\in\mathbb{N}_0} \in A, &(X_n)_{n\in\{0,\dots,L_W\}} \in \mathrm{d}\mathbf{x}, (X_{L_W+n})_{n\in\mathbb{N}_0} \in B\big) \\ &= \mathbb{P}_{x_0}^{(W)}(A)\mathrm{e}_W(x_0)\mathbb{P}_{x_0}((X_n)_{n\in\{0,\dots,N\}} \in \mathrm{d}\mathbf{x})\mathbb{P}_{x_N}^{(W)}(B), \qquad A, B \subset \mathcal{T}_+ \text{ mb.} \end{aligned} \tag{A.5}$$

Proof. We only give some few hints, as the proof is entirely analogous to the one of [Sz10, Theorem 1.1] (which is lengthy and technical). We leave the details to the reader.

The uniqueness in (1) follows from the fact that $\mathcal{T}^*$ is exhausted by the sets $\mathcal{T}^*_{[-R,R]^d}$ with $R \in \mathbb{N}$. The proof of the existence consists of showing that the measure $\Upsilon_W = \pi^* \circ Q_W$ is consistent in W in the sense that, for any $W \subset \widetilde{W} \Subset \mathbb{R}^d$, we have $\mathbb{1}_{\mathcal{T}^*_W}\,\Upsilon_{\widetilde{W}} = \Upsilon_W$. Then Carathéodory's extension theorem yields (1). The proof of this is entirely analogous to the proof of the analogous assertion in [Sz10, Theorem 1.1], with some obvious changes that come only from the continuous nature of $\mathbb{R}^d$: the fact that the equilibrium measure has a Lebesgue density and that trajectories run in $\mathbb{R}^d$ rather than in $\mathbb{Z}^d$. The proof goes via a decomposition into paths that start and end in $\widetilde{W}$ and run within W in between, and it uses the Markov property of the random walk. $\square$

Assertion (1) says that Υ can be described by the explicit formula (A.2) on the set of trajectories that visit W, and Assertion (3) says that every path under Q_W can be decomposed in time in three pieces: one piece of a Gaussian random until the first time of a visit to W (having the equilibrium measure as the 'distribution' at this time), one piece until the last visit to W, and the last piece, which is a Gaussian random walk conditioned on never visiting W anymore. These pieces are independent, given the two locations at the first and the last time of a visit to W.

By Leb we denote the Lebesgue measure on $[0,\infty)$. Now we use the σ-finite measure $\Upsilon \otimes \mathrm{Leb}$ on $\mathcal{T}^+ \times [0,\infty)$ as the intensity measure of a PPP $\varpi = \sum_i \delta_{(X_i^*, v_i)}$, where we are going to use the v-component for controlling the spatial density of the paths. More precisely, for any $v \in (0,\infty)$, we can call the point process $\sum_{i\colon v_i \le v} \delta_{X_i^*}$ the *Gaussian interlacement process with density parameter v.*

Remark A.2 (Construction of the Gaussian interlacement PPP.) From Assertion (3) in Lemma A.1, using the standard construction of PPPs from the intensity measure, the following construction of the PPP $\sum_{i\colon v_i\le v} \delta_{X_i^*}$ emerges. In simple words, one can construct the PPP on the set $\mathcal{T}^*_W$ of paths that visit the box W by first sampling a Poisson-distributed variable N with parameter $v\mathrm{cap}(W) = v\int_W \mathrm{e}_W(x)\,\mathrm{d}x$ and then N random sites $X_0^{(1)}, \dots, X_0^{(N)} \in W$ with distribution $\mathrm{e}_W(x)\,\mathrm{d}x/\mathrm{cap}(W)$. Then one samples independently over $i = 1, \dots, N$ a Gaussian random walk $(X_{-n}^{(i)})_{n\in\mathbb{N}_0}$ conditioned on never visiting W, and another, independent, Gaussian random walk $(X_n^{(i)})_{n\in\mathbb{N}_0}$ (which may return to W never or several times, but eventually vanishes at ∞). Finally, we 'forget' the precise time at which $X^{(i)}$ first enters W, i.e., we put $X_i^*=\pi^*(X^{(i)})$. $\Diamond$

Now we interpolate any Gaussian random walk to a Brownian motion by inserting independent Brownian bridges and scaling them up in time with a factor of β. For this, we will be using the concept of K-marking with a kernel K, which is well-known in PPP-theory. Introduce the map

$$\mathcal{T}_+\ni\mathbf{x}=(x_n)_{n\in\mathbb{N}_0}\mapsto B^{(\mathbf{x})}=(B_t^{(\mathbf{x})})_{t\in[0,\infty)}\in\mathcal{C},\qquad B_t^{(\mathbf{x})}=B_{t-\beta n}^{(x_n,x_{n+1})}\quad\text{if } t\in[\beta n,\beta(n+1)], n\in\mathbb{N}_0,\tag{A.6}$$

and $(B^{(x_n,x_{n+1})})_{n\in\mathbb{N}_0}$ is a family of independent Brownian bridges with time-interval $[0,\beta]$ from x_n to x_{n+1}. Then, if X is a Gaussian random walk with variance 2β of the steps, $B^{(X)}$ is a Brownian motion with generator Δ. We can describe this interpolation operation in terms of distributions via the kernel K from $\mathcal{T}_+$ to $\mathcal{C}$ given by

$$K(\mathbf{x},\mathrm{d}f)=\bigotimes_{n\in\mathbb{N}_0}\mu_{x_n,x_{n+1}}^{(\beta)}\big(\mathrm{d}(\theta_{\beta n}(f|_{[\beta n,\beta(n+1)]}))\big),\qquad \mathbf{x}=(x_n)_{n\in\mathbb{N}_0}\in\mathcal{T}_+, f\in\mathcal{C},\tag{A.7}$$

where we recall from (1.4) the canonical Brownian bridge measures. We use the same symbol K for the variants of this kernel on the sets $\mathcal{T}$ and $\mathcal{T}^*$.

From Υ, we now proceed with the K-marking $\varpi^{(K)}=\sum_i\delta_{(X_i^*,B_i^*,v_i)}$, where the Brownian motions B_i^* derive from the random walks X_i^* via (A.6), independently over i. Its intensity measure is $\Upsilon\otimes K\otimes\mathrm{Leb}$. Observe that X_i^* is a deterministic function of B_i^* and that the Brownian motions are uniquely determined only up to time-shifts in $\beta\mathbb{Z}$, in contrast to the Brownian interlacement process constructed in [Sz13].

Finally, the measure $\mathtt{R}^{(u,\beta)}$ that we announced is now defined as the distribution of the process $\varpi^{(K,u)}=\sum_{i\colon u_i\le v_u}\delta_{B_i^*}$, where $v_u\in[0,\infty)$ is determined by the requirement that $u=\mathbb{E}(\sum_{i\colon u_i\le v_u}\sum_{n\in\mathbb{Z}}\mathbb{1}_{B_i^*(n\beta)\in U})=1$, where $U=[-\frac12,\frac12]^d$ is the unit box. It is only technical to prove that the distribution of the process $\varpi^{(K,u)}$ is invariant under time-reversal and spatial shifts, since the Brownian bridge measures have the relevant properties for that. That is, the distribution $\mathtt{R}^{(u,\beta)}$ of $\varpi^{(K,u)}$ lies in $\mathcal{M}_1^{(\mathrm{s})}(\mathcal{S})$. This finishes the introduction of the Brownian β-spaced interlacement process.

Remark A.3 (Construction of the Brownian β-spaced interlacement PPP.) Referring to the construction of Remark A.2, the B_i^* may be constructed from the $X^{(i)}$ by interpolating them by Brownian bridges as in (A.6), and N independent standard Brownian motions $B^{(i)}=(B_t^{(i)})_{t\in[0,\infty)}$ with generator Δ arise, starting from $B_0^{(i)}=X_0^{(i)}$. Analogously, and using time-reversal properties of the Brownian bridge measures, we interpolate also the negative-time parts and obtain Brownian motions conditioned on never hitting W at any time $\in\beta\mathbb{N}$, independent of the positive-time parts, given the starting sites $B_0^{(i)}$. Finally, we 'forget' the integer-part of $\frac1\beta$ times the time, i.e., we put $B_i^*=\pi^*(B^{(i)})=\{\theta_{\beta n}(B^{(i)})\colon n\in\mathbb{Z}\}$. $\Diamond$

Now we finally give the connection between the interlacement process $\mathtt{R}$ and our analysis of the interacting Bose gas. Recall the kernel $\mathtt{K}_W$ from (2.5) and recall from (2.13) how crucial it is in this connection.

Lemma A.4 (Identification of $\mathtt{K}_W$ in terms of $\mathtt{R}$)**.** *The kernel $\mathtt{K}_W(\mu,\cdot)$ defined in (2.5) is equal to the kernel $\Pi_W(\mathtt{R})_{\mathcal{T}_W\to\mathcal{S}_W}(\mu,\cdot)$.*

Proof. We refer to the construction of the Brownian β-spaced interlacement process ϖ from Remarks A.2 and A.3. The event $\{\partial\Pi_W^{(\mathcal{S})}(\varpi)=\mu\}$ on that we condition depends only on the processes $(B_t^{(i)})_{t\in[0,\infty)}$, i.e., the parts from the time of the first visit to W at a time $\in\beta\mathbb{Z}$ on. Introduce the subsequent entry and exit times (up to the factor β) $0=\tau_1^{(i)}<\sigma_1^{(i)}<\tau_2^{(i)}<\sigma_2^{(i)}<\cdots<\tau_{m_i}^{(i)}<\sigma_{m_i}^{(i)}<\infty$

of $B^{(i)}$, and put $\mu = \sum_{j\in I} \delta_{(x_j,l_j,y_j)}$, then the event $\{\partial\Pi_W^{(\mathcal{S})}(\varpi) = \mu\}$ is equal to the disjoint union over all bijections $h\colon J = \{(i,k) \in [N] \times \mathbb{N}\colon k \leq m_i\} \to I$ of the event

$$\bigcap_{(i,k)\in J} \Big\{B^{(i)}_{\tau_k^{(i)}\beta} = x_{h(i,k)}, \sigma_k^{(i)} - \tau_k^{(i)} = l_{h(i,k)}, B^{(i)}_{\sigma_k^{(i)}\beta} = y_{h(i,k)}\Big\}.$$

On this event, $\Pi_W^{(\mathcal{S})}(\varpi)$ is the empirical measure of the paths $(B_t^{(i)})_{t\in[\beta\tau_k^{(i)},\beta\sigma_k^{(i)}]}$, properly time-shifted, with $(i,k) \in J$. According to the strong Markov property at all these stopping times, these path pieces are conditionally independent, and their distribution is given by $q^{(l_{h(i,k)},W)}_{x_{h(i,k)},y_{h(i,k)}}$. Hence, their joint distribution is the product measure of these, which is $\mathtt{K}_W(\mu,\cdot)$. This finishes the proof. $\square$

Acknowledgement. The research was supported by the DFG-SPP 2265 *Random Geometric Systems*, Project P04 *The Statistical Mechanics of the Interlacement Poisson Point Process*. Many inspiring discussions with Stefan Adams (University of Warwick), Alexander Drewitz (University of Cologne) and Alexander Zass (Weierstrass Institute Berlin) are gratefully acknowledged, as well as some fruitful exchange with Guillaume Bellot (University of Lille) and Quirin Vogel (Universität Klagenfurt).

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