

SELF-REPELLENT BROWNIAN BRIDGES IN AN INTERACTING BOSE GAS

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Abstract: We consider a model of d -dimensional interacting quantum Bose gas, expressed in terms of an ensemble of interacting Brownian bridges in a large box and undergoing the influence of all the interactions between the legs of each of the Brownian bridges. We study the thermodynamic limit of the system and give an explicit formula for the limiting free energy and a necessary and sufficient criterion for the occurrence of a condensation phase transition. For $d \geq 5$ and sufficiently small interaction, we prove that the condensate phase is not empty. The ideas of proof rely on the similarity of the interaction to that of the self-repellent random walk, and build on a lace expansion method conducive to treating *paths* undergoing mutual repulsion within each bridge.

1. INTRODUCTION AND MAIN RESULTS

1.1 Background.

The well-known *interacting Bose gas* can be written, using the *Feynman–Kac formula*, as a random ensemble of many Brownian bridges of random and unbounded lengths (= particle numbers) with mutually repellent interactions between any two legs of any of the bridges. In the thermodynamic limit, the transition from absence to emergence of macroscopically many particles in long bridges in this ensemble as the density increases is a prominent open problem that is strongly linked with the famous phase transition called *Bose–Einstein condensation (BEC)*, if not equal to that.

The starting observation of the present paper is the following: Each of these bridges, under the influence of all the interactions between its legs alone, is a transformation of a Brownian bridge that should lie in the class of the well-known *self-repellent walk* (or *weakly self-avoiding walk*). This class of random motions is prominent in statistical physics and has been studied a lot since several decades, although a lot of problems are left widely open.

However, to the best of our knowledge, a substantial connection between the Bose gas and the self-repellent walk has not been made. In this paper, we establish this connection to study the influence of the interactions between its legs and characterize the free energy of the system explicitly in terms of a variational formula and find criteria for the presence of a condensate phase transition in this model. These criteria lie at the heart of the critical properties of the self-repellent Brownian bridges.

Below in Section 1.2 we will introduce the mathematical layout of the model, in Section 1.3 we will explain our purposes and formulate our main results in Section 1.4. In Section 2 we will discuss the

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necessary background, outline the method of proof and draw conclusions from our main results. The proofs will be given in the subsequent sections.

1.2 The model.

Let us introduce the model that we are interested in. As will be explained in Section 2.1, it is indeed a version of the interacting Bose gas. For the sake of simplicity, we prefer to formulate it as a model of a *random partition* of a positive integer $N \in \mathbb{N}$.

Indeed, denote the set of partitions of $N \in \mathbb{N}$ by

$$\mathfrak{P}_N = \left\{ l = (l_k)_{k \in \mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}} : \sum_{k \in \mathbb{N}} k l_k = N \right\}. \quad (1.1)$$

With a parameter $\beta \in (0, \infty)$, introduce the following probability measure on \mathfrak{P}_N :

$$\mathbf{P}_{\beta, \Lambda}^{(N)}(l) = \frac{1}{Z_N^{(\text{bc})}(\beta, \Lambda)} \prod_{k \in \mathbb{N}} \frac{[|\Lambda| \Gamma_{\Lambda, k}^{(\text{bc})}]^{l_k}}{l_k! k^{l_k}}, \quad l = (l_k)_{k \in \mathbb{N}} \in \mathfrak{P}_N, \quad (1.2)$$

where

$$Z_N^{(\text{bc})}(\beta, \Lambda) = \sum_{l \in \mathfrak{P}_N} \prod_{k \in \mathbb{N}} \frac{[|\Lambda| \Gamma_{\Lambda, k}^{(\text{bc})}]^{l_k}}{l_k! k^{l_k}} \quad (1.3)$$

is the partition function of the model (the normalization). It is well-known that $N / \prod_{k \in \mathbb{N}} l_k! k^{l_k}$ is equal to the number of permutations of $1, \dots, N$ that have precisely l_k cycles for any $k \in \mathbb{N}$. Thus, (1.1)–(1.3) indeed define a model of a random permutation with multiplicative weight $|\Lambda| \Gamma_{\Lambda, k}^{(\text{bc})}$ for any cycle of length k ; see Section 2.4 for further details.

Let us now explain the terms appearing in (1.1) – (1.3). First, $\Lambda \subset \mathbb{R}^d$ denotes a centered box, and $\Gamma_{\Lambda, k}^{(\text{bc})}$ is defined as an integral of an exponential interaction with respect to a Brownian bridge of length k in Λ (cf. (1.5) below). If we denote by $\mu_{x, y}^{(\text{bc}, \beta)}(\cdot)$ the canonical Brownian bridge measure on the time interval $[0, \beta]$ from x to y subject to the boundary condition ‘bc’ in the box Λ (the boundary condition to be specified later), we set

$$\mu_{\Lambda}^{(\text{bc}, \beta)}(df) = \frac{1}{|\Lambda|} \int_{\Lambda} dx \mu_{x, x}^{(\text{bc}, \beta)}(df). \quad (1.4)$$

Thus, $\mu_{\Lambda}^{(\text{bc}, \beta)}(\cdot)$ is a Brownian bridge measure on $\mathcal{C} = \mathcal{C}_1$, where \mathcal{C}_k denotes the set of continuous functions $[0, k\beta] \rightarrow \mathbb{R}^d$, with uniform starting-termination site in the centered box Λ . If we write $\mu(f)$ for the integral $\int f d\mu$ of a function f with respect to a measure μ , the weight $\Gamma_{\Lambda, k}^{(\text{bc})}$ is defined as

$$\Gamma_{\Lambda, k}^{(\text{bc})} = \mu_{\Lambda}^{(\text{bc}, k\beta)} \left[e^{-\sum_{1 \leq i < j \leq k} V(B_i, B_j)} \right], \quad (1.5)$$

where, for any two continuous functions $f, g \in \mathcal{C}$, we write

$$V(f, g) = \int_0^{\beta} v(|f(s) - g(s)|) ds, \quad (1.6)$$

and $v: [0, \infty) \rightarrow [0, \infty)$ is a bounded measurable *pair interaction potential* having compact support. In (1.5), the Brownian bridge B in \mathcal{C}_k has been decomposed into its k *legs*, defined as

$$B_j = (B_j(s))_{s \in [0, \beta]} = (B((j-1)\beta + s))_{s \in [0, \beta]} \in \mathcal{C}, \quad j \in [k] = \{1, \dots, k\}. \quad (1.7)$$

Hence, the exponential interaction term in (1.12) is the pair-interaction sum of all the legs of a Brownian bridge of length k . Therefore, $\Gamma_{\Lambda, k}^{(\text{bc})}$ is the partition function of what can be called a variant

of the *self-repellent* (or *weakly self-avoiding*) Brownian bridge, where the usual δ -interaction between any two $B(s)$ and $B(t)$ is replaced by a smooth interaction between any two legs B_i and B_j . The starting site of this motion is not the origin as usual, but is uniformly distributed over a large box, and the endpoint is conditioned to be the initial site. We will be studying this model in the limit $N \rightarrow \infty$ with the box $\Lambda = \Lambda_N$ having volume $\sim N/\rho$ for some $\rho \in (0, \infty)$, i.e., in the thermodynamic limit.

By the Feynman–Kac formula, the above model can also be represented as an ensemble of N Brownian bridges in Λ with time-horizon $[0, \beta]$ under some symmetrization condition, where each leg has an interaction with each other leg. Due to the symmetrization, one can decompose the ensemble into closed loops (bridges); see Section 2.1 for further details.

We remark that we have kept all features that come from the Bose gas in a large box, in particular the fact that the particles are confined to a box with certain boundary conditions. Most of the existing works on the Bose gas consider periodic boundary conditions, and focus on models of random permutations that dispense with considerations of boundary conditions. However, we find it important to keep them in the model, since they are physically relevant and since they might lead to additional effects. Indeed, for periodic or Dirichlet zero boundary conditions, the asymptotics of the weight $\Gamma_{\Lambda, k}^{(\text{bc})}$ in (1.5) for large k and boxes Λ of volume $\asymp N$ are clearly different for $k \ll N^{2/d}$ and for $k \gg N^{2/d}$.

1.3 Our purpose.

As mentioned above, we will be interested in the *thermodynamic limit* of the model described above. Namely, we fix the inverse temperature $\beta \in (0, \infty)$, a density $\rho \in (0, \infty)$, take the box $\Lambda = \Lambda_N$ of volume $|\Lambda_N| = \frac{N}{\rho}$ and define the corresponding *limiting free energy per volume* as

$$f(\beta, \rho) := - \lim_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log Z_N^{(\text{bc})}(\beta, \Lambda_N), \quad \beta, \rho \in (0, \infty). \quad (1.8)$$

In our first main result, Theorem 1.1, we will see that this limit exists, is independent of boundary conditions and can be expressed in terms of a *characteristic variational formula* that describes the statistics of all the lengths of the Brownian bridges. Next, in this result we prove a large-deviation principle for these statistics and identify the minimizer of the rate function. This implies in particular that the statistics of the cycle lengths converge in distribution towards that minimizer. Now, the characterization of the minimizer involves a possible phase transition in the density ρ – that is, a *non-analyticity* of the map

$$\rho \mapsto f(\beta, \rho) \quad (1.9)$$

emerges at some explicit *critical threshold* $\rho_c(\beta)$, which may or may not be finite in general. This phase transition underlines a *loss of mass* in the statistics of the finite-size cycle lengths in the spirit of the well-known effect in the free Bose gas; it is therefore a signal for a condensation phase transition in a vicinity of the Bose–Einstein condensation.

So far, the above statements are valid for a wide choice of the weights $\Gamma_{\Lambda, k}^{(\text{bc})}$ in (1.3) as well as arbitrary $d \in \mathbb{N}$ and $\beta \in (0, \infty)$. Now for our model, $\rho_c(\beta)$ is shown to be equal to the Green’s function at the origin of a transformed self-interacting Brownian bridge, geometrically weighted with a crucial parameter, the radius of convergence. As a consequence, finiteness of the Green’s function implies the existence of the aforementioned non-analyticity of the map (1.9) at $\rho_c(\beta)$.

Theorem 1.2 is our second main result, where we determine circumstances under which the aforementioned phase transition does occur. Indeed, we show that for $d \geq 5$ and the interaction potential v remaining sufficiently small, the critical density $\rho_c(\beta)$ is actually *finite*. The key idea here is the observation that the self-interacting Brownian bridge lies in the same universality class as the famous *self-avoiding walk* (SAW), the uniform distribution on N -step nearest-neighbour random walks on \mathbb{Z}^d

starting at zero and hitting no site twice. Indeed, since the pair functional v is non-negative, the interaction $\sum_{1 \leq i < j \leq k} V(B_i, B_j)$ repels each two legs from each other, in the spirit of a weak version of the SAW, the *self-repellent random walk*. One main difference to the present model of the interacting quantum Bose gas is that here it is *paths* that undergo a mutual repulsion, not endpoints.

The restrictions to $d \geq 5$ and small β are due to a continuous version of the *lace expansion method* that we develop and employ in the proof of Theorem 1.2. This part is a modification of the method developed in [BHK18]. We refer to Section 2.3 for a short survey on this method, and to Section 2.2 for precise conjectures about the behaviour of the SAW and models in its universality class. Surprisingly, these conjectures imply the finiteness of the Green's function (i.e., the occurrence of the phase transition in our model) even in dimension $d = 2$, not only in dimensions $d \geq 3$, being aligned with conjectures about the BEC phase transition.

1.4 Main results.

The formulation of our results will require setting up some further notation, which will subsequently be used in the entire sequel. As before, \mathcal{C} will denote the space of continuous functions $[0, \beta] \rightarrow \mathbb{R}^d$. The canonical Brownian bridge measure is defined for any measurable $A \subset \mathcal{C}$ as

$$\mu_{x,y}^{(\beta)}(A) = \frac{\mathbb{P}_x(B \in A; B_\beta \in dy)}{dy}. \quad (1.10)$$

Its total mass is given by

$$\mu_{x,y}^{(\beta)}(\mathcal{C}) = \varphi_\beta(x, y) := \frac{\mathbb{P}_x(B_\beta \in dy)}{dy} = (2\pi\beta)^{-d/2} e^{-\frac{1}{2\beta}|x-y|^2}, \quad (1.11)$$

where $B = (B_t)_{t \in [0, \beta]}$ is a Brownian motion in \mathbb{R}^d with generator $\frac{1}{2}\Delta$, starting from x under \mathbb{P}_x . Hence, the total mass of $\mu_{0,0}^{(k\beta)}$ is equal to $(2\pi\beta k)^{-d/2}$.

We fix a bounded measurable function $v: [0, \infty) \rightarrow [0, \infty)$ with compact support. Let us set

$$\begin{aligned} \Gamma_k &= \mu_{0,0}^{(k\beta)} \left[e^{-\sum_{1 \leq i < j \leq k} V(B_i, B_j)} \right] \\ &= \mathbb{E}_0 \left[e^{-\sum_{1 \leq i < j \leq k} V(B_i, B_j)} \mathbb{1}\{B(k\beta) \in dx\} \right] / dx \Big|_{x=0}, \end{aligned} \quad (1.12)$$

with $V(\cdot, \cdot)$ defined in (1.6), compare to (1.5). Then the sequence $(\Gamma_k (2\pi\beta k)^{d/2})_{k \in \mathbb{N}}$ is sub-multiplicative. Consequently, by Fekete's lemma, the limit (known as the *connective constant*)

$$\begin{aligned} \lambda_c(\beta) &= \lim_{k \rightarrow \infty} \Gamma_k^{-1/k} \quad \text{exists and satisfies} \\ \lambda_c(\beta)^k \Gamma_k &\geq (2\pi\beta k)^{-d/2} \quad \forall k \in \mathbb{N}. \end{aligned} \quad (1.13)$$

$\lambda_c(\beta)$ is the radius of convergence of the power series with coefficients Γ_k . It will turn out to capture the limiting behaviour of the coefficients $\Gamma_{\Lambda_{N,k}}^{(\text{bc})}$ of the model. Finally, introduce

$$I(p) = \sum_{k \in \mathbb{N}} p_k \log \frac{p_k k}{\Gamma_k e}, \quad p = (p_k)_{k \in \mathbb{N}} \in [0, \infty)^\mathbb{N}. \quad (1.14)$$

The function I will turn out to play the role of a large-deviation rate function – it is the relative entropy of p with respect to the sequence $(\Gamma_k/k)_{k \in \mathbb{N}}$ plus the sum of the Γ_k/k .

Here is the identification of the free energy $f(\beta, \rho)$ defined in (1.8), which is the first main result of our article.

Theorem 1.1 (Free energy and cycle lengths). *Fix $\beta, \rho \in (0, \infty)$ and consider the model defined in (1.3) and (1.2) in the centred box Λ_N with volume N/ρ with Dirichlet boundary conditions $\text{bc} \in \{\text{Dir}\}$. Then the following hold.*

- (i) *With $\lambda_c(\beta)$ defined in (1.13), the limiting free energy $f(\beta, \rho)$ defined in (1.8) exists and is identified as*

$$f(\beta, \rho) = \inf_{p \in [0, \infty)^{\mathbb{N}}: \sum_k k p_k \leq \rho} \left[I(p) + \left(\rho - \sum_{k \in \mathbb{N}} k p_k \right) \log \lambda_c(\beta) \right]. \quad (1.15)$$

- (ii) *Let $(L_k)_{k \in \mathbb{N}}$ be a random variable under $\mathbb{P}_{\beta, \Lambda_N}^{(N)}$, then the distribution of $(L_k/|\Lambda_N|)_{k \in \mathbb{N}}$ satisfies a large-deviations principle (LDP) ² on the set $\mathcal{X}_\rho = \{p \in [0, \infty)^{\mathbb{N}}: \sum_{k \in \mathbb{N}} k p_k \in [0, \rho]\}$ on the scale $|\Lambda_N|$ with rate function J given by*

$$J(p) = I(p) + \left(\rho - \sum_{k \in \mathbb{N}} k p_k \right) \log \lambda_c(\beta) - f(\beta, \rho). \quad (1.16)$$

- (iii) *Put*

$$\rho_c(\beta) = \sum_{k \in \mathbb{N}} \lambda_c(\beta)^k \Gamma_k \in [0, \infty]. \quad (1.17)$$

Then the unique minimizer p^ of the formula on the right-hand side of (1.15) is given as*

$$p_k^* = \frac{1}{k} \lambda_c(\beta)^k \Gamma_k \times \begin{cases} e^{-c(\rho)k}, & \text{if } \rho \leq \rho_c(\beta), \\ 1 & \text{if } \rho > \rho_c(\beta), \end{cases} \quad k \in \mathbb{N}, \quad (1.18)$$

where $c(\rho) \in [0, \infty)$ is such that

$$\sum_{k \in \mathbb{N}} \lambda_c(\beta)^k \Gamma_k e^{-c(\rho)k} = \rho.$$

In particular, $\rho \mapsto c(\rho)$ is strictly decreasing with $c(\rho_c(\beta)) = 0$. Furthermore, $\sum_{k \in \mathbb{N}} k p_k^ = \rho \wedge \rho_c(\beta)$ for any $\rho \in (0, \infty)$.*

- (iv)

$$f(\beta, \rho) = \rho \log \lambda_c(\beta) - \begin{cases} \rho c(\rho) + \sum_{k \in \mathbb{N}} \frac{1}{k} \Gamma_k \lambda_c(\beta)^k e^{-c(\rho)k} & \text{if } \rho \leq \rho_c(\beta), \\ \sum_{k \in \mathbb{N}} \frac{1}{k} \Gamma_k \lambda_c(\beta)^k & \text{if } \rho > \rho_c(\beta). \end{cases} \quad (1.19)$$

In particular, $f(\beta, \cdot)$ has a phase transition (non-analyticity) in $\rho_c(\beta)$ if this point is finite.

The proof of Theorem 1.1 is provided in Section 3. Let us make some remarks about the assertions appearing there. First note that the rate function of the LDP there has two terms, the entropic term I , which describes the statistics of the finite cycle lengths, and the energy term, $(\rho - \sum_k k p_k) \log \lambda_c(\beta)$, which is the contribution from the condensate part. Unlike in the free Bose gas (see, e.g., [KVZ23]), the condensate yields here an explicit contribution on the leading scale, but not an entropic one. While each cycle of length k comes with an energetic term Γ_k , the condensate comes with the logarithm of the connective constant of the sequence $(\Gamma_k)_{k \in \mathbb{N}}$. The convergence of $(L_k/\Lambda_N)_{k \in \mathbb{N}}$ towards the minimizer p^* shows that the model has a phase transition at the critical density $\rho_c(\beta)$ if this quantity is finite.

² This means the following: A sequence $(X_N)_{N \in \mathbb{N}}$ of random variables taking values in a topological space \mathcal{X} satisfies an LDP on the scale γ_N with rate function $J: \mathcal{X} \rightarrow [0, \infty]$ if J is lower-semicontinuous and for any open subset G and any closed subset F of \mathcal{X} ,

$$\liminf_{N \rightarrow \infty} \frac{1}{\gamma_N} \log \mathbb{P}(X_N \in G) \geq - \inf_G J \quad \text{and} \quad \limsup_{N \rightarrow \infty} \frac{1}{\gamma_N} \log \mathbb{P}(X_N \in F) \leq - \inf_F J.$$

If J has a unique minimizer $x^* \in \mathcal{X}$, then it follows that X_N converges weakly towards x^* . We refer to [DZ98] for the general theory of large deviations.

In (1.17) we see that $\rho_c(\beta)$ is the Green's function of the transformed Brownian motion geometrically weighted with parameter $\lambda = \lambda_c(\beta)$, taken at the origin, and $\lambda_c(\beta)$ is the radius of convergence. If the Green's function is finite at this point, then the free energy $f(\beta, \cdot)$ is not analytic here, since all the coefficients in the series in (1.15) are positive.

Now the question of course arises, under what circumstances this phase transition occurs. We give a positive answer in our next main result.

Theorem 1.2 (Phase transition in $d \geq 5$). *Assume that $d \geq 5$ and that the interaction potential v (cf. (1.5) and (1.6)) is bounded and continuous and has a bounded support. Let $\rho_c(v, \beta) = \rho_c(\beta)$ be the critical density defined in (1.17). Then, for any $\beta \in (0, \infty)$, there is $\alpha_\beta > 0$ such that $\rho_c(\alpha v, \beta) < \infty$ for any $\alpha \in (0, \alpha_\beta]$.*

The proof of Theorem 1.2 is spanned through Section 4 – Section 7. As we will explain in Section 2, the restriction to $d \geq 5$ is related to the question that lies at the heart of a famous and fundamental question that is notoriously difficult to answer in dimensions $d \in \{2, 3, 4\}$: the question whether or not the Green's function of the weakly self-avoiding walk is finite at the critical point.

The organization of the remainder of this paper is as follows. In Section 2 we discuss several aspects of our main results, in Section 3 we prove Theorem 1.1, in Section 4 we give the proof of Theorem 1.2, subject to the proof of two crucial results that we prove in Sections 5 and 6, respectively. In Section 7, we prove some technical estimates, which are used in the two sections preceding to that.

2. BACKGROUND AND DISCUSSION

In this section we discuss the background, proof method and conclusions of our main results. More concretely, in Section 2.1 we explain the relation between the well-known interacting Bose gas and the model that we study in this paper, in Section 2.2 we summarize what is known and what is conjectured about the weakly self-avoiding walk, and we draw conjectural conclusions about the occurrence of the phase transition in those dimensions that we do not handle in this paper. Some elements and outline of our main proof method for Theorem 1.2 is presented and commented on in Section 2.3. Finally, we comment on our model from the viewpoint of random permutations in Section 2.4, and in Section 2.5 we comment on the role of the boundary conditions carried in our model.

2.1 The interacting Bose gas.

The model that we are interested in and described in Section 1.2 is strongly inspired by an investigation of an interacting quantum gas at positive temperature $1/\beta \in (0, \infty)$ in the thermodynamic limit, the setup for which can be briefly described as follows. As before, let $\Lambda \subset \mathbb{R}^d$ be a centered box and

$$\mathcal{H}_{N,\Lambda}^{(\text{bc})} = -\frac{1}{2} \sum_{i=1}^N \Delta_i + \sum_{1 \leq i < j \leq N} v(|x_i - x_j|), \quad x_1, \dots, x_N \in \Lambda,$$

is the Hamiltonian operator for N particles located at x_1, \dots, x_N in $\Lambda \subset \mathbb{R}^d$ with a prescribed boundary condition. Again $v: [0, \infty) \rightarrow [0, \infty)$ is some pair-interaction functional, which is assumed to be bounded with bounded support. We are interested in *bosons*, and in this vein a key object of interest is the *symmetrized trace*

$$\mathbf{Z}_N^{(\text{bc})}(\beta, \Lambda) := \text{Tr}_+(e^{-\beta \mathcal{H}_{N,\Lambda}^{(\text{bc})}}), \quad \beta \in (0, \infty), \quad (2.1)$$

where the index $+$ stands for symmetrization – that is, application of the projection operator on the set of all permutation invariant wave functions.

We are going to rewrite this trace in terms of many interacting Brownian bridges, which is a well-known manipulation since the early 1970s and builds on Feynman's intuition from 1953. We refer to [ACK11, Lemma 2.1] for a proof and further details for the following, and to [U06b] for a non-technical and physics-related introduction to the rewrite of the Bose gas in terms of Brownian bridges.

Using well-known trace formulas [G70], (2.1) can be rewritten in probabilistic terminology using *Brownian bridges* as

$$\mathbf{Z}_N^{(\text{bc})}(\beta, \Lambda) = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \int_{\Lambda^N} dx_1 \cdots dx_N \bigotimes_{i=1}^N \mu_{x_i, x_{\sigma(i)}}^{(\text{bc}, \beta)} \left[\exp \left\{ - \sum_{1 \leq i < j \leq N} V(B^{(i)}, B^{(j)}) \right\} \right]. \quad (2.2)$$

Next, it is well-known that each permutation $\sigma \in \mathfrak{S}_N$ can be decomposed into cycles – that is, each σ can be represented by a partition of N . For each σ there is an $l \in \mathfrak{P}_N$ such that σ consists of precisely l_k cycles of length k , for any k . Using the Markov property of the Brownian motion (or, equivalently, the semigroup property of the family of Brownian bridge measures), we can identify each sequence of legs that lie within one cycle of length k as one Brownian bridge starting and ending at the same site and having the time interval $[0, \beta k]$. Hence, for any $k \in \mathbb{N}$, σ gives rise to l_k bridges of length k , labeled

$$B^{(k,1)}, \dots, B^{(k,l_k)}, \quad (2.3)$$

and any bridge $B^{(k,i)}$ has the legs

$$B_j^{(k,i)} = (B_j^{(k,i)}(s))_{s \in [0, \beta]} = (B^{(k,i)}((j-1)\beta + s))_{s \in [0, \beta]}. \quad (2.4)$$

Then, writing $[m] = \{1, \dots, m\}$, our ensemble of legs is the collection

$$\left\{ (B_j^{(k,i)})_{k,i,j} : k \in [N], i \in [l_k], j \in [k] \right\}$$

sampled under the measure $\bigotimes_{k \in \mathbb{N}} (|\Lambda| \mu_{\Lambda}^{(\text{bc}, k\beta)})^{\otimes l_k}$. Furthermore, observe that, for any $l \in \mathfrak{P}_N$, the number of permutations of $1, \dots, N$ such that l_k is the number of its cycles of length k for any $k \in \mathbb{N}$ is equal to $N! / \prod_{k \in \mathbb{N}} l_k! k^{l_k}$ [C02, Th. 12.1]. As a result, the partition function of the interacting Bose gas may be written

$$\sum_{l \in \mathfrak{P}_N} \left(\prod_{k \in \mathbb{N}} \frac{|\Lambda|^{l_k}}{l_k! k^{l_k}} \right) \left[\bigotimes_{k \in \mathbb{N}} (\mu_{\Lambda}^{(\text{bc}, k\beta)})^{\otimes l_k} \right] [e^{-G_{N,\beta}}], \quad (2.5)$$

where the entire interaction can be written as

$$G_{N,\beta} = \sum_{(k_1, i_1, j_1) \neq (k_2, i_2, j_2)} V(B_{j_1}^{(k_1, i_1)}, B_{j_2}^{(k_2, i_2)}). \quad (2.6)$$

Hence, in the standard form of the interacting Bose gas, every leg of every bridge interacts with any other leg of a bridge. If we drop all interactions between two legs of different bridges, that is, if we replace $G_{N,\beta}$ on the right-hand side of (2.5) by the interaction

$$H_{N,\beta} = \sum_{k=1}^N \sum_{i=1}^{l_k} \sum_{1 \leq j_1 < j_2 \leq k} V(B_{j_1}^{(k,i)}, B_{j_2}^{(k,i)}), \quad (2.7)$$

then, for this interaction, the expectation over the product of the $\mu_{\Lambda}^{(\text{bc}, k\beta)}$ decomposes into a product of expectations, and we see that in (2.5), the partition function of our model, $Z_N^{(\text{bc})}(\beta, \Lambda)$, defined in (1.3) arises. This is our motivation to study this model. Although different cycles do not interact and it is a simplification compared to the scenario of the interacting Bose gas, the self-interactions within each cycle already manifest into an interesting phase transition, as highlighted by Theorem 1.1 and Theorem 1.2, confirming the conjecture that we will now be discussing in Section 2.2.

2.2 Self-repellent random motions.

In our model (1.3) of a partially interacting Bose gas, it turned out in Theorem 1.1 that the occurrence of a condensation phase transition is equivalent to the summability of $\lambda_c(\beta)^k \Gamma_k$ on $k \in \mathbb{N}$. This in turn is equivalent to the finiteness of the Green's function at zero for the corresponding self-repellent random path model in the critical point. This model is of the type of the weakly self-avoiding walk, however, for a Brownian bridge instead of a free (i.e., open-end) path, and with a different type of mutually repellent interaction, which has not yet been studied in the literature on self-interacting path models. Indeed, the interaction $\sum_{1 \leq i < j \leq k} V(B_i, B_j)$ for a Brownian motion B is in the same spirit as the interaction $H_k = \sum_{1 \leq i < j \leq k} \mathbb{1}\{S_i = S_j\}$ (the *self-intersection local time*) for a random walk $(S_i)_{i \in \mathbb{N}_0}$ in \mathbb{Z}^d , since $V > 0$. The two differences between these two models are that (1) precise intersections do almost surely not appear for the Brownian-motion model, and (2) the entire leg $B_i: [0, \beta] \rightarrow \mathbb{R}^d$ is involved in the interaction, not only its endpoint $B_i(\beta)$. But we find it obvious that both models should lie in the same universality class of transformed path measures. Hence let us briefly summarize what is known for the weakly self-avoiding walk.

See [S11] for a brief survey on the weakly self-avoiding walk, the random walk measure on k -step nearest-neighbour paths (S_0, S_1, \dots, S_k) in \mathbb{Z}^d starting from the origin with probability weight $\exp\{-\alpha \sum_{1 \leq i < j \leq k} \mathbb{1}\{S_i = S_j\}\}$ and partition function Z_k . We restrict to dimensions $d \geq 2$. The main conjecture is that the expected end-to-end distance $|S_k - S_0|$ after k steps should be of size $\approx k^{\nu_d}$, with $\nu_d \in [\frac{1}{2}, 1]$ a critical power that is believed to be $\nu_2 = \frac{3}{4}$, $\nu_3 \approx 0.5876$, and $\nu_d = \frac{1}{2}$ in $d \geq 4$, with logarithmic corrections in $d = 4$. Furthermore, Z_k is conjectured to be $\asymp \lambda^{-k} k^{\mu_d-1}$ for some μ_d . The powers ν_d and μ_d are believed to be universal, i.e., not to depend on details of the underlying random walk. Their conjectured values are $\mu_2 = \frac{43}{32}$ and $\mu_3 \approx 1.1568$ and $\mu_4 = 1$ and $\mu_d = 3 - d$ for $d \geq 5$. This implies that the partition function in the critical point should satisfy $Z_k \approx \lambda_c^{-k} k^{\mu_d-1}$ as $k \rightarrow \infty$, where λ_c is the critical parameter (the connective constant, defined analogously to (1.13)).

Conjectures about the finiteness of the Green's function in the critical point can be deduced from the above conjectures. Recall that the prediction is that $|S_k| \approx k^{\nu_d}$ and $Z_k \approx \lambda^{-k} k^{\mu_d-1}$ as $k \rightarrow \infty$, where we remark that Z_k is the expectation of $e^{-\alpha H_k}$ for the random walk with a free end, while Γ_k is the same for the random walk bridge. Assuming even that $S_k k^{-\nu_d}$ converges in distribution under the (free-end) weakly self-avoiding walk model towards some non-degenerate variable (and neglecting the logarithmic corrections that are expected in $d = 4$), we would then obtain that

$$\Gamma_k \lambda^k \approx k^{\mu_d-1} \frac{1}{Z_k} \mathbb{E}[e^{-\alpha H_k} \mathbb{1}_{\{S_k k^{-\nu_d} \in dx\}}] / dx|_{x=0} \approx k^{\mu_d-1-d\nu_d}.$$

That is, if the right-hand side is summable, then the model with bridge instead of free end should have a finite Green's function in the critical point. This is the case precisely if $\mu_d < d\nu_d$. Surprisingly, this should be true in any dimension $d \in \{2, 3, 4\}$, also for $d = 2$.

Hence, we conjecture that the condensation phase transition appears in our model in (1.3) also in $d = 2$. This is surprising, since in the original interacting quantum Bose gas (see Section 2.1) it is generally conjectured that the phase transition exists only in dimensions $d \geq 3$, but not in $d = 2$. For the two-dimensional free (i.e., non-interacting) quantum Bose gas, the non-existence of the phase transition relies on the non-summability of $1/k$, which is 'just at the boundary' of the validity of this statement.

2.3 Lace expansion.

Let us comment on the method that we use in the proof of Theorem 1.2, the *lace expansion*. This is a diagrammatic expansion of the exponential interaction term $e^{-\alpha \sum_{1 \leq i < j \leq k} \mathbb{1}\{S_i = S_j\}}$ in terms of a procedure that reminds of the inclusion-exclusion principle from basic probability theory. Since

diagrammatic illustrations of the method remind of the laces on the edge of a tablecloth, this method got its name “lace expansion”. It was introduced in the mid-eighties in statistical mechanics and was first applied to the (weakly) self-avoiding walk in [HS92] and was several times extended and improved, see [vdHS02, HvdHS03] and, more recently [S22].

The method is perturbative and needs a simple, but non-trivial model to expand around, which is most often simple random walk. Since (at least conjecturally) the weakly self-avoiding walk is similar to the simple random walk only in dimensions $d \geq 5$, the method has been successful only in these dimensions, and only for sufficiently small values of α . For these values of α , rather strong assertions could be proved, among which are the following: the endpoint satisfies a central limit theorem (even a local one), i.e., the distribution of S_k/\sqrt{k} under the transformed measure converges towards a Gaussian distribution with an appropriate non-trivial variance. Furthermore, the Green’s function is finite in the critical parameter value λ_c .

Besides many other extensions, in recent years the lace expansion was extended to models in continuous space, where self-intersections almost surely do not occur. In [ABR16], which gave us the leading idea for the proof in the present paper, the following model is successfully handled in this vein. Here the energy $V(f, g)$ defined in (1.6) is replaced by

$$V_{\text{SAW}}(f, g) = \alpha \mathbb{1}\{|f(\beta) - g(\beta)| \leq r\}, \quad f, g \in \mathcal{C}, \quad (2.8)$$

where $r, \alpha > 0$ are parameters, with a free end of the motion. Then the model is the weakly self-avoiding walk for a Gaussian random walk on \mathbb{R}^d , whose steps have mean zero and variance β . In [ABR16], it was proved that the local central limit theorem holds:

$$\lim_{k \rightarrow \infty} \frac{1}{Z_k} \mathbb{E}_0 \left[e^{-\alpha \sum_{1 \leq i < j \leq k} V(S_i, S_j)} \mathbb{1}\{k^{-1/2} S_k \approx x\} \right] = g_D(0, x), \quad x \in \mathbb{R}^d \setminus U, \quad (2.9)$$

for sufficiently small $\alpha > 0$ and some neighbourhood U of 0, where Z_k is the partition function, g_D is the Gaussian transition kernel with $D = D_\alpha > 0$ a suitable diffusion constant, and “ \approx ” needs to be adjusted to a proper transition from discrete to continuous space. Along the proof, it was also shown that $Z_k \sim C\lambda^{-k}$ for some $C, \lambda > 0$, implying of course that $\lambda = \lambda_c(\gamma)$ is the critical parameter. For $x = 0$, we have only “ \leq ” in (2.9), which says that $\Gamma_k \leq O(\lambda_k^{-k} k^{-d/2})$, which implies that $\Gamma_k \lambda^k$ is summable, i.e., the Green’s function in the critical point is finite. This implies that Theorem 1.2 holds for this model for sufficiently small interaction parameter α .

The novelty of the present paper is to extend it to the interaction V defined in (1.6). This is a substantial non-trivial step since the interaction depends now on the *entire leg* B_i , not only on its endpoint $B_i(\beta)$. The new ideas needed in the present context, for which we will draw on the lace expansion method using Banach algebras introduced in [BHK18], will be discussed along the way of proving Theorem 1.2.

2.4 Random permutations.

The model that we study in this paper is a probability measure on the set \mathfrak{P}_N of all integer partitions of N of the form

$$P_N(l) = \frac{1}{Z_N} \prod_{k \in \mathbb{N}} \frac{[\theta_k^{(N)}]^{l_k}}{l_k! k^{l_k}}, \quad l = (l_k)_{k \in \mathbb{N}} \in \mathfrak{P}_N. \quad (2.10)$$

Here Z_N is the normalisation constant that turns P_N into a probability measure, and $\theta_k^{(N)} \in (0, \infty)$ are positive weights. Such measures are also known as models of *random permutations*, i.e., as a probability measure on the set \mathfrak{S}_N of all permutations σ of $1, \dots, N$, giving the weight $\theta_k^{(N)}$ to any cycle of σ with length k , independently over all cycles. Note that one can freely multiply $\theta_k^{(N)}$ by a factor $e^{\alpha k}$ for fixed k without changing P_N , since this factor drops out in the normalisation.

For $\theta_k^{(N)} = \theta_k$ not depending on N , such distributions have been studied in recent years for many choices of the weights θ_k , and various limiting regimes and limiting distributions have been identified and studied in great detail, producing a rich picture. This line of research goes under the name *random (spatial resp. geometric) permutations* and was initiated in [BU09] as a simplified model for the interacting Bose gas (it appears after interchanging the logarithm with a certain sum in the partition function). See [RZ20] for one of the latest contributions and a host of references on this line of research, which has departed considerably from the spirit of the interacting Bose gas in the last decade.

The weights that we study in this paper, are roughly of the form

$$\theta_k^{(N)} = N(\gamma_{N,k} + o(1)), \quad N \rightarrow \infty, k \in \mathbb{N}, \quad (2.11)$$

with some $\gamma_{N,k} \in (0, \infty)$, whose behaviour as $N \rightarrow \infty$ is *a priori* not easy to determine. Indeed, for fixed k , it converges, as $N \rightarrow \infty$ towards $\gamma_k = \frac{1}{\rho} \Gamma_k$ defined in (1.12), and this is also true for $k \ll N^{2/d}$, but for $k \gg N^{2/d}$, other effects emerge in the asymptotics, and a proof of the necessary asymptotics for deriving an LDP are not so easy to get, and actually in the case of periodic boundary condition we do not. Note that Γ_k is exponentially fast in k , but this exponential terms drops out of the question about the phase transition, and the second term is decisive. For the free Bose gas, $\gamma_k = \frac{1}{\rho} (2\pi\beta k)^{-d/2}$ with $\rho, \beta \in (0, \infty)$ are the particle density and inverse temperature.

2.5 Influence of boundary conditions.

The technical difficulties in deriving an LDP in Theorem 1.1 stem from the precise form of $\gamma_{N,k}$ in (2.11), which depend on the boundary conditions that we use in our interacting Bose gas model.

The by far most often used condition (in the standard interacting Bose gas and in practically all variants) is the periodic one, which is a mathematical idealisation that gives simpler formulas and less technicalities. Many papers on the free Bose gas (e.g. [Sü93, Sü02, BCMP05]) concentrate either on this choice or on the even simpler *free boundary conditions*, where the starting and termination sites of the bridges stay inside the box, but not necessarily the entire path (this does not come from a quantum mechanical model). However, for zero boundary condition, (2.11) is actually not satisfied in the free Bose gas for k coupled with N , since for $k \gg N^{2/d}$, the probability to stay in the box with diameter $\asymp N^{1/d}$ decays to zero exponentially in $kN^{-2/d}$. To the best of our knowledge, this seems to be a fact that has not been noticed nor handled in the study of the free Bose gas before 2023, even though this case is highly relevant from the physics point of view. In the recent paper [KVZ23], this gap is closed by deriving asymptotics for the reduced one-particle matrix of the free Bose gas for all these well-known boundary conditions, also including Neumann conditions, to prove off-diagonal long-range order, which is considered to be the equivalent to the occurrence of BEC.

In our present paper, however, the technicalities arising in the proof of the LDP turned out serious for the case of periodic boundary condition and we decided to complete the proof only for Dirichlet boundary condition. Indeed, we were not able to prove the upper bound $\Gamma_{\Lambda_{N,k}}^{(\text{per})} \leq \lambda_c(\beta)^k (1 + \varepsilon)^k$ for any $k \in \{1, \dots, N\}$ and all sufficiently large N , precisely for the reason mentioned above (the asymptotics for $k \gg N^{2/d}$ are difficult to handle). We elaborate on this point in Remark 1 towards the end of Section 3.1.

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is split into three steps: In Section 3.1 we will show that the limiting free energy exists as $N \rightarrow \infty$ and is independent of the Dirichlet boundary conditions imposed. The

assertions (i) and (ii) of Theorem 1.1 will be shown in Section 3.2, while assertions (iii) and (iv) will be proved in Section 3.3.

3.1 Approximation of $\Gamma_{\Lambda_N, k}^{(\text{bc})}$ with Γ_k .

Recall from (1.12) and from (1.5) that

$$\Gamma_k = \mathbb{E}_0 \left[e^{-H_k(B)} \mathbb{1}\{B(k\beta) \in dx\} \right] / dx \Big|_{x=0}, \quad (3.1)$$

$$\Gamma_{\Lambda_N, k}^{(\text{Dir})} = \frac{1}{|\Lambda_N|} \int_{\Lambda_N} dx \mathbb{E}_x \left[e^{-H_k(B)} \mathbb{1}\{B(s) \in \Lambda_N \forall s \in [0, \beta k]\} \mathbb{1}\{B(k\beta) \in dx\} \right] / dx, \quad (3.2)$$

where we put

$$H_k(B) = \sum_{1 \leq i < j \leq k} V(B_i, B_j) = \sum_{1 \leq i < j \leq k} \int_0^\beta ds v(|B_i(s) - B_j(s)|)$$

and recall that $B_i = (B((i-1) + s))_{s \in [0, \beta]}$ is the i -th leg of the Brownian motion B on $[0, k\beta]$.

In our proof of the LDP in Section 3.2 below, we will need upper bounds for $\Gamma_{\Lambda_N, k}^{(\text{Dir})}$ in terms of Γ_k for any $k \in \{1, \dots, N\}$, but lower bounds only for any $k \leq k_N$ for some $k_N \rightarrow \infty$. In this section, we formulate and prove these assertions in (3.3) and (3.4).

For convenience, we put $\Lambda_N = [-\frac{1}{2}L_N, \frac{1}{2}L_N]^d$. For $x \in \mathbb{R}^d$, we define $z(x) \in \mathbb{Z}^d$ by $x \in z(x)L_N$ and $[x] = x - z(x)L_N \in \Lambda_N$.

First we note that

$$\Gamma_k \geq \Gamma_{\Lambda_N, k}^{(\text{Dir})}, \quad N, k \in \mathbb{N}, \quad (3.3)$$

which follows from dropping the indicator on the event $\{B(s) \in \Lambda_N \forall s \in [0, \beta k]\}$ in (3.2) and noting the shift-invariance of $H_k(B)$.

We now turn to a lower bound for $\Gamma_{\Lambda_N, k}^{(\text{Dir})}$ for k 's that are fixed or diverging slowly. Pick some integer $k_N \rightarrow \infty$ such that $k_N \leq o(N^{1/d})$. Then we claim that for any $\varepsilon \in (0, 1)$, there is N_0 such that

$$\Gamma_{\Lambda_N, k}^{(\text{Dir})} \geq \Gamma_k(1 - \varepsilon), \quad N \geq N_0, k \in \{1, \dots, k_N\}. \quad (3.4)$$

Indeed, with a small $\eta \in (0, 1)$, split the integration in (3.2) into $(1 - \eta)\Lambda_N$ and $\Lambda_N \setminus (1 - \eta)\Lambda_N$ to see that

$$\begin{aligned} \Gamma_{\Lambda_N, k}^{(\text{Dir})} &\geq \Gamma_k - \frac{1}{|\Lambda_N|} \int_{(1-\eta)\Lambda_N} dx \mu_{x,x}^{(\beta k)} \left[e^{-H_k(B)} \mathbb{1}\{B(s) \in \Lambda_N^c \text{ for some } s \in [0, \beta k]\} \right] \\ &\quad - \frac{1}{|\Lambda_N|} \int_{\Lambda_N \setminus (1-\eta)\Lambda_N} dx \mu_{x,x}^{(\beta k)} \left[e^{-H_k(B)} \right] \\ &\geq \Gamma_k - e^{-CN^{2/d}/k} - \frac{|\Lambda_N \setminus (1-\eta)\Lambda_N|}{|\Lambda_N|} \Gamma_k \\ &\geq \Gamma_k - e^{-CN^{2/d}/k} - C\eta \Gamma_k, \end{aligned}$$

where we first used an estimate for the probability that a Brownian bridge on the time interval $[0, k\beta]$ reaches a site that is $\eta N^{1/d}$ away (with some suitable $C > 0$, not depending on N) and then used the maximal growth of k . Now use that $\Gamma_k = \lambda_c(\beta)^{-k(1+o(1))}$ has a nontrivial exponential behaviour in k to see that the latter bound can be lower estimated against $\Gamma_k(1 - \varepsilon)$ for any $N \geq N_0$, if η and N_0 are picked suitably. This proves (3.4).

Remark 1 For the periodic boundary conditions, we are not able to derive the corresponding bounds for the following reason: Indeed, by shift-invariance of the interaction

$$H_{\Lambda_N, k}^{(\text{per})}(B) = \sum_{z \in \mathbb{Z}^d} \sum_{1 \leq i < j \leq k} \int_0^\beta v(|B_i(s) - B_j(s) + zL_N|) \, ds,$$

we see that

$$\Gamma_{\Lambda_N, k}^{(\text{per})} = \sum_{z \in \mathbb{Z}^d} \mathbb{E}_0 \left[e^{-H_{\Lambda_N, k}^{(\text{per})}(B)} \mathbb{1}\{B(k\beta) \in dy\} \right] \Big|_{y=zL_N}.$$

Now we can invoke the lower bound $\Gamma_{\Lambda, k}^{(\text{per})} \geq \Gamma_{\Lambda, k}^{(\text{Dir})}$ and use our lower-bound proof for the latter. We can also estimate $H_{\Lambda_N, k}^{(\text{per})}(B) \geq H_k(B)$. However, the problem that we encounter now is to upper-estimate the expectation density on $\{B(k\beta) \in L_N \mathbb{Z}^d\}$ against the one on $\{B(k\beta) = 0\}$. In fact, this seems to be a rather deep issue, since the behavior of the self-repellent walk depends on the spread of the path in a complicated manner that is currently not understood. However, under the assumptions that we imposed in Theorem 1.2, an extension of that proof might very well be sufficient to derive suitable asymptotics, but we have refrained from pursuing this here.

3.2 Proof of the LDP.

In this section we will prove assertions (i) and (ii) of Theorem 1.1. Recall the set of partitions, $\mathfrak{P}_N = \{l = (l_k)_{k \in \mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}} : \sum_k k l_k = N\}$ and the state space $\mathcal{X}_\rho = \{p \in [0, \infty)^{\mathbb{N}} : \sum_k k p_k \in [0, \rho]\}$. We are going to prove that, for any $p \in \mathcal{X}_\rho$,

$$\lim_{\delta \downarrow 0} \lim_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \mathbb{P}_{\beta, \Lambda_N}^{(N)} \left((L_k / |\Lambda_N|)_{k \in \mathbb{N}} \in B_\delta(p) \right) = -J(p), \quad (3.5)$$

where the rate function $J(p)$ is introduced in (1.16), and $B_\delta(p)$ is the δ -ball around p in some metric that induces the product topology on $[0, \infty)^{\mathbb{N}}$. Since the state space \mathcal{X}_ρ is compact, from (3.5) the LDP in (ii) and the identification of the free energy in (i) of Theorem 1.1 follow along some standard arguments from LDP theory.

Fix $p \in \mathcal{X}_\rho$. From a combinatorial argument [A76] it follows that $\#\mathfrak{P}_N = e^{o(N)}$ as $N \rightarrow \infty$. Hence, the upper bound in (3.5) will follow from

$$\begin{aligned} & \limsup_{\delta \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \left(\sup_{l \in \mathfrak{P}_N : l/|\Lambda_N| \in B_\delta(p)} \prod_{k \in \mathbb{N}} \frac{[|\Lambda_N| \Gamma_{\Lambda_N, k}^{(\text{bc})}]^{l_k}}{l_k! k^{l_k}} \right) \\ & \leq -I(p) - \left(\rho - \sum_{k \in \mathbb{N}} k p_k \right) \log \lambda_c(\beta), \end{aligned} \quad (3.6)$$

and the lower bound will follow from

$$\liminf_{\delta \downarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \left(\prod_{k \in \mathbb{N}} \frac{[|\Lambda_N| \Gamma_{\Lambda_N, k}^{(\text{bc})}]^{l_k^{(\delta, N)}}}{l_k^{(\delta, N)}! k^{l_k^{(\delta, N)}}} \right) \geq -I(p) - \left(\rho - \sum_{k \in \mathbb{N}} k p_k \right) \log \lambda_c(\beta), \quad (3.7)$$

for some $l^{(\delta, N)} \in B_\delta(p)$. Then combining (3.6) and (3.7) will imply (3.5).

Proof of (3.6): Pick $\delta > 0$ and $l \in \mathfrak{P}_N$ such that $l/|\Lambda_N| \in B_\delta(p)$. We need a large auxiliary parameter $L \in \mathbb{N}$. We split the product on k on the left-hand side of (3.6) into the products on $k \leq L$ and on $k > L$. We substitute

$$p_k^{(N, L)} = \frac{l_k}{|\Lambda_N|}, \quad k \in \{1, \dots, L\}. \quad (3.8)$$

Using Stirling's formula and (3.3)-(3.4), the first partial product can be asymptotically identified as

$$\begin{aligned} \prod_{k=1}^L \left[\frac{1}{l_k!} \left(\frac{1}{k} |\Lambda_N| \Gamma_{\Lambda_{n,k}}^{(\text{bc})} \right)^{l_k} \right] &= e^{o(|\Lambda_N|)} \prod_{k=1}^L \frac{(\Gamma_k/k)^{p_k^{(N,L)} |\Lambda_N|}}{(p_k^{(N,L)}/e)^{p_k^{(N,L)} |\Lambda_N|}} \\ &= e^{o(|\Lambda_N|)} \exp \left\{ -|\Lambda_N| I_L(p^{(N,L)}) \right\}, \end{aligned} \quad (3.9)$$

where $I_L(p) = \sum_{k=1}^L p_k \log \frac{p_k k}{\Gamma_k e}$ is the “ L -cutoff” version of the rate function I given in (1.14). Hence, we see from the preceding argument that

$$\limsup_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \left(\sup_{l \in \mathfrak{P}_N : l/|\Lambda_N| \in B_\delta(p)} \prod_{k=1}^L \frac{[|\Lambda_N| \Gamma_{\Lambda_{n,k}}^{(\text{bc})}]^{l_k}}{l_k! k^{l_k}} \right) \leq - \inf_{\tilde{p} \in B_\delta(p)} I_L(\tilde{p}). \quad (3.10)$$

Now the product on $k > L$ on the left-hand side of (3.6) is decomposed into the product of the Γ -terms and the remainder. Indeed, we are going to show, for any $\delta \in (0, 1)$,

$$\limsup_{L \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \left(\sup_{l \in \mathfrak{P}_N : l/|\Lambda_N| \in B_\delta(p)} \prod_{k>L} \left[\frac{1}{l_k!} \left(\frac{1}{k} |\Lambda_N| \right)^{l_k} \right] \right) \leq 0. \quad (3.11)$$

Concerning the remainder product of the Γ -terms, we will show that

$$\limsup_{\delta \downarrow 0} \limsup_{L \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \left(\sup_{l \in \mathfrak{P}_N : l/|\Lambda_N| \in B_\delta(p)} \prod_{k>L} [\Gamma_{\Lambda_{n,k}}^{(\text{bc})}]^{l_k} \right) \leq - \left(\rho - \sum_{k \in \mathbb{N}} k p_k \right) \log \lambda_c(\beta). \quad (3.12)$$

For proving (3.11), we use the Stirling bound

$$\left(\frac{l}{e} \right)^{-l} c \frac{1}{\sqrt{l}} \leq \frac{1}{l!} \leq \left(\frac{l}{e} \right)^{-l} C \frac{1}{\sqrt{l}} \quad l \in \mathbb{N},$$

for some $c, C \in (0, \infty)$. Then we see that the left-hand side of (3.11) can be bounded as follows.

$$\frac{1}{|\Lambda_N|} \log \prod_{k>L} \left[\frac{1}{l_k!} \left(\frac{1}{k} |\Lambda_N| \right)^{l_k} \right] \leq \frac{\rho}{N} \sum_{k>L} l_k \log \frac{N e}{\rho k l_k} + \frac{1}{N} \rho \log C$$

Now use that $\sum_{k>L} \frac{l_k}{N} \leq \frac{1}{L} \sum_{k>L} \frac{k l_k}{N} \leq \frac{1}{L}$ and apply Jensen's inequality to the logarithm and the sum over $l_k / \sum_{n>L} l_n$, to see that (3.11) holds.

Let us turn to a proof of (3.12). Recall from (1.13) that $\lim_{k \rightarrow \infty} \Gamma_k^{-1/k} = \lambda_c(\beta)$. Pick an $\varepsilon > 0$, then we may assume that L is so large that $\Gamma_k \leq (\lambda_c(\beta)^{-1} + \varepsilon)^k$ for any $k > L$. Recall that by (3.3) we have $\Gamma_{\Lambda_{n,k}}^{(\text{Dir})} \leq \Gamma_k$. Thus we can bound the left-hand side of (3.12) as follows, for any $l \in \mathfrak{P}_N$ and for $N > N_0$.

$$\begin{aligned} \frac{1}{|\Lambda_N|} \log \prod_{k>L} [\Gamma_{\Lambda_{n,k}}^{(\text{bc})}]^{l_k} &\leq \frac{1}{|\Lambda_N|} \sum_{k>L} l_k \log \left([(\lambda_c(\beta)^{-1} + \varepsilon)]^k \right) \\ &\leq \log [(\lambda_c(\beta)^{-1} + \varepsilon)] \sum_{k>L} \frac{k l_k}{|\Lambda_N|} \\ &= \log [(\lambda_c(\beta)^{-1} + \varepsilon)] \left(\rho - \sum_{k=1}^L k p_k^{(N,L)} \right), \end{aligned} \quad (3.13)$$

with $p_k^{(N,L)}$ as in (3.8).

Collecting (3.10), (3.11) and (3.13), we see that, for any $L \in \mathbb{N}$ and $\delta, \varepsilon \in (0, 1)$,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \sup_{l \in \mathfrak{P}_N : l/|\Lambda_N| \in B_\delta(p)} \prod_{k \in \mathbb{N}} \frac{[|\Lambda_N| \Gamma_{\Lambda_N, k}^{(\text{bc})}]^{l_k}}{l_k! k^{l_k}} \\ \leq - \inf_{\tilde{p} \in B_\delta(p)} I_L(\tilde{p}) + \left(\rho - \inf_{\tilde{p} \in B_\delta(p)} \sum_{k=1}^L k \tilde{p}_k \right) \log [(\lambda_c(\beta)^{-1} + \varepsilon)]. \end{aligned} \quad (3.14)$$

Letting $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, we obtain, for any $L \in \mathbb{N}$,

$$\limsup_{\delta \downarrow 0} (\text{l.h.s. of (3.14)}) \leq -I_L(p) - \left(\rho - \sum_{k=1}^L k p_k \right) \log \lambda_c(\beta).$$

Passing to the limit $L \rightarrow \infty$, we complete the proof of (3.6). Now let us turn to the

Proof of (3.7). Again, we pick a large auxiliary parameter L and define $l^{(\delta, N)} = (l_k^{(\delta, N)})_{k \in \{1, \dots, N\}}$ (also depending on L) by

$$l_k^{(\delta, N)} = \begin{cases} \lfloor |\Lambda_N| p_k \rfloor & \text{for } k \leq L, \\ A_{p, L, N} & \text{for } k = k_N = \lfloor N^{1/2d} \rfloor, \\ 0 & \text{otherwise,} \end{cases}$$

were $A_{p, L, N}$ is picked such that $\sum_{k=1}^N k l_k^{(\delta, N)} = |\Lambda_N| \rho$. Then $l_k^{(\delta, N)} / |\Lambda_N| \rightarrow p_k$ as $N \rightarrow \infty$ for $k \in \{1, \dots, L\}$, and $l_{k_N}^{(\delta, N)} / |\Lambda_N| = A_{p, L, N} / |\Lambda_N| = (\rho - \sum_{k=1}^L k p_k)(1 + o(1)) \frac{1}{k_N}$ as $N \rightarrow \infty$. We see that $l^{(\delta, N)} / |\Lambda_N| \in B_\delta(p)$ for N and L sufficiently large.

As in (3.9), we see that

$$\lim_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \prod_{k=1}^L \frac{[|\Lambda_N| \Gamma_{\Lambda_N, k}^{(\text{bc})}]^{l_k^{(\delta, N)}}}{l_k^{(\delta, N)}! k^{l_k^{(\delta, N)}}} \geq -I_L(p)$$

Furthermore, it is easy to see that

$$\liminf_{L \rightarrow \infty} \liminf_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \prod_{k > L} \left[\frac{1}{l_k^{(\delta, N)}!} \left(\frac{1}{k} |\Lambda_N| \right)^{l_k^{(\delta, N)}} \right] = 0.$$

Finally, using the assertion around (3.4) (which holds for both bc=per and bc=Dir) and recalling that $\Gamma_k = \lambda_c(\beta)^{-k(1+o(1))}$, we get that

$$\liminf_{L \rightarrow \infty} \liminf_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \log \prod_{k > L} [\Gamma_{\Lambda_N, k}^{(\text{bc})}]^{l_k^{(\delta, N)}} \geq - \left(\rho - \sum_{k \in \mathbb{N}} k p_k \right) \log \lambda_c(\beta).$$

Putting together the last three displays, we see that (3.7) holds.

Hence, (3.5) has been proved, and this finishes the proof of the LDP in the assertions (i) and (ii) of Theorem 1.1.

3.3 Analysis of the rate function

Now we prove assertions (iii) and (iv) of Theorem 1.1. It is clear that the state space \mathcal{X}_ρ is compact and that J is continuous and strictly convex on \mathcal{X}_ρ . Hence, it suffices to derive the Euler–Lagrange equations for minimisers. Abbreviate $\alpha_k = k/\Gamma_k$ and $\alpha = (\alpha_k)_k$, then $J(p) = \rho \log \lambda_c(\beta) + \langle p, \log(p\alpha/e) - \text{id} \lambda_c(\beta) \rangle$ for any $p \in \mathcal{X}_\rho$. Assume that p is a minimizer for J . It is

easy to see that $p_k > 0$ for any k , since otherwise adding a small weight would clearly make the J -value smaller. Then, for any perturbator $\delta = (\delta_k)_k \in \mathbb{R}^{\mathbb{N}}$, we have

$$0 = \partial_\varepsilon \Big|_{\varepsilon=0} J(p + \varepsilon \delta) = \langle \delta, \log(p\alpha/e) - \text{id} \lambda_c(\beta) \rangle + \langle p, \frac{\delta}{p} \rangle = \langle \delta, \log(p\alpha) - \text{id} \lambda_c(\beta) \rangle. \quad (3.15)$$

Let us assume that the minimizer p satisfies $\sum_k k p_k = \rho$. Then we may assume that the perturbator δ satisfies $0 = \sum_k k \delta_k = \langle \delta, \text{id} \rangle$, and the right-hand side of (3.15) is equal to $\langle \delta, \log(p\alpha) \rangle$. Hence, we have derived that any δ that is perpendicular to the vector $\text{id} = (k)_k$ is also perpendicular to the vector $\log(p\alpha)$. Hence, there is some constant C such that $Ck = \log(p_k \alpha_k)$ for any k , which reads

$$k p_k = e^{-Ck} \Gamma_k, \quad k \in \mathbb{N}. \quad (3.16)$$

Now check the constraint $\sum_k k p_k = \rho$. In the first case, where $\sum_k \Gamma_k \lambda_c(\beta)^k$ is finite, the smallest C that can ever make the sum of $e^{-Ck} \Gamma_k$ finite is $C = -\log \lambda_c(\beta)$, and the largest ρ that can ever be realized by that sum is $\rho_c(\beta)$. Substituting $e^{-Ck} = \lambda_c(\beta)^k e^{-c(\rho)k}$ we arrive at the Euler-Lagrange equations in (1.18) with $c = c(\rho)$ as in the text below (1.18). In the case where $\sum_k \Gamma_k \lambda_c(\beta)^k = \infty$, then any $\rho > 0$ can be realized by that choice of $C = c(\rho) + \log \lambda_c(\beta)$.

Now, if the minimizer p satisfies $\sum_k k p_k < \rho$, then the perturbator δ is arbitrary, and from (3.15) we get that $\log(p\alpha) - \text{id} \lambda_c(\beta) = 0$, i.e., that (3.16) holds with e^{-Ck} replaced by $\lambda_c(\beta)^k$, which proves (1.18) as well. In particular, $\sum_k k p_k = \rho_c(\beta)$. The formula in (iv) is proved by substituting what we have derived.

4. PROOF OF THEOREM 1.2: ESTIMATES ON THE GREEN FUNCTION

This section is devoted to the proof of Theorem 1.2. Without loss of generality, we are going to do the proof only for $\beta = 1$ (otherwise, consider a simple rescaled version of v).

In Section 4.1 we introduce some notation, formulate the main steps of the proof and prove Theorem 1.2, subject to these steps. In Section 4.2 we prove explain the main one of them, and prove it finally in Section 4.3.

4.1 Some notation, and the main steps

Recall that $\varphi_t(x) = (2\pi t)^{-d/2} \exp[-\frac{|x|^2}{2t}]$ denotes the Gaussian kernel with variance t , and we write $\varphi := \varphi_1$. We define

$$G(x) := \sum_{n=1}^{\infty} \varphi_n(x), \quad x \in \mathbb{R}^d, \quad (4.1)$$

which is, up to the missing summand for $n = 0$ (which is often formally written as the delta function at x), equal to the Green function of the random walk with standard normal increments. (It is not to be confused with the Green function $G_c(x) = \int_0^\infty \varphi_t(x) dt$ for Brownian motion. See Lemma 7.3 and its proof for some relation between them.) In spite of the missing summand for $n = 0$, we will call G the Green function from now. Note that $G(x)$ is finite in $d \geq 3$, and $G(\cdot)$ is rotational symmetric, bounded, and infinitely often differentiable with all derivatives in the set $\mathcal{C}_0(\mathbb{R}^d)$ of continuous functions $\mathbb{R}^d \rightarrow \mathbb{R}$ vanishing at infinity. Recall that v is a bounded continuous function $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ with compact support, i.e., we assume that there exists $L, R > 0$ such that $v(r) \leq L$ for every r , and $v(r) = 0$ for $r \geq R$.

Let us introduce the main object of the proof, a version of the self-repellent Brownian motion. Recall the Hamiltonian on \mathcal{C}_N :

$$H_N(B) := \sum_{1 \leq i < j \leq N} V(B_i, B_j), \quad \text{where} \quad V(f, g) = \int_0^1 v(|f(s) - g(s)|) ds \text{ for } f, g \in \mathcal{C}_1,$$

where we recall from (1.7) that $B_j = (B_j(s))_{s \in [0,1]} := (B((j-1) + s))_{s \in [0,1]}$ is the j -th leg of the Brownian motion B . Then $0 \leq H_N(B) \leq LN$ since $0 \leq v \leq L$. Also recall that $\mu_{x,y}^{(1)}$ denotes the canonical Brownian bridge measure in the time interval $[0, 1]$. We write \mathbb{P}_N for the standard Wiener measure on \mathcal{C}_N . Then the tilted measure $\mathbb{Q}_{\alpha,N}$ on \mathcal{C}_N is defined by

$$\mathbb{Q}_{\alpha,N}(dB) := e^{-\alpha H_N(B)} \mathbb{P}_N(dB), \quad N \in \mathbb{N}, \alpha \in [0, \infty). \quad (4.2)$$

Note that $\mathbb{Q}_{\alpha,N}$ is not normalized. Its normalized version is the announced variant of the self-repellent Brownian motion. For $N = 1$, there is no interaction, so we have $\mathbb{Q}_{\alpha,1} = \mathbb{P}_1$. The finite dimensional marginals of $\mathbb{Q}_{\alpha,N}$ have continuous Lebesgue densities. In particular, we define

$$\Gamma_{\alpha,N}(x) := \frac{\mathbb{Q}_{\alpha,N}(B_N \in dx)}{dx}. \quad (4.3)$$

Then Γ_k defined in (1.12) for v replaced by αv (and $\beta = 1$) is equal to $\Gamma_{\alpha,k}(0)$. Note that $e^{-\alpha LN} \varphi_N(x) \leq \Gamma_{\alpha,N}(x) \leq \varphi_N(x)$ for any $N \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Now define the Green function for the self-repellent Brownian motion as follows: for $\lambda > 0$, let

$$G_{\alpha,\lambda}^{(\Gamma)}(x) := \sum_{N=1}^{\infty} \lambda^N \Gamma_{\alpha,N}(x) \quad \text{and put} \quad \lambda_{\text{cr}}(\alpha) := \sup \{ \lambda > 0 : \|G_{\alpha,\lambda}^{(\Gamma)}\|_1 < \infty \}. \quad (4.4)$$

Then $\lambda_{\text{cr}}(\alpha)$ is equal to $\lambda_c(1)$ defined in (1.13) for v replaced by αv .

Our main result in the present section is:

Theorem 4.1. *Assume $d \geq 5$. Then there exists $\alpha_0 = \alpha_0(v, d) > 0$ such that for $\alpha \in (0, \alpha_0)$*

$$G_{\alpha,\lambda_{\text{cr}}(\alpha)}^{(\Gamma)}(x) \leq 5G(x), \quad x \in \mathbb{R}^d.$$

In particular, $\sum_{N=1}^{\infty} \lambda_{\text{cr}}(\alpha)^N \Gamma_{\alpha,N}(0) < \infty$ for $\alpha \in (0, \alpha_0)$.

Theorem 4.1 evidently implies Theorem 1.2. The rest of the article is devoted to the proof of Theorem 4.1.

For $\lambda < \lambda_{\text{cr}}(\alpha)$, consider

$$\overline{G}_{\alpha,\lambda}^{(\Gamma)} := G_{\alpha,\lambda}^{(\Gamma)} \star \varphi, \quad (4.5)$$

so that $\overline{G}_{\alpha,\lambda}^{(\Gamma)}$ is evidently integrable, and has bounded and continuous derivatives of all orders. We first observe the following simple fact:

Lemma 4.2. *Fix $\alpha \in (0, \infty)$ such that $e^{\alpha L} \leq 2$. Then the function*

$$f_{\alpha}(\lambda) := \sup_{x \in \mathbb{R}^d} \frac{\overline{G}_{\alpha,\lambda}^{(\Gamma)}(x)}{G(x)}$$

is continuous and increasing in $\lambda \in [0, \lambda_{\text{cr}}(\alpha))$.

Proof. It is clear that f_{α} is increasing. For the continuity, it suffices to prove that $f_{\alpha}(\cdot)$ is continuous on $[\lambda'_0, \lambda_0]$ for any $0 < \lambda'_0 < \lambda_0 < \lambda_{\text{cr}}(\alpha)$.

We first prove that there exists $R(\lambda'_0, \lambda_0)$ such that for $\lambda \in [\lambda'_0, \lambda_0]$

$$f_{\alpha}(\lambda) = \sup_{|x| \leq R(\lambda'_0, \lambda_0)} \frac{\overline{G}_{\alpha,\lambda}^{(\Gamma)}(x)}{G(x)}. \quad (4.6)$$

Since $e^{-\alpha L N} \varphi_N \leq \Gamma_{\alpha, N}$, and since $\sum_n \lambda^n \varphi_n$ does not converge for $\lambda > 1$, we get that $\lambda_{\text{cr}}(\alpha) \leq e^{\alpha L} \leq 2$, by the choice of α . Note that the lower bound $\lambda_{\text{cr}}(\alpha) \geq 1$ is evident as $H_N(B) \geq 0$. Define $\lambda_1 := (\lambda_0 + \lambda_{\text{cr}}(\alpha))/2$. For any $N \in \mathbb{N}$, using that $\lambda_{\text{cr}}(\alpha) \leq 2$ and $\Gamma_{\alpha, n}(\cdot) \leq \varphi_n(\cdot)$,

$$\begin{aligned} \overline{G}_{\alpha, \lambda}^{(\Gamma)}(x) &= \sum_{n=1}^{N-1} \lambda^n (\Gamma_{\alpha, n} \star \varphi)(x) + \sum_{n=N}^{\infty} \left(\frac{\lambda}{\lambda_1} \right)^n \lambda_1^n (\Gamma_{\alpha, n} \star \varphi)(x) \\ &\leq \sum_{n=1}^{N-1} 2^n \varphi_{n+1}(x) + \sum_{n=N}^{\infty} \left(\frac{\lambda_0}{\lambda_1} \right)^n \lambda_1^n (\Gamma_{\alpha, n} \star \varphi)(x) \\ &\leq \sum_{n=1}^{N-1} 2^n \frac{1}{(2\pi(n+1))^{\frac{d}{2}}} \exp[-|x|^2/2(n+1)] + \left(\frac{\lambda_0}{\lambda_1} \right)^N \|\overline{G}_{\alpha, \lambda_1}^{(\Gamma)}\|_{\infty}. \end{aligned} \quad (4.7)$$

Remark that by definition of λ_1 we have $G_{\alpha, \lambda_1}^{(\Gamma)} \in L^1(\mathbb{R}^d)$ and therefore $\|\overline{G}_{\alpha, \lambda_1}^{(\Gamma)}\|_{\infty} = \|G_{\alpha, \lambda_1}^{(\Gamma)} \star \varphi\|_{\infty} < \infty$. By choosing

$$N := \frac{|x|}{2\sqrt{2} \log 2}$$

one sees that there are constants c, C , depending on λ_0 , such that for $\lambda \leq \lambda_0$

$$\overline{G}_{\alpha, \lambda}^{(\Gamma)}(x) \leq C \exp[-c|x|].$$

As $\overline{G}_{\alpha, \lambda}^{(\Gamma)}(0)$ is bounded away from 0 for $\lambda \geq \lambda'_0$, and the Green function $G(x)$ decays like $\text{const} \times |x|^{-d+2}$ as $|x| \rightarrow \infty$, (4.6) follows.

To prove the desired continuity, we appeal to the representation of $\overline{G}_{\alpha, \lambda}^{(\Gamma)}$ in the first identity in (4.7). Then $(\Gamma_{\alpha, n} \star \varphi)(x)$ is continuous, and therefore uniformly continuous on $B_R := \{x : |x| \leq R\}$. Therefore, for any N , the first part $\sum_{n=1}^{N-1} \lambda^n (\Gamma_{\alpha, n} \star \varphi)(\cdot)$ is uniformly continuous, also uniformly in $\lambda \in [\lambda'_0, \lambda_0]$. Now since $\lambda_0 < \lambda_{\text{cr}}$, by definition of $\lambda_{\text{cr}}(\alpha)$, the second summand $\sum_{n=N}^{\infty} \lambda_0^n (\Gamma_{\alpha, n} \star \varphi)(x)$ converges to 0 as $N \rightarrow \infty$ uniformly in x and $\lambda \in [\lambda'_0, \lambda_0]$, we conclude that $\overline{G}_{\alpha, \lambda}^{(\Gamma)}(x)$ is uniformly continuous in $x \in B_R$ and $\lambda \in [\lambda'_0, \lambda_0]$. The same remains true for $\overline{G}_{\alpha, \lambda}^{(\Gamma)}(x)/G(x)$. Therefore $\sup_{x \in B_R} \overline{G}_{\alpha, \lambda}^{(\Gamma)}(x)/G(x)$ is continuous in $\lambda \in [\lambda'_0, \lambda_0]$, as claimed. \square

The key to proof of Theorem 4.1 is the following.

Proposition 4.3. *There exists $\alpha_0 > 0$ such that for $\alpha \leq \alpha_0$, there is no $\lambda < \lambda_{\text{cr}}(\alpha)$ with $f_{\alpha}(\lambda) \in (2, 3]$.*

The proof is based on two main lemmas, namely Lemma 4.4 and Lemma 4.6, which will be proved in the next two sections. Let us first note that the above proposition, together with Lemma 4.2, implies Theorem 4.1:

Proof of Theorem 4.1 (assuming Proposition 4.3). We will show Theorem 4.1 with α_0 replaced by $\min\{\alpha_0, \frac{1}{L} \log 2\}$, where α_0 is from Proposition 4.3.

Fix $\alpha \in (0, \alpha_0)$ so small that $\lambda_{\text{cr}}(\alpha) \leq 2$. As $\Gamma_{\alpha, n} \leq \varphi_n$ for any $n \in \mathbb{N}$, it is evident that $f_{\alpha}(\lambda) \leq 1$ for $\lambda \leq 1$. Then Proposition 4.3 and the continuity in λ shown in Lemma 4.2 imply that $f_{\alpha}(\lambda) \leq 2$ for $\lambda < \lambda_{\text{cr}}(\alpha)$. This means that $\overline{G}_{\alpha, \lambda}^{(\Gamma)}(x) \leq 2G(x)$ for any $x \in \mathbb{R}^d$. As $\Gamma_{\alpha, n}(x) \geq 0$ for all x , it follows that $\overline{G}_{\alpha, \lambda_{\text{cr}}(\alpha)}^{(\Gamma)}(x) \leq 2G(x)$. Observe now that from $H_n(B) \leq H_{n+1}(B)$, it follows that

$$\Gamma_{\alpha, n} \star \varphi \geq \Gamma_{\alpha, n+1}.$$

This implies

$$\overline{G}_{\alpha, \lambda_{\text{cr}}(\alpha)}^{(\Gamma)} \geq \frac{G_{\alpha, \lambda_{\text{cr}}}^{(\Gamma)} - \varphi}{\lambda_{\text{cr}}(\alpha)},$$

and therefore, using $\overline{G}_{\alpha, \lambda_{\text{cr}}(\alpha)}^{(\Gamma)}(x) \leq 2G(x)$, and $\lambda_{\text{cr}}(\alpha) \leq 2$, it follows that

$$G_{\alpha, \lambda_{\text{cr}}(\alpha)}^{(\Gamma)} \leq \lambda_{\text{cr}}(\alpha) \overline{G}_{\alpha, \lambda_{\text{cr}}(\alpha)}^{(\Gamma)} + \varphi \leq 4G + \varphi \leq 5G, \quad (4.8)$$

which implies Theorem 4.1. \square

4.2 Outline of the proof of Proposition 4.3.

We outline the arguments for Proposition 4.3. We have to show that if α is small enough, then the following implication is true for all $\lambda < \lambda_{\text{cr}}(\alpha)$:

$$f_\alpha(\lambda) \leq 3 \implies f_\alpha(\lambda) \leq 2.$$

So, we assume $f_\alpha(\lambda) \leq 3$. For the ease of notation, we drop the parameter α . The proof then splits into three main steps: First, by the argument leading to (4.8) we see that this leads to

$$G_\lambda^{(\Gamma)} \leq \lambda \overline{G}_\lambda^{(\Gamma)} + \varphi \leq 7G(x). \quad (4.9)$$

Next, we build on a modification of the lace expansion technique. This provides a complicated but rather explicit bound for an inverse of $I + G_\lambda^{(\Gamma)}$, under convolution, where I is the identity operator (formally given by the Dirac δ function). This inverse is of the form

$$(I + G_\lambda^{(\Gamma)})^{-1} = I + \lambda\varphi + G_\lambda^{(\Pi)}, \quad \text{where } G_\lambda^{(\Pi)}(x) = \sum_{N=2}^{\infty} \lambda^N \Pi_N(x)$$

for some functions $\Pi_N(x)$ that we will soon describe below. The crucial point is that under the condition $G_\lambda^{(\Gamma)}(x) \leq 7G(x)$ for all $x \in \mathbb{R}^d$, we obtain good decay properties for $G_\lambda^{(\Pi)}(x)$. Also $G_\lambda^{(\Pi)}(x)$ is in a suitable sense small if α is small.

In the final step, which is quite orthogonal to lace expansion used in the preceding step, we just show that if $G_\lambda^{(\Pi)}$ is small, in a sense that will be made precise, we can invert $I + \lambda\varphi + G_\lambda^{(\Pi)}$ under convolution using a Neumann series. This then gives an estimate of $G_\lambda^{(\Gamma)}$ in terms of “smallness properties” of $G_\lambda^{(\Pi)}$, and in particular it gives the estimate $G_\lambda^{(\Gamma)}(x) \leq 2G(x)$ if α is small enough, implying

$$\overline{G}_\lambda^{(\Gamma)} = G_\lambda^{(\Gamma)} \star \varphi \leq 2G(x) \star \varphi \leq 2G.$$

This will subsequently complete the proof of Proposition 4.3.

4.3 The main lemmas, and proof of Proposition 4.3.

For the purposes mentioned above, let us first define the functions Π_N : For $1 \leq k < m$, $k, m \in \mathbb{N}$, let $\mathcal{G}_{k,m}$ be the set of simple graphs on the set $\{k, k+1, \dots, m\}$. We also write $\mathcal{G}_m := \mathcal{G}_{1,m}$ and set

$$U_\alpha(f, g) = 1 - e^{-\alpha V(f, g)} \quad f, g \in \mathcal{C}_1 \quad \text{and} \quad U_{i,j}^{(\alpha)} = U_\alpha(B_i, B_j) \quad i, j \in \mathbb{N}. \quad (4.10)$$

Then with $H_N(B) = \sum_{1 \leq i < j \leq N} V(B_i, B_j)$ we can expand

$$e^{-\alpha H_N(B)} = \prod_{1 \leq i < j \leq N} (1 - U_{i,j}^{(\alpha)}) = \sum_{\mathbf{g} \in \mathcal{G}_N} \prod_{(i,j) \in \mathbf{g}} (-U_{i,j}^{(\alpha)}),$$

where we identify the graph \mathbf{g} with its vertex set. For notational convenience, from now on we will suppress the dependence on α on the function $\Gamma_N = \Gamma_{\alpha, N}$ defined in (4.3), on the tilted measure

$\mathbb{Q}_N = \mathbb{Q}_{\alpha,N}$ defined in (4.2) and on the functions $U = U_\alpha$ and $U_{i,j} = U_{i,j}^{(\alpha)}$ defined in (4.10). Then \mathbb{Q}_N can be rewritten as

$$\mathbb{Q}_N(\mathrm{d}B) = \sum_{\mathbf{g} \in \mathcal{G}_N} \prod_{(i,j) \in \mathbf{g}} (-U_{i,j}) \mathbb{P}_N(\mathrm{d}B). \quad (4.11)$$

We call an index $k \in \{1, \dots, N\}$ a *breakpoint* of a graph $\mathbf{g} \in \mathcal{G}_N$ if there is no edge $(i, j) \in \mathbf{g}$ with $i \leq k < j$. If a graph has no breakpoint, we call it *irreducible*. Remark that we use the notion in a way that is slightly different from the standard one in lace expansions: For instance, if \mathbf{g} contains a bond (k, j) with $j > k$, then k is not a breakpoint. We write \mathcal{J}_N for the set of irreducible graphs in \mathcal{G}_N . This makes sense only for $N \geq 2$. For $\mathbf{g} \in \mathcal{G}_N \setminus \mathcal{J}_N$, we denote by $b(\mathbf{g})$ the smallest breakpoint of \mathbf{g} , and write $\mathcal{G}_N^{(k)}$ for the set of graphs in $\mathcal{G}_N \setminus \mathcal{J}_N$ for which k is the smallest breakpoint. For convenience, we set $b(\mathbf{g}) := N$ for $\mathbf{g} \in \mathcal{J}_N$, and accordingly $\mathcal{G}_N^{(N)} := \mathcal{J}_N$.

If \mathbb{E}_N denotes expectation with respect to the Wiener measure \mathbb{P}_N on \mathcal{C}_N , we define

$$\Pi_N^{(\alpha)}(x) = \Pi_N(x) := \sum_{\mathbf{g} \in \mathcal{J}_N} \mathbb{E}_N \left[\prod_{(i,j) \in \mathbf{g}} (-U_{i,j}) \mathbb{1}_{\{B_N \in \mathrm{d}x\}} \right] / \mathrm{d}x, \quad N \geq 2, \quad (4.12)$$

and define the Green function of Π by

$$G_{\alpha,\lambda}^{(\Pi)}(x) := \sum_{N=2}^{\infty} \lambda^N \Pi_N(x), \quad (4.13)$$

provided the series is absolutely summable. The precise formulation for the second step sketched above is provided by the following lemma.

Lemma 4.4. *Fix $\alpha \in (0, \infty)$ and pick $\lambda > 0$ such that*

$$G_{\alpha,\lambda}^{(\Gamma)}(x) \leq 7G(x), \quad x \in \mathbb{R}^d. \quad (4.14)$$

Then the right-hand side of (4.13) is absolutely summable and there is a $C = C_{\alpha,\lambda} > 0$ such that

$$\sum_{N=2}^{\infty} \lambda^N |\Pi_N(x)| \leq C\alpha(1 + |x|)^{6-3d}, \quad x \in \mathbb{R}^d. \quad (4.15)$$

In particular

$$|G_{\alpha,\lambda}^{(\Pi)}(x)| \leq C\alpha(1 + |x|)^{6-3d}, \quad x \in \mathbb{R}^d. \quad (4.16)$$

The proof of Lemma 4.4 can be found in Section 5. The reason to consider $G_{\lambda}^{(\Pi)}$ is that $I + \lambda\varphi + G_{\lambda}^{(\Pi)}$ is essentially an inverse of $I + G_{\lambda}^{(\Gamma)}$ under convolution. This is based on the following simple result:

Lemma 4.5. *With $\Gamma_0 := \delta_0$ and $\Gamma_1 = \varphi$ we have*

$$\Gamma_N = \varphi \star \Gamma_{N-1} + \sum_{k=2}^N \Pi_k \star \Gamma_{N-k}, \quad N \geq 2. \quad (4.17)$$

Proof. Using (4.11), we have

$$\Gamma_N(x) = \mathbb{E}_N \left[\sum_{\mathbf{g} \in \mathcal{G}_N} \prod_{(i,j) \in \mathbf{g}} (-U_{ij}) \mathbb{1}_{\{B_N \in \mathrm{d}x\}} \right] / \mathrm{d}x = \sum_{k=1}^N \mathbb{E}_N \left[\sum_{\mathbf{g} \in \mathcal{G}_N^{(k)}} \prod_{(i,j) \in \mathbf{g}} (-U_{ij}) \mathbb{1}_{\{B_N \in \mathrm{d}x\}} \right] / \mathrm{d}x.$$

The summand for $k = N$ restricts the summation over \mathbf{g} to $\mathbf{g} \in \mathcal{J}_N$, which gives $\Pi_N = \Pi_N \star \delta$. We next consider the summand for $k = 1$. We remark that 1 is a break point of \mathbf{g} if and only if there is

no edge in \mathfrak{g} starting in 1. In the product $\prod_{(i,j) \in \mathfrak{g}} (-U_{i,j})$ there is then no interaction between the first leg B_1 and any other leg. Using translation invariance, we get

$$\sum_{\mathfrak{g} \in \mathcal{G}_N^{(1)}} \mathbb{E}_N \left[\prod_{(i,j) \in \mathfrak{g}} (-U_{i,j}) \mathbb{1}_{\{B_N \in dx\}} \right] / dx = \int \varphi(y) \Gamma_{N-1}(x-y) dy.$$

This argument can simply be extended to handle the summation over graphs in $\mathcal{G}_N^{(k)}$, for $2 \leq k \leq N-1$. A graph \mathfrak{g} is in $\mathcal{G}_N^{(k)}$ if and only if it is the union of an irreducible graph on $\{1, \dots, k\}$ and an arbitrary graph on $\{k+1, \dots, N\}$. Summing over all the possibilities, we get for $2 \leq k \leq N-1$

$$\sum_{\mathfrak{g} \in \mathcal{G}_N^{(k)}} \mathbb{E}_N \left[\prod_{(i,j) \in \mathfrak{g}} (-U_{i,j}) \mathbb{1}_{\{B_N \in dx\}} \right] / dx = \int \Pi_k(y) \Gamma_{N-k}(x-y) dy.$$

We remark that for $k = N-1$, there is no graph on $\{N\}$, and we get $\Pi_{N-1} \star \varphi$. Summing over k we conclude the proof of the lemma. \square

Remark 2 A consequence of the convolution equation (4.17) is that

$$\begin{aligned} G_\lambda^{(\Gamma)} &= \sum_{N=1}^{\infty} \lambda^N \Gamma_N = \sum_{N=1}^{\infty} \lambda^N \sum_{k=1}^N \Pi_k \star \Gamma_{N-k} = \lambda \varphi + \left(\sum_{k=1}^{\infty} \lambda^k \Pi_k \star \left(\sum_{N=k}^{\infty} \lambda^{N-k} \Gamma_{N-k} \right) \right) \\ &= (\lambda \varphi + G_\lambda^{(\Pi)}) \star (I + G_\lambda^{(\Gamma)}), \end{aligned}$$

that is,

$$(I - \lambda \varphi - G_\lambda^{(\Pi)}) \star (I + G_\lambda^{(\Gamma)}) = I. \quad (4.18)$$

As we want to work only with smooth functions, we write this equation as

$$-(\lambda \varphi + G_\lambda^{(\Pi)}) \star G_\lambda^{(\Gamma)} + G_\lambda^{(\Gamma)} - (\lambda \varphi + G_\lambda^{(\Pi)}) = 0. \quad (4.19)$$

\diamond

The precise formulation of the third step for the proof of Proposition 4.3 is provided by the following lemma.

Lemma 4.6. *Assume that $\alpha \in (0, \infty)$ is such that $G_{\alpha, \lambda}^{(\Pi)}$ satisfies (4.16) for $\lambda < \lambda_{\text{cr}}(\alpha)$, where $C = C_\alpha$ does not depend on λ . If α is small enough, then there exists a unique continuous, bounded, rotational symmetric function F (depending on α) that satisfies*

$$-(\lambda \varphi + G_{\alpha, \lambda}^{(\Pi)}) \star F + F = \lambda \varphi + G_{\alpha, \lambda}^{(\Pi)} \quad (4.20)$$

and

$$F(x) \leq 2G(x), \quad x \in \mathbb{R}^d. \quad (4.21)$$

The proof of Lemma 4.6 can be found in Section 6.

Proof of Proposition 4.3 (assuming Lemma 4.4 and Lemma 4.6). If $f_\alpha(\lambda) \leq 3$, then we have already observed in (4.14) that $G_{\alpha, \lambda}^{(\Gamma)} \leq 7G$. By Lemma 4.4, $G_\lambda^{(\Pi)}$ is finite and satisfies $\|G_{\alpha, \lambda}^{(\Pi)}\| \leq C\alpha$ for some $C = C_\alpha$ that does not depend on λ . By Lemma 4.6, if α is small enough, there is, for any $\lambda < \lambda_{\text{cr}}(\alpha)$, a unique F (depending on λ) that satisfies (4.20). By (4.19), we have $F = G_{\alpha, \lambda}^{(\Gamma)}$, and then by (4.21), $G_\lambda^{(\Gamma)} \leq 2G$. This implies that $f_\alpha(\lambda) \leq 2$ and finishes the proof. \square

5. PROOF OF LEMMA 4.4: LACE EXPANSION

In the standard version of lace expansions, one considers sets of paths that visit a single point on the lattice possibly several times. This is not the case in our situation, since the steps of our random walk have a Lebesgue density. What we do instead is to consider continuous-time paths that on time intervals of order one can be close together. The closeness is measured by the following functions $u_{n,\alpha}$, $n \geq 2$, depending on $2n$ locations:

$$u_n(\mathbf{x}) = u_{n,\alpha}(\mathbf{x}) := \int_{\mathcal{C}_1^n} \prod_{i=1}^{n-1} U(f_i, f_{i+1}) \prod_{i=1}^n \mu_{x_{2i-1}, x_{2i}}^{(1)}(df_i) \quad \text{for } \mathbf{x} \in (\mathbb{R}^d)^{2n}, \quad (5.1)$$

with $U = U_\alpha$ defined in (4.10). In words, this is (up to normalization) the expectation over a vector of n independent Brownian bridges from x_{2i-1} to x_{2i} for $i = 1, \dots, n$ with interactions between each two subsequent ones of them. Clearly, $u_{n,\alpha}$ is invariant under shifting the components of $\mathbf{x} = (x_1, \dots, x_{2n})$ by a single vector $y \in \mathbb{R}^d$. Also, $U(f, g)$ is close to 0 unless f and g are close together. Therefore, $u_{n,\alpha}(\mathbf{x}) \approx 0$ unless the components of \mathbf{x} are close together. The quantitative estimates are expressed in Lemma 7.1 in Section 7.

Again for notational simplicity, we will usually drop the parameter α appearing in $u_{n,\alpha}$ (as we did for the terms $\Gamma_{\alpha,N}$, $\mathbb{Q}_{\alpha,N}$, U_α and $U_{ij}^{(\alpha)}$), but the reader should be aware that the parameter is present in essentially every expression.

5.1 The set of laces \mathcal{L}_N and their characterization. Recall from Section 4.3 that \mathcal{G}_N denotes the set of simple graphs on $\{1, \dots, N\}$ and \mathcal{J}_N denotes the set of irreducible graphs (i.e., graphs with no breakpoint) in \mathcal{G}_N for $N \geq 2$. We also recall the function Π_N defined in (4.12). For the estimation of Π_N , we are going to introduce now a decomposition (called a *lace expansion*) of \mathcal{J}_N according to the minimal number of edges the graph has so that it is still irreducible. Any corresponding minimal subgraph is called a *lace*, i.e., a subgraph that is not irreducible if any edge is removed. Every non-irreducible graph may have many laces, but we are going to construct now for any $\mathbf{g} \in \mathcal{J}_N$ a particular lace $\text{lace}(\mathbf{g})$ that will be crucial in our expansion of Π_N . Indeed, the starting point of our lace expansion is

$$\sum_{\mathbf{g} \in \mathcal{J}_N} \prod_{(i,j) \in \mathbf{g}} (-U_{ij}) = \sum_{\ell \in \mathcal{L}_N} \sum_{\mathbf{g} \in \mathcal{J}_N : \text{lace}(\mathbf{g}) = \ell} \prod_{(i,j) \in \mathbf{g}} (-U_{ij}), \quad (5.2)$$

where \mathcal{L}_N denotes the set of laces.

Evidently, if the graph on $\{1, \dots, N\}$ contains the edge $(1, N)$ then it is irreducible. We denote the set of graphs that contain this particular edge by $\mathcal{J}_N^{(1)}$.

For other irreducible graphs, the construction of its lace is more complicated. If $\mathbf{g} \in \mathcal{J}_N \setminus \mathcal{J}_N^{(1)}$, we define its lace, denoted by $\text{lace}(\mathbf{g})$, as a subgraph of \mathbf{g} in the following way: Let \mathbf{g} contain an edge $(1, j)$ (as otherwise it is not irreducible). We take the largest such j (which is not N , as otherwise we would have $\mathbf{g} \in \mathcal{J}_N^{(1)}$), and denote it by j_1 . This edge $(1, j_1)$ is the first member of $\text{lace}(\mathbf{g})$. For the next, we know that there is an edge (i', j') with $i' \leq j < j'$, as otherwise \mathbf{g} would not be connected. We take the *largest* possible j' and afterwards the smallest possible i' to this j' . We denote this edge by $(i_2, j_2) \in \mathbf{g}$. If $j_2 = N$ we are finished with the construction of $\text{lace}(\mathbf{g})$. Otherwise, we go on in this way. Finally, we end up with a collection

$$\ell = \{(i_1 = 1, j_1), (i_2, j_2), \dots, (i_k, j_k = N)\}. \quad (5.3)$$

This is the uniquely defined $\text{lace}(\mathbf{g})$. We write $\mathcal{J}_N^{(k)}$ for the set of irreducible graphs whose lace has k edges. We write \mathcal{L}_N for the set of laces, i.e., irreducible graphs that are no longer irreducible when leaving out any edge.

Lemma 5.1 (Characterization of laces). *Let $\ell \in \mathcal{L}_N$ and $\mathfrak{g} \in \mathcal{J}_N$. Then $\text{lace}(\mathfrak{g}) = \ell$ if and only if all the edges $(i, j) \in \mathfrak{g} \setminus \ell$ have the property that*

$$\text{lace}(\ell \cup \{(i, j)\}) = \ell.$$

Proof. If $\text{lace}(\mathfrak{g}) = \ell$ then evidently all edges $(i, j) \in \mathfrak{g} \setminus \ell$ satisfy $\text{lace}(\ell \cup \{(i, j)\}) = \ell$.

For the other direction, assume $\ell \in \mathcal{L}_N$ satisfies $\ell \subset \mathfrak{g}$, and $\text{lace}(\ell \cup \{(i, j)\}) = \ell$ for all $(i, j) \in \mathfrak{g} \setminus \ell$. We want to prove that $\text{lace}(\mathfrak{g}) = \ell$. (we remind the reader that \mathfrak{g} may have many subsets which are laces). Write ℓ in the form (5.3). If $(1, j_1) \notin \text{lace}(\mathfrak{g})$, then \mathfrak{g} would have to contain an edge $(1, k)$ with $k > j_1$, and then $\text{lace}(\ell \cup \{(1, k)\}) \neq \ell$. So $(1, j_1) \in \text{lace}(\mathfrak{g})$. Next, if $(i_2, j_2) \notin \text{lace}(\mathfrak{g})$, we would have an edge (i', j') with either $j' > j_2$, or $(j' = j_2 \text{ and } i' < i_2)$. In both cases $\text{lace}(\ell \cup \{(i', j')\}) \neq \ell$, and we conclude $(i_2, j_2) \in \text{lace}(\mathfrak{g})$. In this way it follows that all edges in ℓ are members of $\text{lace}(\mathfrak{g})$, i.e., $\ell \subset \text{lace}(\mathfrak{g})$. But then, if $\text{lace}(\mathfrak{g})$ would contain an edge $\notin \ell$, it would not be a lace, by the minimality property of laces. \square

We write for $\ell \in \mathcal{L}_N$

$$\mathcal{C}_N(\ell) := \{(i, j) \notin \ell : \text{lace}(\ell \cup \{(i, j)\}) = \ell\},$$

and call each bond in $\mathcal{C}_N(\ell)$ *compatible* with ℓ . In this terminology, Lemma 5.1 can therefore be rephrased by saying

$$\text{lace}(\mathfrak{g}) = \ell \iff \mathfrak{g} \setminus \ell \subset \mathcal{C}_N(\ell). \quad (5.4)$$

In particular, (5.4) implies that for a fixed lace $\ell \in \mathcal{L}_N$,

$$\begin{aligned} \sum_{\mathfrak{g} : \text{lace}(\mathfrak{g}) = \ell} \prod_{(i, j) \in \mathfrak{g}} (-U_{ij}) &= \left[\prod_{(i, j) \in \ell} (-U_{ij}) \right] \sum_{\mathfrak{g} \subset \mathcal{C}_N(\ell)} \prod_{(i, j) \in \mathfrak{g}} (-U_{ij}) \\ &= \left[\prod_{(i, j) \in \ell} (-U_{ij}) \right] \prod_{(i, j) \in \mathcal{C}_N(\ell)} (1 - U_{ij}). \end{aligned}$$

Now define

$$\Pi_N^{(k)}(x) := \sum_{\mathfrak{g} \in \mathcal{J}_N^{(k)}} \mathbb{E}_N \left[\prod_{(i, j) \in \mathfrak{g}} (-U_{ij}) \mathbb{1}_{\{B_N \in dx\}} \right] / dx, \quad \text{so that } \Pi_N = \sum_{k=1}^{\infty} \Pi_N^{(k)}. \quad (5.5)$$

Since $\Pi_N^{(k)} = 0$ for large enough k , the above sum is finite, and

$$\sum_{N=1}^{\infty} \lambda^N |\Pi_N(x)| \leq \sum_{k=1}^{\infty} \sum_{N=1}^{\infty} \lambda^N |\Pi_N^{(k)}(x)|.$$

We are going to estimate the summands on the right-hand side individually for every k . To control the higher order terms, we first characterise the structure of a lace. The simple proof of the following lemma is left to the reader.

Lemma 5.2. *Any lace*

$$\ell = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$$

with $n \geq 3$ edges, where $i_1 = 1$, $j_n = N$ and $i_k < j_k$ for all k , satisfies

$$1 = i_1 < i_2 \leq j_1 < i_3 \leq j_2 < \dots < i_{n-1} \leq j_{n-2} < i_n \leq j_{n-1} < j_n = N. \quad (5.6)$$

Remark 3 The statement of the above lemma is in disagreement with the corresponding ones for the standard used lace expansions. In our case, there is the possibility to have $i_{k+1} = j_k$, but there are no other possibilities for equality. To see the point: In the usually used expansions, one can have for a lace with 3 edges (starting with $i_1 = 1$)

$$1 = i_1 < i_2 < j_1 = i_3 < j_2 < j_3 = N.$$

This is not possible in our case, as $\{(i_1, j_1), (i_3, j_3)\}$ would be irreducible, and therefore it would be the lace in this situation. For essentially a similar reasoning, $i_{k+1} = j_k$ is not possible in the standard versions, but it is indeed the case according to our definition. \diamond

5.2 Types of laces and their classification.

We need to make a finer distinction of laces, on base of Lemma 5.2. We say that two neighboring edges $(i_k, j_k), (i_{k+1}, j_{k+1})$ in the lace *touch* if $j_k = i_{k+1}$. Then we say that (i_{k+1}, j_{k+1}) *touches* (i_k, j_k) from the right. Otherwise, i.e., when $i_{k+1} < j_k$, we say that the edges *intersect*.

According to Lemma 5.2, we classify the laces on $\{1, \dots, N\}$ of length n into different types, dictated by (5.6). Given a lace ℓ with n edges, we order the set of edges increasingly according to the left end, writing e_1, \dots, e_n for the edges. We split the set of edges into disjoint chains

$$\chi_j = (e_{n_{j-1}+1}, \dots, e_{n_j}), \quad j = 1, \dots, k, \quad \text{with } 0 = n_0 < n_1 < n_2 < \dots < n_k = n. \quad (5.7)$$

Within a chain, and if a chain has at least two elements, e_{i+1} touches e_i from the right, but e_{n_j+1} intersects e_{n_j} , $j = 1, \dots, k-1$. We write

$$\tau := (n_1, n_2 - n_1, \dots, n_k - n_{k-1})$$

and call it the *type* of the lace. In case $\tau = (1, \dots, 1)$, there are no touching edges, and if $\tau = (n)$, the lace is a chain of n touching edges. The set of types is denoted by \mathfrak{T}_n . We write $\mathcal{L}_N^{(n, \tau)}$ for the set of laces of type τ on $\{1, \dots, N\}$ with n edges. We split $\Pi_N^{(n)}$ defined in (5.5) according to the different types by writing

$$\Pi_N^{(n)} = \sum_{\tau \in \mathfrak{T}_n} \Pi_N^{(n, \tau)},$$

where $\Pi_N^{(n, \tau)}$ is the sum over graphs whose lace have n edges and are of type τ .

Given a lace ℓ with n edges, the set of endpoints of its edges can be written as increasing sequence:

$$\mathbf{s}(\ell) := (s_i)_{i=1, \dots, m}, \quad \text{where } 1 = s_1 < s_2 < \dots < s_{m-1} < s_m = N, \quad (5.8)$$

and m depends on n and the type of the lace. If all neighboring edges intersect, then $m = 2n$. For every touching of neighbor edges, m is reduced by one.²

Any $e \in \ell$ is written $e = (s_{\tau_\star(e)}, s_{\tau^\star(e)})$ with $\tau_\star(e) < \tau^\star(e)$. Then $\ell = ((s_{\tau_\star(e)}, s_{\tau^\star(e)}))_{e \in \ell}$. In an abuse of notation, we write ℓ also as a sequence

$$((\tau_\star(i), \tau^\star(i)))_{i=1, \dots, n} \quad (5.9)$$

where i stands for the i -th edge, these being ordered increasingly.³

We define $\ell_N^{(n, \tau, \mathbf{s})}$ to be the unique lace of type τ that has endpoints $\mathbf{s}(\ell) = \mathbf{s}$ with given $\mathbf{s} = (s_i)_{i=1, \dots, m}$ as in (5.8). We will also write $\mathbf{t} = (t_1, \dots, t_m)$ where $t_i := s_i - s_{i-1}$ and $s_0 = 0$. We will use \mathbf{t} and \mathbf{s} according to what is more convenient, but they are tied in the described way.

²For instance, if the type is $(2, 3, 1, 2)$, then $n = 8$ and $m = 12$.

³For instance if a lace of $n = 5$ edges has type $(1, 3, 1)$ then the sequence of pairs $((\tau_\star(i), \tau^\star(i)))_{i=1}^5$ is $((1, 3), (2, 4), (4, 5), (5, 7), (6, 8))$.

5.3 Proof of Lemma 4.4.

With the notation set up in the previous section, we want to estimate

$$\Pi_N^{(n,\tau)}(y) = \sum_{\ell \in \mathcal{L}_N^{(n,\tau)}} \mathbb{E}_N \left[\left(\prod_{(i,j) \in \ell} (-U_{ij}) \prod_{(i,j) \in \mathcal{C}_N(\ell)} (1 - U_{ij}) \right) \mathbb{1}_{\{B_N \in dy\}} \right] / dy.$$

Recalling that $\mathbf{t} = (t_1, \dots, t_m)$ with $t_i = s_i - s_{i-1}$ and hence $\sum_{i=1}^m t_i = N$, we see that

$$\lambda^N \Pi_N^{(n,\tau)}(y) = \sum_{\mathbf{t}} \left[\prod_{i=1}^m \lambda^{t_i} \right] \Pi_N^{(n,\tau,\mathbf{s})}(y), \quad (5.10)$$

where

$$\Pi_N^{(n,\tau,\mathbf{s})}(y) := \sum_{\ell \in \mathcal{L}_N^{(n,\tau,\mathbf{s})}} \mathbb{E}_N \left[\prod_{(i,j) \in \ell} (-U_{ij}) \prod_{(i,j) \in \mathcal{C}_N(\ell)} (1 - U_{ij}) \mathbb{1}_{\{B_N \in dy\}} \right] / dy.$$

We start with estimating

$$\begin{aligned} |\Pi_N^{(n,\tau,\mathbf{s})}(y)| &\leq \sum_{\ell \in \mathcal{L}_N^{(n,\tau,\mathbf{s})}} \mathbb{E}_N \left[\left(\prod_{(i,j) \in \ell} U_{ij} \right) \prod_{j=1}^{m-1} \prod_{s_j < u < v \leq s_{j+1}-1} (1 - U_{uv}) \mathbb{1}_{\{B_N \in dy\}} \right] / dy \\ &=: g_N^{(n,\tau,\mathbf{s})}(y). \end{aligned} \quad (5.11)$$

We are now going to represent $\Pi_N^{(n,\tau,\mathbf{s})}(y)$ in terms of an integration over $2m-2$ variables $x_2, \dots, x_{2m-1} \in \mathbb{R}^d$ representing the positions of the Brownian motion at times $s_1 = 1, s_2 - 1, s_2, \dots, s_m - 1 = N - 1$. Thus, $s_1 - 1 = 0$ and also, $B_0 = 0$ and at time $s_m = N$, the Brownian motion is fixed to be at y . We define

$$\mathbb{P}_{N,\tau,\mathbf{s}}^{(\mathbf{x},y)}(\cdot) := \mathbb{P}_N \left(\cdot \mid B_{s_i-1} = x_{2i-1}, B_{s_i} = x_{2i} \text{ for } i = 1, \dots, m \right)$$

and integrate over the density of the event in the conditioning. By the time-structure, the Markov inequality can be used at the times $s_i - 1, s_i$ for $i = 1, \dots, m$, which decomposes the path into pieces of time length 1 and $s_i - 1 - s_{i-1}$, with $s_0 = 0$. Recall that φ_t is the Gaussian density with variance t . Then

$$\begin{aligned} g_N^{(n,\tau,\mathbf{s})}(y) &= \int_{(\mathbb{R}^d)^{2m-2}} d\mathbf{x} \mathbb{E}_{N,\tau,\mathbf{s}}^{(\mathbf{x},y)} \left[\prod_{(i,j) \in \ell_N^{(n,\tau,\mathbf{t})}} U_{ij} \right] \\ &\quad \times \prod_{j=1}^m \mathbb{E}_{N,\tau,\mathbf{s}}^{(\mathbf{x},y)} \left[\prod_{s_{j-1} < u < v \leq s_j-1} (1 - U_{ij}) \right] \\ &\quad \times \prod_{j=1}^m \varphi_1(x_{2j} - x_{2j-1}) \prod_{j=2}^m \varphi_{s_j-1-s_{j-1}}(x_{2j-1} - x_{2(j-1)}). \end{aligned} \quad (5.12)$$

Here we followed the convention that $x_1 := 0, x_{2m} = y$ and the integration is only over x_2, \dots, x_{2m-1} and used that under the conditional law $\mathbb{P}_{N,\tau,\mathbf{s}}^{(\mathbf{x},y)}$, the first product and the factors $\prod_{j=1}^{m-1}$ appearing in the definition of $g_N^{(n,\tau,\mathbf{s})}$ on the right hand side of (5.11) are independent. Also, we have made the following convention: it is not excluded that $s_j = s_{j-1} + 1$ in which case, the conditional law would not be defined, except when $x_{2j} = x_{2j-1}$. This is taken care of by interpreting φ_0 as the Dirac function.

Let us now handle factors appearing on the right hand side of (5.12). First we observe that

$$\mathbb{E}_{N,\tau,\mathbf{t}}^{(\mathbf{x},y)} \left[\prod_{s_{j-1} < u < v \leq s_j-1} (1 - U_{ij}) \right] \varphi_{s_j-1-s_{j-1}}(x_{2j} - x_{2j-1}) = \Gamma_{s_j-s_{j-1}-1}(x_{2j} - x_{2j-1}), \quad (5.13)$$

if $s_j \geq s_{j-1} + 2$. In case $s_j \geq s_{j-1} + 1$, the product is over the empty set and therefore equal to 1, and so we have to interpret Γ_0 as δ , too. Actually, also for $s_j = s_{j-1} + 2$, the product is empty, in which case we arrive at $\Gamma_1 = \varphi_1$.

We next express the factor involving the U_{ij} appearing on the right-hand side of (5.12) in terms of the function u_n defined in (5.1). This factor depends on the type $\tau = (v_1, \dots, v_k) \in \mathbb{N}^k$ of the lace. If χ_1, \dots, χ_k are the chains (recall (5.7)) then the terms $\prod_{(i,j) \in \chi_r} U_{ij}$, $r = 1, \dots, k$ are independent under the conditional measure $\mathbb{P}_{N,\tau,\mathbf{t}}^{(\mathbf{x},y)}$. Let a chain χ contain b edges, labelled $(s_{i_1}, s_{i_2}), (s_{i_2}, s_{i_3}), \dots, (s_{i_b}, s_{i_{b+1}})$. Then $i_{k+1} - i_k = 1, 2$ or 3 , depending whether the chain is in relation to the others. In any case, if $b \geq 3$, then $i_{k+1} - i_k = 1$ for $k = 2, \dots, b-1$. Then by recalling the definition of $u_n(\cdot)$ from (5.1),

$$\mathbb{E}_{N,\tau,\mathbf{t}}^{(\mathbf{x},y)} \left[\prod_{(i,j) \in \chi} U_{ij} \right] \prod_{k=1}^{b+1} \varphi_1(x_{i_k} - x_{i_{k-1}}) = u_{b+1}(x_{i_1-1}, x_{i_1}, x_{i_2-1}, x_{i_2}, \dots, x_{i_{b+1}-1}, x_{i_{b+1}}). \quad (5.14)$$

Now we can apply the crucial estimate from Lemma 7.1 and obtain

$$u_{b+1}(x_{i_1-1}, x_{i_1}, \dots, x_{i_{b+1}-1}, x_{i_{b+1}}) \leq C^{b+1} \alpha^b \exp \left[-\frac{1}{8} \sum_{k=1}^{b+1} |x_{i_k-1} - x_{i_k}|^2 - A \sum_{k=1}^b |x_{i_{k+1}} - x_{i_k}|^2 \right]. \quad (5.15)$$

Combining (5.12)-(5.14) we therefore get

$$\begin{aligned} g_N^{(n,\tau,\mathbf{s})}(y) &\leq C^n \alpha^n \int d\mathbf{x} \prod_{j=1}^{m-1} \Gamma_{s_{j+1}-1-s_j}(x_{2j+1} - x_{2j}) \\ &\times \exp \left[-\frac{1}{8} \sum_{j=1}^{m-1} |x_{2j} - x_{2j-1}|^2 - A \sum_{i=1}^n |x_{2\tau^*(i)} - x_{2\tau_*(i)}|^2 \right], \end{aligned} \quad (5.16)$$

where in the last display we used the notation $\tau^*(i)$ and $\tau_*(i)$ defined in (5.9).

We are heading now towards a proof of Lemma 4.4. In the following, we use $C > 0$ as a generic constant that may change its value from appearance to appearance and does not depend on n nor on m or τ and so on, but may depend only on d, α and λ , which we fix now such that (4.14) holds.

From the preceding, we get, for any type τ and any \mathbf{t} , recalling that $N = \sum_{i=1}^m t_i = \sum_{i=1}^m (s_i - s_{i-1})$,

$$\begin{aligned} \sum_N \lambda^N \sum_{\ell \in \mathcal{L}_N^{(n,\tau)}} g_N^{(n,\tau,\mathbf{s})}(y) &\leq C^n \alpha^n \lambda^{m-1} \int d\mathbf{x} \prod_{j=1}^{m-1} \sum_{k=0}^{\infty} \lambda^k \Gamma_k(x_{2j+1} - x_{2j}) \\ &\times \exp \left[-\frac{1}{8} \sum_{j=1}^m |x_{2j} - x_{2j-1}|^2 \right] \exp \left[-A \sum_{i=1}^n |x_{2\tau^*(i)} - x_{2\tau_*(i)}|^2 \right] \\ &\leq C^n \alpha^n \int d\mathbf{x} \prod_{j=2}^m \left[\delta(x_{2j-1} - x_{2j-2}) + G_\lambda^{(\Gamma)}(x_{2j-1} - x_{2j-2}) \right] \\ &\times \exp \left[-\frac{1}{8} \sum_{j=1}^m |x_{2j} - x_{2j-1}|^2 \right] \exp \left[-A \sum_{i=1}^n |x_{2\tau^*(i)} - x_{2\tau_*(i)}|^2 \right], \end{aligned} \quad (5.17)$$

where we remind on $m \leq 2n$. We integrate now over all the x_{2j-1} (except of course x_1 which is 0 by convention). Indeed, using the assumption $G_\lambda^{(\Gamma)} \leq 7G$ from (4.14) of Lemma 4.4, we get

$$\int dx_{2j-1} \left[\delta(x_{2j-1} - x_{2j-2}) + G_\lambda^{(\Gamma)}(x_{2j-1} - x_{2j-2}) \right] \exp \left[-\frac{1}{8} |x_{2j} - x_{2j-1}|^2 \right] \leq C f_d(|x_{2j} - x_{2j-2}|), \quad (5.18)$$

where we abbreviated

$$f_d(x) := (1 + |x|)^{-d+2}. \quad (5.19)$$

Combining the last two estimates we obtain

$$\sum_N \lambda^N \sum_{\ell \in \mathcal{L}_N^{(n,\tau)}} g_N^{(n,\tau,s)}(y) \leq C^n \beta^n (\varphi_1 * X_{n,\tau})(y),$$

where for $n \geq 2$, with renaming $z_i := x_{2i}$, we have written

$$\begin{aligned} X_{n,\tau}(x) &:= \int dz_2 \cdots dz_{m-1} \\ &\times \exp \left[-A \sum_{i=1}^{n-1} |z_{\tau^*(i)} - z_{\tau_*(i)}|^2 \right] \exp \left[-A |z_{\tau_*(n)} - x|^2 \right] \\ &\times f_d(x - z_{m-1}) \prod_{j=2}^{m-1} f_d(z_j - z_{j-1}), \end{aligned} \quad (5.20)$$

where $z_1 := 0$. We also remark that m in the expression above is a function of τ that satisfies $m(\tau) \leq 2n$. Also $m = 2$ for $n = 1$, and $m \geq 3$ for $n \geq 2$.

We will prove by induction on n that for some constant C , depending only on A , and d

$$\sup_{\tau \in \mathfrak{T}_n} X_{n,\tau}(x) \leq C^n f_d(x), \quad n \in \mathbb{N}. \quad (5.21)$$

We remark that from this, it also follows that

$$\sup_{\tau \in \mathfrak{T}_n} (\varphi_1 * X_{n,\tau})(x) \leq C^n f_d(x), \quad n \in \mathbb{N},$$

with a slightly changed constant C . As the number of types of laces with n edges is bounded by 2^n , this will subsequently prove Lemma 4.4.

We will now prove (5.21) by induction. For $n = 1$, there is only one type, and one simply has

$$X_1(x) = \exp[-A|x|^2] f_d(x) \quad (5.22)$$

so (5.21) is evident for $n = 1$.

Let $n = 2$. Then there are the two types $(1, 1)$ and (2) . For $\tau = (2)$

$$\begin{aligned} X_2(x) &= \int dz \exp[-A|z|^2] f_d(z) \exp[-A|x-z|^2] f_d(x-z) \\ &\leq \int dz \exp[-A|z|^2 - A|x-z|^2], \end{aligned}$$

which has Gaussian tails, and so we have only to take care of $\tau = (1, 1)$. There we have $m = 4$, and

$$X_2(x) = \int dz_1 dz_2 \exp[-A(|z_2|^2 + |x - z_1|^2)] f_d(z_1) f_d(z_2 - z_1) f_d(x - z_2),$$

which, according to Lemma 7.2, is bounded by $C(1 + |y|)^{-3d+6}$.

We assume now $n \geq 3$, and that (5.21) holds up to $n - 1$ instead of n . We consider first the case where the type τ starts with $(1, 1, \dots)$. Then the expression for $X_{n,\tau}$ starts with

$$\begin{aligned} &\int d\mathbf{z} \exp[-A(|z_2|^2 + |z_4 - z_1|^2 + |z_5 - z_3|^2 + \cdots)] \\ &\times f_d(z_1) f_d(z_2 - z_1) f_d(z_3 - z_2) f_d(z_4 - z_3) f_d(z_5 - z_4) \cdots \end{aligned}$$

or $|z_5 - z_3|^2$ replaced by $|z_6 - z_3|^2$ if $n > 3$ and the third entry in τ is $\neq 1$. Then

$$\begin{aligned}
& \int dz_1 dz_4 f_d(z_1) f_d(z_2 - z_1) f_d(z_3 - z_2) f_d(z_4 - z_3) f_d(z_5 - z_4) \exp \left[-A |z_4 - z_1|^2 \right] \\
& \leq \sqrt{\int f_d^2(z_1) f_d^2(z_4 - z_3) \exp \left[-A |z_4 - z_1|^2 \right] dz_1 dz_4} \\
& \quad \times \sqrt{\int f_d^2(z_2 - z_1) f_d^2(z_5 - z_4) \exp \left[-A |z_4 - z_1|^2 \right] dz_1 dz_4} \\
& \leq C \sqrt{\int f_d^2(z_1) f_d^2(z_1 - z_3) dz_1} \sqrt{\int f_d^2(z_2 - z_1) f_d^2(z_5 - z_1) dz_1} \\
& \leq C f_d(z_3) f_d(z_5 - z_2).
\end{aligned}$$

To justify the last step in the above bound, we note that $f_d^2(x) = (1 + |x|)^{4-2d}$ is integrable in $d \geq 5$. Then the first integral of the fourth line above is a convolution of f_d^2 with itself, and we can take Fourier transform of this convolution, use a uniform bound on the Fourier coefficient of f_d^2 and then take inverse Fourier transform to obtain the desired bound.

Also, if $n > 3$ and the third entry in τ is $\neq 1$, z_5 is replaced by z_6 . Implementing this inequality then leads to

$$X_{n,\tau} \leq C X_{n-1,\tau'} \quad (5.23)$$

with τ' obtained from τ by replacing $(1, 1, t_3, \dots)$ by $(1, t_3, \dots)$.

We next consider the case where τ is $(1, t_2, t_3, \dots)$ with $t_2 \neq 1$. Here the expression for $X_{n,\tau}$ starts with

$$\begin{aligned}
X_{n,\tau}(y) &= \int d\mathbf{z} \exp \left[-A(|z_2|^2 + |z_1 - z_3|^2 + |z_5 - z_3|^2 + \dots) \right] \\
&\quad \times f_d(z_1) f_d(z_2 - z_1) f_d(z_3 - z_2) f_d(z_4 - z_3) \dots \\
&\leq \int d\mathbf{z} \exp \left[-A(|z_2|^2 + |z_1 - z_3|^2 + |z_5 - z_3|^2 + \dots) \right] \\
&\quad \times f_d(z_1) f_d(z_2 - z_1) f_d(z_4 - z_3) \dots \\
&\leq C \int dz_2 \dots \exp \left[-A(|z_3 - z_1|^2 + |z_6 - z_4|^2 + \dots) \right] \\
&\quad \times f_d(z_3) f_d(z_3 - z_2) f_d(z_4 - z_3) \dots
\end{aligned}$$

That is, we have (5.23) with $\tau' = (1, t_3, \dots)$.

Finally, we have to handle $\tau = (k, t_2, \dots)$ with $k \geq 2$. We first look at the case $\tau = (k)$ in which case $k = n$ if $\tau \in \mathfrak{T}_n$. Then

$$\begin{aligned}
X_{n,\tau}(x) &= \int d\mathbf{z} \exp \left[-A(|z_1|^2 + |z_2 - z_1|^2 + \dots + |x - z_{n-1}|^2) \right] \\
&\quad \times f_d(z_1) f_d(z_2 - z_1) \dots f_d(x - z_{n-1}) \\
&\leq \int d\mathbf{z} \exp \left[-A(|z_1|^2 + |z_2 - z_1|^2 + \dots + |x - z_{n-1}|^2) \right] \\
&\leq c^n \exp \left[-\frac{A}{n} |x|^2 \right]
\end{aligned}$$

for a constant $c > 0$ (depending on A, d only). It is easily checked that $\exp \left[-\frac{A}{n} |x|^2 \right] \leq (c')^n f_d(x)$ for some constant $c' = c'(A) > 0$. (actually, $n^\gamma f_d(x)$ for some $\gamma > 0$ is already an upper bound). So (5.21) is satisfied for $X_{n,(n)}$ for all n , if C is chosen appropriately. We remark that in this case we do not use induction, and estimate the expressions for every n directly. Therefore, we have outside of any inductive assumptions

$$X_{n,(n)} \leq C_1^n f_d, \quad n \in \mathbb{N},$$

for some constant $C_1 > 0$ depending on A and d only.

Let now $\tau = (k, \tau')$ with $k \geq 2$. Take the types $(k-1) \in \mathfrak{T}_{k-1}$ and $(1, \tau') \in \mathfrak{T}_{n-k+1}$. Observe that

$$X_{n,(k,\tau')} = X_{k-1,(k-1)} * X_{n-k+1,(1,\tau')}.$$

As $k \geq 2$, we can use the induction assumption for the second factor and obtain

$$X_{n,(k,\tau')} \leq C_1^{k-1} C^{n-k+1} (f_d * f_d) \leq \gamma C_1^{k-1} C^{n-k+1} f_d,$$

where we use $f_d * f_d \leq \gamma f_d$ for some constant $\gamma = \gamma_d > 0$ (because $3d-6 > d$). Therefore, if $\gamma C_1 \leq C$, we have

$$X_{n,(k,\tau')} \leq C^n f_d,$$

finishing the induction argument for (5.21). Therefore, Lemma 4.4 is proved.

6. PROOF OF LEMMA 4.6: DECONVOLUTION

In this section we prove Lemma 4.6. For this, we introduce a crucial norm on the set of all continuous and bounded functions $\mathbb{R}^d \rightarrow \mathbb{R}$ and derive some properties of the Green's function with respect to this norm, and in Section 6.2 we finish the proof.

6.1 A convolution algebra.

Consider the Banach space \mathbf{B} of bounded continuous functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ that satisfy

$$\|f\| := \max \left\{ \|f\|_1, \|f\|_\infty, \sup_{x \in \mathbb{R}^d} |x|^d |f(x)| \right\} < \infty. \quad (6.1)$$

Lemma 6.1. *If $f, g \in \mathbf{B}$ then also $f \star g \in \mathbf{B}$. Furthermore*

$$\|f \star g\| \leq 2^{d+1} \|f\| \|g\|, \quad f, g \in \mathbf{B}.$$

Proof. Clearly $\|f \star g\|_1 = \|f\|_1 \|g\|_1 \leq \|f\| \|g\|$, and $\|f \star g\|_\infty \leq \|f\|_1 \|g\|_\infty \leq \|f\| \|g\|$. Concerning the third part in the definition of $\|\cdot\|$, let $x \in \mathbb{R}^d$. Then

$$\begin{aligned} |(f \star g)(x)| &= \left| \int f(y) g(x-y) dy \right| \leq \int |f(y)| |g(x-y)| dy \\ &\leq \int_{y: |y| > |x-y|} |f(y)| |g(x-y)| dy + \int_{y: |y| \leq |x-y|} |f(y)| |g(x-y)| dy. \end{aligned}$$

Clearly,

$$\{y: |x-y| < |y|\} \subset \{y: |y| > |x|/2\} \quad \text{and} \quad \{y: |x-y| \geq |y|\} \subset \{y: |x-y| \geq |x|/2\}.$$

Therefore,

$$\begin{aligned} |x|^d |(f \star g)(x)| &\leq |x|^d \int_{y: 2^d |y|^d > |x|^d} |f(y)| |g(x-y)| dy + |x|^d \int_{y: 2^d |x-y|^d \geq |x|^d} |f(y)| |g(x-y)| dy \\ &\leq 2^d \|g\|_1 \sup_{y \in \mathbb{R}^d} |y|^d |f(y)| + 2^d \|f\|_1 \sup_{y \in \mathbb{R}^d} |y|^d |g(y)| \\ &\leq 2^{d+1} \|f\| \|g\|, \end{aligned}$$

which proves the claim. \square

We remind the reader on (4.1): $G(x) = \sum_{n=1}^{\infty} \varphi_n(x)$ is the Green function for the random walk with standard normal increments.

Remark 4 a) With the $C = C_{\alpha, \lambda}$ from (4.16), we obtain from there that $\|G_{\lambda}^{(\Pi)}\| \leq C\alpha$.
 b) There is $C \in (0, \infty)$ (only depending on d) such that, for any $f \in \mathbf{B}$ satisfying $\|f\| \leq 1$, it holds $|f(x)| \leq CG(x)$ for any $x \in \mathbb{R}^d$. (Indeed, use for large $|x|$ that $G(x) \asymp |x|^{2-d}$ and that $|f(x)| \leq |x|^{-d}$ and for small $|x|$ that $G(x)$ is bounded away from zero.)

\diamond

Lemma 6.2. *Assume that $\rho: \mathbb{R}^d \rightarrow \mathbb{R}$ is a rotationally symmetric and continuous function satisfying $|\rho(x)| \leq (1 + |x|)^{-d-4}$ for any $x \in \mathbb{R}^d$ and $\int \rho(x) dx = 0$, then $\|\rho \star G\| < \infty$.*

Proof. Note that G, ρ are both bounded and continuous, and ρ is integrable. Therefore $\rho \star G$ is continuous and bounded and integrable. It remains to show that $|x|^d |(\rho \star G)(x)|$ is bounded in x . It suffices to consider $|x| \geq 1$, which we assume for the rest of the proof.

Evidently

$$(\rho \star G)(x) = \int_{|y| \geq 1} G(y) \rho(x - y) dy + O(|x|^{-d-4}) \quad \text{as } |x| \rightarrow \infty. \quad (6.2)$$

In the following we will be using C as a generic positive constant that does not depend on x nor on y and may change its value from appearance to appearance. According to Lemma 7.3, there is a function $h: \mathbb{R}^d \rightarrow [0, \infty)$ such that

$$G(y) = a_d |y|^{-d+2} + h(y) \quad \text{and } |h(y)| \leq C(1 + |y|)^{-d-2}, \quad |y| \geq 1. \quad (6.3)$$

Using our assumptions on ρ and on h , we see that

$$\begin{aligned} \int_{|y| \geq 1} h(y) \rho(x - y) dy &= \int_{1 \leq |y| \leq |x|/2} h(y) \rho(x - y) dy + \int_{|y| > |x|/2} h(y) \rho(x - y) dy \\ &\leq (1 + |x|)^{-d-4} \|h\|_1 + C(1 + |x|)^{-d-2} \|\rho\|_1 \\ &\leq C(1 + |x|)^{-d-2}. \end{aligned} \quad (6.4)$$

This shows that $|x|^d h \star \rho(x)$ is bounded in x . It remains to show that $\int \rho(y) |x - y|^{-d+2} dy \leq O(|x|^{-d})$ as $|x| \rightarrow \infty$. We split

$$\int \rho(y) |x - y|^{-d+2} dy = \int_{|y| \leq |x|/2} \rho(y) |x - y|^{-d+2} dy + \int_{|y| > |x|/2} \rho(y) |x - y|^{-d+2} dy =: I_1(x) + I_2(x).$$

In the first summand I_1 , we Taylor expand $|x - y|^{-d+2}$ around x :

$$\begin{aligned} |x - y|^{-d+2} &= |x|^{-d+2} + (2 - d) \sum_{i=1}^d y_i x_i |x|^{-d} \\ &\quad + (2 - d) \sum_{i,j=1}^d y_i y_j \left(-d \cdot x_i x_j |x|^{-d-2} + \delta_{ij} |x|^{-d} \right) + O(|y|^3 |x|^{-d-1}). \end{aligned}$$

We split I_1 accordingly into $I_1 = I_{11} + I_{12} + I_{13} + I_{14}$. Now for I_{11} we invoke our assumption $\int \rho(x) dx = 0$ which leads to

$$\begin{aligned} |I_{11}(x)| &= |x|^{-d+2} \left| \int_{|y| \geq |x|/2} \rho(y) dy \right| \leq |x|^{-d+2} \int_{|y| \geq |x|/2} (1 + |y|)^{-d-4} dy \\ &\leq C|x|^{-d} \int_{|y| \geq |x|/2} (1 + |y|)^{-d-2} dy \leq C|x|^{-d}. \end{aligned}$$

We will now use the rotational symmetry of ρ , which means that $\rho(y) = \bar{\rho}(|y|)$ for some $\bar{\rho} : \mathbb{R}_+ \rightarrow \mathbb{R}$. But

$$\int y_i \bar{\rho}(|y|) dy = 0, \quad \int y_i y_j \bar{\rho}(|y|) dy = 0 \quad i \neq j$$

and

$$\int y_i^2 \bar{\rho}(|y|) dy = \int y_j^2 \bar{\rho}(|y|) dy$$

follow by the invariance of Lebesgue measure under rotation and reflection.

Using these observations, we have $I_{12} = 0$ and similarly we have the same statement for the off-diagonal terms $i \neq j$ in $I_{13}(x)$. Now for the diagonal terms in I_{13} we use that $\int y_i^2 \rho(y) dy$ does not depend on i and noting that

$$\sum_{i=1}^d [-d \cdot x_i^2 |x|^{-d-2} + |x|^{-d}] = 0$$

Therefore $I_{13}(x) = 0$. As for I_{14} , we have, by our upper bound on ρ ,

$$|I_{14}(x)| \leq \int_{|y| \leq |x|/2} |\rho(y)| |y|^3 dy |x|^{-d-1} = O(|x|^{-d}).$$

Summarizing, $|x|^d |I_1(x)|$ is bounded. Estimating I_2 is simpler. Indeed, using the decay of ρ imposed in our assumption of the lemma, one easily checks that $|I_2(x)| \leq C(1 + |x|)^{-d-2}$, which is not larger as $O(|x|^{-d})$ as well. This shows that $|x|^d \rho \star G(x)$ is also bounded and proves the lemma. \square

Remark 5 Since for the function ρ in Lemma 6.2 $\|\rho\|$ is finite, also $\|\rho \star G + \rho\|$ is finite. \diamond

Lemma 6.3. *For $\mu \leq 1$ define*

$$G_\mu(x) := \sum_{n=1}^{\infty} \mu^n \varphi_n(x), \quad x \in \mathbb{R}^d.$$

If ρ satisfies the conditions of Lemma 6.2, then there is a $C \in (0, \infty)$ (depending only on ρ and d) such that

$$\|\rho \star G_\mu\| \leq C|\mu|, \quad \mu \in [-1, 1].$$

Proof. Let us first verify that $G_\mu \in \mathbf{B}$ for any $\mu \in (-1, 1)$. We will be using C as a generic positive constant that does not depend on x nor on n and may change its value from appearance to appearance. Since $\sup_x \varphi_n(x) = \varphi_n(0) \leq C$, we have $\|G_\mu\|_\infty \leq \frac{C|\mu|}{1-|\mu|}$. Since the integral of φ_n is one, $\|G_\mu\|_1 \leq \frac{C|\mu|}{1-|\mu|}$. Furthermore,

$$\sup_{x \in \mathbb{R}^d} |x|^d G_\mu(x) \leq \sum_{n=1}^{\infty} |\mu|^n \frac{1}{(2\pi n)^{d/2}} \sup_{x \in \mathbb{R}^d} |x|^d e^{-|x|^2/2n} = d^{d/2} e^{-d/2} \sum_{n=1}^{\infty} |\mu|^n \frac{1}{(2\pi n)^{d/2}} n^{d/2} \leq \frac{C|\mu|}{1-|\mu|}.$$

Hence, $\|G_\mu\| \leq \frac{C|\mu|}{1-|\mu|}$. An elementary calculation shows that

$$\rho \star G_\mu = \mu(\rho \star G) - (1 - \mu)\rho \star G \star G_\mu, \quad \mu \in (-1, 1].$$

Therefore, for any $\mu \in [-\frac{1}{2}, 1]$,

$$\|\rho \star G_\mu\| \leq |\mu| \|\rho \star G\| + (1 - \mu) 2^{d+1} \|\rho \star G\| \|G_\mu\| \leq C|\mu| \|\rho \star G\|$$

by applying Lemma 6.1. Now Lemma 6.2 implies the assertion. \square

6.2 Proof of Lemma 4.6.

We would like to apply Lemma 6.2 and Lemma 6.3 to $\rho = -\lambda\varphi - G_\lambda^{(\Pi)}$. However, for applying Lemma 6.2 we need that $\int \rho(x) dx = 0$. Therefore, we add a suitable multiple of φ , by defining

$$\rho := -G_\lambda^{(\Pi)} + (\mu - \lambda)\varphi, \quad \text{where } \mu = \int [G_\lambda^{(\Pi)}(x) + \lambda\varphi(x)] dx = \int G_\lambda^{(\Pi)}(x) dx + \lambda. \quad (6.5)$$

Then ρ is continuous and bounded and rotationally symmetric with $\int \rho(x) dx = 0$ and satisfies the requested upper bound by (4.16). We remark that from (4.18), we get

$$\left(1 - \lambda - \int G_\lambda^{(\Pi)}(x) dx\right) \left(1 + \int G_\lambda^{(\Gamma)}(x) dx\right) = 1,$$

and as the second factor on the left-hand side is > 0 , we get $\mu \leq 1$. Furthermore, as $|\int G_\lambda^{(\Pi)}(x) dx| = O(\alpha)$ and $\lambda \geq 0$, we have $\mu \geq -1/2$ for small enough α . Hence, $-\frac{1}{2} \leq \mu \leq 1$ and we can therefore apply Lemma 6.3.

Again, we use C to denote a generic constant that may depend only on d and v and may change its value from appearance to appearance. Note that $|\mu| \leq C\alpha$. From Lemma 6.3 we have that $\|\rho \star G_\mu\| \leq C\alpha$. Hence, we have also that $\|\rho \star G_\mu + \rho\| \leq C\alpha$. Defining

$$\tilde{G}_\lambda^{(\Pi)} = \lambda\varphi + G_\lambda^{(\Pi)}$$

one readily checks that

$$-\tilde{G}_\lambda^{(\Pi)} + G_\mu - \tilde{G}_\lambda^{(\Pi)} \star G_\mu = \rho \star G_\mu + \rho.$$

Therefore, $\|-\tilde{G}_\lambda^{(\Pi)} + G_\mu - \tilde{G}_\lambda^{(\Pi)} \star G_\mu\| \leq C\alpha$. Therefore, if α is small enough, the Neumann series

$$Q := \sum_{n=1}^{\infty} (-1)^n (-\tilde{G}_\lambda^{(\Pi)} + G_\mu - \tilde{G}_\lambda^{(\Pi)} \star G_\mu)^{\star n}.$$

converges in \mathbf{B} , and satisfies $\|Q\| \leq C\alpha$. If we define

$$S := Q + G_\mu + Q \star G_\mu. \quad (6.6)$$

then one checks that

$$S - \tilde{G}_\lambda^{(\Pi)} - \tilde{G}_\lambda^{(\Pi)} \star S = 0. \quad (6.7)$$

Formally this means that if δ_0 is the Dirac function at 0, one has

$$(\delta_0 + S) \star (\delta_0 - \tilde{G}_\lambda^{(\Pi)}) = \delta_0$$

meaning that $\delta_0 + S$ is the inverse of $\delta_0 - \tilde{G}_\lambda^{(\Pi)}$ under convolution. The following lemma will complete the argument:

Lemma 6.4. *Under condition (4.16), if $\alpha > 0$ is small enough, then $S = G_\lambda^{(\Gamma)}$.*

Proof of Lemma 6.4: S and $G_\lambda^{(\Pi)}$ are in $L^2(\mathbb{R}^d)$ which then also contains $G_\lambda^{(\Pi)} \star S$. The Fourier transform of (6.7) is given by

$$\widehat{S} - \widehat{\tilde{G}_\lambda^{(\Pi)}} - \widehat{S} \widehat{\tilde{G}_\lambda^{(\Pi)}} = 0, \quad \text{or} \quad (1 + \widehat{S})(1 - \widehat{\tilde{G}_\lambda^{(\Pi)}}) = 1.$$

Note that $G_\lambda^{(\Gamma)}$ satisfies the same equation $G_\lambda^{(\Gamma)} - \tilde{G}_\lambda^{(\Pi)} - \tilde{G}_\lambda^{(\Pi)} \star G_\lambda^\Gamma = 0$, so we get

$$(1 + \widehat{G_\lambda^{(\Gamma)}})(1 - \widehat{\tilde{G}_\lambda^{(\Pi)}}) = 1.$$

This implies $\widehat{S} = \widehat{G_\lambda^{(\Gamma)}}$, so that by Fourier inversion formula, we obtain $S = G_\lambda^{(\Gamma)}$, as required. \square

Completing the proof of Lemma 4.6: By Lemma 6.4 and (6.6) we obtain

$$G_\lambda^{(\Gamma)} = S = Q + G_\mu + Q \star G_\mu.$$

Recall that $G_\mu \leq G$. Also, from the arguments preceding Lemma 6.4 we have that $\|Q + Q \star G_\mu\| \leq C\alpha$, so that for small enough α , by Remark 4 b) we conclude that

$$G_\lambda^{(\Gamma)} = S = Q + G_\mu + Q \star G_\mu \leq Q + G + Q \star G_\mu \leq 2G,$$

as required. \square

7. TECHNICAL ESTIMATES

In the proofs of Lemma 4.4 and Lemma 4.6 we have used some technical estimates, which we state and prove in this section. Recall that φ is the standard Gaussian density and that the interaction function $v: [0, \infty) \rightarrow [0, \infty)$ is continuous and satisfies $v \leq L$ and $v(r) = 0$ for $r > R$ for some parameters $L, R \in (0, \infty)$.

Lemma 7.1 (Gaussian decay of u_n). *Recall the function $u_n = u_{n,\alpha}$ defined in (5.1). Then there exist constants $C, A \in (0, \infty)$, depending only on d, L, R , such that, for any n and $\mathbf{x} = (x_1, x_2, \dots, x_{2n-1}, x_{2n}) \in (\mathbb{R}^d)^{2n}$,*

$$u_n(\mathbf{x}) \leq C^m \alpha^{n-1} \exp \left[-\frac{1}{8} \sum_{i=1}^n |x_{2i} - x_{2i-1}|^2 - A \sum_{i=1}^{n-1} |x_{2i} - x_{2i+2}|^2 \right].$$

Proof. Let us first assume that $n = 2m$ is even (the case of odd n is similar, we drop it in the following), and identify $u_{2m}(\mathbf{x})$ in terms of integrals over standard Brownian bridges $\text{BB}^{(i)}$ on $[0, 1]$ that start and terminate at zero. Indeed, if we define $\bar{x}_t^{(i)} := (1-t)x_{2i-1} + tx_{2i}$, then $(\bar{x}_t^{(i)} + \text{BB}^{(i)}(t))_{t \in [0,1]}$ are independent Brownian bridges from x_{2i-1} to x_{2i} , $i = 1, \dots, m$. That is, they have the same distribution as $\mu_{x_{2i-1}, x_{2i}}^{(1)} / \varphi(x_{2i} - x_{2i-1})$. Hence, we can write

$$u_{2m}(\mathbf{x}) = \prod_{i=1}^{2m} \varphi(x_{2i} - x_{2i-1}) \mathbb{E} \left[\prod_{i=1}^{2m-1} (1 - \exp[-\alpha \varpi_i]) \right], \quad (7.1)$$

where

$$\varpi_i := \int_0^1 dt \, v \left(|\bar{x}_t^{(i)} + \text{BB}^{(i)}(t) - \bar{x}_t^{(i+1)} - \text{BB}^{(i+1)}(t)| \right) \quad \text{and} \quad \bar{x}_t^{(i)} = (1-t)x_{2i-1} + tx_{2i}.$$

We will later upper estimate the expectation on the right-hand side of (7.1) against products of expectations over Brownian bridges, using the Cauchy–Schwarz inequality; hence it will be critical and almost sufficient to handle the case $n = 2$. Hence, first prove the assertion for the case $n = 2$. We will use the well known fact (proved with the help of the reflection principle) that the distribution of $\sup_{t \in [0,1]} |\text{BB}(t)|$ has sub-gaussian tails, i.e., there exist constants $c_1, c_2 > 0$, depending only on d , such that

$$\mathbb{P} \left(\sup_{t \in [0,1]} |\text{BB}(t)| > u \right) \leq c_1 \exp[-c_2 u^2], \quad u \in (0, \infty). \quad (7.2)$$

We first consider the set

$$W := \left\{ \mathbf{x} \in \mathbb{R}^4 : \max_{i=1,2} |x_{2i} - x_{2i-1}| > \frac{1}{4} |x_4 - x_2| \right\}.$$

If $|x_2 - x_1| > |x_4 - x_2|/4$, we have

$$\exp \left[-\frac{1}{2} |x_2 - x_1|^2 \right] \leq \exp \left[-\frac{1}{4} |x_2 - x_1|^2 - \frac{1}{64} |x_4 - x_2|^2 \right], \quad (7.3)$$

and similarly when $|x_4 - x_3| > |x_4 - x_2|/4$. Therefore, for $\mathbf{x} \in W$, we have, also using that $v \leq L$ and that $1 - e^{-\alpha L} \leq \alpha L$,

$$u_2(\mathbf{x}) \leq \frac{L\alpha}{(2\pi)^d} \exp \left[-\frac{1}{4} |x_2 - x_1|^2 - \frac{1}{4} |x_4 - x_3|^2 - \frac{1}{64} |x_4 - x_2|^2 \right],$$

which shows the claim for $n = 2$ on W .

On W^c , and on the event

$$\left\{ \sup_{t \in [0,1]} |\text{BB}^{(i)}(t)| \leq \frac{1}{2} |x_4 - x_2| - 2R \right\}$$

one has $\varpi_1 = 0$. Therefore, on W^c ,

$$\begin{aligned} u_2(\mathbf{x}) &\leq \frac{L\alpha}{(2\pi)^d} \exp \left[-\frac{1}{2} |x_2 - x_1|^2 - \frac{1}{2} |x_4 - x_3|^2 \right] 2\mathbb{P} \left(\sup_{t \in [0,1]} |\text{BB}(t)| > \frac{1}{2} |x_4 - x_2| - 2R \right) \\ &\leq C\alpha \exp \left[-\frac{1}{2} |x_2 - x_1|^2 - \frac{1}{2} |x_4 - x_3|^2 - A|x_4 - x_2|^2 \right], \end{aligned} \quad (7.4)$$

using (7.2), with some constants $C, A > 0$, depending only on d, L, R . This proves the claim for $n = 2$, actually with slightly better coefficients in the exponent. But the ones in the statement comes up in an induction argument.

Let $n \geq 3$. To simplify the notation, we take n even, $n = 2m$. (The case of an odd n follows by a straightforward modification.) We have

$$\begin{aligned} \frac{u_{2m}(\mathbf{x})}{\prod_{i=1}^{2m} \varphi(x_{2i} - x_{2i-1})} &= \mathbb{E} \left[\prod_{j=1}^{2m-1} (1 - \exp[-\alpha \varpi_j]) \right] \\ &= \mathbb{E} \left[\prod_{j=1}^m (1 - \exp[-\alpha \varpi_{2j-1}]) \prod_{j=1}^{m-1} (1 - \exp[-\alpha \varpi_{2j}]) \right] \\ &\leq \left[\mathbb{E} \left(\prod_{j=1}^m (1 - \exp[-\alpha \varpi_{2j-1}])^2 \right) \mathbb{E} \left(\prod_{j=1}^{m-1} (1 - \exp[-\alpha \varpi_{2j}])^2 \right) \right]^{1/2} \\ &\leq \left[\mathbb{E} \left(\prod_{j=1}^m (1 - \exp[-\alpha \varpi_{2j-1}]) \right) \mathbb{E} \left(\prod_{j=1}^{m-1} (1 - \exp[-\alpha \varpi_{2j}]) \right) \right]^{1/2} \\ &= \left[\prod_{j=1}^m \mathbb{E} (1 - \exp[-\alpha \varpi_{2j-1}]) \prod_{j=1}^{m-1} \mathbb{E} (1 - \exp[-\alpha \varpi_{2j}]) \right]^{1/2} \end{aligned}$$

In the third line, we used the Cauchy-Schwarz bound, while in the following inequality, we used that $1 - e^{-\alpha \varpi} \in (0, 1)$, and so we can drop the square. In the fifth line we used independence. Hence, we

get

$$\begin{aligned}
u_{2m}(\mathbf{x}) &\leq \left(\prod_{i=1}^{2m} \sqrt{\varphi(x_{2i} - x_{2i-1})} \right) \sqrt{\varphi(x_2 - x_1)} \sqrt{\varphi(x_{2m} - x_{2m-1})} \\
&\quad \times \sqrt{\prod_{j=1}^m \varphi(x_{4j} - x_{4j-1}) \varphi(x_{4j-2} - x_{4j-3}) \mathbb{E}(1 - \exp[-\alpha \varpi_{2j-1}])} \\
&\quad \times \sqrt{\prod_{j=1}^{m-1} \varphi(x_{4j+2} - x_{4j+1}) \varphi(x_{4j} - x_{4j-1}) \mathbb{E}(1 - \exp[-\alpha \varpi_{2j}])} \\
&\leq \left(\prod_{i=1}^{2m} \sqrt{\varphi(x_{2i} - x_{2i-1})} \right) \sqrt{\varphi(x_2 - x_1)} \sqrt{\varphi(x_{2m} - x_{2m-1})} \\
&\quad \times \sqrt{\prod_{j=1}^m u_2(x_{4j}, x_{4j-1}, x_{4j-2}, x_{4j-3})} \times \sqrt{\prod_{j=1}^{m-1} u_2(x_{4j+2}, x_{4j+1}, x_{4j}, x_{4j-1})}
\end{aligned}$$

Now we apply the estimate in (7.4) for each of terms in the last line and summarize to finish the proof for $n = 2m$. \square

Lemma 7.2. a) For any $m \in \mathbb{N}$ and $h > 0$, there exists $C(m, h) > 0$ such that

$$\frac{1}{C(m, h)(1 + |y|)^m} \leq \int_{\mathbb{R}^d} \frac{1}{(1 + |x|)^m} e^{-h|y-x|^2} dx \leq \frac{C(m, h)}{(1 + |y|)^m}, \quad y \in \mathbb{R}^d.$$

b) For any $m_1, m_2 \in \mathbb{N}$ and $h > 0$, there exists $C(m_1, m_2, h) > 0$ such that

$$\int_{\mathbb{R}^d} \frac{1}{(1 + |y_1 - x|)^{m_1}} \frac{1}{(1 + |y_2 - x|)^{m_2}} e^{-h|x|^2} dx \leq \frac{C(m_1, m_2, h)}{(1 + |y_1|)^{m_1} (1 + |y_2|)^{m_2}}, \quad y_1, y_2 \in \mathbb{R}^d.$$

Proof. a)

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{1}{(1 + |x|)^m} e^{-h|y-x|^2} dx &= \int_{|x-y| \leq |y|/2} \frac{1}{(1 + |x|)^m} e^{-h|y-x|^2} dx + \int_{|x-y| > |y|/2} \frac{1}{(1 + |x|)^m} e^{-h|y-x|^2} dx \\
&\leq \frac{1}{(1 + |y|/2)^m} \int e^{-h|z|^2} dz + \int_{|z| > |y|/2} e^{-h|z|^2} dz \\
&\leq 2^m \pi^{d/2} h^{-d/2} \frac{1}{(1 + |y|)^m} + \int_{|z| > |y|/2} e^{-h|z|^2} dz,
\end{aligned}$$

where we used in the first integration area that $|x - y| \leq |y|/2$ implies that $|x| \geq |y| - |x - y| \geq |y|/2$. As the second summand on the right-hand side has a Gaussian tail bound, the second inequality follows.

For the first,

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{1}{(1 + |x|)^m} e^{-h|y-x|^2} dx &\geq \int_{|x| \leq |y|+1} \frac{1}{(1 + |x|)^m} e^{-h|y-x|^2} dx \\
&\geq \frac{1}{(2 + |y|)^m} \int_{|z| \leq 1} e^{-h|z|^2} dz \geq c(m, h) \frac{1}{(1 + |y|)^m}
\end{aligned}$$

for some small $c(m, h) > 0$ and all $y \in \mathbb{R}^d$.

b) follows by applying the Cauchy–Schwarz inequality and a). \square

Lemma 7.3. *G satisfies*

$$G(x) = a_d |x|^{2-d} + O(|x|^{-d-2}) \quad \text{as } |x| \rightarrow \infty, \quad \text{where } a_d := \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}},$$

and Γ is the standard Gamma function.

Proof. We approximate G by the Green function in continuous time:

$$G_c(x) := \int_0^\infty \varphi_t(x) dt = a_d |x|^{-d+2}.$$

To determine the correction, we consider

$$G_c(x) - G(x) = \sum_{n=1}^\infty \int_{n-1/2}^{n+1/2} [\varphi_t(x) - \varphi_n(x)] dt + \int_0^{1/2} \varphi_t(x) dt.$$

The second summand has Gaussian tails in $|x|$, so we have only to care about the first one. A Taylor expansion gives

$$\varphi_t(x) - \varphi_n(x) = (t - n) \partial_1 \varphi_n(x) + \frac{(t - n)^2}{2} \partial_1^2 \varphi_{n+\theta(t-n)}(x), \quad \theta \in [0, 1]. \quad (7.5)$$

Note that

$$\begin{aligned} \partial_1 \varphi_t(x) &= \left[\frac{|x|^2}{2t^2} - \frac{d}{2t} \prod_{i=1}^{2m} \varphi(x_{2i} - x_{2i-1}) \right] \varphi_t(x) \\ \partial_1^2 \varphi_t(x) &= \prod_{i=1}^{2m} \varphi(x_{2i} - x_{2i-1}) \left[-2 \frac{|x|^2}{2t^3} + \frac{d}{2t^2} \right] \varphi_t(x) + \left[\frac{|x|^2}{2t^2} - \frac{d}{2t} \right]^2 \varphi_t(x) \\ &= \frac{1}{t^2} \left[\left(\frac{|x|^2}{t} \right)^2 - \frac{|x|^2}{t} \left(1 + \frac{d}{2} \right) + \frac{d}{2} \left(1 + \frac{d}{2} \right) \right] \varphi_t(x) \end{aligned}$$

The integration $\int_{n-1/2}^{n+1/2} dt$ over the first summand on the right-hand side of (7.5) is 0. Therefore

$$\begin{aligned} \sum_{n=1}^\infty \left| \int_{n-1/2}^{n+1/2} [\varphi_t(x) - \varphi_n(x)] dt \right| &\leq \frac{1}{24} \sum_{n=1}^\infty \sup_{t \in [n-1/2, n+1/2]} |\partial_1^2 \varphi_t(x)| \\ &\leq \sum_{n=1}^\infty \frac{C}{n^{2+d/2}} \left(\left(\frac{|x|^2}{n} \right)^2 + 1 \right) \exp \left[-\frac{|x|^2}{2(n+1)^2} \right] \\ &= O(|x|^{-d-2}) \end{aligned}$$

as $|x| \rightarrow \infty$. □

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