A VARIATIONAL FORMULA FOR THE FREE ENERGY OF AN INTERACTING MANY-PARTICLE SYSTEM

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Abstract: We consider N bosons in a box in $\mathbb{R}^d$ with volume N/ρ under the influence of a mutually repellent pair potential. The particle density $\rho \in (0,\infty)$ is kept fixed. Our main result is the identification of the limiting free energy, $f(\beta,\rho)$, at positive temperature $1/\beta$, in terms of an explicit variational formula, for any fixed ρ if β is sufficiently small, and for any fixed β if ρ is sufficiently small.

The thermodynamic equilibrium is described by the symmetrised trace of $\mathrm{e}^{-\beta\mathcal{H}_N}$, where $\mathcal{H}_N$ denotes the corresponding Hamilton operator. The well-known Feynman-Kac formula reformulates this trace in terms of N interacting Brownian bridges. Due to the symmetrisation, the bridges are organised in an ensemble of cycles of various lengths. The novelty of our approach is a description in terms of a marked Poisson point process whose marks are the cycles. This allows for an asymptotic analysis of the system via a large-deviations analysis of the stationary empirical field. The resulting variational formula ranges over random shift-invariant marked point fields and optimizes the sum of the interaction and the relative entropy with respect to the reference process.

In our proof of the lower bound for the free energy, we drop all interaction involving 'infinitely long' cycles, and their possible presence is signalled by a loss of mass of the 'finitely long' cycles in the variational formula. In the proof of the upper bound, we only keep the mass on the 'finitely long' cycles. We expect that the precise relationship between these two bounds lies at the heart of Bose-Einstein condensation and intend to analyse it further in future.

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1. Introduction and main results

In this paper, we study a probabilistic model for interacting bosons at positive temperature in the thermodynamic limit with positive particle density. See Section 1.4 for the physical background.

1.1. **The model.** The main object is the following symmetrised sum of Brownian bridge expectations,

$$Z_N^{(\mathrm{bc})}(\beta,\Lambda)=\frac{1}{N!}\sum_{\sigma\in\mathfrak{S}_N}\int_\Lambda \mathrm{d}x_1\cdots\int_\Lambda \mathrm{d}x_N\bigotimes_{i=1}^N\mu_{x_i,x_{\sigma(i)}}^{(\mathrm{bc},\beta)}\Big[\exp\Big\{-\sum_{1\le i<j\le N}\int_0^\beta v(|B_s^{(i)}-B_s^{(j)}|)\,\mathrm{d}s\Big\}\Big]. \quad (1.1)$$

Here $\mu_{x,y}^{(\mathrm{bc},\beta)}$ is the canonical Brownian bridge measure with boundary condition $\mathrm{bc}\in\{\varnothing,\mathrm{per},\mathrm{Dir}\}$, time horizon $\beta>0$ and initial point $x\in\Lambda$ and terminal point $y\in\Lambda$, and the sum is on permutations $\sigma\in\mathfrak{S}_N$ of $1,\dots,N$. (We write $\mu(f)$ for the integral of f with respect to the measure μ.) The *interaction potential* $v\colon\mathbb{R}\to[0,\infty]$ is measurable, decays sufficiently fast at infinity and is possibly infinite close to the origin. Our precise assumptions on v appear prior to Theorem 1.2 below. We assume that Λ is a measurable subset of $\mathbb{R}^d$ with finite volume.

The boundary condition $\mathrm{bc}=\varnothing$ refers to the standard Brownian bridge, whereas for $\mathrm{bc}=\mathrm{Dir}$, the expectation is on those Brownian bridge paths which stay in Λ over the time horizon $[0,\beta]$. In the case of periodic boundary condition, $\mathrm{bc}=\mathrm{per}$, we consider Brownian bridges on the torus $\Lambda=(\mathbb{R}/L\mathbb{Z})^d$ with side length L.

Our main motivation to study the quantity $Z_N^{(\mathrm{bc})}(\beta,\Lambda)$ is the fact that, for both periodic and Dirichlet boundary conditions, it is related to the N-body Hamilton operator

$$\mathcal{H}_{N,\Lambda}^{(\mathrm{bc})}=-\sum_{i=1}^N\Delta_i^{(\mathrm{bc})}+\sum_{1\le i<j\le N}v(|x_i-x_j|),\qquad x_1,\dots,x_n\in\Lambda,\qquad \mathrm{bc}\in\{\mathrm{Dir},\mathrm{per}\} \quad (1.2)$$

where $\Delta_i^{(\mathrm{bc})}$ stands for the Laplacian with bc boundary condition. More precisely, $Z_N^{(\mathrm{bc})}(\beta,\Lambda)$ is equal to the trace of the projection of the operator $\exp\{-\beta\mathcal{H}_{N,\Lambda}^{(\mathrm{bc})}\}$ to the set of symmetric (i.e., permutation invariant) functions $(\mathbb{R}^d)^N\to\mathbb{R}$. This statement is proven via the Feynman-Kac formula, see [G70] or [BR97]. Hence, we call $Z_N^{(\mathrm{bc})}(\beta,\Lambda)$ a partition function.

It is the main purpose of this paper to derive a variational expression for the *limiting free energy*

$$f^{(\mathrm{bc})}(\beta,\rho)=-\frac{1}{\beta}\lim_{N\to\infty}\frac{1}{|\Lambda_{L_N}|}\log Z_N^{(\mathrm{bc})}(\beta,\Lambda_{L_N}), \quad (1.3)$$

where $|\Lambda_{L_N}|=N/\rho$, for any $\beta,\rho\in(0,\infty)$, any $d\in\mathbb{N}$ and any $\mathrm{bc}\in\{\varnothing,\mathrm{per},\mathrm{Dir}\}$. The existence of the thermodynamic limit in (1.3) with $\mathrm{bc}\in\{\mathrm{per},\mathrm{Dir}\}$ under suitable assumptions on the interaction potential v can be shown by standard methods, see, e.g., [Rue69, Th. 3.58] and [Rob71]. However, up to the best of our knowledge, there is no useful identification or characterisation of $f^{(\mathrm{bc})}(\beta,\rho)$ available in the literature. We also give new proofs for the independence of the value of the free energy on the boundary conditions, which is another novelty.

Our approach and the remainder of Section 1 can be summarized as follows. Since any permutation decomposes into cycles, and using the Markov property, the family of the N bridges in (1.1) decomposes into cycles of various lengths, i.e., into bridges that start and end at the same site, which is uniformly distributed over Λ. We conceive these initial-terminal sites as the points of a standard Poisson point process on $\mathbb{R}^d$ and the cycles as marks attached to these points; see Section 1.2 for the relevant notation. In Proposition 1.1 below we rewrite $Z_N^{(\mathrm{bc})}(\beta,\Lambda)$ in terms of an expectation over a reference process, the marked Poisson point process ω_{P}.

In Section 1.3, we present our results on the large-N asymptotics of $Z_N^{(\mathrm{bc})}(\beta,\Lambda)$ when Λ is a centred cube of volume N/ρ. Indeed, in Theorem 1.2, its exponential rate is bounded from above and below in terms of two variational formulas that range over marked shift-invariant point processes and optimise the sum of an energy term and an entropy term. These bounds are shown to coincide for any fixed ρ if β is sufficiently small, and for any fixed β if ρ is sufficiently small. The main value and novelty of these representations is the explicit description of the interplay between entropy, interaction and symmetrisation of the system. We think that these formulas, even in the case where our two bounds do not coincide, are explicit enough to serve as a basis for future deeper investigations of properties like phase transitions.

The physical interpretation, motivation and relevance are discussed in Section 1.4.

1.2. **Representation of the partition function.** In this section, we introduce our representation of the partition function $Z_N^{(\mathrm{bc})}(\beta,\Lambda)$ for each boundary condition bc $\in\{\varnothing,\mathrm{per},\mathrm{Dir}\}$ in terms of an expectation over a marked Poisson point process. The main result of this section is Proposition 1.1. We have to introduce some notation.

We begin with the mark space. The space of marks is defined as

$$E^{(\mathrm{bc})}=\bigcup_{k\in\mathbb{N}}\mathcal{C}_{k,\Lambda}^{(\mathrm{bc})},\qquad \mathrm{bc}\in\{\varnothing,\mathrm{per},\mathrm{Dir}\},\tag{1.4}$$

where, for $k\in\mathbb{N}$, we denote by $\mathcal{C}_k=\mathcal{C}_{k,\Lambda}^{(\varnothing)}$ the set of continuous functions $f\colon[0,k\beta]\to\mathbb{R}^d$ satisfying $f(0)=f(k\beta)$, equipped with the topology of uniform convergence. Moreover, $\mathcal{C}_{k,\Lambda}^{(\mathrm{Dir})}$, resp. $\mathcal{C}_{k,\Lambda}^{(\mathrm{per})}$, is the space of continuous functions in Λ, resp. on the torus $\Lambda=(\mathbb{R}/L\mathbb{Z})^d$, with time horizon $[0,k\beta]$. We sometimes call the marks *cycles*. By $\ell\colon E^{(\mathrm{bc})}\to\mathbb{N}$ we denote the canonical map defined by $\ell(f)=k$ if $f\in\mathcal{C}_{k,\Lambda}^{(\mathrm{bc})}$. We call $\ell(f)$ the *length* of $f\in E$. When dealing with the empty boundary condition, we sometimes drop the superscript $\varnothing$.

We consider spatial configurations that consist of a locally finite set $\xi\subset\mathbb{R}^d$ of particles, and to each particle $x\in\xi$ we attach a mark $f_x\in E^{(\mathrm{bc})}$satisfying $f_x(0)=x$. Hence, a configuration is described by the counting measure

$$\omega=\sum_{x\in\xi}\delta_{(x,f_x)}$$

on $\mathbb{R}^d\times E$ for the empty boundary condition, resp. on $\Lambda\times E^{(\mathrm{bc})}$ for bc $\in\{\mathrm{per},\mathrm{Dir}\}$.

We now introduce three marked Poisson point processes for the three boundary conditions. The one for the empty condition will later serve as a reference process and is introduced separately first.

Reference process.

Consider on $\mathcal{C}=\mathcal{C}_1$ the canonical Brownian bridge measure

$$\mu_{x,y}^{(\varnothing,\beta)}(A)=\mu_{x,y}^{(\beta)}(A)=\frac{\mathbb{P}_x(B\in A;B_\beta\in\mathrm{d}y)}{\mathrm{d}y},\qquad A\subset\mathcal{C}\text{ measurable.}\tag{1.5}$$

Here $B=(B_t)_{t\in[0,\beta]}$ is a Brownian motion in $\mathbb{R}^d$ with generator Δ, starting from x under $\mathbb{P}_x$. Then $\mu_{x,y}^{(\beta)}$ is a regular Borel measure on $\mathcal{C}$ with total mass equal to the Gaussian density,

$$\mu_{x,y}^{(\beta)}(\mathcal{C})=g_\beta(x,y)=\frac{\mathbb{P}_x(B_\beta\in\mathrm{d}y)}{\mathrm{d}y}=(4\pi\beta)^{-d/2}\mathrm{e}^{-\frac{1}{4\beta}|x-y|^2}.\tag{1.6}$$

We write $\mathbb{P}_{x,y}^{(\beta)}=\mu_{x,y}^{(\beta)}/g_\beta(x,y)$ for the normalized Brownian bridge measure on $\mathcal{C}$. Let

$$\omega_{\mathrm{P}}=\sum_{x\in\xi_{\mathrm{P}}}\delta_{(x,B_x)},$$

be a Poisson point process on $\mathbb{R}^d\times E$ with intensity measure equal to ν whose projection onto $\mathbb{R}^d\times\mathcal{C}_k$ is equal to

$$\nu_k(\mathrm{d}x,\mathrm{d}f)=\frac{1}{k}\mathrm{Leb}(\mathrm{d}x)\otimes\mu_{x,x}^{(k\beta)}(\mathrm{d}f),\qquad k\in\mathbb{N}.\tag{1.7}$$

Alternatively, we can conceive $\omega_{\rm P}$ as a marked Poisson point process on $\mathbb{R}^d$, based on some Poisson point process $\xi_{\rm P}$ on $\mathbb{R}^d$, and a family $(B_x)_{x\in\xi_{\rm P}}$ of i.i.d. marks, given $\xi_{\rm P}$. The intensity of $\xi_{\rm P}$ is

$$\overline{q}=\sum_{k\in\mathbb{N}}q_k,\qquad \text{with}\quad q_k=\frac{1}{(4\pi\beta)^{d/2}k^{1+d/2}},\qquad k\in\mathbb{N}. \tag{1.8}$$

Conditionally given $\xi_{\rm P}$, the length $\ell(B_x)$ is an $\mathbb{N}$-valued random variable with distribution $(q_k/\overline{q})_{k\in\mathbb{N}}$, and, given $\ell(B_x)=k$, B_x is in distribution equal to a Brownian bridge with time horizon $[0,k\beta]$, starting and ending at x. Let $\mathtt{Q}$ denote the distribution of $\omega_{\rm P}$ and denote by $\mathtt{E}$ the corresponding expectation. Hence, $\mathtt{Q}$ is a probability measure on the set Ω of all locally finite counting measures on $\mathbb{R}^d\times E$.

Processes for Dirichlet and periodic boundary conditions.

For Dirichlet boundary condition, one restricts the Brownian bridges to not leaving the set Λ. Consider the measure

$$\mu_{x,y}^{({\rm Dir},\beta)}(A)=\frac{\mathbb{P}_x(B\in A;B_\beta\in{\rm d}y)}{{\rm d}y},\qquad A\subset\mathcal{C}_{1,\Lambda}^{({\rm Dir})}\text{ measurable}, \tag{1.9}$$

which has total mass

$$g_\beta^{({\rm Dir})}(x,y)=\mu_{x,y}^{({\rm Dir},\beta)}(\mathcal{C}_{1,\Lambda}^{({\rm Dir})})=\frac{\mathbb{P}_x(B_{[0,\beta]}\subset\Lambda;B_\beta\in{\rm d}y)}{{\rm d}y}. \tag{1.10}$$

For periodic boundary condition, the marks are Brownian bridges on the torus $\Lambda=(\mathbb{R}/L\mathbb{Z})^d$. The corresponding path measure is denoted by $\mu_{x,y}^{({\rm per},\beta)}$; its total mass is equal to

$$g_\beta^{({\rm per})}(x,y)=\mu_{x,y}^{({\rm per},\beta)}(\mathcal{C}_{\Lambda}^{({\rm per})})=\sum_{z\in\mathbb{Z}^d}g_\beta(x,y+zL)=(4\pi\beta)^{-d/2}\sum_{z\in\mathbb{Z}^d}{\rm e}^{-\frac{|x-y-zL|^2}{4\beta}}. \tag{1.11}$$

For periodic and Dirichlet boundary conditions (1.8) is replaced by

$$\overline{q}^{({\rm bc})}=\sum_{k=1}^N q_k^{({\rm bc})},\qquad \text{with } q_k^{({\rm bc})}=\frac{1}{k|\Lambda|}\int_\Lambda{\rm d}x\,g_{k\beta}^{({\rm bc})}(x,x). \tag{1.12}$$

Note that this weight depends on Λ and on N. We introduce the Poisson point process $\omega_{\rm P}=\sum_{x\in\xi_{\rm P}}\delta_{(x,B_x)}$ on $\Lambda\times E^{({\rm bc})}$ with intensity measure $\nu^{({\rm bc})}$ whose projections on $\Lambda\times\mathcal{C}_{k,\Lambda}^{({\rm bc})}$ with $k\le N$ are equal to $\nu_k^{({\rm bc})}({\rm d}x,{\rm d}f)=\frac{1}{k}{\rm Leb}_\Lambda({\rm d}x)\otimes\mu_{x,x}^{({\rm bc},k\beta)}({\rm d}f)$ and are zero on this set for $k>N$. We do not label $\omega_{\rm P}$ nor $\xi_{\rm P}$ with the boundary condition nor with N; $\xi_{\rm P}$ is a Poisson process on Λ with intensity measure $\overline{q}^{({\rm bc})}$ times the restriction ${\rm Leb}_\Lambda$ of the Lebesgue measure to Λ. By $\mathtt{Q}^{({\rm bc})}$ and $\mathtt{E}^{({\rm bc})}$ we denote probability and expectation with respect to this process. Conditionally on $\xi_{\rm P}$, the lengths of the cycles B_x with $x\in\xi_{\rm P}$ are independent and have distribution $(q_k^{({\rm bc})}/\overline{q}^{({\rm bc})})_{k\in\{1,\dots,N\}}$; this process has only marks with lengths $\le N$. A cycle B_x of length k is distributed according to

$$\mathbb{P}_{x,x}^{({\rm bc},k\beta)}({\rm d}f)=\frac{\mu_{x,x}^{({\rm bc},k\beta)}({\rm d}f)}{g_{k\beta}^{({\rm bc})}(x,x)}. \tag{1.13}$$

We now formulate our first main result, a presentation of the partition function defined in (1.1) in $\Lambda\subset\mathbb{R}^d$ with $|\Lambda|<\infty$ and boundary condition ${\rm bc}\in\{\varnothing,{\rm per},{\rm Dir}\}$. We write $\langle P,F\rangle$ for the expectation of a function F with respect to a probability measure P. We introduce a functional on Ω that expresses the interaction between particles in $\Lambda\subset\mathbb{R}^d$, more precisely, between their marks. Define the *Hamiltonian* $H_\Lambda\colon\Omega\to[0,\infty]$ by

$$H_\Lambda(\omega)=\sum_{x,y\in\xi\cap\Lambda}T_{x,y}(\omega),\qquad \text{where } \omega=\sum_{x\in\xi}\delta_{(x,f_x)}\in\Omega, \tag{1.14}$$

where we abbreviate

$$T_{x,y}(\omega)=\frac{1}{2}\sum_{i=0}^{\ell(f_x)-1}\sum_{j=0}^{\ell(f_y)-1}\mathbb{1}_{\{(x,i)\neq(y,j)\}}\int_0^\beta v(|f_x(i\beta+s)-f_y(j\beta+s)|)\,\mathrm{d}s,\qquad \omega\in\Omega,x,y\in\xi. \tag{1.15}$$

The function $H_\Lambda(\omega)$ summarises the interaction between different marks of the point process and between different legs of the same mark; here we call the restriction of a mark f_x to the interval $[i\beta,(i+1)\beta)]$ with $i\in\{0,\dots,\ell(f_x)-1\}$ a leg of the mark. Denote by

$$N_\Lambda^{(\ell)}(\omega)=\sum_{x\in\xi\cap\Lambda}\ell(f_x) \tag{1.16}$$

the total length of the marks of the particles in $\Lambda\subset\mathbb{R}^d$ (whose marks may be not contained in Λ).

Proposition 1.1 (Rewrite in terms of the marked Poisson process)**.** *Fix $\beta\in(0,\infty)$. Let $v\colon[0,\infty)\to(-\infty,\infty]$ be measurable and bounded from below and let $\Lambda\subset\mathbb{R}^d$ be measurable with finite volume (assumed to be a torus for periodic boundary condition). Then, for any $N\in\mathbb{N}$, and* $\mathrm{bc}\in\{\varnothing,\mathrm{per},\mathrm{Dir}\}$,

$$Z_N^{(\mathrm{bc})}(\beta,\Lambda)=\mathrm{e}^{|\Lambda|\overline{q}^{(\mathrm{bc})}}\mathtt{E}^{(\mathrm{bc})}\big[\mathrm{e}^{-H_\Lambda(\omega_{\mathrm{P}})}\mathbb{1}\{N_\Lambda^{(\ell)}(\omega_{\mathrm{P}})=N\}\big]. \tag{1.17}$$

That is, up the non-random term $|\Lambda|\overline{q}^{(\mathrm{bc})}$, the partition function is equal to the expectation over the Boltzmann factor e^{-H_Λ} of a marked Poisson process with fixed total length of marks of the particles.

1.3. **The limiting free energy.** In this section, we present our major result, the identification of the limiting free energy defined in (1.3) in terms of an explicit variational formula, see Theorem 1.2. We first introduce some notation.

Define the shift operator $\theta_y\colon\mathbb{R}^d\to\mathbb{R}^d$ as $\theta_y(x)=x-y$. We extend it to a shift operator on marked configurations by

$$\theta_y(\omega)=\sum_{x\in\xi}\delta_{(x-y,f_x)}=\sum_{x\in\xi-y}\delta_{(x,f_{x+y})},\qquad\text{for }\omega=\sum_{x\in\xi}\delta_{(x,f_x)}.$$

By $\mathcal{P}_\theta$ we denote the set of all shift-invariant probability measures on Ω. The distribution $\mathtt{Q}$ of the above marked Poisson point reference process ω_{P} belongs to $\mathcal{P}_\theta$.

Define $\Phi_\beta\colon\Omega\to[0,\infty]$ by

$$\Phi_\beta(\omega)=\sum_{x\in\xi\cap U}\sum_{y\in\xi}T_{x,y}(\omega), \tag{1.18}$$

where $T_{x,y}(\omega)$ was defined in (1.15), and $U=[-\frac{1}{2},\frac{1}{2}]^d$ denotes the centred unit box. The quantity $\Phi_\beta(\omega)$ describes all the interactions between different legs of marks of ω, when at least one of the marks is attached to a point in U.

Next, we introduce an entropy term. For probability measures μ,ν on some measurable space, we write

$$H(\mu\mid\nu)=\begin{cases}\int f\log f\,\mathrm{d}\nu & \text{if } f=\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\text{ exists,}\\ \infty & \text{otherwise,}\end{cases} \tag{1.19}$$

for the relative entropy of μ with respect ν. It will be clear from the context which measurable space is used. It is easy to see and well-known that $H(\mu\mid\nu)$ is nonnegative and that it vanishes if and only if $\mu=\nu$. Now we set

$$I_\beta(P)=\lim_{N\to\infty}\frac{1}{|\Lambda_N|}H\Big(P_{\Lambda_N}\,\Big|\,\mathtt{Q}_{\Lambda_N}\Big),\qquad P\in\mathcal{P}_\theta, \tag{1.20}$$

where we write P_Λ for the projection of P to Λ, i.e., the image measure of P under

$$\omega\mapsto\omega|_\Lambda=\sum_{x\in\xi\cap\Lambda}\delta_{(x,f_x)},\qquad\text{for }\omega=\sum_{x\in\xi}\delta_{(x,f_x)}. \tag{1.21}$$

The limit in (1.20) is along centred boxes Λ_N with diverging volume. According to [GZ93, Prop. 2.6], the limit in (1.20) exists, and I_β is a lower semicontinuous function with compact level sets in the topology of local convergence, see Lemma 3.3 below. It turns out there that I_β is the rate function of a crucial large-deviations principle for the family of the stationary empirical fields, which is one of the important objects of our analysis and will be introduced at the beginning of Section 3.

Now we introduce two important variational formulas. For any $\beta,\rho\in(0,\infty)$, define

$$\chi^{(\leq)}(\beta,\rho) = \inf\Big\{I_\beta(P)+\langle P,\Phi_\beta\rangle\colon P\in\mathcal{P}_\theta, \langle P,N_U^{(\ell)}\rangle\leq\rho\Big\}, \tag{1.22}$$

$$\chi^{(=)}(\beta,\rho) = \inf\Big\{I_\beta(P)+\langle P,\Phi_\beta\rangle\colon P\in\mathcal{P}_\theta, \langle P,N_U^{(\ell)}\rangle=\rho\Big\}. \tag{1.23}$$

These formulas range over shift-invariant marked processes P. They have three components: the entropic distance $I_\beta(P)$ between P and the reference process $\mathtt{Q}$, the interaction term $\langle P,\Phi_\beta\rangle$ and the condition $\langle P,N_U^{(\ell)}\rangle=\rho$, respectively $\leq\rho$. Obviously, $\chi^{(\leq)}\leq\chi^{(=)}$. Since all the maps $P\mapsto I_\beta(P)$, $P\mapsto\langle P,\Phi_\beta\rangle$ and $P\mapsto\langle P,N_U^{(\ell)}\rangle$ are easily seen to be lower semicontinuous and since the level sets of I_β are compact, it is clear that the infimum on the right-hand side of (1.22) is attained and is therefore a minimum. However, this is not at all clear for (1.23); this question lies much deeper and has some relation to the question about Bose-Einstein condensation, see the discussion in Section 1.4.

Now we specify our assumptions on the particle interaction potential v.

Assumption (v): *We assume that $v\colon[0,\infty)\to[0,\infty]$ is measurable and tempered, that is, there are $h>d, A\geq 0$ and $R_0>0$ such that $v(t)\leq At^{-h}$ for $t\in[R_0,\infty)$. Additionally, we assume that the integral*

$$\alpha(v)=\int_{\mathbb{R}^d}v(|x|)\,\mathrm{d}x$$

is finite and that $\liminf_{r\to 0}v(r)>0$.

We now present variational characterisations for upper and lower bounds for the exponential rate of the partition function. We denote by $\Lambda_L=[-\frac{L}{2},\frac{L}{2}]^d$ the centred box in $\mathbb{R}^d$ with volume L^d.

Theorem 1.2. *Let $L_N=(\frac{N}{\rho})^{1/d}$, such that Λ_{L_N} has volume N/ρ. Let v satisfy Assumption (v). Denote*

$$\mathcal{D}_v=\Big\{(\beta,\rho)\in(0,\infty)^2\colon(4\pi\beta)^{-d/2}\geq\rho\mathrm{e}^{\beta\rho\alpha(v)}\Big\}. \tag{1.24}$$

Then, for any $\beta,\rho\in(0,\infty)$, and for $\mathrm{bc}\in\{\varnothing,\mathrm{Dir},\mathrm{per}\}$,

$$\limsup_{N\to\infty}\frac{1}{|\Lambda_{L_N}|}\log Z_N^{(\mathrm{bc})}(\beta,\Lambda_{L_N}) \leq \frac{\zeta(1+\frac{d}{2})}{(4\pi\beta)^{d/2}}-\chi^{(\leq)}(\beta,\rho), \tag{1.25}$$

$$\liminf_{N\to\infty}\frac{1}{|\Lambda_{L_N}|}\log Z_N^{(\mathrm{bc})}(\beta,\Lambda_{L_N}) \geq \frac{\zeta(1+\frac{d}{2})}{(4\pi\beta)^{d/2}}-\begin{cases}\chi^{(\leq)}(\beta,\rho) & \text{if } (\beta,\rho)\in\mathcal{D}_v,\\ \chi^{(=)}(\beta,\rho) & \text{if } (\beta,\rho)\notin\mathcal{D}_v,\end{cases} \tag{1.26}$$

where $\zeta(m)=\sum_{k=1}^\infty k^{-m}$ denotes the Riemann zeta function.

Note that the first term on the right, $\zeta(1+\frac{d}{2})/(4\pi\beta)^{d/2}$, is equal to the total mass $\overline{q}$, the sum of the q_k defined in (1.8). The proof of Theorem 1.2 is in Sections 3.2 (proof of (1.25)) and 3.3 (proof of (1.26)) for empty boundary conditions, and in Section 3.4 for the other two.

The assumptions $\int_{\mathbb{R}^d}v(|x|)\,\mathrm{d}x<\infty$ and $\liminf_{r\to 0}v(r)>0$ are only necessary for our proof of the lower bound in (1.26). In the proof of the upper bound in (1.25), it is allowed that v takes the value $+\infty$ on a set of positive measure (corresponding to hard core repulsion) and also that $v\equiv 0$ (the non-interacting case); see discussion in Section 1.5.

As an obvious corollary we now identify the free energy defined in (1.3) in the high temperature phase and in the low-density phase.

Corollary 1.3 (**Free energy**). *Fix* $(\beta,\rho)\in\mathcal{D}_v$. *Then, for any* $\mathrm{bc}\in\{\varnothing,\mathrm{Dir},\mathrm{per}\}$*, the free energy introduced in* (1.3) *is given by*

$$f(\beta,\rho)=f^{(\mathrm{bc})}(\beta,\rho)=-\frac{1}{\beta}\frac{\zeta(1+\frac{d}{2})}{(4\pi\beta)^{d/2}}+\frac{1}{\beta}\min\Big\{I_\beta(P)+\langle P,\Phi_\beta\rangle\colon P\in\mathcal{P}_\theta,\langle P,N_U^{(\ell)}\rangle\leq\rho\Big\}. \tag{1.27}$$

A by-product of the proof of the lower bound of (1.26), see Corollary 3.5, we have the following upper bound on the free energy.

Lemma 1.4. *For any* $\beta,\rho\in(0,\infty)$*, and for* $\mathrm{bc}\in\{\varnothing,\mathrm{Dir},\mathrm{per}\}$*,*

$$f^{(\mathrm{bc})}(\beta,\rho)=\limsup_{N\to\infty}-\frac{1}{\beta}\frac{1}{|\Lambda_{L_N}|}\log Z_N^{(\mathrm{bc})}(\beta,\Lambda_{L_N})\leq\frac{\rho}{\beta}\log\Big(\rho(4\pi\beta)^{\frac{d}{2}}\Big)+\rho^2\alpha(v). \tag{1.28}$$

1.4. **Relevance and discussion.** One of the most prominent open problem in mathematical physics is the understanding of *Bose-Einstein condensation (BEC)*, a phase transition in a mutually repellent many-particle system at positive, fixed particle density, if a sufficiently low temperature is reached. That is, a macroscopic part of the system condenses to a state which is highly correlated and coherent. The first experimental realization of BEC was only in 1995, and it has been awarded with a Nobel prize. In spite of an enormous research activity, this phase transition has withstood a mathematical proof yet. Only partial successes have been achieved, like the description of the free energy of the ideal, i.e., non-interacting, system (already contained in Bose's and Einstein's seminal paper in 1925) or the analysis of mean-field models (e.g. [T90, DMP05]) or the analysis of dilute systems at vanishing temperature [LSSY05] or the proof of BEC in lattice systems with half-filling [LSSY05]. However, the original problem for fixed positive particle density and temperature is still waiting for a promising attack. Not even a tractable formula for the limiting free energy was known yet that could serve as a basis for a proof of BEC. The main purpose of the present paper is to provide such a formula.

The mathematical description of bosons is in terms of the symmetrised trace of the negative exponential of the corresponding Hamiltonian times the inverse temperature. The symmetrisation creates long range correlations of the interacting particles making the analysis an extremely challenging endeavour. The Feynman-Kac formula gives, in a natural way, a representation in terms of an expansion with respect to the cycles of random paths. It is conjectured by Feynman [Fe53] that BEC is signalled by the decisive appearance of a macroscopic amount of ‘infinite’ cycles, i.e., cycles whose lengths diverge with the number of particles. This phenomenon is also signalled by a loss of probability mass in the distribution of the ‘finite’ cycles. See [Sü93] and [Sü02] for proofs of this coincidence in the ideal Bose gas and some mean-field models. A different line of research is studying the effect of the symmetrisation in random permutation and random partition models, see [Ver96], [BCMP05], [AD08, AK08, A09], or in spatial random permutation models going back to [F91] and extended in [BU09].

In the present paper, we address the original problem of a mutually repellent many-particle system at fixed positive particle density and temperature and derive an explicit variational expression for the limiting free energy. More precisely, we prove upper and lower bounds, which coincide in the high-temperature phase respectively low density phase. The formula yields deep inside in the cycle structure of the random paths appearing in the Feynman-Kac formula. In particular, it opens up a new way to analyse the structure of the cycles at any temperature and density, also in the low-temperature phase, where our two bounds differ. In future work, we intend to analyse the conjectured phase transition in that variational formula and to link it to BEC.

The methods used in the present paper are mainly probabilistic. Our starting point is the well-known Feynman-Kac formula, which translates the partition function in terms of an expectation over a large symmetrised system of interacting Brownian bridge paths. In a second step, which is also well-known, we reduce the combinatorial complexity by concatenating the bridges, using the symmetrisation. The novelty of the present approach is a reformulation of this system in terms of an expectation with respect to a *marked Poisson point process*, which serves as a reference process. This is a Poisson process in the space $\mathbb{R}^d$ to whose particles we attach cycles called marks, starting and ending at that particle. The symmetrisation is reflected by an *a priori* distribution of cycle lengths. The interaction between the Brownian particles are encoded as interaction between the marks in an exponential functional. The particle density is described by a condition on the total length of the marks in the unit box.

Approaches to Bose gases using point processes have occasionally been used in the past (see [F91] and the references therein) and also recently in [Raf09], but systems with interactions have not yet been considered using this technique, to the best of our knowledge.

The greatest adavantage of this approach is that it is amenable to a large-deviations analysis. The central object here is the *stationary empirical field* of the marked point process, which contains all relevant information and satisfies a large-deviations principle in the thermodynamic limit. For some class of interacting systems, this direction of research was explored in [GZ93, G94]. In the present paper, we apply these ideas to the more difficult case of the interacting Bose gas. The challenge here is that the interaction involves the spatial points and the details of the marks. Modulo some error terms, we express the interaction and the mark length condition in terms of a functional of the stationary empirical field. Formally using Varadhan's lemma, we obtain a variational formula in the limit.

However, due to a lack of continuity in the functionals that describe the interaction and the mark lengths, the upper and lower bounds derived in this way, may differ in general. (At sufficiently high temperature, we overcome this problem by additional efforts and establish a formula for the limit.) This effect is not a technical drawback of the method, but lies at the heart of BEC.

In Theorem 1.2, we formulate the limiting free energy in terms of a minimising problem for random shift-invariant marked point processes with interaction under a constraint on the total length of the marks per unit volume. Both formulas in our upper and lower bounds in Theorem 1.2 are formulated in terms of random point fields having *finitely long* cycles as marks. The concept used in the present paper is not able to incorporate infinitely long cycles nor to quantify their contribution to the interaction. In the proof of our lower bound of the free energy, we drop the interactions involving any cycle longer than a parameter R that is eventually sent to infinity, and in our proof of the upper bound we even drop these cycles in the probability space. As a result, our two formulas register only 'finitely long' cycles. Their total macroscopic contribution is represented by the term $\langle P, N_U^{(\ell)}\rangle$, and the one of the 'infinitely long' cycles by the term $\rho - \langle P, N_U^{(\ell)}\rangle$. In this way, the long cycles are only indirectly present in our analysis: in terms of a 'loss of mass', the difference between the particle density ρ and the total mass of short cycles. Physically speaking, this difference is the total mass of a *condensate* of the particles.

The values of the two formulas $\chi^{(\le)}(\beta,\rho)$ and $\chi^{(=)}(\beta,\rho)$ differ if 'infinitely long' cycles do have some decisive contribution in the sense that the optimal point process(es) P in $\chi^{(\le)}(\beta,\rho)$ satisfies $\langle P, N_U^{(\ell)}\rangle < \rho$. We conjecture that the question whether or not the optimal P in $\chi^{(\le)}(\beta,\rho)$ has a loss of probability mass of infinitely long cycles is intimately related with the question whether or not $\chi^{(\le)}(\beta,\rho) = \chi^{(=)}(\beta,\rho)$ and that this question is in turn decisively connected with the question whether or not BEC appears. This is in accordance with Sütő's work [Sü93, Sü02]. The conjecture is that, for given β and in $d \ge 3$, if ρ is sufficiently small, then it is satisfied, and for sufficiently large ρ it is not satisfied. The latter phase is conjectured to be the BEC phase. Future work will be devoted to an analysis of this question.

Here is an abstract sufficient criterion for $\chi^{(\le)}(\beta,\rho) = \chi^{(=)}(\beta,\rho)$.

Lemma 1.5. *Fix $\beta \in (0,\infty)$. If there exists a minimiser $\widehat{P}$ of the variational problem $\inf_{P\in\mathcal{P}_\theta}(I_\beta(P) + \langle P, \Phi_\beta\rangle)$ satisfying $\widehat{\rho} := \langle \widehat{P}, N_U^{(\ell)}\rangle < \infty$, then, for any $\rho \in (0,\widehat{\rho})$,*

$$\chi^{(\le)}(\beta,\rho) = \chi^{(=)}(\beta,\rho). \tag{1.29}$$

Proof. Pick $\rho < \widehat{\rho}$. Let P be a minimiser in the formula for $\chi^{(\leq)}(\beta,\rho)$, i.e., of $\inf\{I_\beta(P)+\Phi_\beta(P)\colon P \in \mathcal{P}_\theta, \langle P, N_U^{(\ell)}\rangle \leq \rho\}$. If $\langle P, N_U^{(\ell)}\rangle$ would be smaller than ρ, then an appropriate convex combination, $\widetilde{P}$, of P and $\widehat{P}$ would satisfy $\langle \widetilde{P}, N_U^{(\ell)}\rangle \in (\langle P, N_U^{(\ell)}\rangle, \rho]$ and $I_\beta(\widetilde{P}) + \Phi_\beta(\widetilde{P}) < I_\beta(P) + \Phi_\beta(P)$. This would contradict the minimising propery of P. Hence, $\langle P, N_U^{(\ell)}\rangle = \rho$, and therefore P minimises also the formula for $\chi^{(=)}(\beta,\rho)$. □

1.5. **The non-interacting case.** Let us compare our results to the non-interacting case. Indeed, [A09, Thm. 2.1] says that, in the case $v \equiv 0$, the identification of the limiting free energy in (1.27) holds for *any* $\beta, \rho \in (0,\infty)$. To see this, we have to argue a bit, and we will only sketch the argument.

Explicitly, after some elementary manipulations, one sees that [A09, Thm. 2.1] amounts to

$$f(\beta,\rho) = -\frac{1}{\beta}\frac{\zeta(1+\frac{d}{2})}{(4\pi\beta)^{d/2}} + \frac{1}{\beta}\inf_{\lambda\in\ell^1(\mathbb{N})\colon \sum_k k\lambda_k \leq 1} J(\lambda), \tag{1.30}$$

where we recall that q was defined in (1.8), and we put

$$J(\lambda) = \sum_{k\in\mathbb{N}} q_k + \rho H(\lambda \mid q) + \rho\sum_{k\in\mathbb{N}}\lambda_k \log\rho - \rho\sum_{k\in\mathbb{N}}\lambda_k.$$

Now we rewrite the minimum on the right-hand side of (1.27) in a similar form by splitting $N_U^{(\ell)}$ into $\sum_{k\in\mathbb{N}} k\mathcal{N}_k$, where

$$\mathcal{N}_{k,\Lambda}(\omega) = \#\{x \in \xi\cap\Lambda\colon \ell(f_x) = k\} \tag{1.31}$$

and $\mathcal{N}_k = \mathcal{N}_{k,U}$ is the number of particles in the unit box U whose cycles have length k (and are allowed to leave U). Then we may write

$$\inf\Big\{I_\beta(P)\colon P\in\mathcal{P}_\theta, \langle P, N_U^{(\ell)}\rangle \leq \rho\Big\} = \inf_{\lambda\in\ell^1(\mathbb{N})\colon \sum_k k\lambda_k\leq 1}\ \inf_{P\in\mathcal{P}_\theta\colon \lambda(P)=\lambda} I_\beta(P),$$

where $\lambda(P) = \frac{1}{\rho}(\langle P, \mathcal{N}_k\rangle)_{k\in\mathbb{N}}$. In order to see that (1.30) coincides with (1.27) for $v = 0$, one only has to check that $J(\lambda) = \inf_{P\in\mathcal{P}_\theta\colon \lambda(P)=\lambda} I_\beta(P)$ for any $\lambda \in \ell^1(\mathbb{N})$ satisfying $\sum_k k\lambda_k \leq 1$.

We do not offer an analytical proof of this fact, but instead a probabilistic one, which makes use of the large-deviations principle in Lemma 3.3 below for the stationary empirical field $\mathfrak{R}_{\Lambda_L,\omega_{\rm P}}$ introduced in (3.2) with rate function I_β. Observe that the mapping $P \mapsto \lambda(P)$ is continuous as a function from the set of all $P \in \mathcal{P}_\theta$ satisfying $\langle P, N_U^{(\ell)}\rangle \leq \rho$ into the sequence space $\ell^1(\mathbb{N})$. Hence, by the contraction principle (see [DZ98, Thm. 4.2.1]), the sequence $(\lambda(\mathfrak{R}_{\Lambda_L,\omega_{\rm P}}))_{L>0}$ satisfies a large-deviations principle with rate function $\lambda \mapsto \inf_{P\in\mathcal{P}_\theta\colon \lambda(P)=\lambda} I_\beta(P)$. By uniqueness of rate functions, it suffices to show that this sequence satisfies the principle with rate function J. We now indicate how to derive this by explicit calculation.

Introduce

$$M_\Lambda = \Big\{\lambda \in [0,1]^{\mathbb{N}}\colon \sum_k k\lambda_k \leq 1, \forall k\in\mathbb{N}\colon \lambda_k|\Lambda|\rho \in \mathbb{N}_0\Big\},$$

and for $\lambda \in M_\Lambda$, we calculate

$$\mathsf{Q}\Big(\lambda(\mathfrak{R}_{\Lambda,\omega_{\rm P}}) = \lambda\Big) = \mathsf{Q}\Big(\forall k\in\mathbb{N}\colon \langle \mathfrak{R}_{\Lambda,\omega_{\rm P}}, \mathcal{N}_k\rangle = \rho\lambda_k\Big) = \mathsf{Q}\Big(\forall k\in\mathbb{N}\colon \#(\xi_{\rm P}^{(k)}\cap\Lambda) = \rho|\Lambda|\lambda_k\Big),$$

where $\xi_{\rm P}^{(k)} = \{x \in \xi_{\rm P}\colon f_x \in \mathcal{C}_k\}$ is the set of those Poisson points with cycle of length k. Since the Poisson processes $\xi_{\rm P}^{(k)}$, $k\in\mathbb{N}$, are independent with intensity q_k, we can proceed with

$$\mathsf{Q}\Big(\lambda(\mathfrak{R}_{\Lambda,\omega_{\rm P}}) = \lambda\Big) = \prod_{k\in\mathbb{N}} \mathsf{Q}\big(\#(\xi_{\rm P}^{(k)}\cap\Lambda) = \rho|\Lambda|\lambda_k\big) = \prod_{k\in\mathbb{N}}\Big({\rm e}^{-|\Lambda|q_k}\frac{(|\Lambda|q_k)^{\rho|\Lambda|\lambda_k}}{(\rho|\Lambda|\lambda_k)!}\Big).$$

Using Stirling's formula, we get from here that

$$\frac{1}{|\Lambda_L|}\log\mathsf{Q}\Big(\lambda(\mathfrak{R}_{\Lambda_L,\omega_{\rm P}}) = \lambda\Big) \sim -J(\lambda), \qquad \lambda\in M_{\Lambda_L},\ \text{as } L\to\infty.$$

From here, it is easy to finish the proof of the large-deviations principle for $(\lambda(\mathfrak{R}_{\Lambda_L,\omega_{\rm P}}))_{L>0}$ with rate function J. This finishes the proof of (1.27) for *any* $\beta,\rho\in(0,\infty)$ in the noninteracting case $v\equiv 0$.

The well-known Bose-Einstein phase transition in the free energy was made explicit in the analysis of the right-hand side of (1.30) in [A09]. It was shown there that

$$f(\beta,\rho)=-\frac{1}{\beta}\frac{1}{(4\pi\beta)^{d/2}}\times\begin{cases}\sum_{k\in\mathbb{N}}\frac{{\rm e}^{-\alpha k}}{k^{d/2+1}}+(4\pi\beta)^{d/2}\rho\alpha & \text{for } \rho(4\pi\beta)^{d/2}<\zeta(\frac{d}{2}),\\ \zeta(1+\frac{d}{2}) & \text{for } \rho(4\pi\beta)^{d/2}\geq\zeta(\frac{d}{2}),\end{cases}\tag{1.32}$$

where α is the unique root of $\rho=(4\pi\beta)^{-d/2}\sum_{k\in\mathbb{N}}\frac{{\rm e}^{-\alpha k}}{k^{d/2}}$. Note that $\zeta(\frac{d}{2})=\infty$ in $d\in\{1,2\}$, hence there is no phase transition in these dimensions. The first line in (1.32) corresponds to the case where the minimiser λ in (1.30) satifies $\sum_k k\lambda_k=1$, i.e., no 'infinitely long' cycles contribute to the free energy, and the second line to the case $\sum_k k\lambda_k<1$. Hence, the Bose-Einstein phase transition is precisely at the point where the variational formula in (1.30) with '$\leq$' starts differing from the formula with '$=$'.

2. Rewrite of the partition function

In this section, we give the proof of Proposition 1.1.

As a first step, we give a representation of $Z_N^{\rm (bc)}(\beta,\Lambda)$ in terms of an expansion with respect to the cycles of the permutations in (1.1). This is well-known and goes back to Feynman 1955.

We denote the set of all *integer partitions* of N by

$$\mathfrak{P}_N=\Big\{\lambda=(\lambda_k)_k\in\mathbb{N}_0^{\mathbb{N}}\colon\sum_k k\lambda_k=N\Big\}.\tag{2.1}$$

The numbers λ_k are called the *occupation numbers* of the integer partition λ. Any integer partition λ of N defines a conjugacy class of permutations of $1,\dots,N$ having exactly λ_k cycles of length k for any $k\in\mathbb{N}$. The term in (1.1) after the sum on σ depends only on this class. Hence, we replace this sum by a sum on integer partitions $\lambda\in\mathfrak{P}_N$ and count the permutations in that class. For any of these cycles of length k, we integrate out over all but one of the starting and terminating points of all the k Brownian bridges belonging to that cycle and use the Markov property to concatenate them. This gives the i-th (with $i=1,\dots,\lambda_k$) bridge $B^{(k,i)}$ with time horizon $[0,k\beta]$, starting and terminating at a site, which is uniformly distributed over Λ. The family of these bridges $B^{(k,i)}$ is independent, and $B^{(k,i)}$ has distribution $\mathbb{P}_\Lambda^{({\rm bc},k\beta)}$, where we define

$$\mathbb{P}_\Lambda^{({\rm bc},\beta)}({\rm d}f)=\frac{\int_\Lambda{\rm d}x\,\mu_{x,x}^{({\rm bc},\beta)}({\rm d}f)}{\int_\Lambda{\rm d}x\,g_\beta^{({\rm bc})}(x,x)}.\tag{2.2}$$

The expectation will be denoted by $\mathbb{E}_\Lambda^{({\rm bc},\beta)}$.

For $\lambda\in\mathfrak{P}_N$, define

$$\mathcal{G}_{N,\beta}^{(\lambda)}=\frac{1}{2}\sum_{k_1,k_2=1}^{N}\sum_{i_1=1}^{\lambda_{k_1}}\sum_{i_2=1}^{\lambda_{k_2}}\sum_{j_1=0}^{k_1-1}\sum_{j_2=0}^{k_2-1}1\!\!1_{(k_1,i_1,j_1)\neq(k_2,i_2,j_2)}\int_0^\beta{\rm d}s\,v\big(|B^{(k_1,i_1)}(j_1\beta+s)-B^{(k_2,i_2)}(j_2\beta+s)|\big).\tag{2.3}$$

In words, $\mathcal{G}^\lambda_{N,\beta}$ is the total interaction between different bridges $B^{(k_1,i_1)}$ and $B^{(k_2,i_2)}$ and between different legs of the same bridge $B^{(k,i)}$.

Lemma 2.1 (Cycle expansion)**.** *For any $N\in\mathbb{N}$,*

$$Z_N^{({\rm bc})}(\beta,\Lambda)=\sum_{\lambda\in\mathfrak{P}_N}\Big(\prod_{k\in\mathbb{N}}\frac{\big[\int_\Lambda{\rm d}x\,g_{k\beta}^{({\rm bc})}(x,x)\big]^{\lambda_k}}{\lambda_k!\,k^{\lambda_k}}\Big)\bigotimes_{k\in\mathbb{N}}\big(\mathbb{E}_\Lambda^{({\rm bc},k\beta)}\big)^{\otimes\lambda_k}\Big[{\rm e}^{-\mathcal{G}_{N,\beta}^{(\lambda)}}\Big].\tag{2.4}$$

Proof. We are going to split every permutation on the right-hand side of (1.1) into a product of its cycles. Assume that a permutation $\sigma \in \mathfrak{S}_N$ has precisely λ_k cycles of length k, for any $k \in \{1, \dots, N\}$. Then $\sum_{k=1}^N k\lambda_k = N$. The corresponding Brownian bridges may be renumbered $B_j^{(k,i)}$ with $k \in \mathbb{N}$, $i = 1, \dots, \lambda_k$ and $j = 1, \dots, k$. Then the measure $\int_\Lambda \mathrm{d}x_1 \dots \int_\Lambda \mathrm{d}x_N \bigotimes_{i=1}^N \mu_{x_i, x_{\sigma(i)}}^{(\mathrm{bc},\beta)}$ splits into an according product, which can be written, after a proper renumbering of the indices, as

$$\prod_{k=1}^N \prod_{i=1}^{\lambda_k} \prod_{j=0}^{k-1} \int_\Lambda \mathrm{d}x_{k,j+1}^{(i)} \bigotimes_{k\in\mathbb{N}} \bigotimes_{i=1}^{\lambda_k} \bigotimes_{j=0}^{k-1} \mu_{x_{k,j}^{(i)}, x_{k,j+1}^{(i)}}^{(\mathrm{bc},\beta)}, \qquad \text{where } x_{k,0}^{(i)} = x_{k,k}^{(i)}. \tag{2.5}$$

Denote by $f_1 \diamond \dots \diamond f_k$ the concatenation of $f_1, \dots, f_k$, i.e., $f_1 \diamond \dots \diamond f_k((i-1)\beta + s) = f_i(s)$ for $s \in [0, \beta]$. Note that the Markov property of the canonical Brownian bridge measures implies the concatenation formula

$$\mu_{x,x}^{(\mathrm{bc},k\beta)}(\mathrm{d}(f_1 \diamond \dots \diamond f_k)) = \int_{(\Lambda)^{k-1}} \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1} \bigotimes_{i=1}^k \mu_{x_{i-1},x_i}^{(\mathrm{bc},\beta)}(\mathrm{d}f_i), \qquad x_0 = x_k = x. \tag{2.6}$$

Now we integrate out over $x_{k,2}^{(i)}, \dots, x_{k,k}^{(i)}$ for any $k \in \mathbb{N}$ and $i = 1, \dots, \lambda_k$. In this way, we obtain that we may replace the bridges $B_j^{(k,i)}$ under the measure

$$\bigotimes_{k=1}^N \bigotimes_{i=1}^{\lambda_k} \Big(\int_\Lambda \mathrm{d}x_k^{(i)} \mu_{x_k^{(i)}, x_k^{(i)}}^{(\mathrm{bc},k\beta)} \Big)$$

by the bridges $B^{(k,i)} = B_1^{(k,i)} \diamond \dots \diamond B_k^{(k,i)}$ under the measure

$$\bigotimes_{k=1}^N \Big[\int_\Lambda \mathrm{d}x\, g_{k\beta}^{(\mathrm{bc})}(x,x) \Big]^{\lambda_k} \big(\mathbb{E}_\Lambda^{(\mathrm{bc},k\beta)}\big)^{\otimes \lambda_k}.$$

Summarising, we get

$$Z_N^{(\mathrm{bc})}(\beta,\Lambda) = \sum_{\lambda\in\mathfrak{P}_N} \frac{A(\lambda)}{N!} \prod_{k=1}^N \Big[\int_\Lambda \mathrm{d}x\, g_{k\beta}^{(\mathrm{bc})}(x,x) \Big]^{\lambda_k} \bigotimes_{k\in\mathbb{N}} \Big(\mathbb{E}_\Lambda^{(\mathrm{bc},k\beta)}\Big)^{\otimes\lambda_k} \Big[\mathrm{e}^{-\mathcal{G}_{N,\beta}^{(\lambda)}} \Big],$$

where $A(\lambda) = \#\big\{\sigma \in \mathfrak{S}_N : \sigma \text{ has } \lambda_k \text{ cycles of length } k, \forall k \in \mathbb{N}\big\}$ is size of the conjugacy class for the integer partition $\lambda \in \mathfrak{P}_N$. Standard counting arguments (see [C02, Th. 12.1]) give

$$A(\lambda) = \frac{N!}{\prod_{k=1}^N (\lambda_k! k^{\lambda_k})},$$

and conclude the proof. □

Now we explain our rewrite of the partition sum in terms of the marked Poisson point process introduced in Section 1.2, i.e., we prove Proposition 1.1. The main idea is to replace the sum over integer partitions in Lemma 2.1 by an expectation with respect to the marked Poisson point process under conditions on the mark events. We restrict to the case of empty boundary conditions; the other two require only notational changes.

It will be convenient to write the process ω_{P} as the superposition

$$\omega_{\mathrm{P}} = \sum_{k\in\mathbb{N}} \omega_{\mathrm{P}}^{(k)}, \qquad \text{where } \omega_{\mathrm{P}}^{(k)} = \sum_{x\in\xi_{\mathrm{P}}^{(k)}} \delta_{(x,B_x)}, \tag{2.7}$$

and $\omega_{\mathrm{P}}^{(k)}$ is the Poisson process on $\mathbb{R}^d \times \mathcal{C}_k$ with intensity measure ν_k defined in (1.7). The processes $\omega_{\mathrm{P}}^{(k)}$ are independent.

Proof of Proposition 1.1. We start from Lemma 2.1. Pick an integer partition $\lambda \in \mathfrak{P}_N$ with occupation number λ_k satisfying $\sum_{k=1}^N k\lambda_k = N$, and abbreviate the number of cycles of λ by $m =$

$\sum_{k=1}^N \lambda_k$. For any $k\in\mathbb{N}$, the family $(B^{(k,i)})_{i=1,\dots,\lambda_k}$ under the measure $(\mathbb{P}_\Lambda^{(k\beta)})^{\otimes\lambda_k}$ has the same distribution as the family of marks $(B_x)_{x\in\xi_{\rm P}^{(k)}}$ of the conditional Poisson process $\omega_{\rm P}^{(k)}$ given $\{\#(\xi_{\rm P}^{(k)}\cap\Lambda)=\lambda_k\}$. Considering the product measure $\bigotimes_{k\in\mathbb{N}}(\mathbb{P}_\Lambda^{(k\beta)})^{\otimes\lambda_k}$ is equivalent to considering the superposition of the conditional processes $\omega_{\rm P}^{(k)}$ with $k\in\mathbb{N}$.

Hence, we have precisely m Poisson points in Λ. For any $k\in\mathbb{N}$, conditional on $\{\#(\xi_{\rm P}^{(k)}\cap\Lambda)=\lambda_k\}$, the set $\xi_{\rm P}^{(k)}\cap\Lambda$ has the same distribution as the set of starting points, $\{B^{(k,1)}(0),\dots,B^{(k,\lambda_k)}(0)\}$. A comparison of (1.14)-(1.15) with (2.3) shows that the interaction term $\mathcal{G}_{N,\beta}^{(\lambda)}$ must be replaced by the Hamiltonian $H_\Lambda(\omega_{\rm P})$. Hence,

$$\bigotimes_{k\in\mathbb{N}}(\mathbb{E}_\Lambda^{(k\beta)})^{\otimes\lambda_k}\Big[{\rm e}^{-\mathcal{G}_{N,\beta}^{(\lambda)}}\Big]=\mathtt{E}\Big[{\rm e}^{-H_\Lambda(\omega_{\rm P})}\Big|\forall k\in\mathbb{N},\,\#(\xi_{\rm P}^{(k)}\cap\Lambda)=\lambda_k\Big].$$

We see in an elementary way that

$$\begin{aligned}&\mathtt{E}\Big[{\rm e}^{-H_\Lambda(\omega_{\rm P})}\Big|\forall k\in\mathbb{N},\,\#(\xi_{\rm P}^{(k)}\cap\Lambda)=\lambda_k\Big]\\&\quad=\mathtt{E}\Big[{\rm e}^{-H_\Lambda(\omega_{\rm P})}\mathbb{1}\{\forall k\in\mathbb{N},\,\#(\xi_{\rm P}^{(k)}\cap\Lambda)=\lambda_k\}\Big|\#(\xi_{\rm P}\cap\Lambda)=m\Big]\frac{\prod_{k\in\mathbb{N}}\lambda_k!}{m!}\overline{q}^m\prod_{k\in\mathbb{N}}(q_k)^{-\lambda_k},\end{aligned}\tag{2.8}$$

where $\overline{q}$ and the q_k are defined in (1.8). Let us summarise all the terms involving λ_k from (2.4) and (2.8) (noting that $g_\beta(x,x)=(4\pi\beta k)^{-\frac d2}$):

$$\Big(\prod_{k\in\mathbb{N}}\frac{(4\pi\beta k)^{-\frac d2\lambda_k}|\Lambda|^{\lambda_k}}{\lambda_k!\,k^{\lambda_k}}\Big)\times\frac{\prod_{k\in\mathbb{N}}\lambda_k!}{m!}\overline{q}^m\prod_{k\in\mathbb{N}}(q_k)^{-\lambda_k}=|\Lambda|^m\frac{\overline{q}^m}{m!}.$$

We denote by $\mathcal{N}_{k,\Lambda}(\omega)=\#\{x\in\Lambda\colon \ell(f_x)=k\}$ and $N_\Lambda(\omega)=\#(\xi\cap\Lambda)$ the number of particles in Λ (whose marks do not have to be contained in Λ) with mark length equal to k, respectively with arbitrary mark length. Then we get

$$Z_N(\beta,\Lambda)=\sum_{m=1}^N|\Lambda|^m\frac{\overline{q}^m}{m!}\sum_{\substack{\lambda\in\mathfrak{P}_N,\\\sum_k\lambda_k=m}}\mathtt{E}\Big[{\rm e}^{-H_\Lambda(\omega_{\rm P})}\mathbb{1}\{\forall k\in\mathbb{N},\,\mathcal{N}_{k,\Lambda}(\omega_{\rm P})=\lambda_k\}\Big|N_\Lambda(\omega_{\rm P})=m\Big].\tag{2.9}$$

Note that the event $\{N_\Lambda(\omega_{\rm P})=m\}$ has probability $|\Lambda|^m\frac{\overline{q}^m}{m!}\exp\{-|\Lambda|\overline{q}\}$. Hence

$$Z_N(\beta,\Lambda)={\rm e}^{|\Lambda|\overline{q}}\sum_{m=1}^N\sum_{\substack{\lambda\in\mathfrak{P}_N,\\\sum_k\lambda_k=m}}\mathtt{E}\Big[{\rm e}^{-H_\Lambda(\omega_{\rm P})}\mathbb{1}\{\forall k\in\mathbb{N},\,\mathcal{N}_{k,\Lambda}(\omega_{\rm P})=\lambda_k\}\mathbb{1}\{N_\Lambda(\omega_{\rm P})=m\}\Big].\tag{2.10}$$

Note that the events $\{\forall k\in\mathbb{N},\,\mathcal{N}_{k,\Lambda}(\omega_{\rm P})=\lambda_k\}\cap\{N_\Lambda(\omega_{\rm P})=m\}$ are a decomposition of the event $\{N_\Lambda^{(\ell)}(\omega_{\rm P})=N\}$. Hence, the assertion in (1.17) follows.

□

3. Large-deviations arguments: proof of Theorem 1.2

In this section we prove Theorem 1.2 by applying large-deviations arguments to the representation of the partition function in Proposition 1.1. In Sections 3.1–3.3 we carry out the proof for empty boundary condition, and in Section 3.4 we show how to trace the other two boundary conditions back to this case. In Section 3.1 we introduce the main object of our analysis, the stationary empirical field with respect to the marked Poisson process $\omega_{\rm P}$, and we rewrite the partition function in terms of this field. We also formulate and explain the main steps of the proof, among which the crucial large-deviations principle for that field. In Sections 3.2 and 3.3 we prove the upper and lower bounds, respectively, for empty boundary condition.

3.1. **The stationary empirical field.** Our analysis is based on a large-deviations principle for the *stationary empirical field*, defined as follows. For any $\xi \subset \mathbb{R}^d$ and for any centred box $\Lambda \subset \mathbb{R}^d$, let $\xi_{(\Lambda)}$ be the Λ-periodic continuation of $\xi \cap \Lambda$. Analogously, we define the Λ-periodic continuation of the restriction of the configuration ω to Λ as

$$\omega_{(\Lambda)} = \sum_{z\in\mathbb{Z}^d}\sum_{x\in\xi\cap\Lambda}\delta_{(x+Lz,f_x)} \qquad \text{if } \omega = \sum_{x\in\xi}\delta_{(x,f_x)} \in \Omega, \tag{3.1}$$

where L is the side length of the centered cube Λ. Then the stationary empirical field is given by

$$\mathfrak{R}_{\Lambda,\omega} = \frac{1}{|\Lambda|}\int_\Lambda \mathrm{d}y\,\delta_{\theta_y(\omega_{(\Lambda)})}, \qquad \omega\in\Omega, \tag{3.2}$$

where the shift operator $\theta_y\colon \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\theta_y(x) = x-y$. It is clear that $\mathfrak{R}_{\Lambda,\omega}$ is a shift-invariant probability measure on Ω, i.e., it is an element of $\mathcal{P}_\theta$.

Now we express $N_\Lambda^{(\ell)}(\omega)$ in terms of $\mathfrak{R}_{\Lambda,\omega}$. Recall that U denotes the centred unit box.; we write Λ_L for Λ.

Lemma 3.1. *For any centred box $\Lambda \subset \mathbb{R}^d$ with $|\Lambda| > 1$, and any $\omega\in\Omega$,*

$$|\Lambda|\big\langle \mathfrak{R}_{\Lambda,\omega}, N_U^{(\ell)}\big\rangle = N_\Lambda^{(\ell)}(\omega).$$

Proof. The assertion follows from [GZ93, Remark 2.3(1)], however we give a direct proof without using Palm measures. Let $L>1$ be such that $\Lambda = \Lambda_L = [-\frac{L}{2},\frac{L}{2}]^d$. We calculate

$$\begin{aligned}
|\Lambda|\big\langle \mathfrak{R}_{\Lambda,\omega}, N_U^{(\ell)}\big\rangle &= \int_\Lambda \mathrm{d}z\, N_U^{(\ell)}\big(\theta_z(\omega_{(\Lambda)})\big) = \sum_{x\in\xi_{(\Lambda)}}\int_\Lambda \mathrm{d}z\,\mathbb{1}_{U-x}(z)\ell(f_x)\\
&= \sum_{\substack{x\in\xi_{(\Lambda)}\\ x\in\Lambda+U}} \ell(f_x)|\Lambda\cap(U-x)|\\
&= N_\Lambda^{(\ell)}(\omega) + \sum_{x\in\xi_{(\Lambda)}\cap((\Lambda+U)\setminus\Lambda)} \ell(f_x)|\Lambda\cap(U-x)|\\
&\qquad + \sum_{x\in\xi\cap\Lambda}\ell(f_x)\big(|\Lambda\cap(U-x)|-1\big).
\end{aligned}$$

It remains to show that the sum of the two last sums is equal to zero. Note that the last sum can be restricted to $x\in\xi\cap(\Lambda\setminus\Lambda_{L-1})$. We use the fact that for each point $x\in\xi\cap(\Lambda\setminus\Lambda_{L-1})$ there exists a collection of points in $\xi_{(\Lambda)}\cap(\Lambda_{L+1}\setminus\Lambda)$, with the same mark of x. Indeed, there exists a positive integer $m(x)\le d$ and a set $\{x_1',\dots,x_{m(x)}'\}$, such that $x_i'\in\xi_{(\Lambda)}\cap(\Lambda+U)\setminus\Lambda$, $x_i' = x+Lz_i$ for some $z_i\in\mathbb{Z}^d$ and $\sum_{i=1}^{m(x)}|\Lambda\cap(U-x_i')| = 1-|\Lambda\cap(U-x)|$. Notice that

$$\bigcup_{x\in\xi\cap(\Lambda\setminus\Lambda_{L-1})}\ \bigcup_{i=1}^{m(x)} x_i' = \xi_{(\Lambda)}\cap((\Lambda+U)\setminus\Lambda),$$

and $f_x = f_{x_i'}$, for any $i\le m(x)$. Hence

$$\sum_{x\in\xi_{(\Lambda)}\cap((\Lambda+U)\setminus\Lambda)}\ell(f_x)|\Lambda\cap(U-x)| = \sum_{x\in\xi\cap\Lambda}\ell(f_x)\big(1-|\Lambda\cap(U-x)|\big).$$

□

Now we express the interaction Hamiltonian in terms of integrals of the stationary empirical field against suitable functions; more precisely, we give lower and upper bounds. In the following lower bound, it is important that this functional is local and bounded; this will be achieved up to a small error only.

Fix large truncation parameters M, R and K and introduce $\xi^{(\leq K)} = \{x \in \xi \colon \ell(f_x) \leq K\}$ for $\omega \in \Omega$ and

$$\Phi_\beta^{(R,M,K)}(\omega) = \sum_{x \in \xi^{(\leq K)} \cap U} \sum_{y \in \xi^{(\leq K)} \cap \Lambda_R} T_{x,y}^{(M)}(\omega), \tag{3.3}$$

where $\Lambda_R = [-\frac{R}{2}, \frac{R}{2}]^d$ and

$$T_{x,y}^{(M)}(\omega) = \frac{1}{2} \sum_{i=0}^{\ell(f_x)-1} \sum_{j=0}^{\ell(f_y)-1} 1\!\!1_{\{(x,i) \neq (y,j)\}} \int_0^\beta v_M(|f_x(i\beta + s) - f_y(j\beta + s)|)\,\mathrm{d}s,$$

and where $v_M(r) = (v \wedge M)(r) = \min\{v(r), M\}$. Recall that $N_\Lambda(\omega) = \#(\xi \cap \Lambda)$ denotes the particle number in a measurable set $\Lambda \subset \mathbb{R}^d$.

Lemma 3.2 (Hamiltonian bounds). *Fix any centred box $\Lambda = \Lambda_L$.*

(i) *For any $M, R, K, S \in (1, \infty)$, and for $L \geq R + 2$,*

$$H_\Lambda(\omega) \geq |\Lambda| \big\langle \mathfrak{R}_{\Lambda,\omega}, \Phi_\beta^{(R,M,K)} 1\!\!1\{N_{\Lambda_R} \leq S\} \big\rangle - C N_{\Lambda_L \setminus \Lambda_{L-R-2}}(\omega), \quad \omega \in \Omega, \tag{3.4}$$

where $C = 2^d \beta M K^2 r S$, and r depends only on R and d.

(ii)

$$H_\Lambda(\omega) \leq |\Lambda| \big\langle \mathfrak{R}_{\Lambda,\omega}, \Phi_\beta \big\rangle, \quad \omega \in \Omega, \tag{3.5}$$

Proof of (i). Estimate

$$\begin{aligned} |\Lambda| \big\langle \mathfrak{R}_{\Lambda,\omega}, \Phi_\beta^{(R,M,K)} 1\!\!1\{N_{\Lambda_R} \leq S\} \big\rangle &= \int_\Lambda \mathrm{d}z\, \Phi_\beta^{(R,M,K)}(\theta_z(\omega_{(\Lambda)})) 1\!\!1\{N_{\Lambda_R}(\theta_z(\omega_{(\Lambda)})) \leq S\} \\ &\leq \int_\Lambda \mathrm{d}z \sum_{x \in \xi_{(\Lambda)}^{(\leq K)} \cap (U - z)} \sum_{y \in \xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_R - z)} T_{x,y}^{(M)}(\omega_{(\Lambda)}) 1\!\!1\{\#(\xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_R - z)) \leq S\} \\ &= \sum_{\substack{x,y \in \xi_{(\Lambda)}^{(\leq K)}, x \in \Lambda + U, \\ y \in \Lambda + \Lambda_R, x \in \Lambda_{R+1} + y}} T_{x,y}^{(M)}(\omega_{(\Lambda)}) \int_{\Lambda \cap (U - x) \cap (\Lambda_R - y)} \mathrm{d}z\, 1\!\!1\{\#(\xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_R - z)) \leq S\}. \end{aligned} \tag{3.6}$$

Observe that the integral over z is not larger than one. Now we split the last sum into the sums on $(x, y) \in \Lambda^2$ and the remainder. For $(x, y) \in \Lambda^2$, we may replace $T_{x,y}^{(M)}(\omega_{(\Lambda)})$ by $T_{x,y}^{(M)}(\omega)$ and estimate it against $T_{x,y}(\omega)$. Hence,

$$\text{l.h.s. of (3.6)} \leq H_\Lambda(\omega) + \Psi_\Lambda^{(R,M,K,S)}(\omega),$$

where the remainder term is

$$\begin{aligned} &\Psi_\Lambda^{(R,M,K,S)}(\omega) \\ &= \sum_{\substack{x,y \in \xi_\Lambda^{(\leq K)}, x \in \Lambda + U, \\ y \in \Lambda + \Lambda_R, x \in \Lambda_{R+1} + y, (x,y) \notin \Lambda^2}} T_{x,y}^{(M)}(\omega_{(\Lambda)}) \int_{\Lambda \cap (U - x) \cap (\Lambda_R - y)} \mathrm{d}z\, 1\!\!1\{\#(\xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_R - z)) \leq S\} \\ &\leq \frac{1}{2} \beta M K^2 \sum_{\substack{x,y \in \xi_\Lambda^{(\leq K)}, x \in \Lambda + U, \\ y \in \Lambda + \Lambda_R, x \in \Lambda_{R+1} + y, (x,y) \notin \Lambda^2}} 1\!\!1\{\exists\, z \in \Lambda \cap (U - x) \cap (\Lambda_R - y) \colon \#(\xi_\Lambda^{(\leq K)} \cap (\Lambda_R - z)) \leq S\} \\ &\leq \frac{1}{2} \beta M K^2 \sum_{\substack{x,y \in \xi_{(\Lambda)}^{(\leq K)}, x \in \Lambda + U, \\ y \in \Lambda + \Lambda_R, x \in \Lambda_{R+1} + y, (x,y) \notin \Lambda^2}} 1\!\!1\{\#(\xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_{R-1} + x)) \leq S\}. \end{aligned}$$

The sum over $(x,y) \notin \Lambda^2$ is split into the sum over $x \in (\Lambda + U) \setminus \Lambda, y \in \Lambda + \Lambda_R$ and $x \in \Lambda + U, y \in (\Lambda + \Lambda_R) \setminus \Lambda$. Recall that $\Lambda = \Lambda_L$ and that $L \geq R+1$. The condition $x \in \Lambda_{R+1} + y$ implies that in both cases y is summed over a subset of $\Lambda_{L+R+2} \setminus \Lambda_{L-R-1}$. Hence,

$$\begin{aligned}\Psi_\Lambda^{(R,M,K,S)}(\omega) \leq \frac{1}{2}\beta M K^2 \\ \times \sum_{y \in \xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_{L+R+2} \setminus \Lambda_{L-R-1})} \#\{x \in \xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_{R+1} + y) \colon \#(\xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_{R-1} + x)) \leq S\}.\end{aligned}$$

Now we show that the counting factor is not larger than rS, where r depends only on R and the dimension d. Indeed, cover $\Lambda_{R+1} + y$ with r boxes $\Delta_1, \dots, \Delta_r$ of diameter $(R-1)/2$, then

$$\begin{aligned}\#\big\{x \in &\xi_\Lambda^{(\leq K)} \cap (\Lambda_{R+1} + y) \colon \#(\xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_{R-1} + x)) \leq S\big\} \\ &\leq \sum_{i=1}^r \#\{x \in \xi_{(\Lambda)}^{(\leq K)} \cap \Delta_i \colon \#(\xi_{(\Lambda)}^{(\leq K)} \cap (\Lambda_{R-1} + x)) \leq S) \\ &\leq \sum_{i=1}^r \#\{x \in \xi_{(\Lambda)}^{(\leq K)} \cap \Delta_i \colon \#(\xi_{(\Lambda)}^{(\leq K)} \cap \Delta_i) \leq S)\} \\ &\leq rS,\end{aligned}$$

since $\Delta_i \subset \Lambda_{R-1} + x$ if $x \in \Delta_i$. This gives

$$\Psi_\Lambda^{(R,M,K,S)}(\omega) \leq \frac{1}{2}\beta M K^2 r S N_{\Lambda_{L+R+2} \setminus \Lambda_{L-R-1}}(\omega_{(\Lambda)}) \leq 2^d \beta M K^2 r S N_{\Lambda_L \setminus \Lambda_{L-R-2}}(\omega),$$

and finishes the proof of (i).

Proof of (ii). In a similar way as in (3.6), one sees that, for any $\omega \in \Omega$,

$$\begin{aligned}|\Lambda|\big\langle \mathfrak{R}_{\Lambda,\omega}, \Phi_\beta\big\rangle &= \sum_{x,y \in \xi_{(\Lambda)}} T_{x,y}(\omega_{(\Lambda)})\, |\Lambda \cap (U - x)| \\ &= H_\Lambda(\omega) + \sum_{x,y \in \xi \cap \Lambda} T_{x,y}(\omega_{(\Lambda)})\big(|\Lambda \cap (U-x)| - 1\big) \\ &\qquad + \sum_{x,y \in \xi_{(\Lambda)} \colon x \in \Lambda_{L+1}, (x,y) \notin \Lambda^2} T_{x,y}(\omega_{(\Lambda)})\, |\Lambda \cap (U-x)|.\end{aligned} \tag{3.7}$$

It remains to show that the sum of the two last sums is nonnegative. Note that the sum on x in the first sum may be restricted to $x \in \xi \cap (\Lambda \setminus \Lambda_{L-1})$. For each such x and for any $y \in \xi \cap \Lambda$, there exist a positive integer $m(x) \leq d$ and a set $\{x_1', y_1', \dots, x_{m(x)}', y_{m(x)}'\}$, such that $x_i' \in \xi_{(\Lambda)} \cap \Lambda_{L+1}$, $x_i' = x + Lz_i$ and $y_i' = y + Lz_i$ for some $z_i \in \mathbb{Z}^d$, and

$$\sum_{i=1}^{m(x)} |\Lambda \cap (U - x_i')| = |\Lambda \cap (U - x)| - 1.$$

Then $T_{x,y}(\omega_{(\Lambda)}) = T_{x',y'}(\omega_{(\Lambda)})$ by Λ-periodicity of $\omega_{(\Lambda)}$. This shows that the sum of the two last sums in (3.7) is nonnegative, which finishes the proof of (ii). □

Recall that $L_N = (N/\rho)^d$. Applying Lemmas 3.1 and 3.2(i) to the representation in Proposition 1.1, we obtain, for any $R, M, K, S > 0$, the upper bound

$$\begin{aligned}Z_N(\beta, \Lambda_{L_N}) \leq \mathrm{e}^{|\Lambda_{L_N}|\overline{q}}\, \mathrm{E}\Big[\exp\Big\{-|\Lambda_{L_N}|\big\langle \mathfrak{R}_{\Lambda_{L_N},\omega_\mathrm{P}}, \Phi_\beta^{(R,M,K)} 1\!\!1\{N_{\Lambda_R} \leq S\}\big\rangle\Big\} \\ \times \exp\Big\{C N_{\Lambda_{L_N} \setminus \Lambda_{L_N - R-2}}(\omega_\mathrm{P})\Big\} 1\!\!1\{\langle \mathfrak{R}_{\Lambda_{L_N},\omega_\mathrm{P}}, N_U^{(\ell)}\rangle = \rho\}\Big], \qquad N \in \mathbb{N},\end{aligned} \tag{3.8}$$

and, using Lemmas 3.1 and 3.2(ii), the lower bound

$$Z_N(\beta,\Lambda_{L_N})\geq \mathrm{e}^{|\Lambda_{L_N}|\overline{q}}\,\mathtt{E}\Big[\mathrm{e}^{-|\Lambda_{L_N}|\langle \mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}},\Phi_\beta\rangle}\mathbb{1}\{\langle \mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}},N_U^{(\ell)}\rangle=\rho\}\Big],\qquad N\in\mathbb{N}. \tag{3.9}$$

The main point of introducing the stationary empirical field is that the family $(\mathfrak{R}_{\Lambda_L,\omega_{\rm P}})_{L>0}$ satisfies a large-deviations principle on $\mathcal{P}_\theta$, which is known from the work by Georgii and Zessin. On $\mathcal{P}_\theta$ we consider the following topology. A measurable function $g\colon\Omega\to\mathbb{R}$ is called *local* if it depends only on the restriction of ω to some bounded open cube, and it is called *tame* if $|g|\leq c(1+N_\Lambda)$ for some bounded open cube Λ and some constant $c\in\mathbb{R}^+$. We endow the space $\mathcal{P}_\theta$ with the topology $\tau_{\mathcal{L}}$ of *local convergence*, defined as the smallest topology on $\mathcal{P}_\theta$ such that the mappings $P\mapsto\langle P,g\rangle$ are continuous for any $g\in\mathcal{L}$, where $\mathcal{L}$ denotes the linear space of all local tame functions. It is clear that the map $P\mapsto\langle P,N_U\rangle$ is $\tau_{\mathcal{L}}$-continuous; however, the map $P\mapsto\langle P,N_U^{(\ell)}\rangle$ is only lower semicontinuous.

Lemma 3.3 (Large deviations for $\mathfrak{R}_{\Lambda_L,\omega_{\rm P}}$)**.** *The family of measures $\mathfrak{R}_{\Lambda_L,\omega_{\rm P}}$ satisfies, as $L\to\infty$, a large-deviations principle in the topology $\tau_{\mathcal{L}}$ with speed $|\Lambda_L|$ and rate function $I_\beta\colon\mathcal{P}_\theta\to[0,\infty]$ defined in* (1.20)*. The function I_β is affine and lower $\tau_{\mathcal{L}}$-semicontinuous and has $\tau_{\mathcal{L}}$-compact level sets.*

Proof. This is [GZ93, Theorem 3.1]. □

Our goal is to apply Varadhan's lemma to the expectations on the right hand sides of (3.8) and (3.9). In conjunction with the large-deviations principle of Lemma 3.3, this formally suggests that both (1.25) and (1.26) should be valid, as we explain now. Indeed, first consider (3.9) and note that the map $P\mapsto\langle P,\Phi_\beta\rangle$ has the proper continuity property for the application of the lower bound half of Varadhan's lemma. If one neglects the fact that the condition $\langle P,N_U^{(\ell)}\rangle=\rho$ does not define an open set of P's, then one easily formally obtains (1.26) from (3.9).

Now we consider (3.8). Assume that the term $N_{\Lambda_{L_N}\setminus\Lambda_{L_N-R-2}}(\omega_{\rm P})$ is a negligible error term and that taking the truncation parameters R,M,K and S to infinity will finally turn $\Phi_\beta^{(R,M,K)}\mathbb{1}\{N_{\Lambda_R}\leq S\}\rangle$ into Φ_β. The functional $P\mapsto\langle P,\Phi_\beta^{(R,M,K)}\mathbb{1}\{N_{\Lambda_R}\leq S\}\rangle$ has the sufficient continuity property for the application of the upper bound half of Varadhan's lemma. However, the functional $P\mapsto\langle P,N_U^{(\ell)}\rangle$ is not upper semicontinuous. Hence, the *equality* $\langle\mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}},N_U^{(\ell)}\rangle=\rho$ is turned into the *inequality* $\langle P,N_U^{(\ell)}\rangle\leq\rho$ in the resulting variational formula. Therefore, one easily formally obtains (1.25) from (3.8). In particular, our upper and lower bounds in Theorem 1.2 may differ. For small β resp. small ρ, we improve the proof in Lemma 3.4 and achieve a coincidence of upper and lower bounds, but this has nothing to do with large-deviations arguments.

The lack of upper semicontinuity of the functional $P\mapsto\langle P,N_U^{(\ell)}\rangle$ causes serious technical problems in the proof of the lower bound, since the condition $\langle P,N_U^{(\ell)}\rangle=\rho$ must be approximated by some open condition.

In Lemma 3.2, we already estimated away all the interaction involving cycles of length $>K$, and in the proof of the lower bound we will restrict the configuration space to marks with lengths $\leq K$. This is why our variational formulas spot only the presence of 'finitely long' cycles.

3.2. **The upper bound for empty boundary condition.** In this section, we prove the upper bound in (1.25) for bc $=\varnothing$. According to (3.8), it will be sufficient to prove

$$\begin{aligned}\limsup_{R,M,K,S\to\infty}\limsup_{N\to\infty}\frac{1}{|\Lambda_{L_N}|}\log\mathtt{E}\Big[\exp\Big\{-|\Lambda_{L_N}|\big\langle\mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}},\Phi_\beta^{(R,M,K)}\mathbb{1}\{N_{\Lambda_R}\leq S\}\big\rangle\Big\}\\ \times\exp\Big\{CN_{\Lambda_{L_N}\setminus\Lambda_{L_N-R-2}}(\omega_{\rm P})\Big\}\mathbb{1}\{\langle\mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}},N_U^{(\ell)}\rangle=\rho\}\Big]\leq-\chi^{(\leq)}(\beta,\rho).\end{aligned} \tag{3.10}$$

An outline of the proof is as follows. We separate first the two exponential terms from each other with the help of Hölder's inequality. The latter term will turn out to be a negligible error term. The functional that appears in the first exponent turns out to be local and bounded. Since its integral against a probability measure P is a $\tau_{\mathcal{L}}$-continuous and bounded function of P, Varadhan's lemma can be applied and expresses the limit superior in terms of the variational formula for the truncated versions of the interaction functionals. The indicator on the event $\{\langle \mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}}, N_U^{(\ell)}\rangle = \rho\}$ is estimated against the indicator on its closure, which is the same set with '$\leq$' instead of '$=$'. In this way, we obtain an upper bound against a truncated version of the variational formula $-\chi^{(\leq)}(\beta,\rho)$. By letting the truncation parameters go to infinity, this formula converges to $-\chi^{(\leq)}(\beta,\rho)$.

Let us turn to the details. We abbreviate $\mathfrak{R}_N = \mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}}$.

We pick $\eta \in (0,1)$ and start from (3.8), then Hölder's inequality gives

$$
\begin{aligned}
Z_N(\beta,\Lambda_{L_N}) \leq{}& {\rm e}^{|\Lambda_{L_N}|\overline{q}}\, \mathtt{E}\Big[{\rm e}^{-\frac{1}{1-\eta}|\Lambda_{L_N}|\langle \mathfrak{R}_N, \Phi_\beta^{(R,M,K)}1\!\!1\{N_{\Lambda_R}\leq S\}\rangle}\, 1\!\!1\{\langle \mathfrak{R}_N, N_U^{(\ell)}\rangle \leq \rho\}\Big]^{1-\eta}\\
&\times \mathtt{E}\Big[{\rm e}^{\frac{1}{\eta}CN_{\Lambda_{L_N}\setminus\Lambda_{L_N-R-2}}(\omega_{\rm P})}\Big]^{\eta};
\end{aligned}
\tag{3.11}
$$

note that we also estimated '$=\rho$' against '$\leq\rho$' in the indicator. The second term on the right hand side of (3.11) is easily estimated using the fact that $N_{\Lambda_{L_N}\setminus\Lambda_{L_N-R-2}}$ is a Poisson random variable with parameter $\overline{q}\times|\Lambda_{L_N}\setminus\Lambda_{L_N-R-2}|$ and that this parameter is of surface order $L_N^{d-1} = o(|\Lambda_N|)$. Hence, the expectation is estimated

$$
\mathtt{E}\Big[{\rm e}^{\frac{1}{\eta}CN_{\Lambda_{L_N}\setminus\Lambda_{L_N-R-2}}(\omega_{\rm P})}\Big]^{\eta} = {\rm e}^{-\eta\overline{q}|\Lambda_{L_N}\setminus\Lambda_{L_N-R-2}|}\exp\Big\{\eta{\rm e}^{C/\eta}\overline{q}|\Lambda_{L_N}\setminus\Lambda_{L_N-R-2}|\Big\} \leq {\rm e}^{o(|\Lambda_{L_N}|)}.
$$

We turn to the first term on the right hand side of (3.11). It turns out that $\Phi_\beta^{(R,M,K)}1\!\!1\{N_{\Lambda_R}\leq S\}$ is bounded. In fact,

$$
\begin{aligned}
\Phi_\beta^{(R,M,K)}(\omega)1\!\!1\{N_{\Lambda_R}(\omega)\leq S\} &\leq \frac{1}{2}M\beta\Big[\sum_{x\in U\cap\xi}\ell(f_x)\sum_{y\in\Lambda_R\cap\xi}\ell(f_y) + \Big(\sum_{x\in U\cap\xi}\ell(f_x)\Big)^2\Big]1\!\!1\{N_{\Lambda_R}(\omega)\leq S\}\\
&\leq M\beta K^2S^2.
\end{aligned}
\tag{3.12}
$$

Furthermore, it is easily seen that it is also local. Therefore, the map

$$
P \mapsto \Big\langle P, \Phi_\beta^{(R,M,K)}1\!\!1_{\{N_{\Lambda_R}\leq S\}}\Big\rangle
$$

is bounded and continuous on $\mathcal{P}_\theta$ with respect to the topology $\tau_{\mathcal{L}}$. Now we can apply a variant of Varadhan's lemma [DZ98, Thm. 4.3.1] in conjunction with the large-deviations principle of Lemma 3.3, to obtain that

$$
\begin{aligned}
\limsup_{N\to\infty}\frac{1}{|\Lambda_{L_N}|}\log\mathtt{E}\Big[\exp\Big\{-\frac{1}{1-\eta}|\Lambda_{L_N}|\big\langle\mathfrak{R}_N,\Phi_\beta^{(R,M,K)}1\!\!1\{N_{\Lambda_R}\leq S\}\big\rangle\Big\}1\!\!1\{\langle\mathfrak{R}_N,N_U^{(\ell)}\rangle\leq\rho\}\Big]\\
\leq -\inf_{P\in\mathcal{P}_\theta\colon\langle P,N_U^{(\ell)}\rangle\leq\rho}\Big(I_\beta(P)+\frac{1}{1-\eta}\langle P,\Phi_\beta^{(R,M,K)}1\!\!1\{N_{\Lambda_R}\leq S\}\rangle\Big),
\end{aligned}
\tag{3.13}
$$

since the set $\{P\in\mathcal{P}_\theta\colon\langle P,N_U^{(\ell)}\rangle\leq\rho\}$ is closed.

It remains to prove that

$$
\liminf_{R,M,K\to\infty,\eta\downarrow 0}\ \liminf_{S\to\infty}\ \inf_{P\colon\langle P,N_U^{(\ell)}\rangle\leq\rho}\big(I_\beta(P)+F_{M,R,K,S,\eta}(P)\big) \geq \inf_{P\colon\langle P,N_U^{(\ell)}\rangle\leq\rho}\big(I_\beta(P)+F(P)\big),
\tag{3.14}
$$

where we used the abbreviations $F(P)=\langle P,\Phi_\beta\rangle$ and $F_{M,R,K,S,\eta}(P)=\frac{1}{1-\eta}\langle P,\Phi_\beta^{(R,M,K)}1\!\!1\{N_{\Lambda_R}\leq S\}\rangle$. Fix $M,R,K>0$ and $\eta\in(0,1)$ and pick a sequence $S_n\to\infty$ and some Q_n satisfing $\langle Q_n,N_U^{(\ell)}\rangle\leq\rho$ such that

$$
I_\beta(Q_n)+F_{M,R,K,S_n,\eta}(Q_n) < \inf_{P\colon\langle P,N_U^{(\ell)}\rangle\leq\rho}\big(I_\beta(P)+F_{M,R,K,S_n,\eta}(P)\big)+\frac{1}{n}.
\tag{3.15}
$$

By compactness of the level sets of I_β, we may assume that the limiting measure $Q=\lim_{n\to\infty} Q_n$ exists in $\mathcal{P}_\theta$, where the limit is taken along some suitable subsequence. Notice further that $\langle Q, N_U^{(\ell)}\rangle \le \rho$ by Fatou's lemma. Fix any large $S>0$, then for n sufficiently large,

$$\begin{aligned}\inf_{P\colon \langle P,N_U^{(\ell)}\rangle\le\rho}\Big(I_\beta(P)+F_{M,R,K,S_n,\eta}(P)\Big) &> I_\beta(Q_n)+F_{M,R,K,S_n,\eta}(Q_n)-\frac{1}{n}\\ &\ge I_\beta(Q_n)+F_{M,R,K,S,\eta}(Q_n)-\frac{1}{n},\end{aligned}\tag{3.16}$$

where the second inequality uses the monotonicity of $F_{M,R,K,S,\eta}$ in S. Now send $n\to\infty$ and use the lower semi-continuity of I_β and the continuity of $F_{M,R,K,S,\eta}$, to get that the limit inferior of the right hand side of (3.16) is larger or equal to $I_\beta(Q)+F_{M,R,K,S,\eta}(Q)$. Sending $S\to\infty$ and using the monotone convergence theorem, we arrive at

$$\liminf_{S\to\infty}\inf_{P\colon \langle P,N_U^{(\ell)}\rangle\le\rho}(I_\beta(P)+F_{M,R,K,S,\eta}(P))\ge \inf_{P\colon \langle P,N_U^{(\ell)}\rangle\le\rho}(I_\beta(P)+F_{M,R,K,\infty,\eta}(P)).\tag{3.17}$$

In a similar way one proves that

$$\liminf_{R,M,K\to\infty,\eta\downarrow 0}\inf_{P\colon \langle P,N_U^{(\ell)}\rangle\le\rho}(I_\beta(P)+F_{M,R,K,\infty,\eta}(P))\ge \inf_{P\colon \langle P,N_U^{(\ell)}\rangle\le\rho}\big(I_\beta(P)+F(P)\big),$$

which implies (3.14) and ends the proof of (3.10).

3.3. **The lower bound for empty boundary condition.** In this section, we prove the lower bound in (1.26) for bc $=\varnothing$. According to (3.9), it will be sufficient to prove

$$\liminf_{N\to\infty}\frac{1}{|\Lambda_{L_N}|}\log \mathtt{E}\Big[\mathrm{e}^{-|\Lambda_{L_N}|\langle \mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}},\Phi_\beta\rangle}\mathbb{1}\{\langle \mathfrak{R}_{\Lambda_{L_N},\omega_{\rm P}}, N_U^{(\ell)}\rangle=\rho\}\Big]\ge -\chi^{(=)}(\beta,\rho).\tag{3.18}$$

We follow the standard strategy of changing the measure so that untypical events become typical, and controlling the Radon-Nikodym density by means of McMillan's theorem. However, for our problem we have to overcome two major difficulties. First, the map $P\mapsto\langle P,\Phi_\beta\rangle$ is not upper semicontinuous, and second, the set $\{P\in\mathcal{P}_\theta\colon \langle P,N_U^{(\ell)}\rangle=\rho\}$ appearing in the indicator is not open. This set induces long-range correlations not only between the points of the process, but also between their marks. Therefore, the results of [GZ93] cannot be applied directly, but some ideas of [G94] can be adapted.

Our strategy is as follows. In Lemma 3.7, we replace the condition $\langle P,N_U^{(\ell)}\rangle=\rho$ by the condition $|\langle P,N_U^{(\ell)}\rangle-\rho|<\delta$ for some small δ and control the replacement error. This condition becomes an open condition when restricting the mark space E to a cut-off version. A restriction of $\mathcal{P}_\theta$ in Lemma 3.8 makes the map $P\mapsto\langle P,\Phi_\beta\rangle$ continuous. In order to apply McMillan's theorem to the transformed point process, an ergodic approximation is carried out in Lemma 3.10.

Let us turn to the details. First, we prepare for relaxing the condition '$=\rho$' to '$\approx\rho$' in the following step, which is of independent interest. Bounding the quotient Z_{N+1}/Z_N of partition functions is often the key step to prove the equivalence of the canonical ensemble with the grand canonical ensemble, where the particle number is not fixed but governed by the mean. In the following, we give a lower bound in our case, which will also imply a non-trivial upper bound for the limiting free energy. Our proof is carried out in the setting of the cycle expansion introduced in Section 2 and is independent of the reformulation in terms of the marked Poisson point process.

Lemma 3.4. *For any $N\in\mathbb{N}$ and any measurable set $\Lambda\subset\mathbb{R}^d$,*

$$\frac{Z_{N+1}(\beta,\Lambda)}{Z_N(\beta,\Lambda)}\ge(4\pi\beta)^{-\frac{d}{2}}\frac{|\Lambda|}{N+1}\,\mathrm{e}^{-N\beta\alpha(v)/|\Lambda|},\tag{3.19}$$

where we recall that $\alpha(v)=\int_{\mathbb{R}^d}v(|x|)\,\mathrm{d}x$.

Proof. The strategy is as follows. We start with the cycle expression for the partition function Z_l. We then add a particle, i.e., an additional cycle of length one, and control the changes in the combinatorial factor and in the energy. Here our assumption $\int_{\mathbb{R}^d} v(|x|)\,\mathrm{d}x < \infty$ allows to bound the additional interaction energy.

We abbreviate $Z_N(\beta,\Lambda)$ by Z_N in this proof. Recall (2.1). According to Lemma 2.1, the cycle representation of the partition function reads

$$Z_N = \sum_{\lambda\in\mathfrak{P}_N} F_1(\lambda)F_2(\lambda), \tag{3.20}$$

with the combinatorial and interaction part

$$F_1(\lambda) = \prod_{k=1}^N \frac{(4\pi\beta k)^{-d\lambda_k/2}|\Lambda|^{\lambda_k}}{\lambda_k! k^{\lambda_k}} \qquad \text{and} \qquad F_2(\lambda) = \Big(\bigotimes_{k=1}^N \Big(\mathbb{E}_\Lambda^{(k\beta)}\Big)^{\otimes\lambda_k}\Big)\Big[\mathrm{e}^{-\mathcal{G}_{N,\beta}^{(\lambda)}}\Big].$$

Define the injection

$$T\colon \mathfrak{P}_N \to \mathfrak{P}_{N+1}, \qquad T(\lambda) = \widetilde{\lambda} \quad \text{with } \widetilde{\lambda}_k = \begin{cases} \lambda_1 + 1 & \text{if } k = 1 \\ \lambda_k & \text{if } k \geq 2. \end{cases}$$

All the terms in (3.20) are nonnegative, hence we may estimate

$$\begin{aligned} Z_{N+1} &\geq \sum_{\widetilde{\lambda}\in\mathfrak{P}_{N+1}\colon \widetilde{\lambda}_1\geq 1} F_1(\widetilde{\lambda})F_2(\widetilde{\lambda}) = \sum_{\lambda\in\mathfrak{P}_N} F_1(T(\lambda))F_2(T(\lambda)) \\ &= \sum_{\lambda\in\mathfrak{P}_N} \frac{F_1(T(\lambda))}{F_1(\lambda)}\frac{F_2(T(\lambda))}{F_2(\lambda)}\,F_1(\lambda)F_2(\lambda). \end{aligned} \tag{3.21}$$

The first quotient on the right hand side of (3.21) is bounded from below as follows

$$\frac{F_1(T(\lambda))}{F_1(\lambda)} = (4\pi\beta)^{-d/2}\frac{|\Lambda|}{\lambda_1+1} \geq (4\pi\beta)^{-d/2}\frac{|\Lambda|}{N+1}. \tag{3.22}$$

The second quotient is estimated via Jensen's inequality as follows. Recall that $B^{(k,i)}_{(j-1)\beta+s}$ is the Brownian bridge of the j-th leg of the i-th cycle of length k, $1\leq i\leq\lambda_k$.

$$\begin{aligned} F_2(T(\lambda)) &= \mathbb{E}_\Lambda^{(\beta)} \otimes \Big(\bigotimes_{k=1}^N \Big(\mathbb{E}_\Lambda^{(k\beta)}\Big)^{\otimes\lambda_k}\Big)\Big[\mathrm{e}^{-\mathcal{G}_{N,\beta}^{(\lambda)}} \exp\Big\{-\sum_{k\in\mathbb{N}}\sum_{i=1}^{\lambda_k}\sum_{j=1}^{k}\int_0^\beta v(|B_s - B^{(k,i)}_{(j-1)\beta+s}|)\,\mathrm{d}s\Big\}\Big] \\ &\geq \Big(\bigotimes_{k=1}^N \Big(\mathbb{E}_\Lambda^{(k\beta)}\Big)^{\otimes\lambda_k}\Big)\Big[\mathrm{e}^{-\mathcal{G}_{N,\beta}^{(\lambda)}} \exp\Big\{-\sum_{k\in\mathbb{N}}\sum_{i=1}^{\lambda_k}\sum_{j=1}^{k}\int_0^\beta \mathbb{E}_\Lambda^{(\beta)}\big[v(|B_s - B^{(k,i)}_{(j-1)\beta+s}|)\big]\,\mathrm{d}s\Big\}\Big]. \end{aligned} \tag{3.23}$$

Given $\lambda\in\mathfrak{P}_N$ and $k\in\mathbb{N}, i\in\{1,\dots,\lambda_k\}, j\in\{1,\dots,k\}$, we write $f(s) := B^{(k,i)}_{(j-1)\beta+s}$, and we estimate the expectation in the exponent as follows.

$$\begin{aligned} \mathbb{E}_\Lambda^{(\beta)}(v(|B_s - f(s)|)) &= \frac{1}{|\Lambda|}\int_\Lambda \mathrm{d}x \int_\Lambda \mathrm{d}y\, \frac{g_s(x,y)v(|y-f(s)|)g_{\beta-s}(y,x)}{g_\beta(x,x)} \\ &= \frac{1}{|\Lambda|}\int_\Lambda \mathrm{d}y\, v(|y-f(s)|)\int_\Lambda \mathrm{d}x\,\Big(\frac{g_{\beta-s}(y,x)g_s(x,y)}{g_\beta(y,y)}\Big)\frac{g_\beta(y,y)}{g_\beta(x,x)} \\ &= \frac{1}{|\Lambda|}\int_\Lambda \mathrm{d}y\, v(|y-f(s)|), \end{aligned} \tag{3.24}$$

since, because of $g_\beta(x,x) = g_\beta(y,y)$, the integral over x is exactly 1. An upper bound follows easily because the interaction potential is nonnegative, i.e.,

$$\mathbb{E}_\Lambda^{(\beta)}(v(|B_s - f(s)|)\,\mathrm{d}s = \frac{1}{|\Lambda|}\int_\Lambda \mathrm{d}y\, v(|y-f(s)|) \leq \frac{1}{|\Lambda|}\int_{\mathbb{R}^d} v(|x|)\,\mathrm{d}x = \frac{1}{|\Lambda|}\alpha(v). \tag{3.25}$$

Using this in (3.23), we get

$$F_2(T(\lambda)) \geq \Big(\bigotimes_{k=1}^N \Big(\mathbb{E}_\Lambda^{(k\beta)}\Big)^{\otimes\lambda_k}\Big)\Big[\mathrm{e}^{-\mathcal{G}_{N,\beta}^{(\lambda)}}\mathrm{e}^{-\sum_{k\in\mathbb{N}}\sum_{i=1}^{\lambda_k}\sum_{j=1}^k \beta\frac{1}{|\Lambda|}\alpha(v)}\Big] \quad = F_2(\lambda)\mathrm{e}^{-\frac{N\beta}{|\Lambda|}\alpha(v)}.$$

Using this and (3.22) in (3.21), the assertion follows. □

Now we draw two corollaries. First, we give an upper bound for the free energy, introduced in (1.3). Recall that Λ_{L_N} is the centred box with volume N/ρ.

Corollary 3.5 (Upper bound for the free energy)**.** *For any $\beta,\rho\in(0,\infty)$,*

$$\limsup_{N\to\infty} -\frac{1}{\beta}\frac{1}{|\Lambda_{L_N}|}\log Z_N(\beta,\Lambda_{L_N}) \leq \frac{\rho}{\beta}\log\Big(\rho(4\pi\beta)^{\frac{d}{2}}\Big)+\rho^2\alpha(v).$$

Proof. We use Lemma 3.4 iteratively, to get

$$Z_N(\beta,\Lambda_{L_N}) = \prod_{l=0}^{N-1}\frac{Z_{l+1}(\beta,\Lambda_{L_N})}{Z_l(\beta,\Lambda_{L_N})} \geq \prod_{l=0}^{N-1}\Big((4\pi\beta)^{-\frac{d}{2}}\frac{1}{\rho}\mathrm{e}^{-\beta\alpha(v)\rho}\Big) = \Big((4\pi\beta)^{-\frac{d}{2}}\frac{1}{\rho}\mathrm{e}^{-\beta\alpha(v)\rho}\Big)^N$$

The assertion follows by taking $\limsup_{N\to\infty}-\frac{1}{\beta}\frac{1}{|\Lambda_{L_N}|}\log$. □

Corollary 3.6. *Fix $(\beta,\rho)\in\mathcal{D}_v$. Then, for any $N,\widetilde{N}\in\mathbb{N}$ satisfying $\widetilde{N}\leq N$,*

$$\mathtt{E}\Big[\mathrm{e}^{-H_{\Lambda_{L_N}}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{\Lambda_{L_N}}^{(\ell)}(\omega_{\mathrm{P}})=N\}\Big] \geq \mathtt{E}\Big[\mathrm{e}^{-H_{\Lambda_{L_N}}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{\Lambda_{L_N}}^{(\ell)}(\omega_{\mathrm{P}})=\widetilde{N}\}\Big].$$

In particular, the map $\widetilde{N}\mapsto Z_{\widetilde{N}}(\beta,\Lambda_{L_N})$ is increasing in $\widetilde{N}\in\{1,\dots,N\}$.

Proof. Observe that, for $l<N$, by Lemma 3.4,

$$\frac{Z_{l+1}(\beta,\Lambda_{L_N})}{Z_l(\beta,\Lambda_{L_N})} \geq (4\pi\beta)^{-\frac{d}{2}}\frac{|\Lambda_{L_N}|}{l+1}\mathrm{e}^{-l\beta\alpha(v)/|\Lambda_{L_N}|} \geq (4\pi\beta)^{-\frac{d}{2}}\frac{1}{\rho}\mathrm{e}^{-\beta\rho\alpha(v)} \geq 1,$$

where the last step follows from $(\beta,\rho)\in\mathcal{D}_v$. Hence, for any $\widetilde{N}\in\mathbb{N}$ satisfying $\widetilde{N}\leq N$, we have $Z_N(\beta,\Lambda_{L_N})\geq Z_{\widetilde{N}}(\beta,\Lambda_{L_N})$. Now use Proposition 1.1 to finish. □

Openness

As we already mentioned, some of the technical difficulties for the application of Varadhan's lemma come from the fact that the set $\{P\in\mathcal{P}_\theta\colon\langle P,N_U^{(\ell)}\rangle=\rho\}$ is not open. This problem will be taken care of in the following lemma: we derive a lower bound for the right-hand side in (3.9) in terms of the same expectation, where the strict condition $=\rho$ is replaced by the condition $\in(\rho-\delta,\rho+\delta)$, for some $\delta>0$. Though this set is not open in $\mathcal{P}_\theta$, it will be open after restricting Ω to some cut-off version $\Omega^{(K,R)}$, which we will introduce a bit later.

Lemma 3.7. *Fix $\beta,\rho\in(0,\infty)$. We abbreviate $\mathfrak{R}_N(\omega)=\mathfrak{R}_{\Lambda_{L_N},\omega}$ for $\omega\in\Omega$. Fix $\delta\in(0,\rho)$. Then for any $N\in\mathbb{N}$,*

$$\begin{aligned}\mathtt{E}\Big[&\mathrm{e}^{-H_{\Lambda_{L_N}}(\omega_{\mathrm{P}})}\mathbb{1}\{\langle\mathfrak{R}_N(\omega_{\mathrm{P}}),N_U^{(\ell)}\rangle=\rho\}\Big]\\ &\geq\frac{(C_1\wedge C_2)^{\delta|\Lambda_{L_N}|}}{2\delta|\Lambda_{L_N}|+2}\mathtt{E}\Big[\mathrm{e}^{-|\Lambda_{L_N}|\langle\mathfrak{R}_N(\omega_{\mathrm{P}}),\Phi_\beta\rangle}\mathbb{1}\{\langle\mathfrak{R}_N(\omega_{\mathrm{P}}),N_U^{(\ell)}\rangle\in(\rho-\delta,\rho+\delta)\}\Big],\end{aligned} \tag{3.26}$$

where $C_1=1\wedge\big(\mathrm{e}^{-(\rho+\delta)\beta\alpha(v)}(4\pi\beta)^{-d/2}\frac{1}{\rho+\delta}\big)$ and $C_2=\mathrm{e}^{-\frac{\overline{q}}{\rho-\delta}}$.

Proof. Define the subset

$$\mathcal{P}_l = \Big\{ P \in \mathcal{P}_\theta \colon \langle P, N_U^{(\ell)} \rangle = \frac{l}{|\Lambda_{L_N}|} \Big\}$$

of probability measures. Abbreviate

$$Y_l^{(1)} = \mathtt{E}\Big[\mathrm{e}^{-H_{\Lambda_{L_N}}(\omega_\mathrm{P})} 1\!\!1_{\mathcal{P}_l}(\mathfrak{R}_N(\omega_\mathrm{P}))\Big], \tag{3.27}$$

$$Y_l^{(2)} = \mathtt{E}\Big[\mathrm{e}^{-|\Lambda_{L_N}|\langle \mathfrak{R}_N(\omega_\mathrm{P}), \Phi_\beta\rangle} 1\!\!1_{\mathcal{P}_l}(\mathfrak{R}_N(\omega_\mathrm{P}))\Big]. \tag{3.28}$$

Notice that, since $N/|\Lambda_{L_N}| = \rho$, the left-hand side of (3.26) is equal to $Y_N^{(1)}$, while the expectation on the right-hand side is equal to

$$\sum_{l \in \mathbb{N} \colon (\rho-\delta)|\Lambda_{L_N}| < l < (\rho+\delta)|\Lambda_{L_N}|} Y_l^{(2)}.$$

We now estimate the quotients $Y_{l+1}^{(1)}/Y_l^{(1)}$, respectively $Y_{l+1}^{(2)}/Y_l^{(2)}$, from below and above. More precisely, we show, for any $l \in \mathbb{N}_0$,

$$Y_{l+1}^{(1)} \geq C_1 Y_l^{(1)} \qquad \text{if } (\rho-\delta)|\Lambda_{L_N}| < l \leq \rho|\Lambda_{L_N}|, \tag{3.29}$$

and

$$Y_l^{(2)} \geq C_2 Y_{l+1}^{(2)} \qquad \text{if } \rho|\Lambda_{L_N}| \leq l < (\rho+\delta)|\Lambda_{L_N}|. \tag{3.30}$$

The proof of (3.29) follows from Lemma 3.4, combined with Proposition 1.1. Now we prove (3.30).

We find a map $\mathcal{T} \colon \mathcal{P}_{l+1} \to \mathcal{P}_l$ that describes a thinning procedure with the parameter $p = \frac{l}{l+1}$. To this end, we introduce a probability kernel K from Ω to Ω by putting $K(\omega, \cdot)$ equal to the distribution of $\omega^{(\eta)} = \sum_{x\in\xi} \eta_x \delta_{(x,f_x)} = \sum_{x \in \xi^{(\eta)}} \delta_{(x,f_x)}$, where $\omega = \sum_{x\in\xi} \delta_{(x,f_x)} \in \Omega$, and, given ω, $(\eta_x)_{x\in\xi}$ is a Bernoulli sequence with parameter p. The mapping

$$\mathcal{T} \colon \mathcal{P}_{l+1} \to \mathcal{P}_l, \qquad \mathcal{T}(P) = PK, \tag{3.31}$$

describes the distribution of what is left from a configuration with distribution P after deleting each particle independently with probability p. Given $P \in \mathcal{P}_{l+1}$, it follows, writing $\mathtt{E}_\eta$ for the expectation with respect to $(\eta_x)_{x\in\xi}$,

$$\begin{aligned} \langle \mathcal{T}(P), N_U^{(\ell)} \rangle &= \int_\Omega P(\mathrm{d}\omega) \int_\Omega K(\omega, \mathrm{d}\widetilde{\omega})\, N_U^{(\ell)}(\widetilde{\omega}) = \int_\Omega P(\mathrm{d}\omega)\, \mathtt{E}_\eta \Big[N_U^{(\ell)}\Big(\sum_{x\in\xi} \eta_x \delta_{(x,f_x)}\Big)\Big] \\ &= \int_\Omega P(\mathrm{d}\omega)\, \mathtt{E}_\eta \Big[\sum_{x \in \xi\cap U} \eta_x \ell(f_x)\Big] = p \langle P, N_U^{(\ell)}\rangle = \frac{l}{l+1} \langle P, N_U^{(\ell)} \rangle = \frac{l}{|\Lambda_{L_N}|}, \end{aligned}$$

which shows that $\mathcal{T} \colon \mathcal{P}_{l+1} \to \mathcal{P}_l$ is well defined. Since $\mathcal{T}$ removes particles, and therefore energy, the estimate

$$\langle P, \Phi_\beta \rangle \geq \langle \mathcal{T}(P), \Phi_\beta \rangle, \qquad P \in \mathcal{P}_{l+1}, \tag{3.32}$$

follows easily. Inequality (3.32) gives the estimate

$$\begin{aligned} Y_{l+1}^{(2)} &\leq \mathtt{E}\Big[\mathrm{e}^{-|\Lambda_{L_N}|\langle \mathcal{T}(\mathfrak{R}_N(\omega_\mathrm{P})), \Phi_\beta\rangle} 1\!\!1_{\mathcal{P}_l}(\mathcal{T}(\mathfrak{R}_N(\omega_\mathrm{P})))\Big] \\ &= \int_{\mathcal{P}_l} \mathrm{e}^{-|\Lambda_{L_N}|\langle P, \Phi_\beta \rangle} \frac{\mathrm{d}\mathtt{Q} \circ \mathfrak{R}_N^{-1} \circ \mathcal{T}^{-1}}{\mathrm{d}\mathtt{Q} \circ \mathfrak{R}_N^{-1}}(P)\, \mathtt{Q} \circ \mathfrak{R}_N^{-1}(\mathrm{d}P), \end{aligned} \tag{3.33}$$

where we recall that $\mathtt{Q}$ and $\mathtt{E}$ are the distribution of and expectation with respect to the marked Poisson process ω_P, and we conceive $\mathfrak{R}_N$ as a map $\Omega \to \mathcal{P}_\theta$; note that $\mathfrak{R}_N$ depends only on the configuration in Λ_{L_N}.

Now we identify the corresponding Radon-Nikodym density $\varphi_N = \mathrm{d}\mathtt{Q} \circ \mathfrak{R}_N^{-1} \circ \mathcal{T}^{-1} / \mathrm{d}\mathtt{Q} \circ \mathfrak{R}_N^{-1}$ on the image $\mathfrak{R}_N(\Omega)$. We claim that

$$\varphi_N(\mathfrak{R}_N(\omega)) = p^{\#(\xi\cap\Lambda_{L_N})} \mathrm{e}^{(1-p)\overline{q}|\Lambda_{L_N}|}, \qquad \omega \in \Omega. \tag{3.34}$$

This is shown as follows. Note that φ_N is the density of $\mathcal{T}(\mathfrak{R}_N(\omega_{\rm P}))$ with respect to $\mathfrak{R}_N(\omega_{\rm P})$ and that $\mathcal{T}(\mathfrak{R}_N(\omega_{\rm P}))$ has the distribution of $\mathfrak{R}_N(\omega_{\rm P}^{(\eta)})$. Recall that the particle process $\xi_{\rm P}\cap\Lambda_{L_N}$ is a standard Poisson process on Λ_{L_N} with intensity $\overline{q}|\Lambda_{L_N}|$, and $\xi_{\rm P}^{(\eta)}\cap\Lambda_{L_N}$ has intensity $p\overline{q}|\Lambda_{L_N}|$. It is standard that the right-hand side of (3.34) is the density of $\xi_{\rm P}^{(\eta)}\cap\Lambda_{L_N}$ with respect to $\xi_{\rm P}\cap\Lambda_{L_N}$. But this implies that (3.34) holds, as we have, for any nonnegative measurable test function $g\colon\mathcal{P}\to[0,\infty]$,

$$\begin{aligned}\int g(P)\,\mathtt{Q}\circ\mathcal{T}(\mathfrak{R}_N)^{-1}({\rm d}P)&=\mathtt{E}\big[g(\mathcal{T}(\mathfrak{R}_N(\omega_{\rm P})))\big]=\mathtt{E}\Big[\mathtt{E}_\eta\big[g(\mathfrak{R}_N(\omega_{\rm P}^{(\eta)}))\big]\Big]\\&=\mathtt{E}\big[p^{\#(\xi\cap\Lambda_{L_N})}{\rm e}^{(1-p)\overline{q}|\Lambda_{L_N}|}g(\mathfrak{R}_N(\omega_{\rm P}))\big]\\&=\int g(P)p^{\#(\xi\cap\Lambda_{L_N})}{\rm e}^{(1-p)\overline{q}|\Lambda_{L_N}|}\,\mathtt{Q}\circ\mathfrak{R}_N^{-1}({\rm d}P).\end{aligned}$$

Note that, for $(\rho-\delta)|\Lambda_{L_N}|<l\le\rho|\Lambda_{L_N}|$,

$$\varphi_N(\mathfrak{R}_N(\omega))\le{\rm e}^{(1-p)\overline{q}|\Lambda_{L_N}|}={\rm e}^{\frac{\overline{q}}{l+1}|\Lambda_{L_N}|}\le{\rm e}^{\frac{\overline{q}}{\rho-\delta}},\qquad\omega\in\Omega.$$

Hence, from (3.33) we have

$$Y_{l+1}^{(2)}\le{\rm e}^{\frac{\overline{q}}{\rho-\delta}}\int_{\mathcal{P}_l}{\rm e}^{-|\Lambda_{L_N}|\langle P,\Phi_\beta\rangle}\mathtt{Q}\circ\mathfrak{R}_N^{-1}({\rm d}P)={\rm e}^{\frac{\overline{q}}{\rho-\delta}}Y_l^{(2)},$$

and thus the estimate (3.30).

Now we finish the proof of the lemma subject to (3.29) and (3.30). By Lemma 3.2(ii), we have $Y_N^{(1)}\ge Y_N^{(2)}$ and therefore

$$\text{l.h.s. of (3.26)}=Y_N^{(1)}\ge\frac{1}{2\delta|\Lambda_{L_N}|+2}\Big(\sum_{(\rho-\delta)|\Lambda_{L_N}|<l\le\rho|\Lambda_{L_N}|}Y_N^{(1)}+\sum_{\rho|\Lambda_{L_N}|<l<(\rho+\delta)|\Lambda_{L_N}|}Y_N^{(2)}\Big).$$

For $(\rho-\delta)|\Lambda_{L_N}|<l\le\rho|\Lambda_{L_N}|$ the estimate (3.29) gives

$$Y_N^{(1)}\ge C_1Y_{N-1}^{(1)}\ge\dots\ge C_1^{\delta|\Lambda_{L_N}|}Y_l^{(1)}\ge C_1^{\delta|\Lambda_{L_N}|}Y_l^{(2)},$$

because $C_1\le1$, where we again used Lemma 3.2(ii). On the other hand, for $\rho|\Lambda_{L_N}|<l<(\rho+\delta)|\Lambda_{L_N}|$ the estimate (3.30) gives

$$Y_N^{(2)}\ge C_2Y_{N+1}^{(2)}\ge\dots\ge C_2^{\delta|\Lambda_{L_N}|}Y_l^{(2)},$$

where we used $C_2<1$. Therefore

$$Y_N^{(1)}\ge\frac{(C_1\wedge C_2)^{\delta|\Lambda_{L_N}|}}{2\delta|\Lambda_{L_N}|+2}\sum_{(\rho-\delta)|\Lambda_{L_N}|<l<(\rho+\delta)|\Lambda_{L_N}}Y_l^{(2)}\quad=\text{r.h.s. of (3.26)},\tag{3.35}$$

which finishes the proof of the lemma. □

As a conclusion of Lemma 3.7 we have the following lower bound, for any sufficiently large $N\in\mathbb{N}$.

$$Z_N(\beta,\Lambda_{L_N})\ge{\rm e}^{|\Lambda_{L_N}|(\overline{q}-C\delta)}\mathtt{E}\Big[{\rm e}^{-|\Lambda_{L_N}|\langle\mathfrak{R}_N(\omega_{\rm P}),\Phi_\beta\rangle}1\!\!1\{\langle\mathfrak{R}_N(\omega_{\rm P}),N_U^{(\ell)}\rangle\in(\rho-\delta,\rho+\delta)\}\Big],\tag{3.36}$$

for any $\delta\in(0,\frac{\rho}{2})$ and some C depending only on β,ρ and v. Furthermore, if $(\beta,\rho)\in\mathcal{D}_v$, then we can combine Lemma 3.7 with Corollary 3.6 to get, for any $\widetilde{\rho}\in(0,\rho]$ and any $\delta\in(0,\frac{\widetilde{\rho}}{2})$, for any sufficiently large $N\in\mathbb{N}$,

$$Z_N(\beta,\Lambda_{L_N})\ge{\rm e}^{|\Lambda_{L_N}|(\overline{q}-C\delta)}\mathtt{E}\Big[{\rm e}^{-|\Lambda_{L_N}|\langle\mathfrak{R}_N(\omega_{\rm P}),\Phi_\beta\rangle}1\!\!1\{\langle\mathfrak{R}_N(\omega_{\rm P}),N_U^{(\ell)}\rangle\in(\widetilde{\rho}-\delta,\widetilde{\rho}+\delta)\}\Big].\tag{3.37}$$

Hence, in order to prove both bounds in (1.26), it is enough to prove

$$\liminf_{\delta\downarrow0}\liminf_{N\to\infty}\frac{1}{|\Lambda_{L_N}|}\log\mathtt{E}\Big[{\rm e}^{-|\Lambda_{L_N}|\langle\mathfrak{R}_N(\omega_{\rm P}),\Phi_\beta\rangle}1\!\!1\{\langle\mathfrak{R}_N(\omega_{\rm P}),N_U^{(\ell)}\rangle\in(\rho-\delta,\rho+\delta)\}\Big]\ge-\chi^{(=)}(\beta,\rho),\tag{3.38}$$

for any $\beta,\rho\in(0,\infty)$, since $\chi^{(\le)}(\beta,\rho)=\inf_{\widetilde{\rho}\in(0,\rho)}\chi^{(=)}(\beta,\rho)$.

Restriction of the mark space

We will approximate the mark space E by the cut-off version

$$E^{(K,R)} := \bigcup_{k=1}^{K} \mathcal{C}_{k,R}, \qquad \text{where } \mathcal{C}_{k,R} := \Big\{ f \in \mathcal{C}_k \colon \sup_{s\in[0,k\beta]} |f(s) - f(0)| \le R \Big\}.$$

Let $\Omega^{(K,R)}$ denote the set of locally finite point measures on $\mathbb{R}^d \times E^{(K,R)}$. Define the canonical projection $\pi_{K,R} \colon \Omega \to \Omega^{(K,R)}$ by

$$\pi_{K,R}(\omega) = \omega^{(K,R)} = \sum_{x\in\xi\colon f_x \in E^{(K,R)}} \delta_{(x,f_x)}.$$

On $\Omega^{(K,R)}$ we consider the Poisson point process

$$\omega_{\mathrm{P}}^{(K,R)} = \pi_{K,R}(\omega_{\mathrm{P}}) = \sum_{x\in\xi_{\mathrm{P}}\colon B_x \in E^{(K,R)}} \delta_{(x,B_x)} \tag{3.39}$$

as the reference process. The distribution of $\omega_{\mathrm{P}}^{(K,R)}$ is denoted $\mathtt{Q}^{(K,R)}$, its intensity measure is $\nu^{(K,R)} = \sum_{k=1}^{K} \nu_k^{(K,R)}$, where $\nu_k^{(K,R)}$ is the restriction of ν_k to $\Omega^{(K,R)}$; see (1.7). By $I_\beta^{(K,R)}$ we denote the rate function with respect to $\omega_{\mathrm{P}}^{(K,R)}$, that is, $I_\beta^{(K,R)}$ is defined as I_β in (1.20) with ω_{P} replaced by $\omega_{\mathrm{P}}^{(K,R)}$. If there is no confusion possible, we identify the set $\mathcal{P}_\theta(\Omega^{(K,R)})$ of shift-invariant marked random point fields on $\Omega^{(K,R)}$ with the set of those $P \in \mathcal{P}_\theta = \mathcal{P}_\theta(\Omega)$ that are concentrated on $\Omega^{(K,R)}$. A variant of Lemma 3.3 gives that $(\mathfrak{R}_{\Lambda_L,\omega_{\mathrm{P}}^{(K,R)}})_{L>0}$ satisfies the large-deviations principle with rate function $I_\beta^{(K,R)}$. Observe that $\mathfrak{R}_{\Lambda_L,\omega_{\mathrm{P}}^{(K,R)}} = \mathfrak{R}_{\Lambda_L,\omega_{\mathrm{P}}} \circ \pi_{K,R}^{-1}$. Hence, according to the contraction principle, we have the identification

$$I_\beta^{(K,R)}(P) = \inf\{ I_\beta(Q) \colon Q \in \mathcal{P}_\theta, Q \circ \pi_{K,R}^{-1} = P \}, \tag{3.40}$$

since the map $Q \mapsto Q \circ \pi_{K,R}^{-1}$ is continuous.

For a while, we keep K and R fixed. Now we work on the expectation on the right-hand side of (3.9). We obtain a lower bound by requiring that $\mathfrak{R}_{\Lambda_{L_N},\omega_{\mathrm{P}}}$ be concentrated on $\Omega^{(K,R)}$. On this event, we may replace $\mathfrak{R}_{\Lambda_{L_N},\omega_{\mathrm{P}}}$ by $\mathfrak{R}_{\Lambda_{L_N},\omega_{\mathrm{P}}^{(K,R)}}$, and we may replace the expectation $\mathtt{E}$ with respect to the Poisson process ω_{P} by the expectation $\mathtt{E}^{(K,R)}$ with respect to $\omega_{\mathrm{P}}^{(K,R)}$. We write $\mathfrak{R}_N$ for $\mathfrak{R}_{\Lambda_{L_N},\omega_{\mathrm{P}}^{(K,R)}}$ in the following. Hence, we can extend (3.36) by

$$Z_N(\beta,\Lambda_{L_N}) \ge \mathrm{e}^{|\Lambda_{L_N}|(\overline{q}-C\delta)} \mathtt{E}^{(K,R)}\Big[\mathrm{e}^{-|\Lambda_{L_N}|\langle \mathfrak{R}_N,\Phi_\beta\rangle} 1\!\!1\{ \langle \mathfrak{R}_N, N_U^{(\ell)}\rangle \in (\rho-\delta,\rho+\delta)\} \Big]. \tag{3.41}$$

Notice that $\{P \in \mathcal{P}_\theta(\Omega^{(K,R)}) \colon \langle P, N_U^{(\ell)}\rangle \in (\rho-\delta,\rho+\delta)\}$ is an open set. In order to apply the lower bound of Varadhan's lemma to the right-hand side, we need to have that the map $P \mapsto \langle P, \Phi_\beta\rangle$ is upper semicontinuous. This will be achieved by a further restriction procedure.

Continuity

We prove the continuity of the map $P \mapsto \langle P, \Phi_\beta\rangle$ on the following suitable subset of measures. For $r \in (0,\infty)$, put

$$\Gamma_r = \Big\{ \omega \in \Omega^{(K,R)} \colon T_{x,y}(\omega) \le r \quad \forall x,y \in \xi, \text{ and } |x-y| \ge \frac{1}{r} \text{ for all distinct } x,y\in\xi \Big\}, \tag{3.42}$$

where $T_{x,y}(\omega)$ was defined in (1.15). Denote

$$\mathcal{P}_{\theta,r} := \big\{ P \in \mathcal{P}_\theta(\Omega^{(K,R)}) \colon P(\Gamma_r) = 1 \big\}.$$

In the following lemma we use that the map $t \mapsto t^{d-1}\sup_{s\ge t-2R} v(s)$ is integrable, which easily follows from the temperedness assumption in Assumption (v).

Lemma 3.8. *For any $r > 0$, the map $P \mapsto \langle P, \Phi_\beta\rangle$ is continuous on the set $\mathcal{P}_{\theta,r}$.*

Proof. We adapt the proof of the lower bound in [G94, Thm. 2]. Recall that $\pi_n\colon \Omega\to\Omega_{2n}$ denotes the projection $\pi_n(\omega)=\sum_{x\in\xi\cap\Lambda_{2n}}\delta_{(x,f_x)}$ on the box $\Lambda_{2n}=[-n,n]^d$. For any P let $P_n:=P\circ\pi_n^{-1}$. Let P and a net $(P^{(\alpha)})_{\alpha\in D}$ be in $\mathcal{P}_{\theta,r}$ such that $P^{(\alpha)}$ converges to P (in the topology $\tau_{\mathcal{L}}$). Then we have, for any $n\in\mathbb{N}$ and $\alpha\in D$,

$$\begin{aligned}|\langle P,\Phi_\beta\rangle-\langle P^{(\alpha)},\Phi_\beta\rangle|&\\ &\le|\langle P,\Phi_\beta-\Phi_\beta\circ\pi_n\rangle|+|\langle P^{(\alpha)}-P,\Phi_\beta\circ\pi_n\rangle|+\sup_{\alpha\in D}|\langle P^{(\alpha)},\Phi_\beta-\Phi_\beta\circ\pi_n\rangle|\\ &\le|\langle P^{(\alpha)}-P,\Phi_\beta\circ\pi_n\rangle|+2\sup_{\widetilde{P}\in\mathcal{P}_{\theta,r}}\langle\widetilde{P},|\Phi_\beta-\Phi_\beta\circ\pi_n|\rangle.\end{aligned}\tag{3.43}$$

Observe that the last term on the right-hand side vanishes as $n\to\infty$ since $\Phi_\beta\circ\pi_n$ converges to Φ_β uniformly on Γ_r. Indeed, for $\omega\in\Gamma_r$ estimate

$$\Phi_\beta(\omega)-\Phi_\beta\big(\pi_n(\omega)\big)=\sum_{x\in U\cap\xi}\sum_{y\in\xi\cap\Lambda_{2n}^{\rm c}}T_{x,y}(\omega)\le\frac12\sum_{x\in U\cap\xi}\sum_{y\in\xi\cap\Lambda_{2n}^{\rm c}}K^2\beta\sup_{s\ge|x-y|-2R}v(s),\tag{3.44}$$

where we also used that $\ell(f_x)\le K$ and $\sup_{s\in[0,\beta\ell(f_x)]}|f_x(s)-f_x(0)|\le R$ for any $x\in\xi$, since $\omega\in\Omega^{(K,R)}$. Since $|x-y|\ge\frac1r$ for any distinct $x,y\in\xi$, the upper bound is not larger than

$$K^2\beta C_{r,R}\int_n^\infty t^{d-1}\sup_{s\ge t-2R}v(s)\,{\rm d}t,$$

for some $C_{r,R}$ depending only on r and R. Now use that map $t\mapsto t^{d-1}\sup_{s\ge t-2R}v(s)$ is integrable.

For any n, the first term on the right-hand side of (3.43) vanishes asymptotically since the net $(P^{(\alpha)})_{\alpha\in D}$ converges to P, and $\Phi_\beta\circ\pi_n$ is local and bounded on Γ_r. □

Ergodic approximation

As a preparation for the construction of an ergodic approximation, we now show that any P with finite energy is tempered, that is, the expectation of the square of the mean-particle density is finite. Here we use the assumption that $\liminf_{r\downarrow 0}v(r)>0$, which is part of Assumption (v). Hence, we may pick $R^*>0$ and $\zeta>0$ such that $v(|x|)\ge\zeta$ for all $|x|\le R^*$.

Lemma 3.9 (**Temperedness**). *Fix $K,R\in\mathbb{N}$, and let $P\in\mathcal{P}_\theta(\Omega^{(K,R)})$ with $\langle P,\Phi_\beta\rangle<\infty$. Then*

$$\langle P,N_U^2\rangle<\infty\qquad\textit{and}\qquad\langle P,(N_U^{(\ell)})^2\rangle<\infty.$$

Proof. We may assume that $R^*<\frac12$. Therefore, we obtain a lower bound for $\langle P,\Phi_\beta\rangle$ by restricting the sums on x,y to $x,y\in\Lambda_{R^*/4}=[-\frac{R^*}{4},\frac{R^*}{4}]^d$ and by dropping all the parts of the cycles except for the first one:

$$\begin{aligned}\langle P,\Phi_\beta\rangle&=\frac12\int P({\rm d}\omega)\sum_{x\in\xi\cap U,y\in\xi}\sum_{i=0}^{\ell(f_x)-1}\sum_{j=0}^{\ell(f_y)-1}1\!\!1_{\{(x,i)\neq(y,j)\}}\int_0^\beta v(|f_x(i\beta+s)-f_y(j\beta+s)|)\,{\rm d}s\\ &\ge\frac12\int P({\rm d}\omega)\sum_{x,y\in\xi\cap\Lambda_{R^*/4}}1\!\!1\{x\neq y\}\int_0^\beta v(|f_x(s)-f_y(s)|)\,{\rm d}s.\end{aligned}\tag{3.45}$$

Define, for any $\omega\in\Omega^{(K,R)}$ and $x\in\xi$,

$$\tau_x(\omega)=\inf\{s\in[0,\beta]\colon|f_x(s)-x|>R^*/4\}\wedge\delta.\tag{3.46}$$

Note that $|x-y|\le R^*/2$ on the right-hand side of (3.45). Since $v(|x|)\ge\zeta$ for all $|x|\le R^*$, each integral on the right hand side of (3.45) can be estimated from below as follows.

$$\int_0^\beta v(|f_x(s)-f_y(s)|)\,{\rm d}s\ge\int_0^{\tau_x(\omega)\wedge\tau_y(\omega)}v(|f_x(s)-f_y(s)|)\,{\rm d}s\ge\zeta\,(\tau_x(\omega)\wedge\tau_y(\omega)),\qquad x\in\xi^{(k)},y\in\xi^{(k')}.$$

We get a further lower bound in (3.45) by inserting the indicator on the event $\{\tau_x = \delta = \tau_y\}$:

$$\langle P, \Phi_\beta\rangle \geq \frac{\delta\zeta}{2}\int P(\mathrm{d}\omega)\,\#\Big\{(x,y)\in \big(\xi\cap\Lambda_{R^*/4}\big)^2 \colon x\neq y, \tau_x = \delta = \tau_y\Big\}.$$

Since the event $\{\tau_x = \delta\}$ is decreasing for decreasing δ and its probability tends to one as $\delta \downarrow 0$, the above counting variable tends to the number of distinct pairs in $\xi\cap\Lambda_{R^*/4}$. Hence, for some sufficiently small $\delta > 0$, we have

$$\langle P, \Phi_\beta\rangle \geq \frac{\delta\zeta}{4}\int P(\mathrm{d}\omega)\,\#\Big\{(x,y)\in \big(\xi\cap\Lambda_{R^*/4}\big)^2 \colon x\neq y\Big\} \geq \frac{\delta\zeta}{8}\langle P, N^2_{\Lambda_{R^*/4}}\rangle.$$

Hence, if $\langle P, \Phi_\beta\rangle$ is finite, then, by shift-invariance of P, also $\langle P, N_\Lambda^2\rangle$ is finite for any bounded box Λ. Since P is concentrated on configurations with bounded leg length, also $\langle P, (N_\Lambda^{(\ell)})^2\rangle$ is finite for any bounded box Λ. □

Now we approximate any probability measure on $\Omega^{(K,R)}$ with an ergodic measure. Define

$$\psi_R(t) := \begin{cases} \sup_{s\geq t-2R} v(s) & \text{if } t\geq 3R,\\ v(R) & \text{if } t\in[0,3R].\end{cases} \tag{3.47}$$

Recall from Assumption (v) that $\psi_R(t) = O(t^{-h})$ for some $h > d$.

Lemma 3.10 (**Ergodic approximation**). *Fix $K, R\in\mathbb{N}$ and $\varepsilon > 0$. Then, for any $P\in\mathcal{P}_\theta(\Omega^{(K,R)})$ satisfying $I_\beta^{(K,R)}(P) + \Phi_\beta(P) < \infty$ and for any neighborhood V of P in $\mathcal{P}_\theta(\Omega^{(K,R)})$, there exists an ergodic measure $\widetilde{P}\in V$ and some $r > 0$ such that $\widetilde{P}(\Gamma_r) = 1$, and $\langle\widetilde{P}, \Phi_\beta\rangle \leq \langle P, \Phi_\beta\rangle + \varepsilon$ and $I_\beta^{(K,R)}(\widetilde{P}) \leq I_\beta^{(K,R)}(P) + \varepsilon$.*

Proof. This is similar to [G94, Lemma 5.1]. Recall that P_n denotes the projection of P on Ω_n, the configuration space on the box $\Lambda_{2n} = [-n,n]^d$. Since $\langle P, \Phi_\beta\rangle < \infty$, and as $\Phi_\beta \geq 0$, we have $\langle P_n, \Phi_\beta\rangle < \infty$. Hence $\lim_{r\to\infty} P_n(\Gamma_r) = 1$, for any $n\in\mathbb{N}$. Therefore, we can choose a sequence $r(n)\to\infty$ such that $\lim_{n\to\infty} P_n(\Gamma_{r(n)}) = 1$. Set $m = n+3R$. Denote by $\widehat{P}^{(n)}$ the probability measure under which the particle configurations in the (up to the boundary, disjoint) boxes $\Lambda_m + 2mk$, with $k\in\mathbb{Z}^d$, are independent and distributed as $P_n' := P_n(\cdot \mid \Gamma_{r(n)})$. In particular, no points are contained in the corridors $(\Lambda_m\backslash\Lambda_n) + 2mk$.

We now put

$$P^{(n)} = \frac{1}{|\Lambda_m|}\int_{\Lambda_m}\widehat{P}^{(n)}\circ\theta_z\,\mathrm{d}z.$$

It is then clear that $P^{(n)}\in\mathcal{P}_\theta$. A standard argument shows that $P^{(n)}$ is ergodic; see, e.g., [G88, Theorem 14.12]. Since $\Gamma_{r(n)}$ is shift invariant, and $\widehat{P}^{(n)}(\Gamma_{r(n)}) = 1$, it also follows that $P^{(n)}(\Gamma_{r(n)}) = 1$. We claim that $\widetilde{P} = P^{(n)}$ with n sufficiently large, satisfies the requirements. For this, we have to show that (1) $\limsup_{n\to\infty} I_\beta(P^{(n)}) \leq I_\beta(P)$, (2) $\limsup_{n\to\infty}\langle P^{(n)}, \Phi_\beta\rangle \leq \langle P, \Phi_\beta\rangle$, and finally (3) the net $(P^{(n)})_{n\in\mathbb{N}}$ converges to P (in the topology $\tau_{\mathcal{L}}$).

The proof of (1) can be found in the proof of [G94, Lemma 5.1].

Now we turn to the proof of (2). First note that

$$\langle P^{(n)}, \Phi_\beta\rangle = \frac{1}{|\Lambda_m|}\int_{\Lambda_m}\mathrm{d}z\int\widehat{P}^{(n)}(\mathrm{d}\omega\,)\sum_{x\in\xi\cap(U-z)}\sum_{y\in\xi}T_{x,y}(\omega), \tag{3.48}$$

where we recall the notation in (1.15). The sum on y in (3.48) will be split in the sum over $y\in\xi\cap\Lambda_n$ and the remainder. The first sum is handled as follows. As x, y both belong to Λ_n, the measure $\widehat{P}^{(n)}$

can be replaced by P'_n. Furthermore, since $T_{x,y}(\omega) \geq 0$, the integration with respect to P'_n may be estimated against the integration with respect to $P(\cdot)/P_n(\Gamma_{r(n)})$. This gives

$$\begin{aligned}
&\frac{1}{|\Lambda_m|}\int_{\Lambda_m} \mathrm{d}z \int \widehat{P}^{(n)}(\mathrm{d}\omega) \sum_{x\in\xi\cap(U-z)} \sum_{y\in\xi\cap\Lambda_n} T_{x,y}(\omega)\\
&\qquad \leq \frac{1}{P_n(\Gamma_{r(n)})}\frac{1}{|\Lambda_m|}\int_{\Lambda_m} \mathrm{d}z \int P(\mathrm{d}\omega) \sum_{x\in\xi\cap(U-z)} \sum_{y\in\xi} T_{x,y}(\omega).
\end{aligned}$$

Now use the shift invariance of P and recall that $\lim_{n\to\infty} P_n(\Gamma_{r(n)}) = 1$ to see that the last expression approaches $\langle P, \Phi_\beta\rangle$.

Now we consider the remainder sum in (3.48), where y is summed over $\xi \cap \Lambda_m^{\rm c}$. Observe that $|x-y| \geq 3R$, hence we may estimate

$$T_{x,y}(\omega) \leq \beta K^2 \psi_R(|x-y|) \leq \beta K^2 \sup_{x\colon |x|\leq |z|+1} \psi_R(|x-y|) \leq \beta K^2 \psi_R(|y|-|z|-1)$$

where in the last inequality we used the fact that $|x-y| \geq |x|-|y|$ and that $\psi_R(\cdot)$ is non-increasing. Now we distinguish to which of the boxes $\Lambda_n + 2km$, with $k \in \mathbb{Z}^d$, the point y belongs (recall that the configurations in these boxes are independent). Hence for any $z \in \Lambda_m$, we have that

$$\begin{aligned}
&\int \widehat{P}^{(n)}(\mathrm{d}\omega) \sum_{x\in\xi\cap(U-z)} \sum_{y\in\xi\cap\Lambda_m^{\rm c}} T_{x,y}(\omega)\\
&\quad \leq \beta K^2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \int_{\Omega_n} P'_n(\mathrm{d}\omega^{(1)}) \int_{\Omega_n} P'_n(\mathrm{d}\omega^{(2)}) \#\big(\xi^{(1)} \cap \big(U-z\big)\big) \sum_{y\in(\xi^{(2)}\cap\Lambda_n)+2km} \psi_R(|y|-|z|-1)\\
&\quad \leq \frac{\beta K^2}{P_n(\Gamma_{r(n)})^2} \langle P, N_U\rangle \langle P, N_{\Lambda_n}\rangle \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_R(2|k|m - m - |z| - 1),
\end{aligned}$$

where we estimated integrals with respect to P'_n against integrals with respect to $P/P_n(\Gamma_{r(n)})$ twice, and used the shift invariance of P. Now we use Assumption (v) and obtain a constant C (depending only on R) such that $\psi_R(t) \leq Ct^{-h}$ for any $t \geq 0$. Using this in the last display gives that

$$\begin{aligned}
&\int \widehat{P}^{(n)}(\mathrm{d}\omega) \sum_{x\in\xi\cap(U-z)} \sum_{y\in\xi\cap\Lambda_m^{\rm c}} T_{x,y}(\omega)\\
&\qquad \leq \frac{\beta K^2 C 2^d}{P_n(\Gamma_{r(n)})^2} \langle P, N_U\rangle^2 n^d \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (2|k|m - m - |z| - 1)^{-h}.
\end{aligned}$$

Now add the factor $1/|\Lambda_m|$ and integrate over $z \in \Lambda_m$. Pick some $l = l(n)$ such that $l \sim n$ and $n^d(n-l)^{-h} \to 0$ as $n \to \infty$ and split the integral on $z \in \Lambda_m$ into the integrals on $z \in \Lambda_l$ and on the remainder. Then it is easy to see that

$$\lim_{n\to\infty} \frac{1}{|\Lambda_m|}\int_{\Lambda_m} \mathrm{d}z \int \widehat{P}^{(n)}(\mathrm{d}\omega) \sum_{x\in\xi\cap(U-z)} \sum_{y\in\xi\cap\Lambda_m^{\rm c}} T_{x,y}(\omega) = 0.$$

Now we have shown (2), i. e., that $\limsup_{n\to\infty} \langle \widehat{P}^{(n)}, \Phi_\beta\rangle \leq \langle P, \Phi_\beta\rangle$.

For the proof of (3), we pick $f \in \mathcal{L}$. Using an affine transformation, if necessary, we may assume that $f = f(\cdot \cap \Delta)$ and $|f| \leq N_\Delta$ for some bounded measurable $\Delta \subset \mathbb{R}^d$. To estimate the difference of

$|P^{(n)}(f) - P(f)|$ we integrate over the box Λ_m and get

$$\begin{aligned}|P^{(n)}(f) - P(f)| \leq \frac{1}{|\Lambda_m|}\int_{\Lambda_m} \mathrm{d}x\, \mathbb{1}\{x+\Delta \subset \Lambda_m\}\big|P_n(f\circ\theta_x \mid \Gamma_{r(n)}) - P(f\circ\theta_x)\big| \\ + \frac{1}{|\Lambda_m|}\int_{\Lambda_m} \mathrm{d}x\, \mathbb{1}\{x+\Delta \not\subset \Lambda_m\}\big|\widehat{P}^{(n)}(N_{\Delta+x}) + P(N_{\Delta+x})\big|.\end{aligned} \tag{3.49}$$

Now $P(N_{\Delta+x}) \leq \frac{|\Delta|\mu(P)}{P_n(\Gamma_{r(n)})}$, where $\mu(P) < \infty$ is the intensity of P. In the same way we obtain

$$\widehat{P}^{(n)}(N_{\Delta+x}) = P_n(N_{\Delta+x \bmod 2m+1} \mid \Gamma_{r(n)}) \leq \frac{|\Delta|\mu(P)}{P_n(\Gamma_{r(n)})}.$$

Hence the second term on the right hand side of (3.49) is not larger than the volume of $\{x\in\Lambda_m\colon x+\Delta\not\subset\Lambda_m\}$ (which is of surface order of Λ_m) times $O(|\Lambda_m|^{-1})$, i.e., it vanishes. Concerning the first term on the right hand side of (3.49), we estimate

$$\begin{aligned}&\big|P_n(f\circ\theta_x \mid \Gamma_{r(n)}) - P(f\circ\theta_x)\big| \\ &\quad\leq \Big|\frac{1}{P_n(\Gamma_{r(n)})} - 1\Big|\, P_n(N_{\Delta+x};\Gamma_{r(n)}) + P_n(N_{\Delta+x};\Gamma_{r(n)}^{\mathrm{c}}) \\ &\quad\leq |\Delta|\mu(P)\Big|\frac{1}{P_n(\Gamma_{r(n)})} - 1\Big| + P(N_\Delta^2)^{1/2}(1-P_n(\Gamma_{r(n)}))^{1/2}.\end{aligned}$$

By Lemma 3.9, $P(N_\Delta^2)$ is finite, hence the right-hand side vanishes as $n\to\infty$. Therefore, also the first term on the right hand side of (3.49) vanishes, and we conclude that (3) holds. □

Final step: proof of the lower bound in (1.26)**:**

Now we can finish the proof of the lower bound in (1.26). Recall that it is sufficient to prove (3.38) for any $\beta,\rho\in(0,\infty)$, to get both lower bounds in (1.26). Fix $K,R\in\mathbb{N}$ and $\delta\in(0,\rho)$. We start from the right-hand side of (3.41). Fix $\varepsilon>0$, and pick $P\in\mathcal{P}_\theta(\Omega^{(K,R)})$ satisfying $I_\beta^{(K,R)}(P) + \langle P,\Phi_\beta\rangle < \infty$ and $|\langle P, N_U^{(\ell)}\rangle - \rho| < \delta$. By Lemma 3.10, we may fix some $r>0$ and some ergodic measure $\widetilde{P}\in\mathcal{P}_\theta(\Omega^{(K,R)})$ satisfying $|\langle P, N_U^{(\ell)}\rangle - \rho| < \delta$ and $\langle \widetilde{P},\Phi_\beta\rangle \leq \langle P,\Phi_\beta\rangle + \varepsilon$ and $I_\beta^{(K,R)}(\widetilde{P}) \leq I_\beta^{(K,R)}(P) + \varepsilon$ and $\widetilde{P}(\Gamma_r) = 1$. Since $I_\beta^{(K,R)}(\widetilde{P}) < \infty$, for N large enough there is a density $f_N^{(K,R)}$ of the projection $\widetilde{P}_{L_N}$ of $\widetilde{P}$ to $\Omega_{L_N}^{(K,R)}$ with respect to the projection $\mathtt{Q}_{L_N}^{(K,R)}$ of the restricted marked Poisson point process $\mathtt{Q}^{(K,R)}$ to Ω_{L_N}, where we recall that Ω_{L_N} is the set of restrictions of configurations in Ω to Λ_{L_N}, and $\Omega_{L_N}^{(K,R)}$ is defined analogously. We conceive $\mathfrak{R}_N$ as a map $\mathfrak{R}_{N,\cdot}\colon \Omega_{L_N}\to\mathcal{P}_\theta(\Omega^{(K,R)})$. Now introduce the event

$$C_N = \Big\{\omega\in\Omega_{L_N}^{(K,R)}\colon \langle\mathfrak{R}_{N,\omega},\Phi_\beta\rangle \leq \langle\widetilde{P},\Phi_\beta\rangle + \varepsilon, \frac{1}{|\Lambda_{L_N}|}\log f_N^{(K,R)}(\omega) \leq I_\beta^{(K,R)}(\widetilde{P}) + \varepsilon\Big\}. \tag{3.50}$$

Then we can estimate

$$\begin{aligned}\mathtt{E}^{(K,R)}\Big[\mathrm{e}^{-|\Lambda_N|\langle\mathfrak{R}_N,\Phi_\beta\rangle}\mathbb{1}\{|\langle\mathfrak{R}_N,N_U^{(\ell)}\rangle - \rho| < \delta\}\Big] &= \int_{\Omega_{L_N}^{(K,R)}} \mathrm{d}\mathtt{Q}_{L_N}^{(K,R)}\, \mathrm{e}^{-|\Lambda_N|\langle\mathfrak{R}_N,\Phi_\beta\rangle}\mathbb{1}\{|\langle\mathfrak{R}_N,N_U^{(\ell)}\rangle - \rho| < \delta\} \\ \geq \int_{C_N} \widetilde{P}_{L_N}(\mathrm{d}\omega)\,\frac{1}{f_N^{(K,R)}(\omega)}\mathrm{e}^{-|\Lambda_N|\langle\mathfrak{R}_N,\Phi_\beta\rangle}\mathbb{1}\{|\langle\mathfrak{R}_N,N_U^{(\ell)}\rangle - \rho| < \delta\} \\ \geq \mathrm{e}^{-|\Lambda_{L_N}|(I_\beta^{(K,R)}(\widetilde{P})+\varepsilon)}\mathrm{e}^{-|\Lambda_{L_N}|(\langle\widetilde{P},\Phi_\beta\rangle+\varepsilon)}\widetilde{P}_{L_N}\big(C_N\cap\{\omega\in\Omega_{L_N}^{(K,R)}\colon |\langle\mathfrak{R}_N,N_U^{(\ell)}\rangle - \rho| < \delta\}\big).\end{aligned} \tag{3.51}$$

The continuity of the map $P \mapsto \langle P, \Phi_\beta \rangle$ (see Lemma 3.8), the law of large numbers and McMillan's theorem imply that

$$\widetilde{P}_{L_N}\Big(\{\omega \in \Omega_{L_N}^{(K,R)} \colon |\langle \mathfrak{R}_{N,\omega}, N_U^{(\ell)}\rangle - \rho| < \delta, \langle \mathfrak{R}_{N,\omega}, \Phi_\beta\rangle \leq \langle \widetilde{P}, \Phi_\beta \rangle + \varepsilon,$$
$$\frac{1}{|\Lambda_{L_N}|} \log f_N^{(K,R)}(\omega) \leq I_\beta^{(K,R)}(\widetilde{P}) + \varepsilon\}\Big) \to 1 \qquad \text{as } N \to \infty.$$

Using this in (3.51) and this in (3.41), we arrive at

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_{L_N}|} \log Z_N(\beta, \Lambda_{L_N}) \geq \overline{q} - \delta - I_\beta^{(K,R)}(\widetilde{P}) - \varepsilon - \langle \widetilde{P}, \Phi_\beta\rangle - \varepsilon. \tag{3.52}$$

Now recall that $\langle \widetilde{P}, \Phi_\beta \rangle \leq \langle P, \Phi_\beta\rangle + \varepsilon$ and $I_\beta^{(K,R)}(\widetilde{P}) \leq I_\beta(P) + \varepsilon$. Now we can let $\varepsilon \downarrow 0$ and take the infimum over P, to obtain

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_{L_N}|} \log Z_N(\beta, \Lambda_{L_N}) \geq \overline{q} - \delta - \inf_{P\in\mathcal{P}_\theta(\Omega^{(K,R)}) \colon |\langle P, N_U^{(\ell)}\rangle - \rho| < \delta} \big\{ I_\beta^{(K,R)}(P) + \langle P, \Phi_\beta \rangle\big\}$$

Our last step is to approach the variational formula $\chi^{(=)}(\beta,\rho)$ on the right-hand side of (1.26) by the finite-K and finite-R versions.

Lemma 3.11 (**Removing the cut-off**). *For any $\delta \in (0,\rho)$,*

$$\begin{aligned} \limsup_{K,R\to\infty} \inf_{P\in\mathcal{P}_\theta(\Omega^{(K,R)}) \colon |\langle P, N_U^{(\ell)}\rangle - \rho| < \delta} &\big\{ I_\beta^{(K,R)}(P) + \langle P, \Phi_\beta \rangle\big\} \\ \leq \inf_{P\in\mathcal{P}_\theta(\Omega) \colon \langle P, N_U^{(\ell)}\rangle = \rho} &\big\{ I_\beta(P) + \langle P, \Phi_\beta\rangle\big\} = \chi^{(=)}(\beta,\rho). \end{aligned} \tag{3.53}$$

Proof. Fix $P \in \mathcal{P}_\theta$ satisfying $\langle P, N_U^{(\ell)}\rangle = \rho$ and $I_\beta(P) + \Phi_\beta(P) < \infty$. For $K, R \in \mathbb{N}$, consider $P_{K,R} = P \circ \pi_{K,R}^{-1}$. Then we have $P_{K,R}(\Omega^{(K,R)}) = 1$ and $\langle P_{K,R}, N_U^{(\ell)}\rangle = \langle P, \pi_{K,R} \circ N_U^{(\ell)}\rangle \uparrow \langle P, N_U^{(\ell)}\rangle$ for $K, R \to \infty$ by the monotonous convergence theorem. Hence, for K and R sufficiently large, $|\langle P_{K,R}, N_U^{(\ell)}\rangle - \rho| < \delta$. Observe that $\langle P_{K,R}, \Phi_\beta\rangle \leq \langle P, \Phi_\beta\rangle$ since $\Phi_\beta \geq 0$. By (3.40), we have $I_\beta^{(K,R)}(P_{K,R}) \leq I_\beta(P)$. Finally, observe that the infimum over P such that $|\langle P, N_U^{(\ell)}\rangle - \rho| < \delta$ is obviously not larger than the infimum over P satisfying $\langle P, N_U^{(\ell)}\rangle = \rho$. □

3.4. **Proof of Theorem 1.2 for Dirichlet and periodic boundary conditions.** In this section, we show how to adapt the proof of Theorem 1.2 for empty boundary conditions to obtain the proof for Dirichlet and periodic boundary conditions. Let us make a couple of obvious observations. First, the restriction of the periodised Brownian bridge measure on paths that do not leave the box Λ equals the Brownian bridge measure with Dirichlet boundary conditions, i.e.,

$$\mu_{x,x}^{(\mathrm{per},k\beta)}\big|_{\mathcal{C}_{k,\Lambda}^{(\mathrm{Dir})}} = \mu_{x,x}^{(\mathrm{Dir},k\beta)}.$$

Hence, it is easy to see that $\overline{q}^{(\mathrm{Dir})} \leq \overline{q}^{(\mathrm{per})}$ and that

$$Z_N^{(\mathrm{Dir})}(\beta,\Lambda) \leq Z_N(\beta,\Lambda) \leq Z_N^{(\mathrm{per})}(\beta,\Lambda), \tag{3.54}$$

since the Feynman-Kac formula for $Z_N^{(\mathrm{Dir})}$ contains only those paths that stay in Λ all the time with the same distribution as under which they appear in the formula for $Z_N^{(\mathrm{per})}$. Hence, it will be sufficient to prove the upper bound in (1.25) for $Z_N^{(\mathrm{per})}$ and the lower bound in (1.26) for $Z_N^{(\mathrm{Dir})}$ only.

We start with the representation of $Z_N^{(\mathrm{Dir})}$ and $Z_N^{(\mathrm{per})}$ given in Proposition 1.1. The first step is to show that the weights $\overline{q}^{(\mathrm{bc})}$ converge to $\overline{q} = \sum_{k\in\mathbb{N}} q_k$. For notational reasons, we now write $\overline{q}_\Lambda^{(\mathrm{bc})}$ for $\overline{q}^{(\mathrm{bc})}$; however notice that it depends on N. Recall that Λ_{L_N} is the centred box with side length $L_N = (N/\rho)^{1/d}$.

Lemma 3.12. *Let* bc $\in$ {Dir, per}. *Then*

$$\lim_{N\to\infty} \overline{q}^{(\mathrm{bc})}_{\Lambda_{L_N}} = \overline{q}. \tag{3.55}$$

Proof. (a) First we consider periodic boundary conditions. Then we have

$$\overline{q}^{(\mathrm{per})}_{\Lambda_{L_N}} = (4\pi\beta)^{-d/2} \sum_{k=1}^{N} \frac{1}{k^{1+d/2}} \sum_{z\in\mathbb{Z}^d} \mathrm{e}^{-\frac{|z|^2}{4k\beta}L_N^2}. \tag{3.56}$$

Since the sum on $k = 1,\dots,N$ and $z = 0$ converges towards $(4\pi\beta)^{-d/2}\sum_{k=1}^{\infty}\frac{1}{k^{1+d/2}} = \overline{q}$, we only have to show that $\sum_{k=1}^{N}\frac{1}{k^{1+d/2}}\sum_{z\in\mathbb{Z}^d\setminus\{0\}} \mathrm{e}^{-\frac{|z|^2}{4k\beta}L_N^2}$ vanishes as $N\to\infty$.

Using an approximation with an integral, one sees that, for some $c\in(0,\infty)$, only depending on d,

$$\sum_{z\in\mathbb{Z}^d\setminus\{0\}} \mathrm{e}^{-a|z|^2} \leq c a^{-d/2} \qquad \text{for all } a\in(0,\infty).$$

Using this with $a = L_N^2/(4\beta k)$, we see that $\sum_{z\in\mathbb{Z}^d\setminus\{0\}} \mathrm{e}^{-\frac{|z|^2}{4k\beta}L_N^2}$ is of order $k^{d/2}L_N^{-d}$. Using that N is of order L_N^d and applying the harmonic series, we see that $\sum_{k=1}^{N}\frac{1}{k^{1+d/2}}\sum_{z\in\mathbb{Z}^d\setminus\{0\}} \mathrm{e}^{-\frac{|z|^2}{4k\beta}L_N^2}$ is of order $L_N^{-d}\log L_N$ and therefore vanishes as $N\to\infty$.

(b) Now we consider Dirichlet boundary conditions. For any $M\in\mathbb{N}$ and $\delta\in(0,1)$, we get, for any sufficiently large N,

$$\overline{q}^{(\mathrm{Dir})}_{\Lambda_{L_N}} = \frac{1}{|\Lambda_{L_N}|}\sum_{k=1}^{N}\frac{1}{k}\int_{\Lambda_{L_N}} \mathrm{d}x\, \mu^{(k\beta)}_{x,x}\big(B_{[0,k\beta]}\subset\Lambda_{L_N}\big) \geq \sum_{k=1}^{M}\frac{1}{k}\frac{1}{|\Lambda_{L_N}|}\int_{(1-\delta)\Lambda_{L_N}} \mathrm{d}x\, \mu^{(k\beta)}_{x,x}\big(B_{[0,k\beta]}\subset\Lambda_{L_N}\big). \tag{3.57}$$

It is easy to see that, in the limit $N\to\infty$, the integrand $\mu^{(k\beta)}_{x,x}(B_{[0,k\beta]}\subset\Lambda_{L_N})$ tends to $\mu^{(k\beta)}_{0,0}(\mathbb{1}) = (4\pi k\beta)^{-d/2}$, uniformly in $x\in(1-\delta)\Lambda_{L_N}$ and $k\in\{1,\dots,M\}$. Hence,

$$\liminf_{N\to\infty} \overline{q}^{(\mathrm{Dir})}_{\Lambda_{L_N}} \geq \sum_{k=1}^{M}\frac{1}{k}(4\pi k\beta)^{-d/2}\frac{|(1-\delta)\Lambda_{L_N}|}{|\Lambda_{L_N}|},$$

which tends to $\overline{q}$ as $M\to\infty$ and $\delta\downarrow 0$.

□

Proof of the upper bound for periodic boundary condition.

We continue to write Λ for Λ_{L_N}, where $L_N = (N/\rho)^{1/d}$. We adapt the proof of the upper bound in Section 3.2 for periodic boundary conditions. The main idea is to drop all the paths that reach the boundary of the box Λ and to use that their distribution is equal to the one under the free Brownian bridge measure. Let us introduce, for parameters $r\in(0,1)$ and $\widetilde{R}\in(0,\infty)$, the random variable

$$N^{(\ell,\widetilde{R})}_{r\Lambda}(\omega) = \sum_{x\in\xi\cap r\Lambda} \ell(f_x)\mathbb{1}\Big\{\sup_{s\in[0,\beta\ell(f_x)]}|f_x(s)-f_x(0)|\leq\widetilde{R}\Big\}, \tag{3.58}$$

the total length of the marks of particles starting in $r\Lambda$ that stay within distance $\leq\widetilde{R}$ from their starting sites. Furthermore, let

$$H^{(\widetilde{R})}_{r\Lambda}(\omega) = \sum_{x,y\in\xi\cap r\Lambda} T_{x,y}(\omega)\mathbb{1}\Big\{\sup_{s\in[0,\beta\ell(f_x)]}|f_x(s)-f_x(0)|\leq\widetilde{R}\Big\}\mathbb{1}\Big\{\sup_{s\in[0,\beta\ell(f_y)]}|f_y(s)-f_y(0)|\leq\widetilde{R}\Big\},$$

be the Hamiltonian in (1.14) restricted to paths starting in $r\Lambda$ and traveling no further than $\widetilde{R}$. Note that, for N large enough (depending only on r and $\widetilde{R}$), such paths do never reach the boundary of Λ

and therefore have the same distribution under the periodised Brownian bridge measure as under the free one or the one with Dirichlet boundary condition. Hence, we estimate

$$\begin{aligned}\mathtt{E}^{(\mathrm{per})}\Big[\mathrm{e}^{-H_\Lambda(\omega_{\mathrm{P}})}\mathbb{1}\{N_\Lambda^{(\ell)}(\omega_{\mathrm{P}})=N\}\Big] &\le \mathtt{E}^{(\mathrm{per})}\Big[\mathrm{e}^{-H_{r\Lambda}^{(\widetilde R)}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{r\Lambda}^{(\ell,\widetilde R)}(\omega_{\mathrm{P}})\le N\}\Big]\\ &= \mathtt{E}^{(\mathrm{Dir})}\Big[\mathrm{e}^{-H_{r\Lambda}^{(\widetilde R)}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{r\Lambda}^{(\ell,\widetilde R)}(\omega_{\mathrm{P}})\le N\}\Big]\\ &\le \mathtt{E}\Big[\mathrm{e}^{-H_{r\Lambda}^{(\widetilde R)}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{r\Lambda}^{(\ell,\widetilde R)}(\omega_{\mathrm{P}})\le N\}\Big],\end{aligned} \tag{3.59}$$

where '(per)' and '(Dir)' refer to the box Λ. Therefore, we can use the same method as in Section 3.2, the only two differences being that Λ is replaced by $r\Lambda$ and that we deal solely with paths that do not travel further than $\widetilde R$. That is, we have two additional truncation parameters r and $\widetilde R$. It is straightforward to see that adapted versions of Lemmas 3.1 and 3.2 hold and that the proof given in Section 3.2 applies verbatim as well. Finally, one takes the limits $\widetilde R\to\infty$ and $r\uparrow 1$ in the resulting variational formula, which is the same as the proof of (3.17).

Proof of the lower bound for Dirichlet boundary conditions.

We continue to write Λ for Λ_{L_N}, where $L_N=(N/\rho)^{1/d}$. The strategy for Dirichlet boundary conditions is as follows. First we pick some $\varepsilon\in(0,\frac12)$ and consider $\widetilde\Lambda=(1-\varepsilon)\Lambda$ and $\partial\Lambda=\Lambda\setminus\widetilde\Lambda$. The idea is to require that $\partial\Lambda$ receives no particle and that the marks of all particles in $\widetilde\Lambda$ have length $\le K$ and spatial extension $\le R$. In this way, we get a lower estimate against the truncated version of the Poisson process on $\widetilde\Lambda$ rather than on L. The only difference to the proof for empty boundary condition is then that Lemma 3.7, which was given before the introduction of the truncation, now has to be proved with the presence of the truncation, which requires some adaptation. Every other step of the proof is literally the same for Λ instead of $\widetilde\Lambda$, which means that in the end of the proof, the parameter ε has to be sent to 0, which is extremely simple.

Let us come to the details. We first show that there exist $c>0$ and $C_{K,R}>0$ such that, for any $N,R,K\in\mathbb N$,

$$\mathtt{E}^{(\mathrm{Dir})}\Big[\mathrm{e}^{-H_\Lambda(\omega_{\mathrm{P}})}\mathbb{1}\{N_\Lambda^{(\ell)}(\omega_{\mathrm{P}})=N\}\Big]\ge \mathrm{e}^{-\varepsilon c|\Lambda|}\mathrm{e}^{-C_{K,R}|\Lambda|}\mathtt{E}^{(K,R)}\Big[\mathrm{e}^{-H_{\widetilde\Lambda}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{\widetilde\Lambda}^{(\ell)}(\omega_{\mathrm{P}})=N\}\Big], \tag{3.60}$$

where $C_{K,R}\to 0$ as $R\to\infty$ and afterwards $K\to\infty$. This is done as follows. Estimate

$$\begin{aligned}&\mathtt{E}^{(\mathrm{Dir})}\Big[\mathrm{e}^{-H_\Lambda(\omega_{\mathrm{P}})}\mathbb{1}\{N_\Lambda^{(\ell)}(\omega_{\mathrm{P}})=N\}\Big]\\ &\quad= \mathtt{E}\Big[\mathrm{e}^{-H_\Lambda(\omega_{\mathrm{P}})}\mathbb{1}\{N_\Lambda^{(\ell)}(\omega_{\mathrm{P}})=N\}\mathbb{1}\{\forall x\in\xi_{\mathrm{P}}\cap\Lambda\colon B_x([0,\beta\ell(B_x)])\subset\Lambda\}\Big]\\ &\quad\ge \mathtt{E}\Big[\mathrm{e}^{-H_\Lambda(\omega_{\mathrm{P}})}\mathbb{1}\{N_\Lambda^{(\ell)}(\omega_{\mathrm{P}})=N\}\mathbb{1}\{\forall x\in\xi_{\mathrm{P}}\cap\widetilde\Lambda\colon B_x\in E^{(K,R)}\}\\ &\qquad\qquad\times\mathbb{1}\{\forall x\in\xi_{\mathrm{P}}\cap\Lambda\colon B_x([0,\beta\ell(B_x)])\subset\Lambda\}\mathbb{1}\{N_{\partial\Lambda}(\omega_{\mathrm{P}})=0\}\Big]\\ &\quad= \mathtt{E}\Big[\mathrm{e}^{-H_{\widetilde\Lambda}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{\widetilde\Lambda}^{(\ell)}(\omega_{\mathrm{P}})=N\}\mathbb{1}\{N_{\partial\Lambda}(\omega_{\mathrm{P}})=0\}\mathbb{1}\{\omega_{\mathrm{P}}\big(\widetilde\Lambda\times\big(E^{(K,R)}\big)^{\mathrm c}\big)=0\}\Big].\end{aligned} \tag{3.61}$$

Independence of the events in the indicators gives

$$\begin{aligned}\text{r.h.s. of (3.61)} &= \mathtt{E}^{(K,R)}\Big[\mathrm{e}^{-H_{\widetilde\Lambda}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{\widetilde\Lambda}^{(\ell)}(\omega_{\mathrm{P}})=N\}\Big]\mathtt{Q}(N_{\partial\Lambda}(\omega_{\mathrm{P}})=0)\mathtt{Q}\big(\omega_{\mathrm{P}}(\widetilde\Lambda\times\big(E^{(K,R)}\big)^{\mathrm c})=0\big)\\ &= \mathtt{E}^{(K,R)}\Big[\mathrm{e}^{-H_{\widetilde\Lambda}(\omega_{\mathrm{P}})}\mathbb{1}\{N_{\widetilde\Lambda}^{(\ell)}(\omega_{\mathrm{P}})=N\}\Big]\mathrm{e}^{-\overline q|\partial\Lambda|}\mathrm{e}^{-\nu(\widetilde\Lambda\times(E^{(K,R)})^{\mathrm c})},\end{aligned} \tag{3.62}$$

since $N_{\widetilde\Lambda}(\omega_{\rm P})$ and $\omega_{\rm P}(\widetilde\Lambda\times\big(E^{(K,R)}\big)^{\rm c})$ are Poisson distributed with respective parameters $\overline{q}|\partial\Lambda|$ and $\nu(\widetilde\Lambda\times(E^{(K,R)})^{\rm c})$. We estimate $\overline{q}|\partial\Lambda|\leq c\varepsilon|\Lambda|$ for some $c>0$ and

$$\nu(\widetilde\Lambda\times\big(E^{(K,R)}\big)^{\rm c})\leq|\widetilde\Lambda|\sum_{k=K+1}^\infty\frac{q_k}{k}+|\widetilde\Lambda|\sum_{k=1}^K\mu_{0,0}^{(k\beta)}\Big(\max_{s\in[0,\beta k]}|B_s|>R\Big)\leq|\Lambda|C_{K,R},\tag{3.63}$$

with some $C_{K,R}$ that vanishes as $R\to\infty$ and afterwards $K\to\infty$. Hence, we have got (3.60).

Now we need a version of Lemma 3.7 for truncated point processes, i.e., we need to show that, for any $R,K\in\mathbb N$ and for any $\delta\in(0,\rho)$, for all sufficiently large N,

$$\begin{aligned}&{\tt E}^{(K,R)}\Big[{\rm e}^{-H_\Lambda(\omega_{\rm P})}1\!\!1\{\langle\mathfrak{R}_N(\omega_{\rm P}),N_U^{(\ell)}\rangle=\rho\}\Big]\\&\qquad\geq\frac{(C_1\wedge C_2)^{\delta|\Lambda|}}{2\delta|\Lambda|+2}{\tt E}^{(K,R)}\Big[{\rm e}^{-|\Lambda|\langle\mathfrak{R}_N(\omega_{\rm P}),\Phi_\beta\rangle}1\!\!1\{\langle\mathfrak{R}_N(\omega_{\rm P}),N_U^{(\ell)}\rangle\in(\rho-\delta,\rho+\delta)\}\Big],\end{aligned}\tag{3.64}$$

where C_1 and C_2 may depend on R and K.

Since Lemma 3.4 was used in the proof of Lemma 3.7, we first need a truncated version of Lemma 3.4. For this we consider the truncated version of $Z_N(\beta,\Lambda)$:

$$Z_N^{(K,R)}(\beta,\Lambda)=\sum_{\lambda\in\mathfrak{P}_N\colon\sum_{k=1}^K k\lambda_k=N}\prod_{k=1}^K\frac{(\overline{q}_{k,\Lambda}^{(R)})^{\lambda_k}|\Lambda|^{\lambda_k}}{\lambda_k!k^{\lambda_k}}\bigotimes_{k=1}^K\big(\mathbb E_\Lambda^{(R,k\beta)}\big)^{\otimes\lambda_k}\big[{\rm e}^{-\mathcal G_{N,\beta}^{(\lambda)}}\big],\tag{3.65}$$

where

$$\overline{q}_{k,\Lambda}^{(R)}=\frac{1}{|\Lambda|}\int_\Lambda{\rm d}x\,\mu_{x,x}^{(k\beta)}\Big(\max_{s\in[0,\beta k]}|B_s-B_0|\leq R\Big),$$

and where $\mathbb E_\Lambda^{(R,k\beta)}$ is the expectation with respect to the probability measure

$$\mathbb P_\Lambda^{(R,k\beta)}({\rm d}f)=\frac{\int_\Lambda{\rm d}x\mu_{x,x}^{(k\beta)}\big({\rm d}f1\!\!1\{\max_{s\in[0,\beta k]}|f_s-f_0|\leq R\}\big)}{|\Lambda|\overline{q}_\Lambda^{(R)}}.$$

All steps in the proof of Lemma 3.4 are easily adapted, but the estimate in (3.25) needs a slightly different argument. We now estimate

$$\begin{aligned}\mathbb E_\Lambda^{(R,\beta)}(v(|B_s-f(s)|)&=\frac{1}{\overline{q}_\Lambda^{(R)}|\Lambda|}\int_\Lambda{\rm d}x\,\mathbb E_x\big[v(|B_s-f(s)|)1\!\!1\{\max_{0\leq s\leq\beta}|B_s-B_0|\leq R\},B_\beta\in{\rm d}x\big]/{\rm d}x\\&\leq\frac{(4\pi\beta)^{-d/2}}{\overline{q}_\Lambda^{(R)}|\Lambda|}\int_\Lambda{\rm d}x\int_\Lambda{\rm d}y\,\frac{g_s(x,y)v(|y-f(s)|)g_{\beta-s}(y,x)}{g_\beta(x,x)}.\end{aligned}$$

Now we can proceed as in (3.24)-(3.25) and obtain that $\mathbb E_\Lambda^{(R,\beta)}(v(|B_s-f(s)|)\leq\frac{\alpha(v)(4\pi\beta)^{-d/2}}{\overline{q}_\Lambda^{(R)}|\Lambda|}$. Hence, we get the following truncated version of Lemma 3.4:

$$\frac{Z_{N+1}^{(K,R)}(\beta,\Lambda)}{Z_N^{(K,R)}(\beta,\Lambda)}\geq\frac{|\Lambda|}{N+1}\exp\Big(-\frac{N\beta\alpha(v)(4\pi\beta)^{-d/2}}{|\Lambda|\overline{q}_\Lambda^{(R)}}\Big).\tag{3.66}$$

Using this instead of Lemma 3.4 in the proof of Lemma 3.7, we get the truncated version (3.64) of Lemma 3.7 with C_2 as before and with C_1 replaced by

$$C_1^{(R)}=1\wedge\frac{\overline{q}_\Lambda^{(R)}}{\rho+\delta}\exp\Big(-\frac{(\rho+\delta)\beta\alpha(v)(4\pi\beta)^{-d/2}}{\overline{q}_\Lambda^{(R)}}\Big).$$

The remaining proof of the lower bound is exactly as in the case of empty boundary condition, with $\widetilde\Lambda$ instead of Λ. This slight difference vanishes in the end when taking $\varepsilon\downarrow0$.

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