LES and VMS Methods for the Simulation of Incompressible Turbulent Flows

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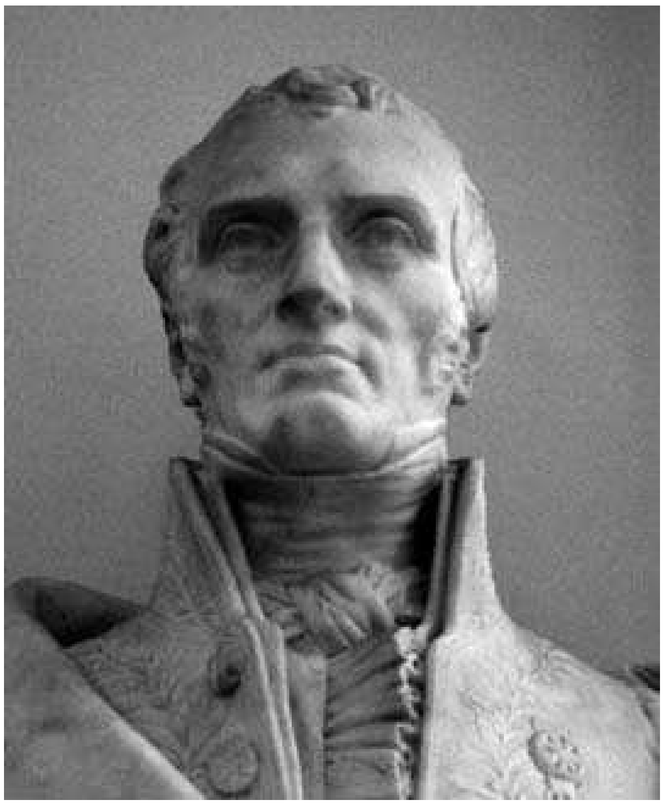
Contents

1 Incompressible Turbulent Flows

- Navier–Stokes equations: fundamental equations of fluid dynamics
- Claude Louis Marie Henri Navier (1785 – 1836), George Gabriel Stokes (1819 – 1903)



The Incompressible Navier–Stokes Equations

- conservation laws
  - conservation of linear momentum
  - conservation of mass

$$
\begin{aligned}
\mathbf{u}_t - 2Re^{-1}\nabla\cdot\mathbb{D}(\mathbf{u}) + \nabla\cdot(\mathbf{u}\mathbf{u}^T) + \nabla p &= \mathbf{f} && \text{in } (0,T]\times\Omega\\
\nabla\cdot\mathbf{u} &= 0 && \text{in } [0,T]\times\Omega\\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0 && \text{in } \Omega
\end{aligned}
$$

+ boundary conditions

- given:
  - $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$: domain
  - T: final time
  - $\mathbf{u}_0$: initial velocity
  - boundary conditions
- to compute:
  - velocity $\mathbf{u}$, where

$$
\mathbb{D}(\mathbf{u}) = \frac{\nabla\mathbf{u} + \nabla\mathbf{u}^T}{2},
$$

    is the velocity deformation tensor
  - pressure p
- parameter:
  - Reynolds number Re

The Incompressible Navier–Stokes Equations (cont.)

- Reynolds number

$$Re = \frac{LU}{\nu}$$

  - L $[m]$ – characteristic length scale (diameter of a channel, diameter of a body in the flow)
  - U $[ms^{-1}]$ – characteristic velocity scale (inflow velocity)
  - ν $[m^2 s^{-1}]$ – kinematic viscosity (water: $\nu = 10^{-6}\ m^2 s^{-1}$)
- rough classification of flows:
  - Re small: steady–state flow field (if data do not depend on time)
  - Re larger: time–dependent flow field
  - Re very large: turbulent flows

The Incompressible Navier–Stokes Equations (cont.)

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- rough classification of flows:
  - Re small: steady–state flow field (if data do not depend on time)
  - Re larger: time–dependent flow field
  - Re very large: turbulent flows
- There is no exact definition of what is a turbulent flow !

The Incompressible Navier–Stokes Equations (cont.)

- mathematical analysis
  - 2d: existence and uniqueness of weak solution, Leary (1933), Hopf (1951)
  - 3d: existence of weak solution, Leary (1933), Hopf (1951)

Uniqueness of weak solution of 3d Navier–Stokes equations is open problem !

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Uniqueness of weak solution of 3d Navier–Stokes equations is open problem !

- difficulty in numerical analysis of methods for simulating turbulent flows
  - assumption of sufficient regularity of solution such that uniqueness is given
  - How regular are turbulent flow fields ?

Characteristics of Turbulent Flows

- posses flow structures of very different size
  - hurricane Katrina (2005)

  - some large eddies (scales), many very small eddies (scales)

Characteristics of Turbulent Flows (cont.)

- Richardson energy cascade: energy is transported in the mean from large to smaller eddies

  - start of cascade: kinetic energy introduced into flow by productive mechanisms at largest scale
  - inner cascade: transmitting energy to smaller and smaller scales by processes not depending on molecular viscosity
  - end of cascade: molecular viscosity enforcing dissipation of kinetic energy at smallest scales

- smallest scales important for physics of the flow

Characteristics of Turbulent Flows (cont.)

- Kolmogorov (1941):

  energy is dissipated from eddies of size λ (Kolmogorov scale) such that

$$Re(\lambda) = \frac{\lambda u_\lambda}{\nu} = 1, \ \lambda = \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \ [m]$$

Characteristics of Turbulent Flows (cont.)

- Kolmogorov (1941):

  

  energy is dissipated from eddies of size λ (Kolmogorov scale) such that

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- size of the smallest eddies
  - rate of dissipation of turbulent energy (from theoretical and experimental studies)

    $$\epsilon := 2\nu \left\langle \mathbb{D}(\mathbf{u})' : \mathbb{D}(\mathbf{u})' \right\rangle \sim \frac{U^3}{L} \ [m^2 s^{-3}]$$

    $\langle \cdot \rangle$ – mean value, $\mathbf{u}' = \mathbf{u} - \langle \mathbf{u} \rangle$ fluctuation
  - $\Longrightarrow$

    $$\frac{\lambda}{L} \sim \left(\frac{\nu^3}{L^3 U^3}\right)^{1/4} = Re^{-3/4} \quad \Longleftrightarrow \quad \lambda \sim Re^{-3/4}$$

Impact on Numerical Simulations

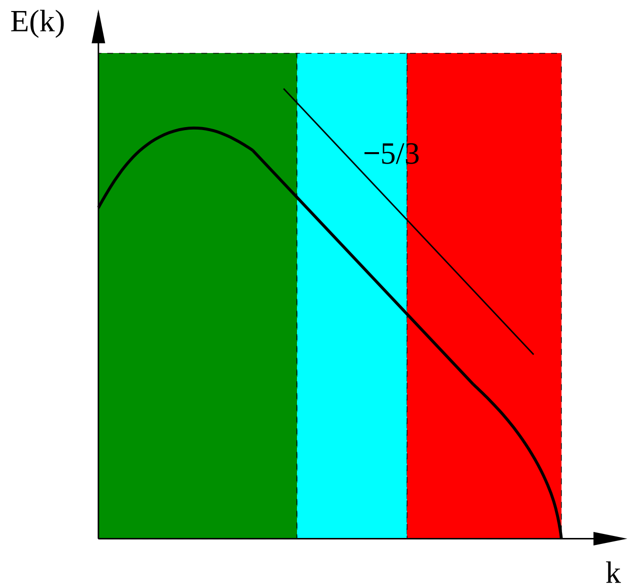
- Galerkin method aims to simulate all persisting eddies, Direct Numerical Simulation (DNS)
  - $\Omega = (0,1)^3 \implies L = 1$
  - approx 10^7 cubic mesh cells ($\approx 215^3$)
  - low order method (mesh width $\approx$ resolution of discretization)
  - $\implies \lambda \approx 1/215$
  - $\implies Re \approx 1290$
- applications: Reynolds numbers larger by orders of magnitude

Direct Numerical Simulation not feasible !

- only resolvable scales can be simulated

The Kolmogorov Energy Spectrum

- energy of scales in wave number space (Fourier space)



- logarithmic axes
- resolved scales
  - large scales
  - resolved small scales
- unresolved scales, subgrid scales

- k – wave number
- $E(k)$ – turbulent kinetic energy of modes with wave number k
- $k^{-5/3}$ – law of energy spectrum: $E(k) \sim \epsilon^{2/3} k^{-5/3}$

Remarks to 3d vs. 2d

- smallest scales in 2d flows, Kraichnan (1967)

$$\lambda = \mathcal{O}\left(Re^{-1/2}\right)$$

- vortex stretching
  - vorticity: $\boldsymbol{\omega} = \nabla \times \mathbf{u}$
  - neglect viscous term for large Reynolds numbers

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} \approx \boldsymbol{\omega} \cdot \nabla \mathbf{u}$$

  - equation of infinitesimal line element of material
  - if $\nabla \mathbf{u}$ acts to stretch the line element than $|\boldsymbol{\omega}|$ will be stretched, too $\Longrightarrow$ vortex stretching, important feature of turbulent flows
  - 2d: right hand side vanishes $\Longrightarrow$ no vortex stretching

2d flows at high Reynolds number are qualitatively different from 3d turbulent flows

Summary

- DNS impossible
- (very) small scales important, have to be taken into account
- 3d simulations necessary
- literature
  - P.A. Davidson, *Turbulence*, Oxford University Press, 2004
  - U. Frisch, *Turbulence*, Cambridge University Press, 1995
  - S.B. Pope, *Turbulent Flows*, Cambridge University Press, 2000

Summary

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Impact on numerical simulations

- only large scales of a turbulent flows possible to simulate, two approaches
  - Large Eddy Simulation (LES)
  - Variational Multiscale (VMS) Methods
- impact of the small scales has to be modelled

2 Large Eddy Simulation (LES)

2.1 The Space–Averaged Navier–Stokes Equations

- two–scale decomposition of the flow: large and unresolved scales
- main idea in LES: large scales are defined by averages in space
  - equations for the large scales necessary

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- two–scale decomposition of the flow: large and unresolved scales
- main idea in LES: large scales are defined by averages in space
  - equations for the large scales necessary
- starting point: incompressible Navier–Stokes equations

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\begin{aligned}
\mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + \nabla\cdot(\mathbf{u}\mathbf{u}^T) + \nabla p &= \mathbf{f} && \text{in } (0,T]\times\Omega \\
\nabla\cdot\mathbf{u} &= 0 && \text{in } [0,T]\times\Omega \\
\mathbf{u} &= \mathbf{0} && \text{in } [0,T]\times\partial\Omega \\
\mathbf{u}(0,\mathbf{x}) &= \mathbf{u}_0 && \text{in } \Omega \\
\int_\Omega p\,d\mathbf{x} &= 0 && \text{in } (0,T]
\end{aligned}
$$

- $\Omega\subset\mathbb{R}^d, d=2,3$: bounded domain, with Lipschitz boundary $\partial\Omega$

Space–Averaged Navier–Stokes Equations (cont.)

- assumptions :
  - regularity :

$$
\begin{aligned}
&\mathbf{u} \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^d && \text{for } t \in [0, T] \\
&\mathbf{u} \in \left(H^1((0, T))\right)^d && \text{for } \mathbf{x} \in \overline{\Omega} \\
&p \in H^1(\Omega) \cap \mathrm{L}_0^2(\Omega) && \text{for } t \in (0, T]
\end{aligned}
$$

  - weak solution is unique

Space–Averaged Navier–Stokes Equations (cont.)

- decompose velocity and pressure

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \quad p = \overline{p} + p'$$

  - $\overline{\mathbf{u}}, \overline{p}$: large scales
  - $\mathbf{u}', p'$: subgrid scales

Space–Averaged Navier–Stokes Equations (cont.)

- decompose velocity and pressure

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  - $\overline{\mathbf{u}}$, $\overline{p}$: large scales
  - $\mathbf{u}', p'$: subgrid scales

- large scales defined by averaging in space (convolution with filter function)
  - filter out small flow structures
  - damp high wave numbers
- goal of LES : approximate $\overline{\mathbf{u}}$, $\overline{p}$ $\Longrightarrow$ one needs equations for $\overline{\mathbf{u}}$, $\overline{p}$

Space–Averaged Navier–Stokes Equations (cont.)

- derivation of space averaged Navier–Stokes equations (literature) :
  - filter Navier–Stokes equations with filter function g

$$g * (\nabla \cdot \mathbf{u}) = \overline{\nabla \cdot \mathbf{u}}$$

  - assume that convolution and differentiation commute

$$g * (\nabla \cdot \cdot) = \nabla \cdot (g * \cdot)$$

  - commute both operators

$$g * (\nabla \cdot \mathbf{u}) = \nabla \cdot (g * \mathbf{u}) = \nabla \cdot \overline{\mathbf{u}}$$

$\Longrightarrow$ equation for $\overline{\mathbf{u}}$

Space–Averaged Navier–Stokes Equations (cont.)

- application of convolution well defined in $\mathbb{R}^d \Longrightarrow$ extend all functions to $\mathbb{R}^d$:

$$\mathbf{u} = \mathbf{0}, \quad \mathbf{u}_0 = \mathbf{0}, \quad p = 0, \quad \mathbf{f} = \mathbf{0} \quad \text{for } \mathbf{x} \notin \overline{\Omega}$$

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- resulting regularities :

$$\begin{aligned} &\mathbf{u} \in \left(H_0^1(\mathbb{R}^d)\right)^d && \text{for } t \in [0,T] \\ &\mathbf{u} \in \left(H^1((0,T))\right)^d && \text{for } \mathbf{x} \in \mathbb{R}^d \\ &p \in L_0^2(\mathbb{R}^d) && \text{for } t \in (0,T] \end{aligned}$$

$\Longrightarrow$ well defined in $\mathbb{R}^d$

$$\mathbf{u}_t, \quad \nabla \cdot (\mathbf{u}\mathbf{u}^T), \quad \nabla \mathbf{u}, \quad \nabla \cdot \mathbf{u}$$

Space–Averaged Navier–Stokes Equations (cont.)

- define pressure term and viscous term in the sense of distributions :
  - $\varphi \in C_0^\infty(\mathbb{R}^d)$
  - pressure term

$$
\begin{aligned}
(\nabla p)(\varphi)(t) \quad &:= \quad -\int_{\mathbb{R}^d} p(t,\mathbf{x})\nabla\varphi(\mathbf{x})\,d\mathbf{x} \\
&= \quad \int_\Omega \varphi(\mathbf{x})\nabla p(t,\mathbf{x})\,d\mathbf{x} - \int_{\partial\Omega} \varphi(\mathbf{s})p(t,\mathbf{s})\mathbf{n}(\mathbf{s})\,d\mathbf{s}
\end{aligned}
$$

$\mathbf{n}$ - outward pointing unit normal on $\partial\Omega$

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  - viscous term

$$\begin{aligned}\nabla\cdot\mathbb{D}(\mathbf{u})\,(\varphi)\,(t) &:= -\int_{\mathbb{R}^d}\mathbb{D}(\mathbf{u})\,(t,\mathbf{x})\,\nabla\varphi\,(\mathbf{x})\,d\mathbf{x} \\ &= \int_\Omega \varphi\,(\mathbf{x})\,\nabla\cdot\mathbb{D}(\mathbf{u})\,(t,\mathbf{x})\,d\mathbf{x} - \int_{\partial\Omega}\varphi\,(\mathbf{s})\,\mathbb{D}(\mathbf{u})\,(t,\mathbf{s})\,\mathbf{n}\,(\mathbf{s})\,d\mathbf{s}\end{aligned}$$

Space–Averaged Navier–Stokes Equations (cont.)

- convolve distributional form of momentum equation with a filter function $g(x) \in C^\infty(\mathbb{R}^d)$

Convolution and differentiation commute !

Space–Averaged Navier–Stokes Equations (cont.)

- convolve distributional form of momentum equation with a filter function $g(x) \in \mathrm{C}^\infty(\mathbb{R}^d)$

  Convolution and differentiation commute !

- space averaged momentum equation in $(0,T] \times \mathbb{R}^d$

$$
\begin{aligned}
\overline{\mathbf{u}}_t - 2\nu\nabla\cdot\mathbb{D}(\overline{\mathbf{u}}) + \nabla\cdot\left(\overline{\mathbf{u}\mathbf{u}^T}\right) + \nabla\overline{p} \\
= \quad \overline{\mathbf{f}} + \int_{\partial\Omega} g\left(\mathbf{x}-\mathbf{s}\right)\mathbb{S}\left(\mathbf{u},p\right)\left(t,\mathbf{s}\right)\mathbf{n}\left(\mathbf{s}\right)d\mathbf{s}
\end{aligned}
$$

with the stress tensor

$$
\mathbb{S}\left(\mathbf{u},p\right) = 2\nu\mathbb{D}(\mathbf{u}) - p\mathbb{I}
$$

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$$

with the stress tensor

$$
\mathbb{S}\left(\mathbf{u},p\right) = 2\nu\mathbb{D}(\mathbf{u}) - p\mathbb{I}
$$

- regularity of the normal stress

$$
\mathbb{S}\left(\mathbf{u},p\right)\mathbf{n} \in \left(L^q(\partial\Omega)\right)^d, \quad d=2: q\in[1,\infty), \quad d=3: q\in[1,4]
$$

- usual practice: neglect term with normal stress (does not appear if $\Omega = \mathbb{R}^d$)

Space–Averaged Navier–Stokes Equations (cont.)

- closure problem: space averaged Navier–Stokes equations not yet equations for $(\overline{\mathbf{u}}, \overline{p})$

$$\nabla \cdot \left(\overline{\mathbf{u}\mathbf{u}^T} \right) = \nabla \cdot \left(\overline{\mathbf{u}}\, \overline{\mathbf{u}}^T \right) - \nabla \cdot \left(\overline{\mathbf{u}}\, \overline{\mathbf{u}}^T - \overline{\mathbf{u}\mathbf{u}^T} \right)$$

last term (divergence of Reynolds stress tensor) depends on all scales

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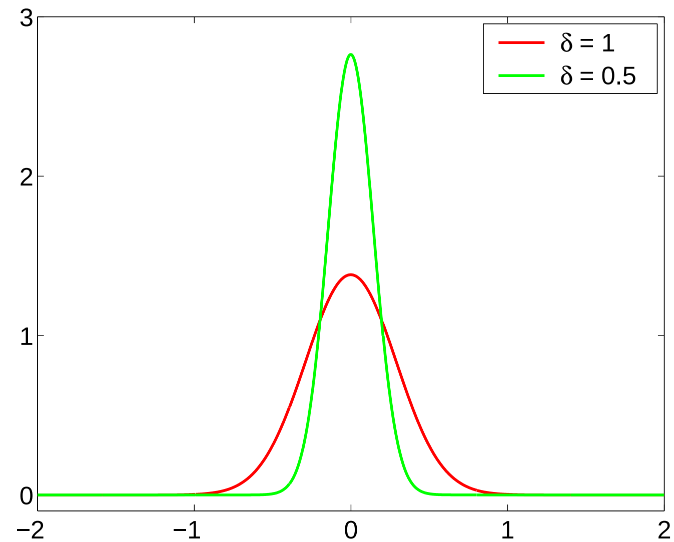
  last term (divergence of Reynolds stress tensor) depends on all scales

- open problems:
  - modelling of Reynolds stress tensor, main topic in LES
  - analysis of commutation error term

2.2 Commutation Errors

- standard filter function: Gaussian filter

$$g_\delta(\mathbf{x}) = \left(\frac{6}{\delta^2\pi}\right)^{d/2} \exp\left(-\frac{6}{\delta^2}\|\mathbf{x}\|_2^2\right)$$



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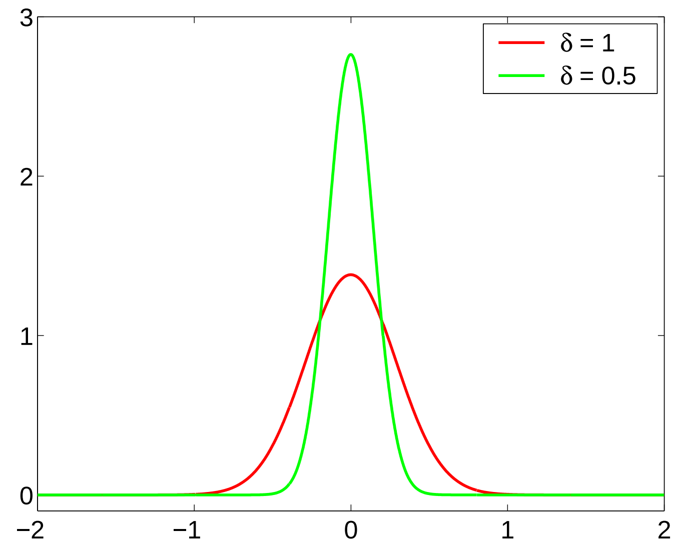

- δ – filter width, larger than mesh width
- properties for $\delta = const.$:
  - regularity : $g_\delta \in C^\infty(\mathbb{R}^d)$,
  - positivity : $0 < g_\delta(\mathbf{x}) \le \left(\frac{6}{\delta^2\pi}\right)^{\frac{d}{2}}$,
  - integrability : $g_\delta \in L^p(\mathbb{R}^d), p \in [1,\infty]$, $\|g_\delta\|_{L^1(\mathbb{R}^d)} = 1$,
  - symmetry : $g_\delta(\mathbf{x}) = g_\delta(-\mathbf{x})$,
  - monotonicity : $g_\delta(\mathbf{x}) \ge g_\delta(\mathbf{y})$ if $\|\mathbf{x}\|_2 \le \|\mathbf{y}\|_2$

Commutation Errors – Constant Filter Width (cont.)

- per definition : $\delta \to 0$ implies $\overline{\mathbf{u}} \to \mathbf{u}$
- questions :
  - Implies $\delta \to 0$ in a certain sense

$$\int_{\partial\Omega} g_\delta \left(\mathbf{x}-\mathbf{s}\right) \mathbb{S}\left(\mathbf{u},p\right)\left(t,\mathbf{s}\right) \mathbf{n}\left(\mathbf{s}\right) d\mathbf{s} \to 0 \quad ?$$

  - How fast is the convergence w.r.t. δ ?
- analyse terms of the form

$$\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})d\mathbf{s}$$

  with $\psi(\mathbf{s}) \in L^q(\partial\Omega)$, $1 \leq q \leq \infty$
- Dunca, J., Layton (2004), J. (2004)

Commutation Errors – Constant Filter Width (cont.)

- strong form of the commutation error
- one can show that :
  - regularity :

$$\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})d\mathbf{s} \in L^p(\mathbb{R}^d), \quad 1 \le p \le \infty$$

  - in general : no convergence for $\delta \to 0$:

$$\lim_{\delta\to 0} \left\| \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})d\mathbf{s} \right\|_{L^p(\mathbb{R}^d)} = 0,$$

$1 \le p \le \infty$, if and only if

$$\psi(\mathbf{s}) = 0 \text{ a.e. on } \partial\Omega$$

Commutation Errors – Constant Filter Width (cont.)

- proof: assumption $\lim_{\delta\to 0}\left\|\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\psi(\mathbf{s})d\mathbf{s}\right\|_{L^p(\mathbb{R}^d)} = 0$
- then follows for every $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned}
&\lim_{\delta\to 0}\left|\int_{\mathbb{R}^d}\varphi(\mathbf{x})\left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\,\psi(\mathbf{s})\,d\mathbf{s}\right)d\mathbf{x}\right| \\
&\quad\leq \quad \lim_{\delta\to 0}\|\varphi\|_{L^q(\mathbb{R}^d)}\left\|\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\,\psi(\mathbf{s})\,d\mathbf{s}\right\|_{L^p(\mathbb{R}^d)} = 0
\end{aligned}$$

- for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ is

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{\mathbb{R}^d}\varphi(\mathbf{x})\left(\int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\,\psi(\mathbf{s})\,d\mathbf{s}\right)d\mathbf{x} \\
&\quad = \quad \lim_{\delta\to 0}\int_{\partial\Omega}\psi(\mathbf{s})\left(\int_{\mathbb{R}^d} g_\delta(\mathbf{x}-\mathbf{s})\,\varphi(\mathbf{x})\,d\mathbf{x}\right)d\mathbf{s} = \int_{\partial\Omega}\psi(\mathbf{s})\,\varphi(\mathbf{s})\,d\mathbf{s}
\end{aligned}$$

- $\Longrightarrow$ for every $\varphi \in C_0^\infty(\mathbb{R}^d)$: $0 = \left|\int_{\partial\Omega}\psi(\mathbf{s})\,\varphi(\mathbf{s})\,d\mathbf{s}\right|$
- $\Longrightarrow \psi = 0$ a.e. on $\partial\Omega$

Commutation Errors – Constant Filter Width (cont.)

- implications :
  - $$S(\mathbf{u}, p)(t, \mathbf{s})\,\mathbf{n}(\mathbf{s}) = \mathbf{0} \text{ on } \partial\Omega$$
    $$\Longleftrightarrow$$
    fluid and boundary exert exactly zero force on each other
  - commutation error does not vanish asymptotically for discretisations which rely upon a strong form of the space averaged Navier–Stokes equations, e.g., finite difference methods !!

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- $H^{-1}(\Omega)$ norm of the commutation error
  - estimate
    $$\left\| \int_{\partial\Omega} g_\delta(\mathbf{x}-\mathbf{s})\,\psi(\mathbf{s})\,d\mathbf{s} \right\|_{H^{-1}(\Omega)} \le C\delta^{1/2}\,\|\psi\|_{L^2(\partial\Omega)}$$
    for each $\delta > 0 \Longrightarrow$ order of convergence at least $1/2$
  - commutation error vanishes asymptotically for discretisations which rely upon a weak form of the space averaged NSE, e.g., finite element methods !!

Commutation Errors – Nonconstant Filter Width

- observation: difficulties arise from non–smooth extensions of functions off Ω
- goal: use filter with support always in Ω (bounded)

Commutation Errors – Nonconstant Filter Width

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- goal: use filter with support always in Ω (bounded)
  - non–uniform box filter

$$\overline{u}(\mathbf{y}) = \frac{1}{8\delta(x)\delta(y)\delta(z)} \int_{x-\delta(x)}^{x+\delta(x)} \int_{y-\delta(y)}^{y+\delta(y)} \int_{z-\delta(z)}^{z+\delta(z)} u(\mathbf{x})\, d\mathbf{x}$$

  - non–constant filter width s.t. $\delta \to 0$ as $\mathbf{x} \to \partial\Omega$

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  - non–constant filter width s.t. $\delta \to 0$ as $\mathbf{x} \to \partial\Omega$
- implications
  - no extension of functions necessary for filter operation to be well defined
  - commutation error because of non–constant filter width
- concrete formulas in Berselli, Grisanti, J. (2007)
  - asymptotic vanishing of commutation errors requires very small filter widths at the boundary
  - filter width depends on regularity of the filtered function
  - implication: resolution of the flow at the boundary becomes necessary

Commutation Errors – Nonconstant Filter Width (cont.)

- extra terms in space averaged Navier–Stokes equations

$$\text{commutation error } + \nabla \cdot \left(\overline{\mathbf{u}}\, \overline{\mathbf{u}}^T - \overline{\mathbf{u}\mathbf{u}^T} \right)$$

importance of both terms studied in Berselli, J. (2006)

Commutation Errors – Nonconstant Filter Width (cont.)

- extra terms in space averaged Navier–Stokes equations

$$\text{commutation error } + \nabla \cdot \left(\overline{\mathbf{u}}\, \overline{\mathbf{u}}^T - \overline{\mathbf{u}\mathbf{u}^T} \right)$$

  importance of both terms studied in Berselli, J. (2006)
- away from boundary: divergence of Reynolds stress tensor more important
- modelling the unknown flow field near the boundary with wall laws (mean flow), e.g. $1/\alpha$th power law, $+$ fluctuations
  - mean flow responsible for leading order terms in commutation errors
  - commutation error and divergence of Reynolds stress tensor are asymptotically of same order

modelling of commutation error at boundary as important as
modelling of divergence of Reynolds stress tensor

Commutation Errors – Nonconstant Filter Width (cont.)

- extra terms in space averaged Navier–Stokes equations

$$\text{commutation error } + \nabla \cdot \left(\overline{\mathbf{u}}\, \overline{\mathbf{u}}^T - \overline{\mathbf{u}\mathbf{u}^T} \right)$$

  importance of both terms studied in Berselli, J. (2006)

- away from boundary: divergence of Reynolds stress tensor more important
- modelling the unknown flow field near the boundary with wall laws (mean flow), e.g. $1/\alpha$th power law, $+$ fluctuations
  - mean flow responsible for leading order terms in commutation errors
  - commutation error and divergence of Reynolds stress tensor are asymptotically of same order

modelling of commutation error at boundary as important as
modelling of divergence of Reynolds stress tensor

- numerical studies
  - van der Bos, Geurts (2005) observed important commutation errors for some kinds of filters

Commutation Errors, Summary

- some open problems
  - optimal order of convergence for $H^{-1}(\Omega)$ commutation error
  - commutation error analysis for other filters than Gaussian and box filter
  - ...

Commutation Errors, Summary

- some open problems
  - optimal order of convergence for $H^{-1}(\Omega)$ commutation error
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  - ...

- Summary
  - commutation error give important contributions in the derivation of the space averaged Navier–Stokes equations
  - they are important at and near the boundary
  - they are simply neglected in practice, practinioners do not care about the analytical results

2.3 Models

- space averaged Navier–Stokes equations in $(0,T] \times \mathbb{R}^d$

$$\begin{aligned} \overline{\mathbf{u}}_t - 2\nu\nabla\cdot\mathbb{D}(\overline{\mathbf{u}}) + \nabla\cdot\left(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T\right) + \nabla\overline{p} &= \overline{\mathbf{f}} + \nabla\cdot\left(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T - \overline{\mathbf{u}\mathbf{u}^T}\right) \\ \nabla\cdot\overline{\mathbf{u}} &= 0 \end{aligned} \qquad (1)$$

- closure problem :
  - $d+1$ space averaged unknowns in (1) and $d(d+1)/2$ unknown values in $\overline{\mathbf{u}\mathbf{u}^T}$
  - only $d+1$ equations in (1)

2.3 Models

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\end{aligned} \qquad (1)
$$

- closure problem :
  - $d+1$ space averaged unknowns in (1) and $d(d+1)/2$ unknown values in $\overline{\mathbf{u}\mathbf{u}^T}$
  - only $d+1$ equations in (1)
- main issue in LES : model $\overline{\mathbf{u}\mathbf{u}^T}$ with $(\overline{\mathbf{u}}, \overline{p})$

2.3.1 Models Based on an Approximation in Fourier Space

- derivation is mainly based on mathematical arguments (not physical)
- $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$ implies

$$\overline{\mathbf{u}\mathbf{u}^T} = \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T} + \overline{\overline{\mathbf{u}}\,\mathbf{u}'^T} + \overline{\mathbf{u}'\,\overline{\mathbf{u}}^T} + \overline{\mathbf{u}'\mathbf{u}'^T} \qquad (2)$$

- $\overline{\mathbf{u}}$ defined with Gaussian filter
- derivation:
  - transform each term of (2) to the Fourier space
  - replace Fourier transform of $\mathbf{u}'$ by Fourier transform of $\overline{\mathbf{u}}$
  - approximate Fourier transform of the Gaussian filter by a simpler function (2nd order approximations)
  - neglect all terms which are in certain sense of higher order (formally δ^4)
  - apply inverse Fourier transform

Models Based on an Approximation in Fourier Space (cont.)

- transform to Fourier space
  - large scale advective term

$$\mathcal{F}\left(\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T}\right) = \mathcal{F}(g_\delta)\,\mathcal{F}\left(\overline{\mathbf{u}}\,\overline{\mathbf{u}}^T\right)$$

  - cross terms

$$\mathcal{F}\left(\overline{\overline{\mathbf{u}}\,\mathbf{u}'^T}\right) = \mathcal{F}(g_\delta)\left(\mathcal{F}(\overline{\mathbf{u}}) * \mathcal{F}(\mathbf{u}')^T\right)$$

replace $\mathcal{F}(\mathbf{u}')$, use $\mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$ and $\mathcal{F}(g_\delta) \neq 0$, use $\mathcal{F}(\overline{\mathbf{u}}) = \mathcal{F}(g_\delta)\,\mathcal{F}(\mathbf{u})$

$$\mathcal{F}(\mathbf{u}') = \mathcal{F}(\mathbf{u}) - \mathcal{F}(\overline{\mathbf{u}}) = \left(\frac{1}{\mathcal{F}(g_\delta)} - 1\right)\mathcal{F}(\overline{\mathbf{u}})$$

gives

$$\mathcal{F}\left(\overline{\overline{\mathbf{u}}\,\mathbf{u}'^T}\right) = \mathcal{F}(g_\delta)\left(\mathcal{F}(\overline{\mathbf{u}}) * \left(\frac{1}{\mathcal{F}(g_\delta)} - 1\right)\mathcal{F}(\overline{\mathbf{u}})^T\right)$$

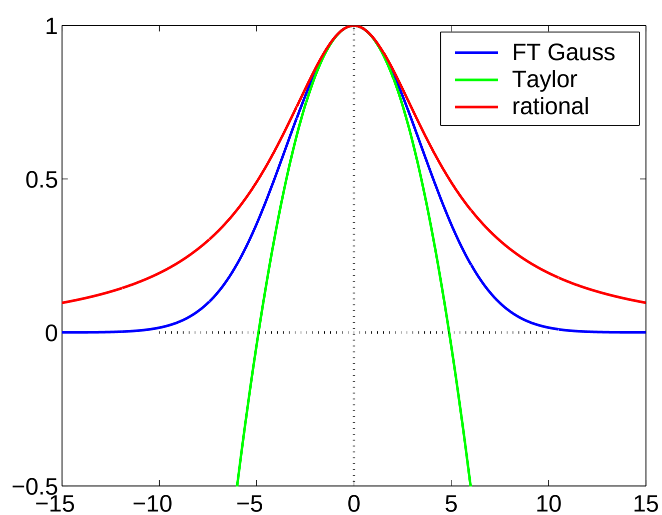
- no modeling up to here

Models Based on an Approximation in Fourier Space (cont.)

- Approximation of the Fourier transform of the Gaussian filter $\mathcal{F}(g_\delta)$
  - Taylor series (Leonard (1974), Clark, Reynolds, Ferziger (1979)), Taylor LES model, gradient method

    Damping of highly oscillating components is not preserved !!!
  - approximation with rational function (Galdi, Layton (2000)), rational LES model

$$\mathcal{F}(g_\delta)(\delta, \mathbf{y}) = 1 - \frac{\|\mathbf{y}\|_2^2}{4\gamma}\delta^2 + \mathcal{O}\left(\delta^4\right) \text{ vs. } \mathcal{F}(g_\delta)(\delta, \mathbf{y}) = \frac{1}{1 + \frac{\|\mathbf{y}\|_2^2}{4\gamma}\delta^2} + \mathcal{O}\left(\delta^4\right)$$



Models Based on an Approximation in Fourier Space (cont.)

- subgrid scale term
  - both approaches $\overline{\mathbf{u}'\mathbf{u}'^T} \approx \mathbf{0}$
  - blow–up in finite time in numerical simulations, J. (2004)
- use instead
  - Smagorinsky model (1963), see later for details

$$\overline{\mathbf{u}'\mathbf{u}'^T} \approx -c_S\delta^2 \left\|\mathbb{D}(\,\overline{\mathbf{u}}\,)\right\|_F \mathbb{D}(\,\overline{\mathbf{u}}\,)$$

    formally of order δ^2

  - Iliescu–Layton model (1998)

$$\overline{\mathbf{u}'\mathbf{u}'^T} \approx -c_S\delta \left\|\,\overline{\mathbf{u}} - g_\delta * \overline{\mathbf{u}}\,\right\|_2 \mathbb{D}(\,\overline{\mathbf{u}}\,)$$

    formally of order δ^3

Models Based on an Approximation in Fourier Space (cont.)

- find approximation $(\mathbf{w}, r)$ to $(\overline{\mathbf{u}}, \overline{p})$ such that in $(0,T] \times \mathbb{R}^d$

$$\begin{aligned} \mathbf{w}_t - 2\nabla \cdot ((\nu + \nu_T)\mathbb{D}(\mathbf{w})) + (\mathbf{w} \cdot \nabla)\mathbf{w} & & \\ +\nabla r + \nabla \cdot \frac{\delta^2}{12}\left(A\left(\nabla \mathbf{w} \nabla \mathbf{w}^T\right)\right) &=& \overline{\mathbf{f}} \\ \nabla \cdot \mathbf{w} &=& 0 \\ \mathbf{w}(0,\mathbf{x}) &=& \overline{\mathbf{u}_0} \end{aligned}$$

- turbulent viscosity, eddy viscosity

$$\nu_T = c_S \delta^2 \left\|\mathbb{D}(\mathbf{w})\right\|_F \quad \text{or} \quad \nu_T = c_S \delta \left\|\mathbf{w} - g_\delta * \mathbf{w}\right\|_2$$

- LES model

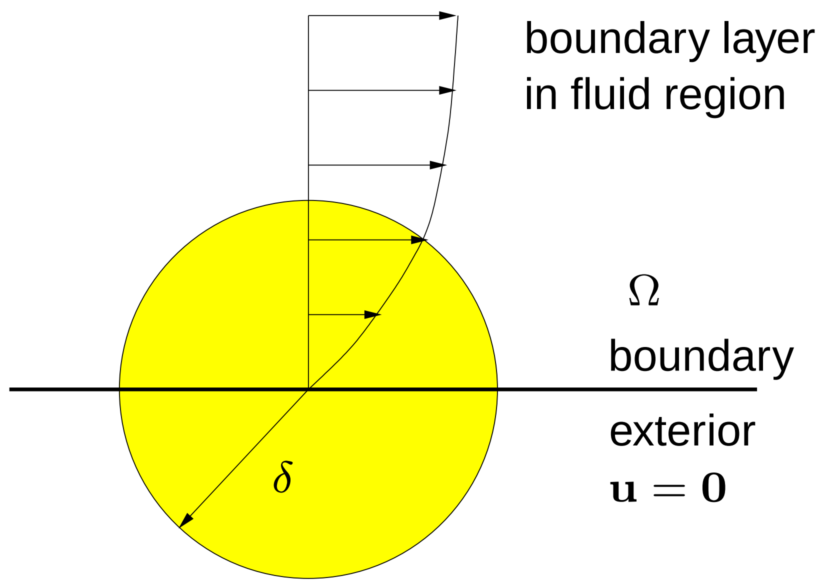
  $A = 0$: Smagorinsky model ($\nu_T = 0$: Navier–Stokes equations)

  $A = I$: Taylor LES model

  $A = (I - \delta^2/(24)\Delta)^{-1}$: rational LES model
    - inverse of a Helmholtz operator, differential filter
    - approximation of convolution with Gaussian filter

2.3.2 Bounded Domain

- restrict equations to Ω
  - unknown error is committed
- boundary conditions for large scales
  - unresolved problem
  - boundary conditions of $(\mathbf{u}, p)$ for $(\mathbf{w}, r)$: wrong



Boundary Conditions

- slip with friction and no penetration, Galdi, Layton (2000)
  - problem : determination of friction coefficient
  - can be given for model problems, J., Layton, Sahin (2004)
  - numerical experiences: Hoffman (2005, . . .)

Boundary Conditions

- slip with friction and no penetration, Galdi, Layton (2000)
  - problem : determination of friction coefficient
  - can be given for model problems, J., Layton, Sahin (2004)
  - numerical experiences: Hoffman (2005, . . .)
- boundary treatment in practice, Piomelli, Balaras (2002)
  - impose some form of law of the wall
  - solve simplified equations in boundary layer regions

2.3.3 The Smagorinsky Model

- starting point Boussinesq hypothesis

Turbulent fluctuations are dissipative in the mean.

$$\Longrightarrow \quad \nabla \cdot \left(\overline{\mathbf{u}\mathbf{u}^T} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}^T \right) \approx -\nabla \cdot (\nu_T \mathbb{D}(\overline{\mathbf{u}})) + \text{terms inc. in } \overline{p}$$

ν_T – eddy viscosity, turbulent viscosity

The Smagorinsky Model – the Turbulent Viscosity Coefficient

- rate of dissipation of turbulent energy

$$\epsilon \sim \frac{U_{\text{int}}^3}{L_{\text{int}}}$$

  L_{int} – integral length scale (characterize the distance over which the fluctuating velocity field is correlated)
  U_{int} – corresponding velocity scale

- same ansatz for scales of size δ

$$\epsilon \sim \frac{U_\delta^3}{\delta}$$

-

$$\Longrightarrow \quad U_\delta \sim U_{\text{int}} \left(\frac{\delta}{L_{\text{int}}} \right)^{1/3}$$

The Smagorinsky Model – the Turbulent Viscosity Coefficient (cont.)

- goal of eddy viscosity model: capture dissipation of eddies of size δ

$$Re(\delta) = \frac{\delta U_\delta}{\nu_T} = 1 \quad \Longrightarrow \quad \epsilon \sim \frac{U_\delta^3}{\delta} \sim \delta U_\delta \frac{U_\delta^2}{\delta^2} \sim \nu_T \frac{U_\delta^2}{\delta^2}$$

- $$\Longrightarrow \quad \nu_T \sim \epsilon \frac{\delta^2}{U_\delta^2} \sim \frac{U_\delta^3}{\delta} \frac{\delta^2}{U_\delta^2} \sim U_\delta \delta \sim U_{\text{int}} L_{\text{int}}^{-1/3} \delta^{4/3}$$

- assumption

$$U_{\text{int}} \sim L_{\text{int}} \left\| \mathbb{D}(\,\overline{\mathbf{u}}\,) \right\|_F$$

- replacing similarity by equality with an unknown constant

$$\Longrightarrow \quad \nu_T = c L_{\text{int}}^{2/3} \delta^{4/3} \left\| \mathbb{D}(\,\overline{\mathbf{u}}\,) \right\|_F$$

- L_{int} is hard to determine, approximate $L_{\text{int}} \sim \delta$

$$\nu_T = c_S \delta^2 \left\| \mathbb{D}(\,\overline{\mathbf{u}}\,) \right\|_F \quad \text{often} \quad \nu_T = (c_S^* \delta)^2 \left\| \mathbb{D}(\,\overline{\mathbf{u}}\,) \right\|_F$$

The Smagorinsky Model – Choice of c_S

- Lilly (1967)
- idea: consider ideal situation and set

$$\langle \epsilon \rangle = \langle \epsilon_{\mathrm{Sma}} \rangle \,, \quad \text{time averages}$$

  with

$$\epsilon_{\mathrm{Sma}} \approx \int_\Omega \nu_T \, \|\mathbb{D}(\mathbf{w})\|_F^2 \; d\mathbf{x} = \int_\Omega c_S \delta^2 \, \|\mathbb{D}(\mathbf{w})\|_F^3 \; d\mathbf{x} = c_S \delta^2 \, \|\mathbf{w}\|_{L^3}^3$$

- further assumptions, details in Berselli, Iliescu, Layton (2006):
  - neglect time averages
  - ideal turbulence (homogeneous, isotropic)
- use Kolmogorov law
- result:

$$\sqrt{c_S} \approx 0.17, \quad c_S^* = 0.17$$

- practice: constant too large, results too dissipative

The Smagorinsky Model – Choice of c_S (cont.)

- dynamic Smagorinsky model $c_S = c_S(t, \mathbf{x})$, Germano, Piomelli, Moin, Cabot (1991), Lilly (1992)
- idea (more details in J. (2004)):
  - use two filters, e.g. δ and 2δ (coarse grid)
  - filter Navier–Stokes equations with both filters
  - make Smagorinsky model ansatz for both filtered equations
  - assume same $c_S(t, \mathbf{x})$ for both filters

The Smagorinsky Model – Choice of c_S (cont.)

- dynamic Smagorinsky model $c_S = c_S(t, \mathbf{x})$, Germano, Piomelli, Moin, Cabot (1991), Lilly (1992)
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  - filter Navier–Stokes equations with both filters
  - make Smagorinsky model ansatz for both filtered equations
  - assume same $c_S(t, \mathbf{x})$ for both filters
- result:
  - 6 equations for $c_S(t, \mathbf{x})$, coefficients depend on (doubled) filtered velocities with both filters
  - solve equations in least squares sense

The Smagorinsky Model – Choice of c_S (cont.)

- practice:
  - hard to implement
  - expensive
  - smoothing in space and time necessary, otherwise very oscillating and negative turbulent viscosities $\Longrightarrow$ blow up
  - backscatter of energy possible, since $\nu_T < 0$ possible
    - on the average: energy transferred from large to small scales
    - inverse transfer (backscatter) might be significant

    $\Longrightarrow$ backscatter should be included in model
  - values are far from being optimal, Meyers, Sagaut (2006)
  - very popular until some years ago

The Smagorinsky Model – Variational Formulation

- velocity and pressure space

$$V = \left\{ \mathbf{v} \in (W^{1,3}(\Omega))^d, \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega \right\}, \quad Q = L_0^2(\Omega)$$

- find $(\mathbf{w}, r) \in V \times Q$ such that
  i) for all $t \in (0, T]$ and all $(\mathbf{v}, q) \in V \times Q$

$$\begin{aligned} (\mathbf{w}_t, \mathbf{v}) + a(\mathbf{w}, \mathbf{w}, \mathbf{v}) + b_s(\mathbf{w}, \mathbf{w}, \mathbf{v}) & \\ +(q, \nabla \cdot \mathbf{w}) - (r, \nabla \cdot \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \end{aligned}$$

  with

$$\begin{aligned} a(\mathbf{u}, \mathbf{w}, \mathbf{v}) &= \left((2\nu + c_S \delta^2 \left\| \mathbb{D}(\mathbf{u}) \right\|_F) \mathbb{D}(\mathbf{w}), \mathbb{D}(\mathbf{v}) \right) \\ b_s(\mathbf{u}, \mathbf{w}, \mathbf{v}) &= \frac{1}{2} \left(b(\mathbf{u}, \mathbf{w}, \mathbf{v}) - b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \right) \\ b(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= ((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w}) \end{aligned}$$

  c_S – constant

  ii) $\mathbf{w}(0, \mathbf{x}) = \mathbf{w}_0(\mathbf{x})$

The Smagorinsky Model – Analysis

- weak equation posseses unique solution

$$\nabla \mathbf{w} \in L^3(0, T, L^3(\Omega))$$

  in 2d and 3d for large data and large time intervals, Ladyzhenskaya (1967)
  - more known than for Navier–Stokes equations (uniqueness in 3d)

The Smagorinsky Model – Analysis

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  in 2d and 3d for large data and large time intervals, Ladyzhenskaya (1967)
  - more known than for Navier–Stokes equations (uniqueness in 3d)
- proof by Galerkin method, Hopf (1951):
  - consider equation in finite dimensional space
  - show solvability of this equation
  - extract a subsequence of the finite dimensional solutions which converges to a solution of the continuous problem

  $\Longrightarrow$ existence of weak solution

The Smagorinsky Model – Analysis (cont.)

- main analytical tools for uniqueness :
  - strong monotonicity

$$\left(\|\mathbb{D}(\mathbf{u})\|_F\,\mathbb{D}(\mathbf{u}) - \|\mathbb{D}(\mathbf{v})\|_F\,\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{u}-\mathbf{v})\right) \geq \underline{C}\|\mathbb{D}(\mathbf{u}-\mathbf{v})\|_{L^3(\Omega)}^3$$

  - local Lipschitz continuity

$$\left(\|\mathbb{D}(\mathbf{u})\|_F\,\mathbb{D}(\mathbf{u}) - \|\mathbb{D}(\mathbf{v})\|_F\,\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{w})\right) \leq C\|\mathbb{D}(\mathbf{u}-\mathbf{v})\|_{L^3(\Omega)}\|\mathbb{D}(\mathbf{w})\|_{L^3(\Omega)}$$

  - Sobolev imbeddings of $W^{1,3}(\Omega)$
    - more regular function space as for Navier–Stokes equations ($W^{1,2}(\Omega)$)
      $\Longrightarrow$ more Sobolev imbeddings for Smagorinsky model

2.3.4 The $k - \epsilon$ Model

- very popular in engineering community
- additional quantities to compute
  - k – kinetic energy of the turbulence

$$k = \frac{1}{2}\left\langle \left\| \mathbf{u}' \right\|_2^2 \right\rangle, \quad \langle\cdot\rangle \text{ space average (filter)}$$

  - ϵ – rate of dissipation of turbulent energy

$$\epsilon = \frac{\nu}{2}\left\langle \left\| \nabla \mathbf{u}' + \left(\nabla \mathbf{u}'\right)^T \right\|_F^2 \right\rangle$$

- a number of hypotheses, see Mohammadi, Pironneau (1994)

The $k - \epsilon$ Model (cont.)

- find approximation $(\mathbf{w}, r)$ to $(\overline{\mathbf{u}}, \overline{p})$ and (k, ϵ) such that in $(0,T] \times \Omega$

$$
\begin{aligned}
\mathbf{w}_t - 2\nabla\cdot(\nu\mathbb{D}(\mathbf{w})) + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla r - c_k \nabla\cdot\left(\frac{k^2}{\epsilon}\left(\nabla\mathbf{w} + \nabla\mathbf{w}^T\right)\right) &= \overline{\mathbf{f}} \\
\nabla\cdot\mathbf{w} &= 0 \\
k_t + \mathbf{w}\nabla k - \frac{c_k}{2}\frac{k^2}{\epsilon}\left\|\nabla\mathbf{w} + \nabla\mathbf{w}^T\right\|_F^2 - \nabla\cdot\left(c_k\frac{k^2}{\epsilon}\nabla k\right) + \epsilon &= 0 \\
\epsilon_t + \mathbf{w}\nabla\epsilon - \frac{c_1}{2}\left\|\nabla\mathbf{w} + \nabla\mathbf{w}^T\right\|_F^2 - \nabla\cdot\left(c_\epsilon\frac{k^2}{\epsilon}\nabla k\right) + c_2\frac{\epsilon^2}{k} &= 0
\end{aligned}
$$

+ boundary and initial conditions

- $c_k, c_\epsilon, c_1, c_2$ – appropriate constants
- coupled system of equations
  - Navier–Stokes type equations
  - 2 convection–dominated convection–diffusion equations

The $k - \epsilon$ Model – Remarks

- original proposal by Launder and Spalding (1972)
- standard model of (almost) all commercial CFD codes
- model not valid near solid walls
- correct boundary conditions for all equations are open problem
- accurate and efficient numerical solution of convection–dominated scalar equations is active field of research
- only initial steps for numerical analysis available
- a lot of variants proposed, newer proposals by the Cottet, Jiroveanu, Michaux (2003)

2.3.5 Approximate Deconvolution Models

- Adams, Stolz, et al. (1999 – 2001)
- approximate deconvolution operator D_N of order N:

$$\varphi = D_N\left(\overline{\varphi}\right) + \mathcal{O}\left(\delta^{2N+2}\right)$$

  for smooth functions φ

  - $$D_0\left(\overline{\varphi}\right) = \overline{\varphi} + \mathcal{O}\left(\delta^2\right)$$

- closure approximation

$$\overline{\mathbf{u}\mathbf{u}^T} \approx \overline{D_N(\overline{\mathbf{u}})D_N(\overline{\mathbf{u}})^T}$$

- N–th order ADM: find approximation $(\mathbf{w}, r)$ to $(\overline{\mathbf{u}}, \overline{p})$ such that in $(0,T] \times \Omega$

$$\begin{aligned} \mathbf{w}_t - 2\nabla\cdot(\nu\mathbb{D}(\mathbf{w})) + \nabla\cdot\overline{D_N(\mathbf{w})D_N(\mathbf{w})^T} + \nabla r &= \overline{\mathbf{f}} \\ \nabla\cdot\mathbf{w} &= 0 \\ \mathbf{w}(0,\mathbf{x}) &= \overline{\mathbf{u}_0} \end{aligned}$$

$+$ boundary conditions

Approximate Deconvolution Models (cont.)

- van Cittert approximate deconvolution operator (1931)
  - define average by differential filter (Helmholtz filter) A

$$A\,\overline{\varphi} := -\delta^2 \Delta \overline{\varphi} + \overline{\varphi} = \varphi \quad \text{in } \Omega$$

  + periodic (or homogeneous) boundary conditions
  - recursive definition of approximate deconvolution

    $\mathbf{u}_0 = \overline{\mathbf{u}}$
    for $n = 1, \ldots, N-1$
    $\quad \mathbf{u}_{n+1} = \mathbf{u}_n + \left(\overline{\mathbf{u}} - A^{-1}\mathbf{u}_n\right)$
    end
  - results

$$\begin{aligned} D_0\,\overline{\mathbf{u}} &= \overline{\mathbf{u}} \\ D_1\,\overline{\mathbf{u}} &= 2\,\overline{\mathbf{u}} - \overline{\overline{\mathbf{u}}} \\ D_2\,\overline{\mathbf{u}} &= 3\,\overline{\mathbf{u}} - 3\,\overline{\overline{\mathbf{u}}} + \overline{\overline{\overline{\mathbf{u}}}} \end{aligned}$$

Approximate Deconvolution Models (cont.)

- numerical studies: Adams, Stolz, et al. (1999 –)
  - zeroth–order ADM
- analysis
  - existence and uniqueness of solution, energy inequalities, Dunca, Epshteyn (2006), Layton, Lewandowski (2006)
  - finite element error analysis, Ervin, Layton, Neda (2007), Manica, Merdan (2007)
  - energy dissipation, Layton (2007)
  - conservation laws, Layton, Manica, Neda, Rebholz (2008)
  - ...

2.3.6 Other Models

- scale similarity models, Bardina, Ferziger, Reynolds (1980)
- Leray regularization model, Leray (1933)
  - analysis of ADM regularization, Layton, Manica, Neda, Rebholz (2008)
  - numerical study of different regularizations Geurts, Kuczaj, Titi (2008)
- Navier–Stokes α–model, Camassa–Holm model
  - analysis and numerical studies Foias, Holm, Titi (2001, 2002)
- Navier–Stokes ω–model
  - theory of continuous model, Layton, Stanculescu, Trenchea (2008)
  - finite element analysis, Layton, Manica, Neda, Rebholz (2009)

2.4 Finite Element Discretizations

- finite element code MooNMD (Mathematics and object–oriented Numerics in MagDeburg)
- J., Matthies (2004)

2.4 Finite Element Discretizations

- finite element code MooNMD (Mathematics and object–oriented Numerics in MagDeburg)
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- discretization strategy :
  - discretization in time
  - variational formulation and iterative solution of the algebraic equations in each discrete time
  - discretization of the linear saddle point problems in each step of the iteration with an inf–sup stable finite element method

Temporal Discretization

- second order implicit schemes
  - Crank–Nicolson scheme (A–stable)
  - fractional–step θ–scheme (strongly A–stable, more expensive)
- much more accurate than first order schemes, J., Matthies, Rang (2006) for laminar Navier–Stokes equations

Temporal Discretization

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  - fractional–step θ–scheme (strongly A–stable, more expensive)
- much more accurate than first order schemes, J., Matthies, Rang (2006) for laminar Navier–Stokes equations
- future: certain Rosenbrock schemes might be of interest
  - even more accurate
  - allow simple time step control with imbedded schemes
  - studies for laminar Navier–Stokes equations: J., Rang (2009, in preparation)

Linearization of Nonlinear Terms

- convective term
  - with fixed point iteration (Picard iteration)

$$(\mathbf{w}^{(n)}\nabla\cdot)\mathbf{w}^{(n)} \approx (\mathbf{w}^{(n-1)}\nabla\cdot)\mathbf{w}^{(n)}$$

    $\mathbf{w}^{(n-1)}$ – current velocity approximation

  - more efficient than Newton's method, J. (2006) for laminar Navier–Stokes equations

$$(\mathbf{w}^{(n)}\nabla\cdot)\mathbf{w}^{(n)} \approx (\mathbf{w}^{(n-1)}\nabla\cdot)\mathbf{w}^{(n)} + (\mathbf{w}^{(n)}\nabla\cdot)\mathbf{w}^{(n-1)} - (\mathbf{w}^{(n-1)}\nabla\cdot)\mathbf{w}^{(n-1)}$$

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$$(\mathbf{w}^{(n)}\nabla\cdot)\mathbf{w}^{(n)} \approx (\mathbf{w}^{(n-1)}\nabla\cdot)\mathbf{w}^{(n)} + (\mathbf{w}^{(n)}\nabla\cdot)\mathbf{w}^{(n-1)} - (\mathbf{w}^{(n-1)}\nabla\cdot)\mathbf{w}^{(n-1)}$$

- turbulent viscosity, e.g. Smagorinsky term

$$\nu_T\left(\mathbf{w}^{(n)}\right)\mathbf{w}^{(n)} \approx \nu_T\left(\mathbf{w}^{(n-1)}\right)\mathbf{w}^{(n)}$$

- LES term, explicit

$$A\left(\nabla\mathbf{w}_k\left(\nabla\mathbf{w}_k\right)^T\right) \approx A\left(\nabla\mathbf{w}_{k-1}\left(\nabla\mathbf{w}_{k-1}\right)^T\right)$$

Discretization of Linear Saddle Point Problems

- inf–sup stable pairs of finite elements (Babuška–Brezzi condition): there is a constant C independent of the mesh size parameter h s.t.

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in V^h} \frac{\left(\nabla \cdot \mathbf{v}^h, q^h\right)}{\left\|\nabla \mathbf{v}^h\right\|_{L^2} \left\|q^h\right\|_{L^2}} \geq C$$

V^h – velocity finite element space
Q^h – pressure finite element space
$(\cdot, \cdot)$ – inner product in $L^2(\Omega)$

Discretization of Linear Saddle Point Problems

- inf–sup stable pairs of finite elements (Babuška–Brezzi condition): there is a constant C independent of the mesh size parameter h s.t.

$$\inf_{q^h \in Q^h} \sup_{\mathbf{v}^h \in V^h} \frac{\left(\nabla \cdot \mathbf{v}^h, q^h\right)}{\left\|\nabla \mathbf{v}^h\right\|_{L^2} \left\|q^h\right\|_{L^2}} \geq C$$

  V^h – velocity finite element space
  Q^h – pressure finite element space
  $(\cdot, \cdot)$ – inner product in $L^2(\Omega)$

  - no pressure stabilization necessary
  - V^h, Q^h have to be different finite element spaces

Inf–Sup Stable Pairs of Finite Elements

- experiences with different finite element spaces, J.,Matthies (2001), J. (2002), J. (2004), J. (2006)
- spaces with continuous pressure (Taylor–Hood spaces)
  - divergence constraint very inaccurate
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- spaces with discontinuous pressure
  - more accurate
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  - best ratio between accuracy and efficiency: second order velocity, first order discontinuous pressure $\Longrightarrow$ recommendations:
    - hexahedral grids Q_2/P_1^{disc}
    - tetrahedral grids $P_2^{\mathrm{bubble}}/P_1^{\mathrm{disc}}$, Bernardi, Raugel (1985)

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    - tetrahedral grids $P_2^{\mathrm{bubble}}/P_1^{\mathrm{disc}}$, Bernardi, Raugel (1985)
- lowest order elements (P_1^{nc}/P_0 Crouzeix, Raviart (1973), Q_1^{rot}/Q_0 Rannacher, Turek (1992))
  - very inaccurate
  - important to construct an efficient multigrid solver

2.5 Finite Element Error Analysis

- Smagorinsky model, J., Layton (2002)
- find $(\mathbf{w}, r) \approx (\,\overline{\mathbf{u}}\,,\,\overline{p}\,)$ such that

$$\begin{array}{rcll} \mathbf{w}_t - \nabla\cdot((2\nu + \nu_T)\mathbb{D}(\mathbf{w})) + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla r & = & \mathbf{f} & \text{in } (0,T]\times\Omega \\ \nabla\cdot\mathbf{w} & = & 0 & \text{in } [0,T]\times\Omega \\ \mathbf{w} & = & \mathbf{0} & \text{in } [0,T]\times\partial\Omega \\ \mathbf{w}(0,\mathbf{x}) & = & \mathbf{w}_0 & \text{in } \Omega \\ \int_\Omega r\,d\mathbf{x} & = & 0 & \text{in } (0,T] \end{array}$$

with

$$\nu_T = a_0(\delta) + c_S\delta^2\,\|\mathbb{D}(\mathbf{w})\|_F\,, \quad a_0(\delta) > 0$$

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\mathbf{w}(0,\mathbf{x}) & = & \mathbf{w}_0 & \text{in } \Omega \\
\displaystyle\int_\Omega r\,d\mathbf{x} & = & 0 & \text{in } (0,T]
\end{array}
$$

with

$$
\nu_T = a_0(\delta) + c_S\delta^2\,\|\mathbb{D}(\mathbf{w})\|_F\,, \quad a_0(\delta) > 0
$$

- observations in computations : error independent of ν
- $(\mathbf{w}, r) \in V\times Q$ – weak solution of the continuous problem
- $(\mathbf{w}^h, r^h) \in V^h\times Q^h \subset V\times Q$ – finite element solution

Goals and Sketch of the Proof

- goals of the analysis :
  - error estimates for $\|\mathbf{w} - \mathbf{w}^h\|$ in appropriate norms independent of ν
  - use only minimal regularity of solution $\mathbf{w} \in L^3(0, T; W_0^{1,3}(\Omega))$

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- Sketch of the proof :
  1. prove stability of $\mathbf{w}, \mathbf{w}^h$ with constants independent of ν in various norms
     - use $\mathbf{w}, \mathbf{w}^h$ as test functions
     - standard estimates (Poincaré, Korn, Young)

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     - use $\mathbf{w}, \mathbf{w}^h$ as test functions
     - standard estimates (Poincaré, Korn, Young)
  2. introduce appropriate approximation $\tilde{\mathbf{w}} \in V^h$ of $\mathbf{w}$ and split the error

$$\mathbf{w} - \mathbf{w}^h = (\mathbf{w} - \tilde{\mathbf{w}}) + (\tilde{\mathbf{w}} - \mathbf{w}^h) = \eta - \phi^h$$

$\Longrightarrow \eta$: approximation error (independent of the problem)

Sketch of the Proof (cont.)

3. prove differential inequality

$$\frac{d}{dt}\|\phi^h(t)\| + \text{ non-negative terms } \le g(t) + c(t)\|\phi^h(t)\|^\gamma$$

$g(t), c(t)$ bounded by approximation errors and data independent of ν

- long estimates (here is the work)
- standard estimates
- Sobolev imbeddings
- strong monotonicity
- local Lipschitz continuity

Sketch of the Proof (cont.)

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- long estimates (here is the work)
- standard estimates
- Sobolev imbeddings
- strong monotonicity
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4. Gronwall's lemma
   - show $g(t), c(t) \in L^1(0,T)$
   - show $\gamma = 1$
   - Gronwall's lemma

$$\|\phi^h(t)\| + \text{ non-negative terms } \le \exp\left(\int_0^t c(\tau)d\tau\right)\left(\|\phi^h(0)\| + \int_0^t g(\tau)d\tau\right)$$

$\Longrightarrow \|\phi^h(t)\|$ bounded by data and approximation errors independent of ν

Final Estimate

$$\begin{aligned}
&\|\mathbf{w}-\mathbf{w}^h\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\cdot(\mathbf{w}-\mathbf{w}^h)\|^2_{L^2(0,T;L^2(\Omega))}\\
&\qquad + (\nu + Ca_0(\delta))\,\|\mathbb{D}(\mathbf{w}-\mathbf{w}^h)\|^2_{L^2(0,T;L^2(\Omega))} + \delta^2\|\mathbb{D}(\mathbf{w}-\mathbf{w}^h)\|^3_{L^3(0,T;L^3(\Omega))}\\
&\quad\leq \quad C\exp\left(\|c(t,\delta)\|_{L^1(0,T)}\right)\|(\mathbf{w}-\mathbf{w}^h)(0,\mathbf{x})\|^2_{L^2(\Omega)}\\
&\qquad +C\inf_{\tilde{\mathbf{w}}\in V^h_{\mathrm{div}}\cap W^{1,3}(\Omega),q^h\in Q^h}\mathcal{F}(\mathbf{w}-\tilde{\mathbf{w}},r-q^h,\delta)
\end{aligned}$$

with approximation error $\mathcal{F}(\mathbf{w}-\tilde{\mathbf{w}},r-q^h,\delta)$

error $\leq$ constant $\cdot$ approximation error

where the constant is independent of ν

Finite Element Error Analysis – Remarks

- analysis possible for different boundary conditions (slip with friction and no penetration)
- similar finite element error analysis possible for Taylor LES model (Iliescu, J., Layton (2002))
  - strong monotonicity if c_S sufficiently large
  - Taylor LES model only a local perturbation
- numerical examples support analysis

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  - Taylor LES model only a local perturbation
- numerical examples support analysis
- open problems
  - ν–independent estimate for $a_0(\delta) = 0$
    - only global estimates used so far
    - $c_S\delta^2 \|\mathbb{D}(\mathbf{w})\|_F > 0$ cannot be ensured for each point in Ω
    - if there are points with $c_S\delta^2 \|\mathbb{D}(\mathbf{w})\|_F = 0$, global estimates cannot be better than for Navier–Stokes equations
  - finite element error analysis for rational LES model
    - difficulty: strong monotonicity open, rational LES model is a global perturbation
    - result for small data/time, Berselli, Galdi, Iliescu, Layton (2002)

2.6 A Numerial Study

- Goal : LES models have been derived for the approximation of $(\overline{\mathbf{u}}, \overline{p})$ – how good is that achieved ?
  - Smagorinsky model
  - rational LES model with different subgrid scale terms

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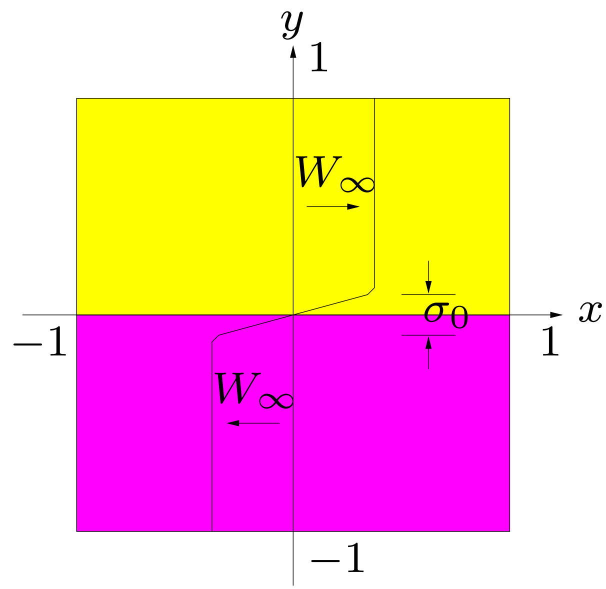
- mixing layer problem in 2d



Setup of the Mixing Layer Problem

- initial velocity

$$\mathbf{w}_0 = \begin{pmatrix} W_\infty \text{tanh}\left(\frac{2y}{\sigma_0}\right) \\ 0 \end{pmatrix} + \text{ noise}$$

  with

$$\sigma_0 = 1/14, \qquad W_\infty = 1, \qquad \text{viscosity } \nu^{-1} = 140000$$

- $Re = \dfrac{\sigma_0 W_\infty}{\nu} = 10000$
- Galerkin FEM : appr. 3 000 000 d.o.f. in space
- LES : appr. 45 000 d.o.f. in space
- Q_2/P_1^{disc} finite element discretisation

Comparison of the Vorticity of $\overline{\mathbf{u}^h}$ *and* $\mathbf{w}^h$

- Smagorinsky model
  - first vortex pairing too late
  - second vortex pairing somewhat too late, symmetric
  - delay before the last vortex pairing, much too late
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rational LES model with Iliescu-Layton sgs term best in this example

Further Numerical Studies with Rational LES model

- turbulent driven cavity 2D, 3D: Iliescu, J., Layton, Matthies, Tobiska (2003)
- turbulent channel flows: Fischer, Iliescu (2003, 2004)
- turbulent flow around a cylinder: J., Kindl, Suciu (2009, preprint)
- geophysical flows: Fischer, Iliescu, Ozgokmen (2009)
- rational LES model seldom used

2.7 Summary

- analysis and modeling
  - commutation errors arise, partially analyzed, important near boundaries
  - boundary conditions open
  - Smagorinsky model with constant c_S well analyzed (existence, uniqueness of solution, finite element errors)

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- analysis and modeling
  - commutation errors arise, partially analyzed, important near boundaries
  - boundary conditions open
  - Smagorinsky model with constant c_S well analyzed (existence, uniqueness of solution, finite element errors)
- practical application of LES models
  - many models proposed, used
  - Smagorinsky model (and variants) very popular, but often too diffusive
- Literatur
  - best reference: Sagaut (2006)
  - more mathematical: J. (2004), Berselli, Iliescu, Layton (2006)

3 Variational Multiscale (VMS) Methods

3.1 Motivation and Derivation

- motivation: definition of large scales by spatial averaging leads to problems (in particular at the boundary)
- goal: define large scales in a different way

3.1 Motivation and Derivation

- motivation: definition of large scales by spatial averaging leads to problems (in particular at the boundary)
- goal: define large scales in a different way
- ideas:
  - define scales by projections into function spaces
    VMS methods are based on variational formulation of underlying equation
  - model for influence of unresolved small scales acts directly only on resolved small scales (many VMS models)
- three scale decomposition of the flow (many VMS models)
  - (resolved) large scales, should be simulated
  - resolved small scales, should be simulated, too
  - unresolved small scales
- based on ideas for simulation of multiscale problems from Hughes (1995), Guermond (1999)
- first connection to turbulent flows: Hughes, Mazzei, Jansen (2000)

Motivation and Derivation (cont.)

- three scale VMS method
- starting point: variational form of the Navier–Stokes equations
- $V = \left(H_0^1\left(\Omega\right)\right)^3$, $Q = L_0^2\left(\Omega\right)$
- find $\mathbf{u} : [0,T] \to V,\ p : (0,T] \to Q$ satisfying for all $(\mathbf{v}, q) \in V \times Q$

$$(\mathbf{u}_t, \mathbf{v}) + \left(2Re^{-1}\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v})\right) + \left(\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}, \mathbf{v}\right) - (p, \nabla\cdot\mathbf{v}) + (q, \nabla\cdot\mathbf{u}) = (\mathbf{f}, \mathbf{v})$$

and $\mathbf{u}\left(0,\mathbf{x}\right) = \mathbf{u}_0\left(\mathbf{x}\right) \in V$

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  and $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \in V$
- short form of variational equation

$$A(\mathbf{u}; (\mathbf{u}, p), (\mathbf{v}, q)) = F(\mathbf{v})$$

- three scale decomposition
  - large scales $(\overline{\mathbf{u}}, \overline{p})$
  - resolved small scales $(\tilde{\mathbf{u}}, \tilde{p})$
  - unresolved small scales $(\mathbf{u}', p')$

Motivation and Derivation (cont.)

- find $\mathbf{u} = \overline{\mathbf{u}} + \tilde{\mathbf{u}} + \mathbf{u}' \; : \; [0,T] \to V, \; p = \overline{p} + \tilde{p} + p' \; : \; (0,T] \to Q$ s.t. for all $(\mathbf{v}, q) \in V \times Q$

$$
\begin{aligned}
A\left(\mathbf{u};(\overline{\mathbf{u}},\overline{p}),(\overline{\mathbf{v}},\overline{q})\right) + A\left(\mathbf{u};(\tilde{\mathbf{u}},\tilde{p}),(\overline{\mathbf{v}},\overline{q})\right) + A\left(\mathbf{u};\left(\mathbf{u}',p'\right),(\overline{\mathbf{v}},\overline{q})\right) &= F\left(\overline{\mathbf{v}}\right),\\
A\left(\mathbf{u};(\overline{\mathbf{u}},\overline{p}),(\tilde{\mathbf{v}},\tilde{q})\right) + A\left(\mathbf{u};(\tilde{\mathbf{u}},\tilde{p}),(\tilde{\mathbf{v}},\tilde{q})\right) + A\left(\mathbf{u};\left(\mathbf{u}',p'\right),(\tilde{\mathbf{v}},\tilde{q})\right) &= F\left(\tilde{\mathbf{v}}\right),\\
A\left(\mathbf{u};(\overline{\mathbf{u}},\overline{p}),\left(\mathbf{v}',q'\right)\right) + A\left(\mathbf{u};(\tilde{\mathbf{u}},\tilde{p}),\left(\mathbf{v}',q'\right)\right) + A\left(\mathbf{u};\left(\mathbf{u}',p'\right),\left(\mathbf{v}',q'\right)\right) &= F\left(\mathbf{v}'\right)
\end{aligned}
$$

Motivation and Derivation (cont.)

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$$
\begin{aligned}
A\left(\mathbf{u}; (\overline{\mathbf{u}}, \overline{p}), (\overline{\mathbf{v}}, \overline{q})\right) + A\left(\mathbf{u}; (\tilde{\mathbf{u}}, \tilde{p}), (\overline{\mathbf{v}}, \overline{q})\right) + A\left(\mathbf{u}; (\mathbf{u}', p'), (\overline{\mathbf{v}}, \overline{q})\right) &= F(\overline{\mathbf{v}}), \\
A\left(\mathbf{u}; (\overline{\mathbf{u}}, \overline{p}), (\tilde{\mathbf{v}}, \tilde{q})\right) + A\left(\mathbf{u}; (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})\right) + A\left(\mathbf{u}; (\mathbf{u}', p'), (\tilde{\mathbf{v}}, \tilde{q})\right) &= F(\tilde{\mathbf{v}}), \\
A\left(\mathbf{u}; (\overline{\mathbf{u}}, \overline{p}), (\mathbf{v}', q')\right) + A\left(\mathbf{u}; (\tilde{\mathbf{u}}, \tilde{p}), (\mathbf{v}', q')\right) + A\left(\mathbf{u}; (\mathbf{u}', p'), (\mathbf{v}', q')\right) &= F(\mathbf{v}')
\end{aligned}
$$

- ideas and assumptions
  - neglect equation with test function $(\mathbf{v}', q')$
  - assume

$$
A\left(\mathbf{u}; (\mathbf{u}', p'), (\overline{\mathbf{v}}, \overline{q})\right) = 0
$$

    (direct influence of unresolved scales onto the large scales negligible)
  - model influence of the unresolved scales onto the small resolved scales:

$$
A\left(\mathbf{u}; (\mathbf{u}', p'), (\tilde{\mathbf{v}}, \tilde{q})\right) \approx B\left(\mathbf{u}; (\overline{\mathbf{u}}, \overline{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})\right)
$$

Motivation and Derivation (cont.)

- find $(\overline{\mathbf{u}}, \tilde{\mathbf{u}}, \overline{p}, \tilde{p}) \in \overline{V} \times \tilde{V} \times \overline{Q} \times \tilde{Q}$ s.t. for all $(\overline{\mathbf{v}}, \tilde{\mathbf{v}}, \overline{q}, \tilde{q}) \in \overline{V} \times \tilde{V} \times \overline{Q} \times \tilde{Q}$

$$
\begin{aligned}
A\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\overline{\mathbf{v}}, \overline{q})\right) + A\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\tilde{\mathbf{u}}, \tilde{p}), (\overline{\mathbf{v}}, \overline{q})\right) &= F(\overline{\mathbf{v}}) \\
A\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\tilde{\mathbf{v}}, \tilde{q})\right) + A\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})\right) & \\
+ B\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})\right) &= F(\tilde{\mathbf{v}})
\end{aligned}
$$

Motivation and Derivation (cont.)

- find $(\overline{\mathbf{u}}, \tilde{\mathbf{u}}, \overline{p}, \tilde{p}) \in \overline{V} \times \tilde{V} \times \overline{Q} \times \tilde{Q}$ s.t. for all $(\overline{\mathbf{v}}, \tilde{\mathbf{v}}, \overline{q}, \tilde{q}) \in \overline{V} \times \tilde{V} \times \overline{Q} \times \tilde{Q}$

$$\begin{aligned} A\left(\overline{\mathbf{u}}+\tilde{\mathbf{u}};\left(\overline{\mathbf{u}},\overline{p}\right),\left(\overline{\mathbf{v}},\overline{q}\right)\right)+A\left(\overline{\mathbf{u}}+\tilde{\mathbf{u}};\left(\tilde{\mathbf{u}},\tilde{p}\right),\left(\overline{\mathbf{v}},\overline{q}\right)\right) &= F\left(\overline{\mathbf{v}}\right) \\ A\left(\overline{\mathbf{u}}+\tilde{\mathbf{u}};\left(\overline{\mathbf{u}},\overline{p}\right),\left(\tilde{\mathbf{v}},\tilde{q}\right)\right)+A\left(\overline{\mathbf{u}}+\tilde{\mathbf{u}};\left(\tilde{\mathbf{u}},\tilde{p}\right),\left(\tilde{\mathbf{v}},\tilde{q}\right)\right) & \\ +B\left(\overline{\mathbf{u}}+\tilde{\mathbf{u}};\left(\overline{\mathbf{u}},\overline{p}\right),\left(\tilde{\mathbf{u}},\tilde{p}\right),\left(\tilde{\mathbf{v}},\tilde{q}\right)\right) &= F\left(\tilde{\mathbf{v}}\right) \end{aligned}$$

- parameters:
  - spaces $(\overline{V}, \overline{Q})$
  - spaces $(\tilde{V}, \tilde{Q})$
  - model $B\left(\mathbf{u};\left(\overline{\mathbf{u}},\overline{p}\right),\left(\tilde{\mathbf{u}},\tilde{p}\right),\left(\tilde{\mathbf{v}},\tilde{q}\right)\right)$

Motivation and Derivation (cont.)

- find $(\overline{\mathbf{u}}, \tilde{\mathbf{u}}, \overline{p}, \tilde{p}) \in \overline{V} \times \tilde{V} \times \overline{Q} \times \tilde{Q}$ s.t. for all $(\overline{\mathbf{v}}, \tilde{\mathbf{v}}, \overline{q}, \tilde{q}) \in \overline{V} \times \tilde{V} \times \overline{Q} \times \tilde{Q}$

$$\begin{aligned} A\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\overline{\mathbf{v}}, \overline{q})\right) + A\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\tilde{\mathbf{u}}, \tilde{p}), (\overline{\mathbf{v}}, \overline{q})\right) &= F(\overline{\mathbf{v}}) \\ A\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\tilde{\mathbf{v}}, \tilde{q})\right) + A\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})\right) & \\ + B\left(\overline{\mathbf{u}} + \tilde{\mathbf{u}}; (\overline{\mathbf{u}}, \overline{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})\right) &= F(\tilde{\mathbf{v}}) \end{aligned}$$

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  - spaces $(\overline{V}, \overline{Q})$
  - spaces $(\tilde{V}, \tilde{Q})$
  - model $B\left(\mathbf{u}; (\overline{\mathbf{u}}, \overline{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})\right)$
- often $B\left(\mathbf{u}; (\overline{\mathbf{u}}, \overline{p}), (\tilde{\mathbf{u}}, \tilde{p}), (\tilde{\mathbf{v}}, \tilde{q})\right)$ – Smagorinsky model
  - influence of Smagorinsky model is controlled with appropriate choice of spaces
  - in contrast to control with $c_S(t, \mathbf{x})$ as in dynamic Smagorinsky LES model

3.2 Practical Realizations and Experiences

- Hughes, Mazzei, Oberai, Wray (2001)
  - Fourier spectral method
  - scale separation by wave numbers
  - two static Smagorinsky–type models
  - homogeneous isotropic turbulence
  - Result: VMS in better agreement with DNS data than, e.g., dynamic Smagorinsky LES method

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  - Result: VMS in better agreement with DNS data than, e.g., dynamic Smagorinsky LES method
- Hughes, Oberai, Mazzei (2001)
  - streamwise and spanwise: Fourier spectral method
    wall–normal: spectral method (Legendre polynomials)
  - two static Smagorinsky–type models
  - turbulent channel flow at $Re_\tau \in \{180, 395\}$
  - Result: VMS produced in general better results than dynamic Smagorinsky LES method

Practical Realizations and Experiences (cont.)

- Holmen, Hughes, Oberai, Wells (2004)
  - Aim: sensitivity of VMS method with respect to scale partition (in terms of wave numbers)
  - static and dynamic Smagorinsky model
  - turbulent channel flow at $Re_\tau \in \{180, 395\}$
  - Results:
    - VMS methods better than dynamic Smagorinsky LES method
    - static VMS methods are highly accurate at appropriate partition ratios
    - dynamic VMS method relatively insensitive to the scale separation

Practical Realizations and Experiences (cont.)

- Ramakrishnan, Collis (2004)
  - streamwise and spanwise: Fourier spectral method
    wall–normal: second order FVM
  - scale separation only streamwise and spanwise: planar VMS (PVMS)
  - static Smagorinsky model
  - turbulent channel flow at $Re_\tau \in \{180, 590\}$
  - Result: PVMS consistently outperformed the dynamic Smagorinsky LES method

Practical Realizations and Experiences (cont.)

- Ramakrishnan, Collis (2004)
  - streamwise and spanwise: Fourier spectral method
    wall–normal: second order FVM
  - scale separation only streamwise and spanwise: planar VMS (PVMS)
  - static Smagorinsky model
  - turbulent channel flow at $Re_\tau \in \{180, 590\}$
  - Result: PVMS consistently outperformed the dynamic Smagorinsky LES method
- Ramakrishnan, Collis (2004); Collis, Ramkrishnan (2005)
  - discontinuous Galerkin discretization: local VMS method
  - scale separation by polynomial degree
  - no turbulence model used (grids fine enough)
  - turbulent channel flow at $Re_\tau \in \{180, 395\}$
  - Result: p–refinement better than h–refinement

Practical Realizations and Experiences (cont.)

- Gravemeier, Wall, Ramm (2004, 2005)
  - finite elements
  - bubble functions for $\tilde{\mathbf{u}}$, bubble VMS method, see Section 3.3
  - additional grad–div stabilization in large scale momentum equation
  - dynamic Smagorinsky model
  - driven cavity at $Re = 10000$
  - Results: good mean velocity; less good second order statistics

Practical Realizations and Experiences (cont.)

- Gravemeier, Wall, Ramm (2004, 2005)
  - finite elements
  - bubble functions for $\tilde{\mathbf{u}}$, bubble VMS method, see Section 3.3
  - additional grad–div stabilization in large scale momentum equation
  - dynamic Smagorinsky model
  - driven cavity at $Re = 10000$
  - Results: good mean velocity; less good second order statistics
- Gravemeier (2006, 2007)
  - projection–based VMS method, see Section 3.4, with second–order FVM
  - two–level method
  - additional viscous term in the momentum equation for the large scales
  - turbulent channel flow at $Re_\tau \in \{180, 590\}$
    turbulent flow in a diffuser
  - Result: VMS with constant Smagorinsky model better than VMS with dynamic Smagorinsky model and dynamic Smagorinsky LES method

Practical Realizations and Experiences (cont.)

- two–scale VMS method, Calo (2004), Bazilevs, Calo, Cottrell, Hughes, Reali, Scovazzi (2007)
- scale decomposition with projector $\overline{P} \,:\, V \to \overline{V}$, $U = (\mathbf{u}, p)$ into two scales

$$\overline{U} := \overline{P}(U), \quad \tilde{U} := (I - \overline{P})(U), \quad V = \overline{V} \oplus \tilde{V}$$

- write coupled equation as in three–scale VMS method
- rewrite (and linearize) equation for small scale test functions

$$\begin{aligned} A\left(\tilde{U}; \tilde{U}, \tilde{V}\right) + A\left(\overline{U}; \tilde{U}, \tilde{V}\right) + A\left(\tilde{U}; \overline{U}, \tilde{V}\right) &= F\left(\tilde{V}\right) - A\left(\overline{U}; \overline{U}, \tilde{V}\right) \\ &= \left(\mathrm{res}(\overline{U}), \tilde{V}\right) \end{aligned}$$

- formal representation

$$\tilde{U} = \tilde{F}\left(\overline{U}, \mathrm{res}(\overline{U})\right)$$

How to approximate $\tilde{F} \,:\, V^{\prime *} \to V^{\prime}$?

A Two–Scale VMS Method (cont.)

- assume $\varepsilon = \left\| \mathrm{res}(\overline{U}) \right\|_{\tilde{V}^*}$ small, perturbation series $\tilde{U} = \sum_{k=1}^{\infty} \varepsilon^k \tilde{U}_k$
- inserting in linearized equation with small scale test functions

$$
\begin{aligned}
A\left(\tilde{U}_1; \overline{U}, \tilde{V}\right) + A\left(\overline{U}; \tilde{U}_1, \tilde{V}\right) &= (\varepsilon^{-1}\mathrm{res}(\overline{U}), \tilde{V}) \\
A\left(\tilde{U}_k; \overline{U}, \tilde{V}\right) + A\left(\overline{U}; \tilde{U}_k, \tilde{V}\right) &= -\sum_{i=1}^{k-1} A\left(\tilde{U}_i; \tilde{U}_{k-i}, \tilde{V}\right), \ k \geq 2
\end{aligned}
$$

How to solve this recursively defined system?

A Two–Scale VMS Method (cont.)

- assume $\varepsilon = \left\|\mathrm{res}(\overline{U})\right\|_{\tilde{V}^*}$ small, perturbation series $\tilde{U} = \sum_{k=1}^{\infty} \varepsilon^k \tilde{U}_k$
- inserting in linearized equation with small scale test functions

$$\begin{aligned} A\left(\tilde{U}_1;\overline{U},\tilde{V}\right) + A\left(\overline{U};\tilde{U}_1,\tilde{V}\right) &= (\varepsilon^{-1}\mathrm{res}(\overline{U}),\tilde{V}) \\ A\left(\tilde{U}_k;\overline{U},\tilde{V}\right) + A\left(\overline{U};\tilde{U}_k,\tilde{V}\right) &= -\sum_{i=1}^{k-1} A\left(\tilde{U}_i;\tilde{U}_{k-i},\tilde{V}\right), \ k \ge 2 \end{aligned}$$

How to solve this recursively defined system?

- formally with small–scale Green's operator $\tilde{G}_{\overline{U}} \ : \ \tilde{V}^* \to \tilde{V}$

$$\tilde{U}_1 = \tilde{G}_{\overline{U}}\left((\varepsilon^{-1}\mathrm{res}(\overline{U}),\tilde{V})\right), \quad \tilde{U}_k = \tilde{G}_{\overline{U}}\left(-\sum_{i=1}^{k-1} A\left(\tilde{U}_i;\tilde{U}_{k-i},\tilde{V}\right)\right), \ k \ge 2$$

- $\tilde{G}_{\overline{U}}$ can be represented by $\overline{P}$ and classical Green's operator (Hughes, Sangalli (2007))

A Two–Scale VMS Method (cont.)

- practice: truncate series at $k = 1$

$$\tilde{U} \approx \varepsilon \tilde{U}_1 = \tilde{G}_{\overline{U}}\left(\left(\mathrm{res}(\overline{U}), \tilde{V}\right)\right)$$

A Two–Scale VMS Method (cont.)

- practice: truncate series at $k = 1$

$$\tilde{U} \approx \varepsilon \tilde{U}_1 = \tilde{G}_{\overline{U}}\left(\left(\mathrm{res}(\overline{U}), \tilde{V}\right)\right)$$

- practice: approximate small–scale Green's operator
    - large scales defined by finite element function $U^h = \overline{U}$
    - K – mesh cell

$$\tilde{U} \approx \tilde{G}_{\overline{U}}\left(\left(\mathrm{res}(U^h), \tilde{V}\right)\right)|_K \approx \boldsymbol{\tau}_K \mathrm{res}(U^h)|_K, \quad \boldsymbol{\tau}|_K \in \mathbb{R}^{4\times 4}$$

  linear, local approximation

- small scales are approximated by product of $\boldsymbol{\tau}$ and the large scale residual

A Two–Scale VMS Method (cont.)

- additional terms in momentum equation
  - Streamline–Upwind Petrov–Galerkin (SUPG) term

$$\sum_{K\in\mathcal{T}^h}\left(\boldsymbol{\tau}_m \mathrm{res}_m(U^h), (\mathbf{u}^h\cdot\nabla)\mathbf{v}^h + \nabla q^h\right)_K$$

  - grad–div term

$$\sum_{K\in\mathcal{T}^h}\left(\boldsymbol{\tau}_c \nabla\cdot\mathbf{u}^h, \nabla\cdot\mathbf{v}^h\right)_K$$

  - two further terms with residuals
- features:
  - no eddy viscosity model
  - parameters in the additional terms
- extension of residual–based stabilized methods for Navier–Stokes equations
- numerical studies
  - Calo, Hughes et al. (2004 –)
  - Gravemeier, Wall (2008), some improvements of two–scale VMS method compared to standard stabilized finite element methods

Practical Realizations and Experiences (cont.)

- Algebraic Multigrid VMS method, Gravemeier (2008 –),
  - algebraic scale separation by an algebraic multigrid
  - turbulent flow around a cylinder, see Section 3.5:
    better than dynamic Smagorinsky LES model and two–scale VMS method

Practical Realizations and Experiences (cont.)

- Algebraic Multigrid VMS method, Gravemeier (2008 –),
  - algebraic scale separation by an algebraic multigrid
  - turbulent flow around a cylinder, see Section 3.5:
    better than dynamic Smagorinsky LES model and two–scale VMS method
- projection–based finite element VMS method, J., Kaya, Kindl (2005 –)
  - see Section 3.4

Practical Realizations and Experiences (cont.)

- Algebraic Multigrid VMS method, Gravemeier (2008 –),
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    better than dynamic Smagorinsky LES model and two–scale VMS method
- projection–based finite element VMS method, J., Kaya, Kindl (2005 –)
  - see Section 3.4

Summary

- VMS methods in general better than dynamic Smagorinsky LES method
- static VMS methods work well

3.3 Bubble VMS Methods

- standard finite element spaces $(\overline{V}^h, \overline{Q}^h)$ for large scales
- finite element spaces $(\tilde{V}^h, \tilde{Q}^h)$ for resolved small scales necessary
  - higher resolution than $(\overline{V}^h, \overline{Q}^h)$ since small scales
    $\Longrightarrow$ higher order finite elements or refined mesh
  - solution of equations for resolved small scales must not be too expensive
    $\Longrightarrow$ decouple finite elements to get local problems

  $\Longrightarrow$ use bubble functions for velocity

$$\tilde{V}^h \subset \left\{ \mathbf{v} \ : \ \mathbf{v} \in \left(H_0^1\right)^3, \mathbf{v} = \mathbf{0} \text{ on faces of the mesh cells} \right\}$$

- idea in Hughes, Mazzei, Jansen (2000)

Bubble VMS Methods (cont.)

- Gravemeier (2003), Gravemeier, Wall, Ramm (2004,2005), ...
- strategy:
  1. simplify resolved small scale equation:
     - model

       $$\tilde{p} = \tau_K \nabla \cdot \overline{\mathbf{u}}^h \implies \sum_{K \in \mathcal{T}^h} (\tau_K \nabla \cdot \overline{\mathbf{u}}^h, \nabla \cdot \overline{\mathbf{v}}^h)$$

       grad–div stabilization in large scale equation
     - neglect resolved small scale continuity equation
  2. solve resolved small scale equation for velocity with residual free bubble (RFB) functions, using current approximation on $(\overline{\mathbf{u}}, \overline{p})$
  3. solve large scale equations with result from step 2 and grad–div stabilization
- implementation:
  - $(\overline{V}^h, \overline{Q}^h) = Q_1/Q_1$ or P_1/P_1
  - solve bubble equations using Q_1 finite elements
  - $\approx 4 \times 4 \times 4$ local meshes
- similar method studied in J., Kindl (2009)

Bubble VMS Methods (cont.)

- bubble VMS method without model for small scale pressure blows up in finite time, J., Kindl (2009)
- high effort in implementation
- our conclusions:
  - bubble VMS methods not worth to be considered
  - two–scale VMS method seems to be more attractive alternative

Bubble VMS Methods (cont.)

- bubble VMS method without model for small scale pressure blows up in finite time, J., Kindl (2009)
- high effort in implementation
- our conclusions:
  - bubble VMS methods not worth to be considered
  - two–scale VMS method seems to be more attractive alternative
- Unphysical property of bubble–VMS methods
  - resolved small scales bound to the mesh cells, no interactions between resolved small scales across mesh cell boundaries

    does not reflect physical reality
  - impact of this property on numerical results not known

3.4 A Finite–Element Projection–Based VMS Method

- J., Kaya (2005), based on ideas from Layton (2002)
- (V^h, Q^h) – conform velocity–pressure finite element spaces fulfilling the inf–sup condition for all resolved scales
- L^H – finite dimensional space of symmetric tensor–valued functions in $L^2(\Omega)^{d\times d}$ (large scale space)

3.4 A Finite–Element Projection–Based VMS Method

- J., Kaya (2005), based on ideas from Layton (2002)
- (V^h, Q^h) – conform velocity–pressure finite element spaces fulfilling the inf–sup condition for all resolved scales
- L^H – finite dimensional space of symmetric tensor–valued functions in $L^2(\Omega)^{d\times d}$ (large scale space)
- find $\mathbf{u}^h : [0,T] \to V^h$, $p^h : (0,T] \to Q^h$, $\mathbb{G}^H : [0,T] \to L^H$:

$$
\begin{array}{rcll}
(\mathbf{u}_t^h, \mathbf{v}^h) + (2Re^{-1}\mathbb{D}(\mathbf{u}^h), \mathbb{D}(\mathbf{v}^h)) + ((\mathbf{u}^h \cdot \nabla)\mathbf{u}^h, \mathbf{v}^h) & & & \\
-(p^h, \nabla \cdot \mathbf{v}^h) + (\nu_T(\mathbb{D}(\mathbf{u}^h) - \mathbb{G}^H), \mathbb{D}(\mathbf{v}^h)) & = & (\mathbf{f}, \mathbf{v}^h) & \forall\, \mathbf{v}^h \in V^h \\
(q^h, \nabla \cdot \mathbf{u}^h) & = & 0 & \forall\, q^h \in Q^h \\
(\mathbb{G}^H - \mathbb{D}(\mathbf{u}^h), \mathbb{L}^H) & = & 0 & \forall\, \mathbb{L}^H \in L^H
\end{array}
$$

$\nu_T(t, \mathbf{x}) \geq 0$ – turbulent viscosity, turbulence model
$\mathbb{G}^H = P_{L^H}\mathbb{D}(\mathbf{u}^h)$ – L^2–projection

- same idea as in local projection stabilization (LPS) schemes for stabilizing convection–dominated equations

Properties of the Projection–Based VMS Method

- three scale decomposition:
  - (resolved) large scales
  - resolved small scales
  - unresolved small scales
- turbulence model acts directly only on the resolved small scales
  modeling the influence of unresolved small scales
- indirect influence onto large scales by coupling of resolved small and large scales

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- parameters of the VMS method
  - L^H
  - ν_T

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  modeling the influence of unresolved small scales
- indirect influence onto large scales by coupling of resolved small and large scales
- parameters of the VMS method
  - L^H
  - ν_T
- finite element error analysis: J., Kaya (2008); J., Kaya, Kindl (2008)
  follows the approach of the analysis for Smagorinsky model
- similar approach with finite volume methods by Gravemeier (2006)

How to Choose the Large Scale Space L^H ?

- standard bases for velocity–pressure finite element spaces
- here: L^H defined on the same grid:

$$L^H = \text{span}\left\{ \begin{pmatrix} l_j^H & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & l_j^H & 0 \\ l_j^H & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & l_j^H \\ 0 & 0 & 0 \\ l_j^H & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & l_j^H & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & l_j^H \\ 0 & l_j^H & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & l_j^H \end{pmatrix} \right\}$$

$j = 1, \ldots, n_L$

- two–level method (for convection–diffusion equations), J., Kaya, Layton (2006)

How to Choose the Large Scale Space L^H ? (cont.)

- coupled system

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & B_1^T & \tilde{G}_{11} & \tilde{G}_{12} & \tilde{G}_{13} & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & B_2^T & 0 & \tilde{G}_{22} & 0 & \tilde{G}_{24} & \tilde{G}_{25} & 0 \\ A_{31} & A_{32} & A_{33} & B_3^T & 0 & 0 & \tilde{G}_{33} & 0 & \tilde{G}_{35} & \tilde{G}_{36} \\ B_1 & B_2 & B_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ G_{11} & 0 & 0 & 0 & M & 0 & 0 & 0 & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 & 0 & \frac{M}{2} & 0 & 0 & 0 & 0 \\ G_{31} & 0 & G_{33} & 0 & 0 & 0 & \frac{M}{2} & 0 & 0 & 0 \\ 0 & G_{42} & 0 & 0 & 0 & 0 & 0 & M & 0 & 0 \\ 0 & G_{52} & G_{53} & 0 & 0 & 0 & 0 & 0 & \frac{M}{2} & 0 \\ 0 & 0 & G_{63} & 0 & 0 & 0 & 0 & 0 & 0 & M \end{pmatrix} \begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \\ p^h \\ g_{11}^H \\ g_{12}^H \\ g_{13}^H \\ g_{22}^H \\ g_{23}^H \\ g_{33}^H \end{pmatrix} = \begin{pmatrix} f_1^h \\ f_2^h \\ f_3^h \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- 7 additional matrices

How to Choose the Large Scale Space L^H ? (cont.)

- condensation

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & B_1^T \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & B_2^T \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & B_3^T \\ B_1 & B_2 & B_3 & 0 \end{pmatrix} \begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \\ p^h \end{pmatrix} = \begin{pmatrix} f_1^h \\ f_2^h \\ f_3^h \\ 0 \end{pmatrix}$$

$$\begin{aligned} \tilde{A}_{11} &= A_{11} - \tilde{G}_{11} M^{-1} G_{11} - \frac{1}{2} \tilde{G}_{24} M^{-1} G_{42} - \frac{1}{2} \tilde{G}_{36} M^{-1} G_{63} \\ &\vdots \\ \tilde{A}_{33} &= A_{33} - \tilde{G}_{36} M^{-1} G_{63} - \frac{1}{2} \tilde{G}_{11} M^{-1} G_{11} - \frac{1}{2} \tilde{G}_{24} M^{-1} G_{42} \end{aligned}$$

- goal: sparsity pattern of $\tilde{A}_{\alpha\beta}$ same like $A_{\alpha\beta}$

How to Choose the Large Scale Space L^H ? (cont.)

- conditions on L^H:
  - support of each basis function of L^H only one mesh cell
  - basis of L^H is L^2–orthogonal

  $\Longrightarrow$ discontinuous finite element spaces with bases of piecewise Legendre polynomials
- simulations found in the literature: J., Kaya (2005), J., Roland (2007), J., Kindl (2008)
  - $L^H(K) = P_0(K)$ for all mesh cells K
  - $L^H(K) = P_1^{\mathrm{disc}}(K)$ for all mesh cells K

How to Choose the Large Scale Space L^H ? (cont.)

- conditions on L^H:
  - support of each basis function of L^H only one mesh cell
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  $\Longrightarrow$ discontinuous finite element spaces with bases of piecewise Legendre polynomials
- simulations found in the literature: J., Kaya (2005), J., Roland (2007), J., Kindl (2008)
  - $L^H(K) = P_0(K)$ for all mesh cells K
  - $L^H(K) = P_1^{\mathrm{disc}}(K)$ for all mesh cells K
- goal: method should determine local coarse space $L^H(K)$ a posteriori such that
  - $L^H(K)$ is a small space where flow is strongly turbulent
    $\Longleftrightarrow$ turbulence model has large influence
  - $L^H(K)$ is a large space where flow is less turbulent
    $\Longleftrightarrow$ turbulence model has little influence

Adaptive Large Scale Space

- assumption: local turbulence intensity reflected by size of local resolved small scales
  - size of resolved small scales large $\Longrightarrow$ many unresolved scales can be expected
  - size of resolved small scales small $\Longrightarrow$ little unresolved scales can be expected
- compute the deformation tensor of the large scales G^H
  - computation is not necessary for static L^H
  - additional matrices to assemble in comparison to static L^H

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- compute the deformation tensor of the large scales G^H
  - computation is not necessary for static L^H
  - additional matrices to assemble in comparison to static L^H
- define indicator of the size of the resolved small scales in mesh cell K

$$\eta_K = \frac{\|\mathbb{G}^H - \mathbb{D}(\mathbf{u}^h)\|_{L^2(K)}}{\|1\|_{L^2(K)}} = \frac{\|\mathbb{G}^H - \mathbb{D}(\mathbf{u}^h)\|_{L^2(K)}}{|K|^{1/2}}, \quad K \in \mathcal{T}^h$$

  - size of the resolved small scales does not depend on size of mesh cell
  - size of the mesh cell scales out

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  - size of the resolved small scales does not depend on size of mesh cell
  - size of the mesh cell scales out
- compare η_K to some reference value
  - similar to a posteriori error estimation and mesh refinement

Adaptive Large Scale Space (cont.)

- reference values
  - mean value at current time $\overline{\eta} := \frac{1}{\text{no. of cells}} \sum_{K \in \mathcal{T}^h} \eta_K$
  - time average of mean values $\overline{\eta}^t := \frac{1}{\text{no. of time steps}} \sum_{\text{time steps}} \overline{\eta}$
  - linear combination $\overline{\eta}^{t/2} := \frac{\overline{\eta} + \overline{\eta}^t}{2}$

Adaptive Large Scale Space (cont.)

- reference values
  - mean value at current time $\overline{\eta} := \dfrac{1}{\text{no. of cells}} \displaystyle\sum_{K\in\mathcal{T}^h} \eta_K$
  - time average of mean values $\overline{\eta}^t := \dfrac{1}{\text{no. of time steps}} \displaystyle\sum_{\text{time steps}} \overline{\eta}$
  - linear combination $\overline{\eta}^{t/2} := \dfrac{\overline{\eta} + \overline{\eta}^t}{2}$

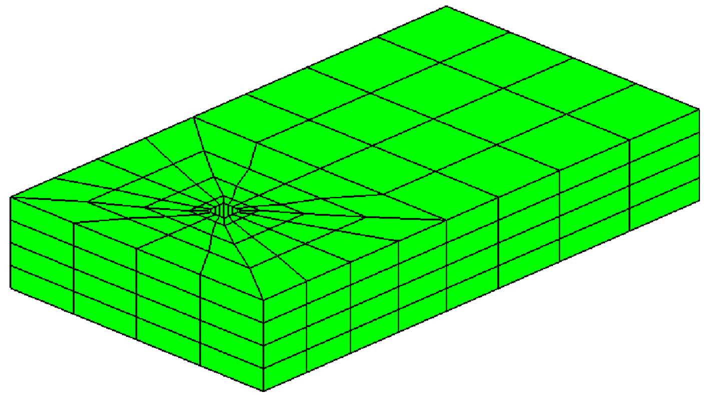
- local spaces ($V^h = Q_2$ or $V^h = P_2^{\text{bubble}}$)
  - $\mathbb{L}^H(K) = 0 = P_{00}(K)$ turbulence model influences locally all resolved scales
  - $\mathbb{L}^H(K) = P_0(K)$
  - $\mathbb{L}^H(K) = P_1(K)$
  - $\mathbb{L}^H(K) = P_2(K)$ set $\nu_T(K) = 0$, locally no turbulence model

Adaptive Large Scale Space (cont.)

- procedure:
  - choose three values
$$0 \le C_1 \le C_2 \le C_3$$
  - choose a mean value η
  - choose a frequency of updating the large scale space
$$n_{\mathrm{update}}$$
  - in every n_{update}–th step:
    compute η_K and determine the local large scale space
$$\begin{array}{ll} L^H(K) = P_2^{\mathrm{disc}}(K),\ \nu_T(K) = 0 & \text{if } \eta_K \le C_1\eta \\ L^H(K) = P_1^{\mathrm{disc}}(K) & \text{if } C_1\eta < \eta_K \le C_2\eta \\ L^H(K) = P_0(K) & \text{if } C_2\eta < \eta_K \le C_3\eta \\ L^H(K) = P_{00}(K) & \text{if } C_3\eta < \eta_K \end{array}$$

3.5 A Numerical Study – the Turbulent Flow Around a Cylinder

- domain and coarse grid



- vortex street (iso–surfaces of the velocity)
- statistically periodic flow
- $Re = 22000$, (inflow, diameter of cylinder, viscosity)

3.5 A Numerical Study – the Turbulent Flow Around a Cylinder

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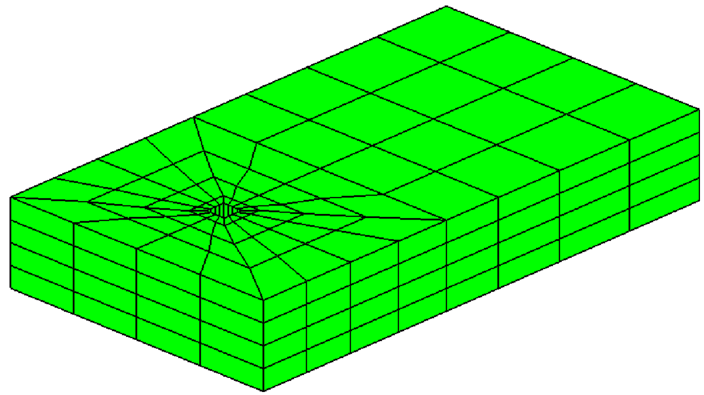


- vortex street (iso–surfaces of the velocity)
- statistically periodic flow
- $Re = 22000$, (inflow, diameter of cylinder, viscosity)
- Q_2/P_1^{disc}, no. of d.o.f.: 522 720 velocity, 81 920 pressure
- Crank–Nicolson scheme with $\Delta t = 0.005$
- static Smagorinsky model for ν_T:

$$\nu_T = 0.01(2h_{K,\min})^2 \|\mathbb{D}(\mathbf{u}^h)\|_F$$

$h_{K,\min}$ shortest edge of K

Turbulent Flow Around a Cylinder at $Re = 22000$ *(cont.)*

- characteristic values of the flow
  - lift coefficient c_l
  - drag coefficient c_d
  - Strouhal number St

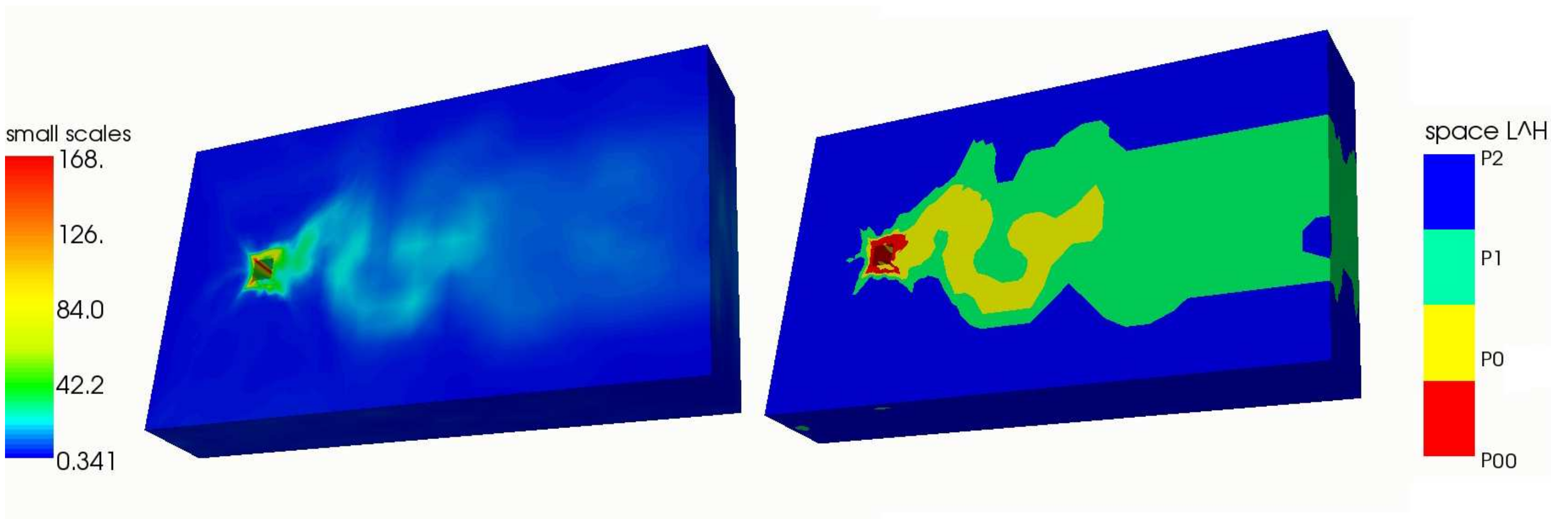
Turbulent Flow Around a Cylinder at $Re = 22000$ (cont.)

- characteristic values of the flow
  - lift coefficient c_l
  - drag coefficient c_d
  - Strouhal number St
- time–averaged values and rms values (25 periods)

C_1	C_2	C_3	mean	n_{update}	$\bar{c}_l$	$c_{l,\text{rms}}$	$\bar{c}_d$	$c_{d,\text{rms}}$	St
VMS with $L^H = P_0$					0.010	1.10	2.55	0.16	0.138
VMS with $L^H = P_1^{\text{disc}}$					-0.007	1.24	2.57	0.21	0.137
0.2	0.5	2	$\overline{\eta}$	1	-0.018	1.12	2.49	0.20	0.135
0.2	0.75	2	$\overline{\eta}$	1	-0.009	1.29	2.57	0.14	0.142
0.2	1.0	2	$\overline{\eta}$	1	0.016	1.05	2.49	0.15	0.136
0.2	1.25	2	$\overline{\eta}$	1	0.005	1.33	2.57	0.20	0.138
0.2	0.5	2	$\overline{\eta}^{t/2}$	1	-0.042	1.36	2.54	0.23	0.144
0.2	0.75	2	$\overline{\eta}^{t/2}$	1	-0.003	1.24	2.60	0.12	0.136
0.2	1.0	2	$\overline{\eta}^{t/2}$	1	-0.055	1.36	2.59	0.13	0.140
0.2	1.25	2	$\overline{\eta}^{t/2}$	1	0.011	1.53	2.62	0.22	0.143
experiments						0.7–1.4	1.9–2.1	0.1–0.2	0.132

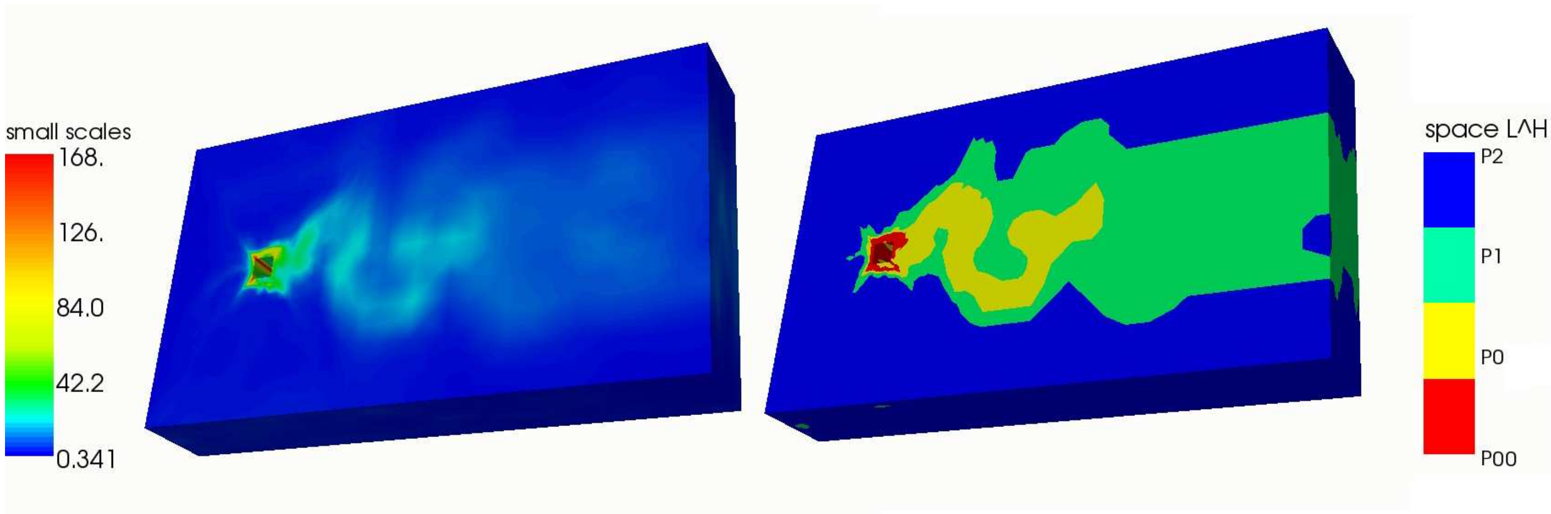
Turbulent Flow Around a Cylinder at $Re = 22000$ *(cont.)*

- typical snapshots



Turbulent Flow Around a Cylinder at $Re = 22000$ (cont.)

- typical snapshots



- results with adaptive large scale space and good parameters better than with uniform large scale space

$$C_1 \approx 0.2, \quad C_2 \in \{0.5, 1\}, \quad C_3 \in \{2, 3\}, \quad \overline{\eta}, \quad n_{\mathrm{update}} = 1$$

- different mean values lead to rather different results
- increase of n_{update} leads to worse results

3.6 Summary and Outlook

- VMS is attractive alternative to LES
- bubble VMS methods not to be recommended
- current approaches:
  - two–scale VMS method (Hughes et al.)
  - algebraic Multigrid VMS method (Gravemeier et al.)
  - projection–based finite element VMS method (J. et al.)
- literature: review Gravemeier (2006)

3.6 Summary and Outlook

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  - projection–based finite element VMS method (J. et al.)
- literature: review Gravemeier (2006)
- projection–based finite element VMS method
  - adaptive VMS method is able to adapt large scale space to local intensity of the turbulence
  - further studies of parameters of the method (C_1, C_2, C_3, n_{update}) necessary
  - mathematical analysis for supporting parameter choice necessary
  - method can be extended to tetrahedral meshes and $P_2^{\mathrm{bubble}}/P_1^{\mathrm{disc}}$ finite element (J., Kindl, Suciu (Preprint 2009))
  - add adaptivity in space and time

4 Further Aspects

4.1 Analysis of Temporal Mean Values for Turbulent Flows

- J., Manica, Layton (2007)
- homogeneous Dirichlet boundary conditions
- temporal mean value of a quantity q

$$< q >= \limsup_{T\to\infty} \frac{1}{T}\int_0^T q(t)\,dt$$

- studied for energy dissipation rate per volume

$$\varepsilon(\mathbf{u}) = \frac{Re^{-1}}{|\,\Omega\,|}\|\,\nabla\mathbf{u}(\cdot,t)\,\|^2$$

  and kinetic energy

$$k(\mathbf{u}) = \frac{1}{2|\,\Omega\,|}\|\mathbf{u}(\cdot,t)\|^2$$

- Galerkin finite element discretization (continuous–in–time) with inf–sup stable finite element spaces (V^h, Q^h)

Analysis of Temporal Mean Values for Turbulent Flows: Large Data

- initial results for $\mathbf{u}$ obtained with Hopf construction

$$\lim_{T\to\infty}\frac{1}{T}\|\mathbf{u}(T)\|^2 = 0,\ \lim_{T\to\infty}\frac{1}{T}\|\mathbf{u}^h(T)\|^2 = 0 \implies \lim_{T\to\infty}\frac{1}{T}\|(\mathbf{u}-\mathbf{u}^h)(T)\|^2 = 0$$

  proof for continuous solution:
  - for finite–dimensional subspaces take solution in this subspace as test function
  - stability estimate $\Longrightarrow$ solution in this subspace has finite kinetic energy
  - limit: dimension of subspace $\to \infty$

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  proof for continuous solution:
  - for finite–dimensional subspaces take solution in this subspace as test function
  - stability estimate $\Longrightarrow$ solution in this subspace has finite kinetic energy
  - limit: dimension of subspace $\to \infty$
- energy dissipation rate

$$\begin{aligned} < \varepsilon(\mathbf{u}) > \quad &\le \quad \frac{1}{|\Omega|} < (\mathbf{f},\mathbf{u}) > \le \frac{Re}{|\Omega|}\|f\|^2_{L^\infty(0,\infty;H^{-1})} \\ < \varepsilon(\mathbf{u}^h) > \quad &= \quad \frac{1}{|\Omega|} < (\mathbf{f},\mathbf{u}^h) > \le \frac{Re}{|\Omega|}\|f\|^2_{L^\infty(0,\infty;H^{-1})} \end{aligned}$$

  - energy (in)equality
  - results from above

Analysis of Temporal Mean Values for Turbulent Flows: Large Data

- estimate for pressure error

$$\| < p - p^h > \| \quad \leq \quad \frac{Re^{-1}}{\beta^h} \Big(1 + 2M\ Re^2 \|\mathbf{f}\|_{L^\infty(0,\infty;H^{-1})} \Big) < \|\nabla(\mathbf{u} - \mathbf{u}^h)\|^2 >^{1/2}$$
$$+ \left(1 + \frac{\sqrt{3}}{\beta^h} \right) \inf_{q^h \in \mathbb{Q}^h} \limsup_{T \to \infty} \| < p - q^h >_T \|$$

  - β^h – discrete inf–sup constant
  - M – norm of the convective operator

- estimate for pressure error

$$\| < p - p^h > \| \quad \leq \quad \frac{Re^{-1}}{\beta^h}\Big(1 + 2M\ Re^2 \|\mathbf{f}\|_{L^\infty(0,\infty;H^{-1})}\Big) < \|\nabla(\mathbf{u}-\mathbf{u}^h)\|^2 >^{1/2}$$
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  - β^h – discrete inf–sup constant
  - M – norm of the convective operator
  - error equation
  - take test function independent of time in the error equation
  - standard estimates

Analysis of Temporal Mean Values for Turbulent Flows: Large Data

- estimate for the velocity error

$$
\begin{aligned}
&< \varepsilon(\mathbf{u}-\mathbf{u}^h) > \\
&\leq \quad C \inf_{\tilde{u}\in Y} \Big[< \varepsilon(\mathbf{u}-\tilde{\mathbf{u}}) > + Re < \|(\mathbf{u}-\tilde{\mathbf{u}})_t\|_{-1}^2 > \\
&\qquad + Re^3 < \|\mathbf{u}-\mathbf{u}^h\|^{2/3} \|\nabla \mathbf{u}\|^{4/3} \|\nabla(\mathbf{u}-\tilde{\mathbf{u}})\|^{4/3} > \\
&\qquad + Re^3 < \|\mathbf{u}-\mathbf{u}^h\|^2 \|\nabla(\mathbf{u}-\tilde{\mathbf{u}})\|^4 > + < Re\|\mathbf{u}\| \, \|\nabla \mathbf{u}\| \, \|\nabla(\mathbf{u}-\tilde{\mathbf{u}})\|^2 > \Big] \\
&\qquad + C \inf_{q^h\in\mathbb{Q}^h} \Big[\nu^{-1} < \|p-q^h\|^2 > \Big] + C Re^3 < \|\nabla \mathbf{u}\|^4 \|\mathbf{u}-\mathbf{u}^h\|^2 > .
\end{aligned}
$$

  - $Y \subset V^h$
  - estimate not closed because of last term on the right hand side
  - proof based on error equation, error splitting and standard estimates

Analysis of Temporal Mean Values for Turbulent Flows: Large Data

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\end{aligned}
$$

  - $Y \subset V^h$
  - estimate not closed because of last term on the right hand side
  - proof based on error equation, error splitting and standard estimates
  - closed estimate possible with higher regularity assumptions (then uniqueness of weak solution)

Analysis of Temporal Mean Values for Turbulent Flows: Small Body Forces

- closed estimate possible with assumption that solution becomes stationary for $T \to \infty$
  uniqueness of the way to reach the stationary limit not assumed

$$< \varepsilon(\mathbf{u} - \mathbf{u}^h) > \le C \left[\inf_{\mathbf{v}^h \in \mathbb{V}^h} Re^{-1} \|\nabla(\mathbf{u}^* - \mathbf{v}^h)\|^2 + \inf_{q^h \in \mathbb{Q}^h} Re \|p^* - q^h\|^2 \right]$$

$(\mathbf{u}^*, p^*)$ – solution of stationary problem

Analysis of Temporal Mean Values for Turbulent Flows: Small Body Forces

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  $(\mathbf{u}^*, p^*)$ – solution of stationary problem

- additional estimates for:
  - drag and lift coefficient for flows around obstacles
  - energy dissipation rate for shear flows
    behaves accordingly to the Kolmogorov law

$$< \varepsilon(\mathbf{u}^h) > \le C \frac{U^3}{L}$$

    if grid at the wall is sufficiently fine
    U – characteristic velocity scale
    L – characteristic length scale

4.2 Solving the Algebraic Systems – Coupled Multigrid Methods

- multigrid methods for the (coupled) saddle point problems

$$\mathcal{A}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

- components of a multigrid method
  - smoother: simple iterative method for damping the highly oscillating error components
  - restriction: restricts the residual from level l to level $l-1$
  - prolongation: prolongation of correction from level $l-1$ to level l
  - coarse grid solver: direct or iterative solver on level 0

Smoother for Saddle Point Problems

- main difficulty: smoother because of zero–block in the system matrix
- Vanka smoother
  - based on solution of local problems, Vanka (1986)
  - smoothing property provable if A s.p.d., $C = B^T$ for additive Vanka smoother (block Jacobi method), Zulehner (2002)
  - multiplicative Vanka smoother (block Gauss–Seidel method) more efficient
  - no theory for multiplicative Vanka smoother

Multiplicative Vanka Smoother

- decomposition of velocity d.o.f.$\mathcal{V}_h$ and pressure d.o.f. $\mathcal{Q}_h$

$$\mathcal{V}_h = \cup_{j=1}^{J} \mathcal{V}_{hj}, \quad \mathcal{Q}_h = \cup_{j=1}^{J} \mathcal{Q}_{hj}$$

- $\mathcal{A}_j$ matrix block $\mathcal{A}$ which is connected to $\mathcal{W}_{hj} = \mathcal{V}_{hj} \cup \mathcal{Q}_{hj}$

$$\mathcal{A}_j = \begin{pmatrix} A_j & B_j \\ C_j & 0 \end{pmatrix} \in \mathbb{R}^{\dim(\mathcal{W}_{hj}) \times \dim(\mathcal{W}_{hj})}$$

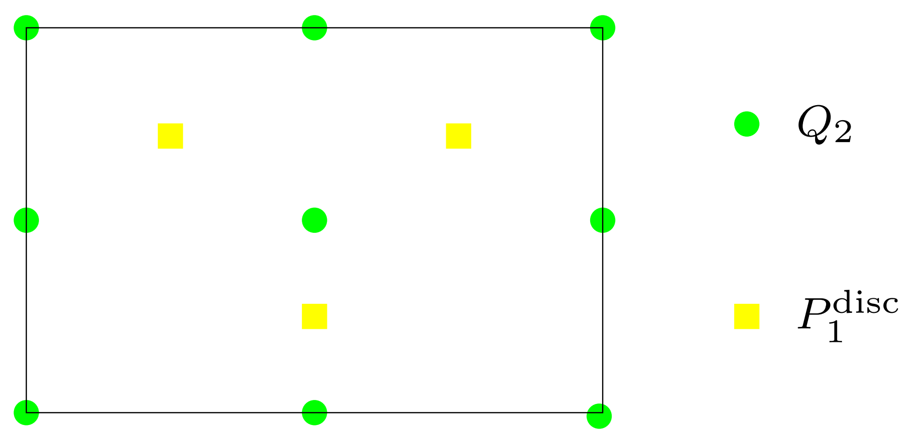
- one application of multiplicative Vanka smoother: for $j = 1, \ldots, J$

$$\begin{pmatrix} u \\ p \end{pmatrix}_j := \begin{pmatrix} u \\ p \end{pmatrix}_j + \mathcal{A}_j^{-1} \left(\begin{pmatrix} f \\ g \end{pmatrix} - \mathcal{A} \begin{pmatrix} u \\ p \end{pmatrix} \right)_j$$

- strategy:
  - choose $\mathcal{Q}_{hj}$
  - $\mathcal{V}_{hj}$ all velocity d.o.f. which are connected to pressure d.o.f. in $\mathcal{Q}_{hj}$

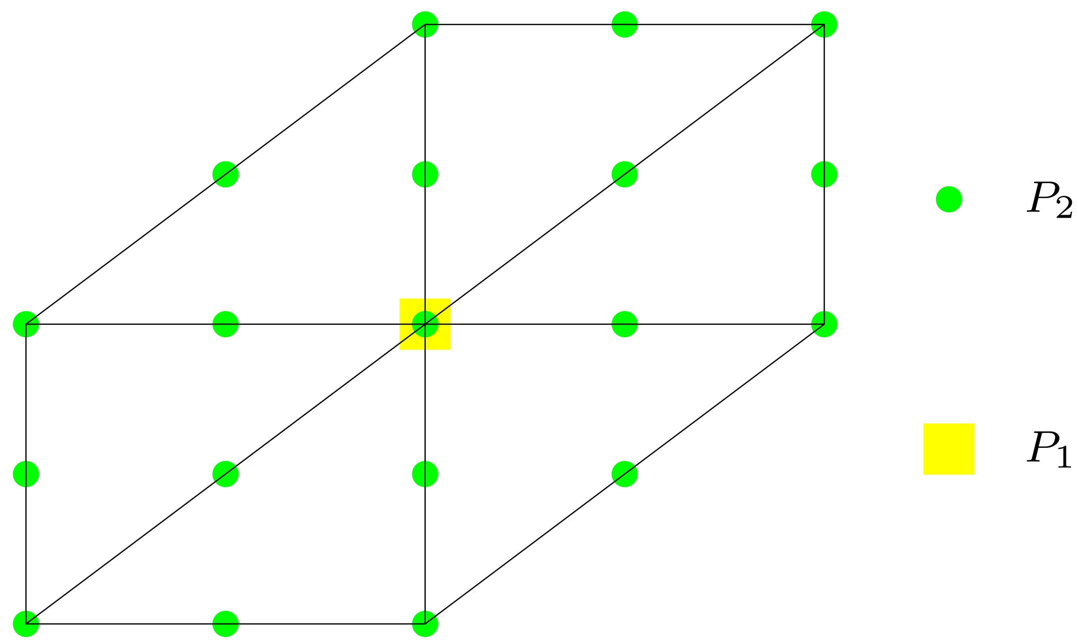
Mesh Cell Oriented Vanka Smoother

- discontinuous pressure approximation
- $\mathcal{W}_{hj}$: all d.o.f. which are connected to one mesh cell
- J : number of mesh cells



Pressure Node Oriented Vanka Smoother

- continuous pressure approximation
- $\dim \mathcal{Q}_{hj} = 1$ for all j
- J : number of pressure d.o.f.



Size of Local Systems

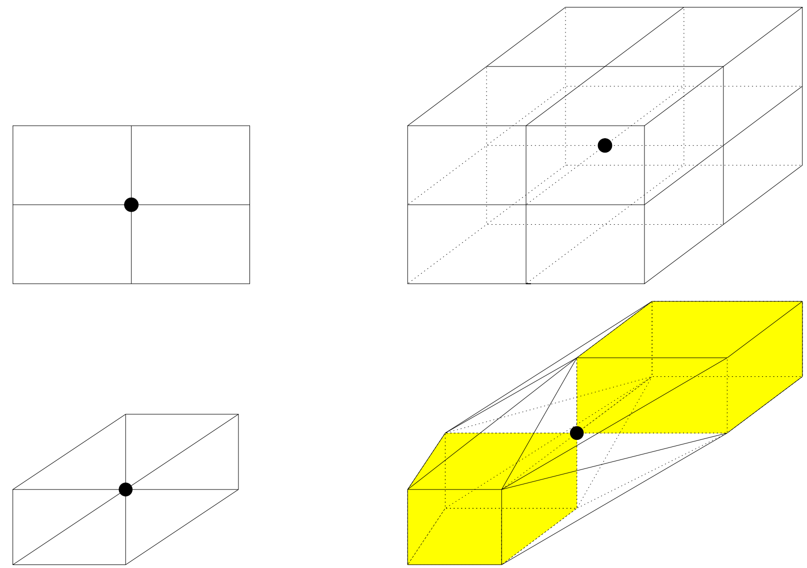
- mesh cell oriented Vanka smoother

	2d			3d		
	velo	pressure	total	velo	pressure	total
Q_1^{nc}/Q_0 (R/T)	4	1	9	6	1	19
Q_2/P_1^{disc}	9	3	21	27	4	85
Q_3/P_2^{disc}	16	6	38	64	10	202
P_1^{nc}/P_0 (C/R)	3	1	7	4	1	13

- same size for all mesh cells

Size of Local Systems (cont.)

- pressure node oriented Vanka smoother



	2d			3d		
	velo	pressure	total	velo	pressure	total
Q_2/Q_1	25	1	51	125	1	376
Q_3/Q_2	49	1	99	343	1	1030
P_2/P_1	19	1	39	65	1	196
P_3/P_2	37	1	75	175	1	526

The Multiple Discretization Multilevel Method (md ml)

- coupled multigrid methods with Vanka smoother:
  - very efficient for lowest order non–conforming finite element discretizations (Turek (1999); J., Tobiska (2000))
  - less efficient for higher order finite element discretizations (J., Matthies (2001))

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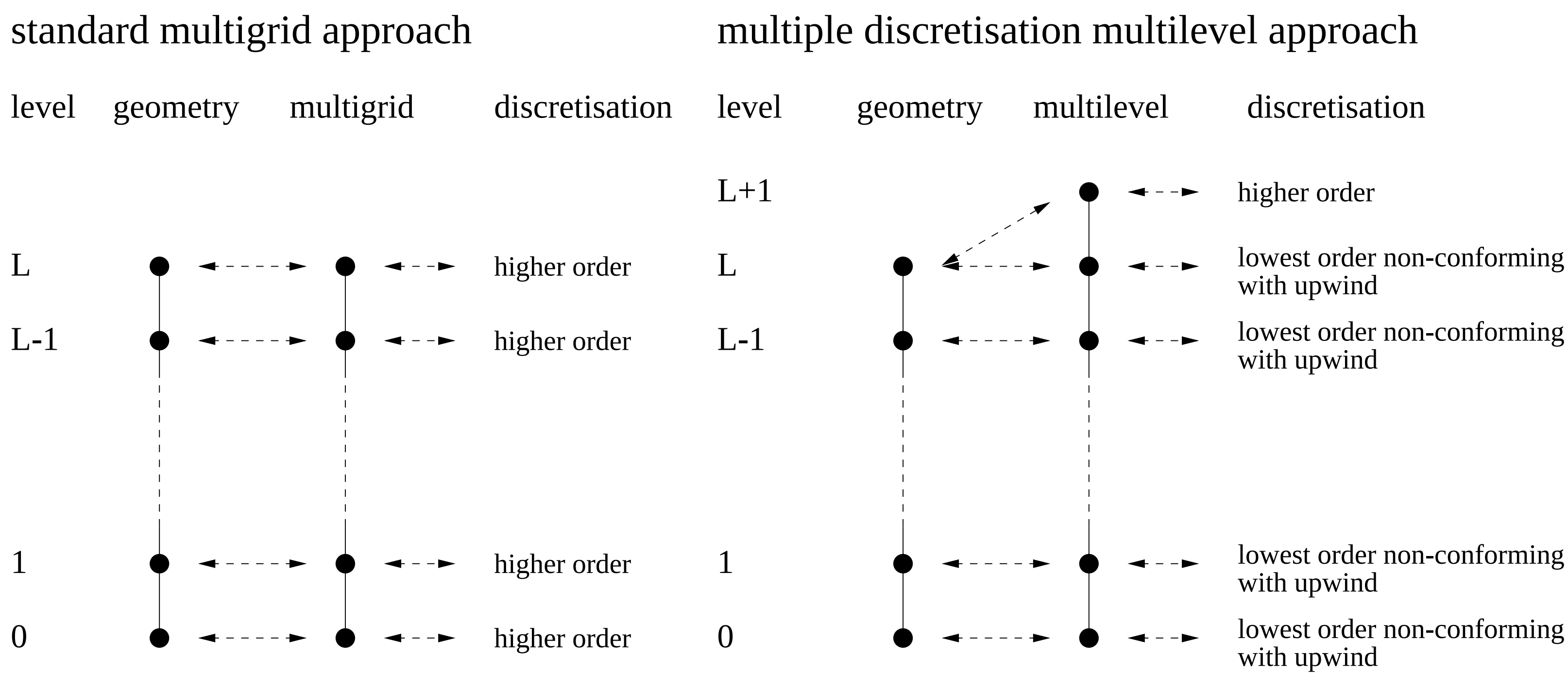
  - very efficient for lowest order non–conforming finite element discretizations (Turek (1999); J., Tobiska (2000))
  - less efficient for higher order finite element discretizations (J., Matthies (2001))
- Construct a multigrid method for higher order finite element discretizations which is based on lowest order non–conforming finite element discretizations !!

The Multiple Discretization Multilevel Method (cont.)

- both multilevel approaches



- md ml: convergence of W–cycle for A s.p.d., $C = B^T$, Braess–Sarazin smoother: J., Knobloch, Matthies, Tobiska (2002)

Summary of Our Experiences

- higher order discretizations in space and time necessary for accurate simulations
- low order discretizations are important tools in the construction of multilevel solvers for systems coming from higher order discretizations
- system are much more complicated to solve for higher order discretizations
- multiple discretization multilevel methods as preconditioner of stable Krylov subspace method currently an efficient approach
- flexible Krylov subspace methods necessary (flexible GMRES, Saad (1993))
- systems easier to solve for discontinuous pressure approximations
- similar observations for the steady state Navier–Stokes equations, J. (2002)

Thank you for your attention !

http://www.math.uni-sb.de/ag/john/