Isogeometric Variational Multiscale Methods in Turbulence Modeling: Progress and Challenges

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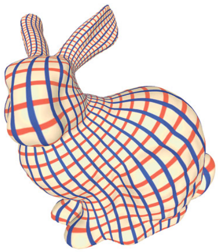
Outline

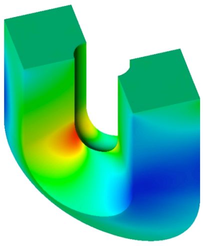
- Two ways to improve a variational method:
  - Improve the variational formulation, e.g., stabilization, VMS
  - Improve the discrete spaces, e.g., FEM, NURBS
- Isogeometric Analysis
- Incompressible Navier-Stokes Equations:
  - Large Eddy Simulation inspired VMS turbulence *theory*
  - Residual-based turbulence modeling
- Numerical Examples
  - Forced isotropic turbulence (Re_λ=164)
  - Channel flow (Re_τ= 395 to 950)
  - Weak versus strong Dirichlet boundary conditions
  - Fluid-structure interaction
- Concluding Remarks

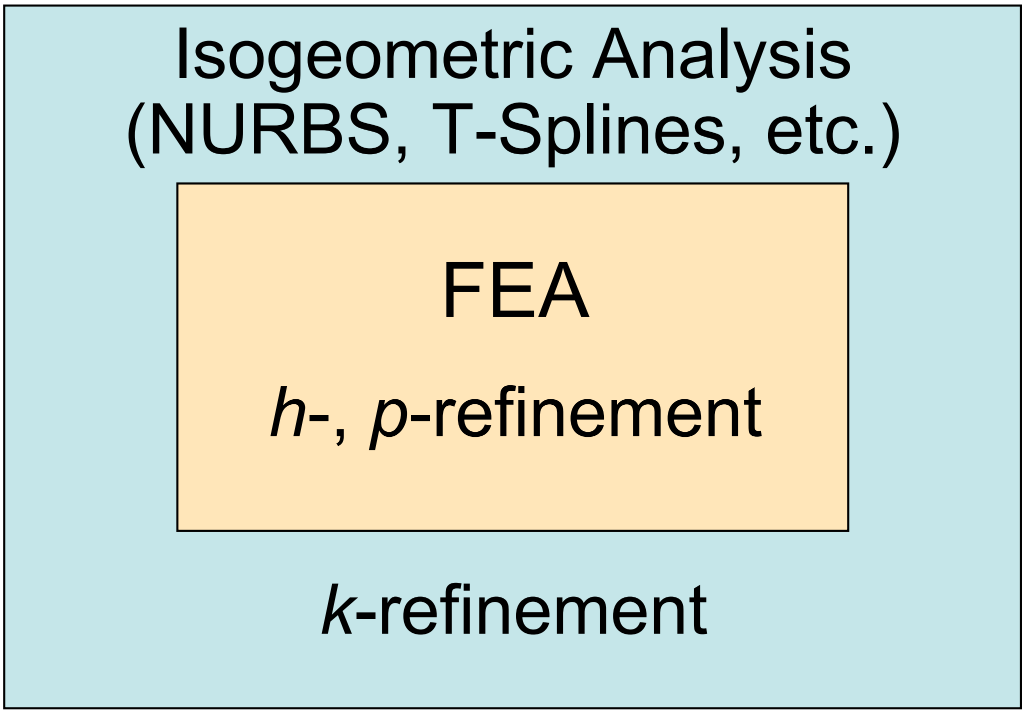
Isogeometric Analysis

- Based on technologies (e.g., NURBS) from *computational geometry* used in:
  - Design
  - Animation
  - Graphic art
  - Visualization



- Includes standard FEA as a special case, but offers other possibilities:
  - Precise and efficient geometric modeling
  - Simplified mesh refinement
  - Superior approximation properties
  - *Integration* of design and analysis

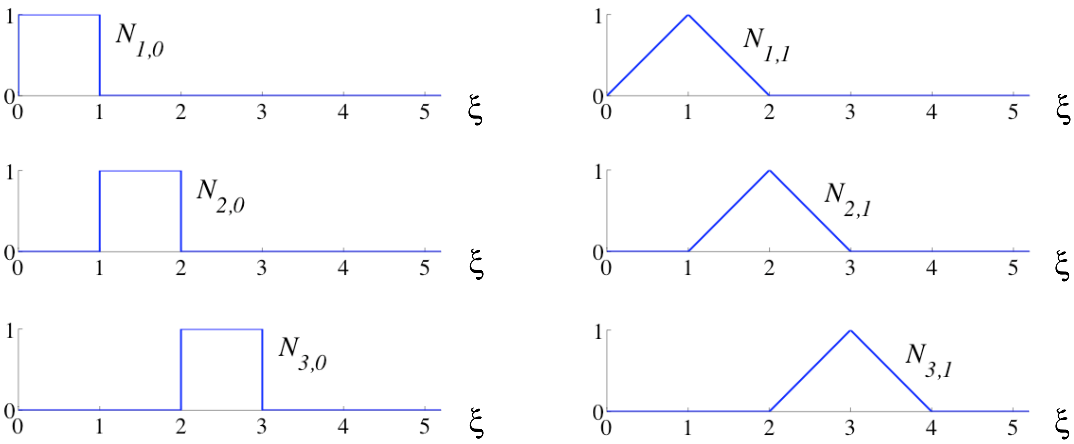


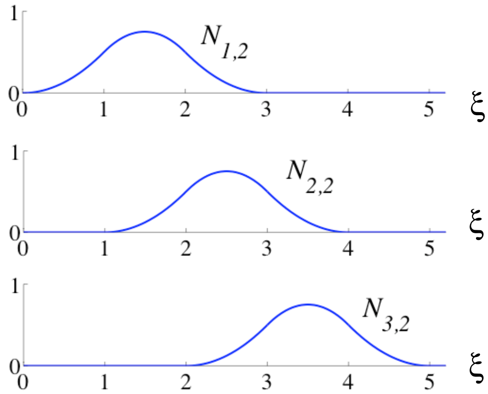


B-Splines

B-spline Basis Functions

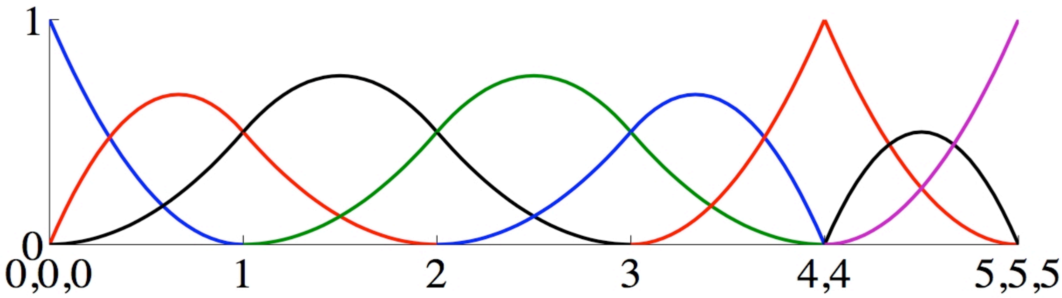
- $$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise} \end{cases}$$
- $$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) \; + \\ \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$





B-spline basis functions of order 0, 1, 2 for a *uniform knot vector:*

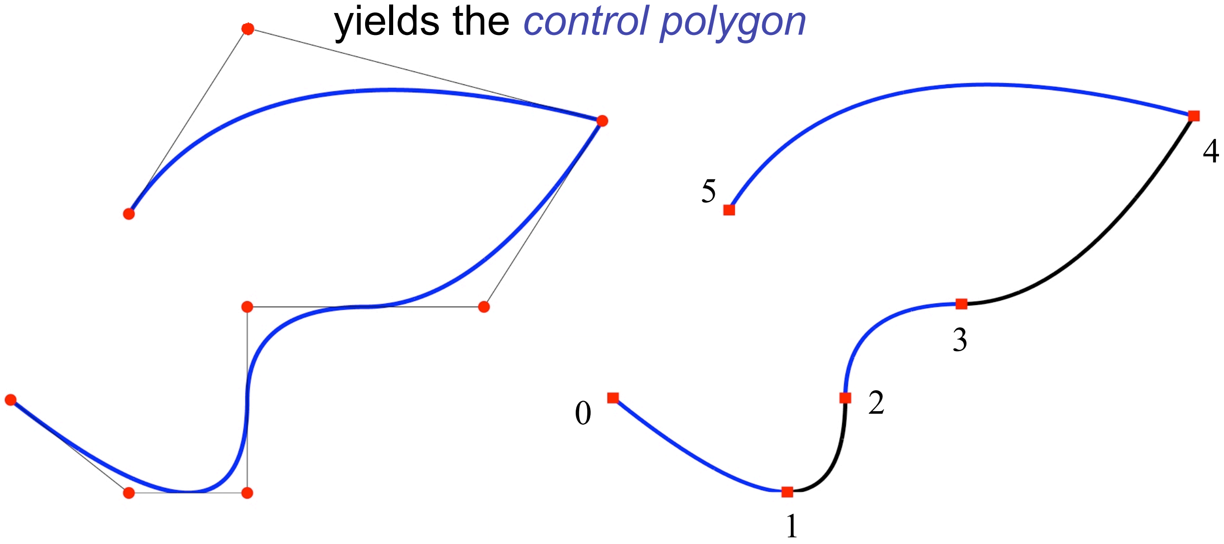
$$\Xi = \{0,1,2,3,4,\ldots\}$$



Quadratic (p=2) basis functions for an *open, non-uniform knot vector:*

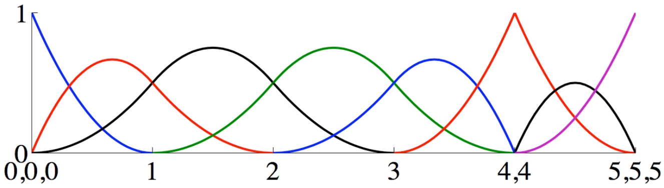
$$\Xi = \{0,0,0,1,2,3,4,4,5,5,5\}$$

Linear interpolation of control points yields the *control polygon*

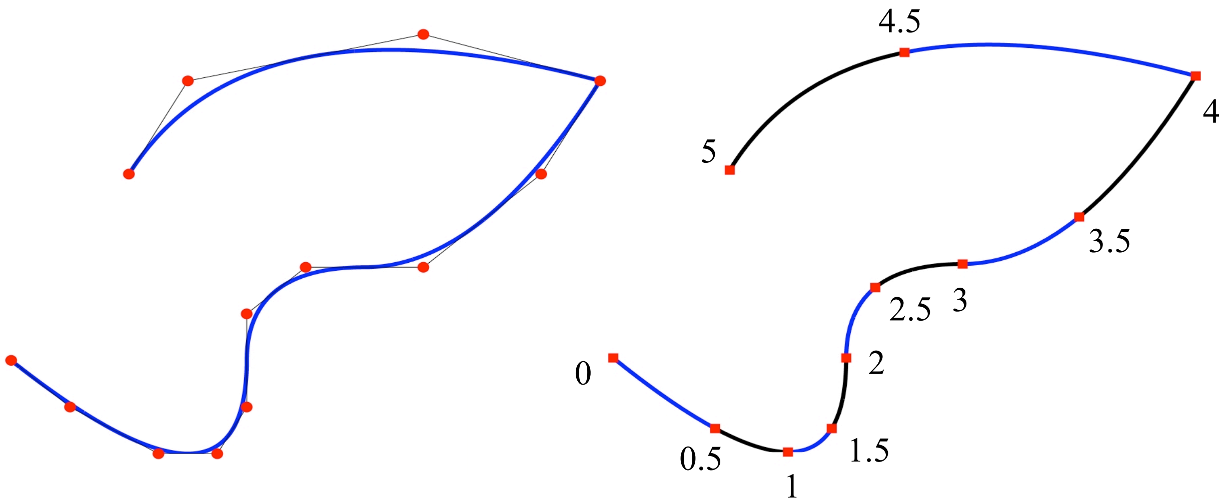


● - control points ■ - knots

Quadratic basis

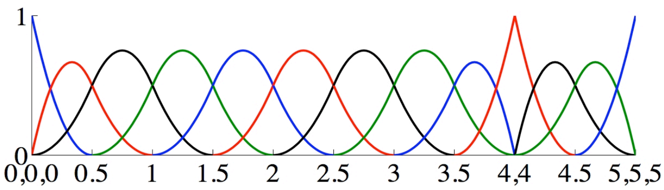


h-refined Curve

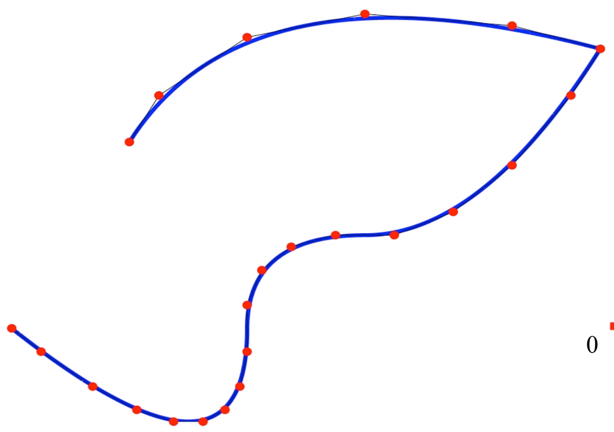


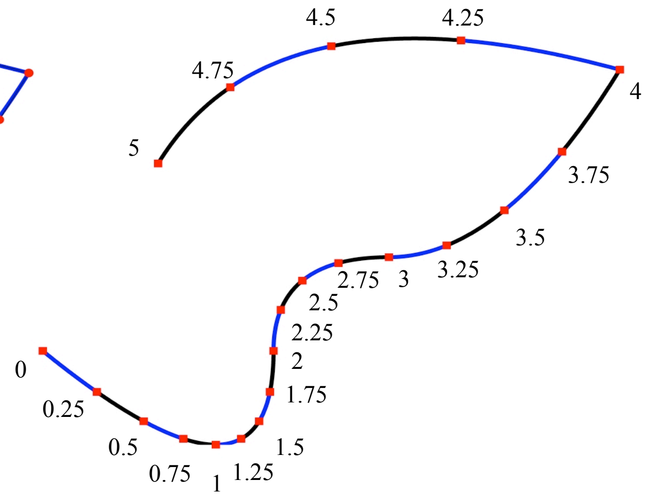
● - control points ■ - knots

Quadratic basis



Further *h*-refined Curve

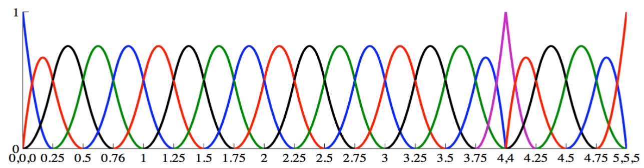




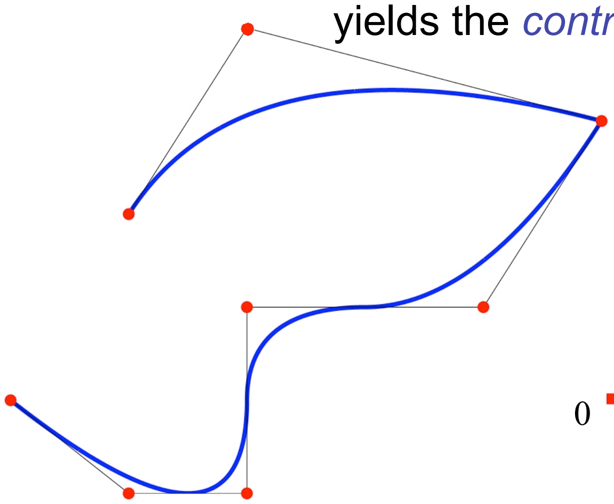
● - control points

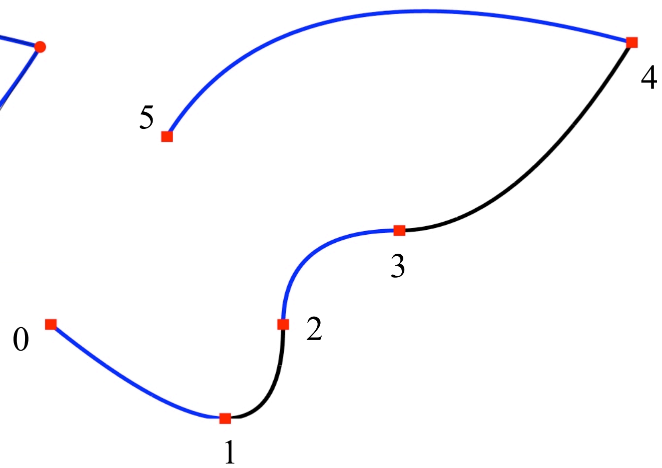
■ - knots

Quadratic basis



Linear interpolation of control points yields the *control polygon*

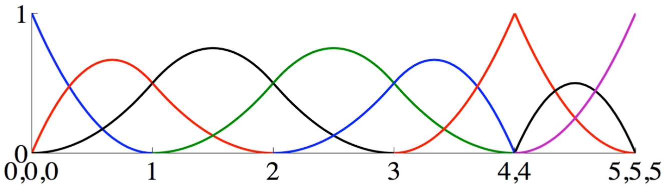




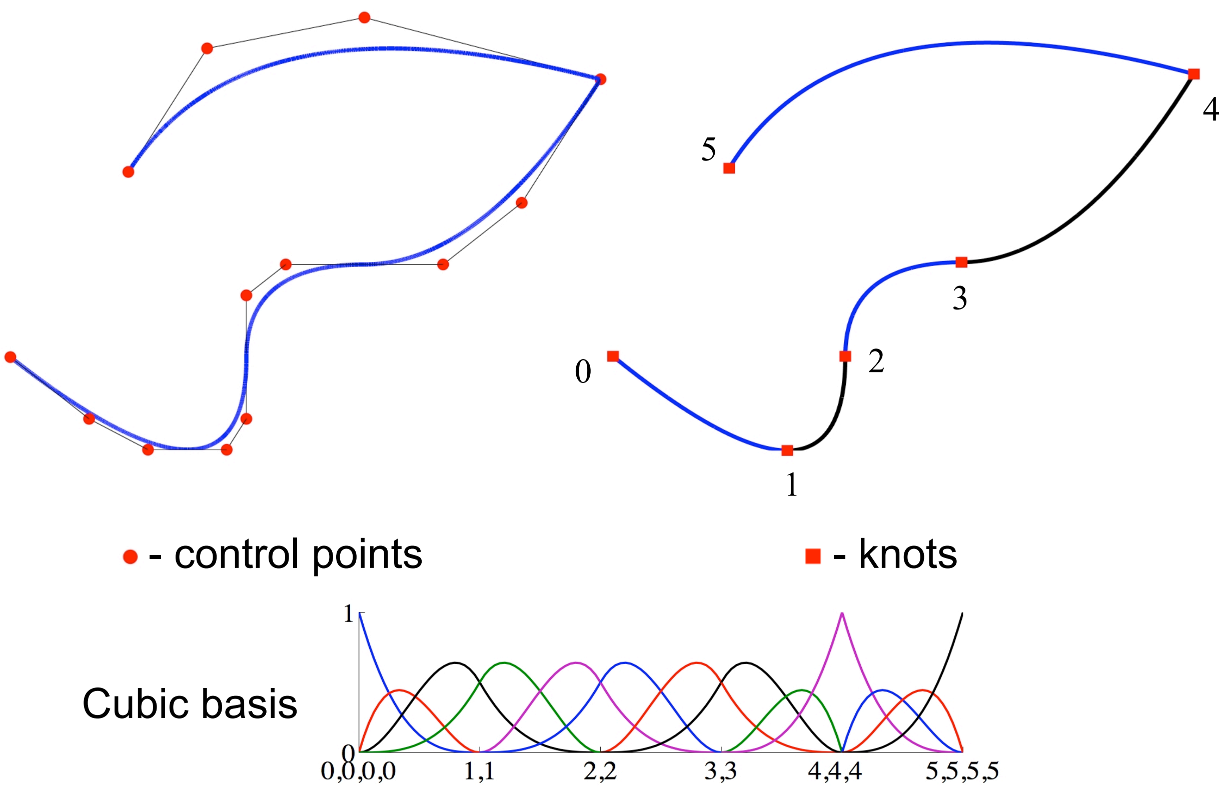
● - control points

■ - knots

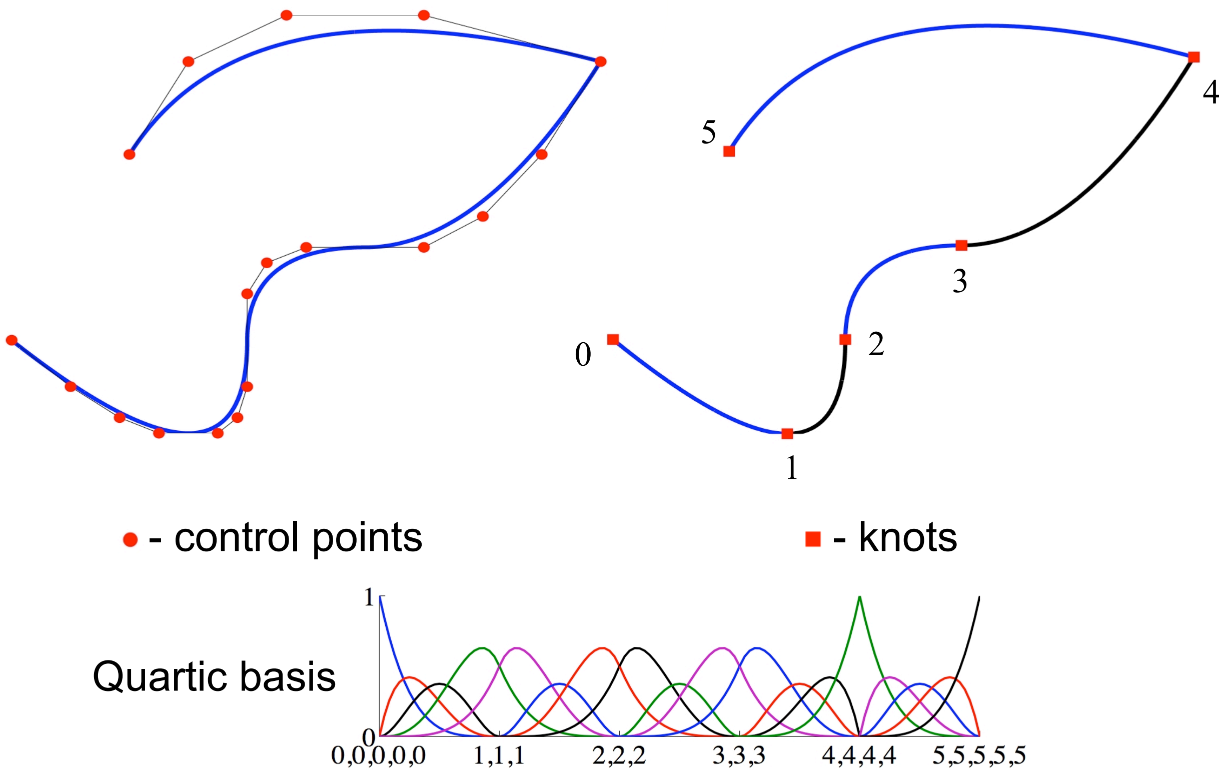
Quadratic basis



Cubic *p*-refined Curve

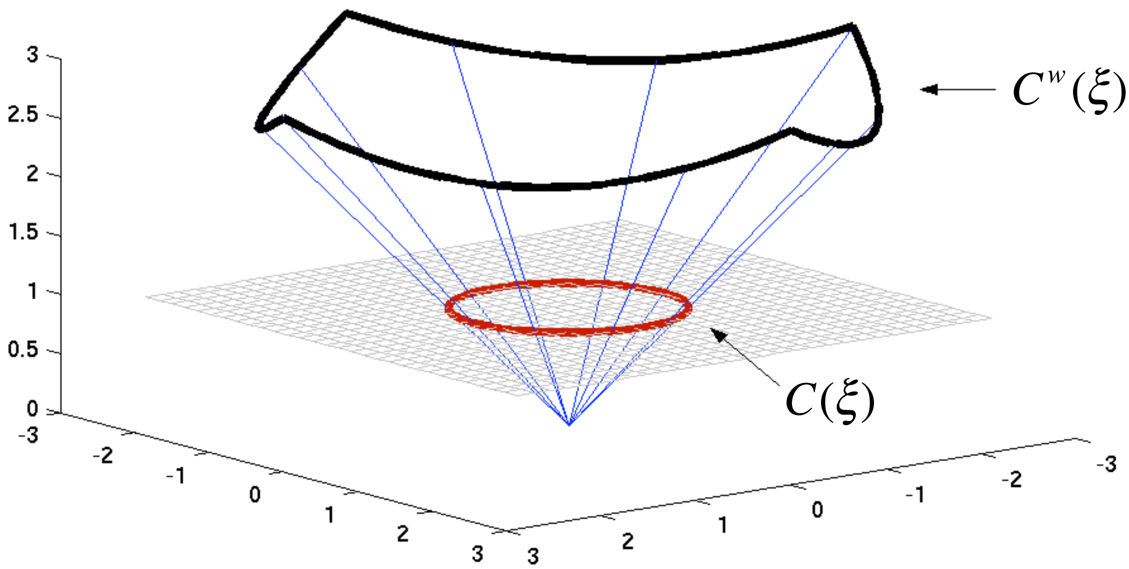


Quartic *p*-refined Curve

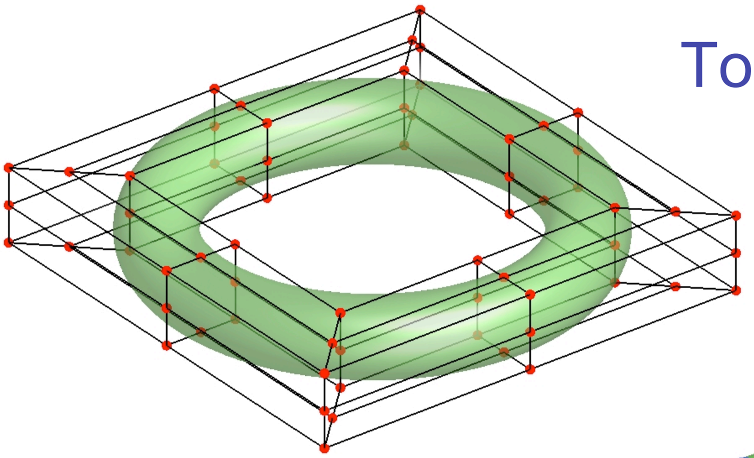


Non-Uniform Rational B-splines

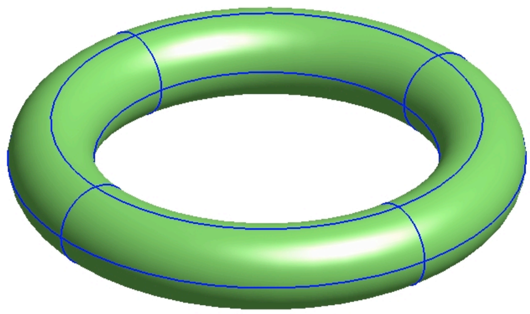
Circle from 3D Piecewise Quadratic Curves



Toroidal Surface

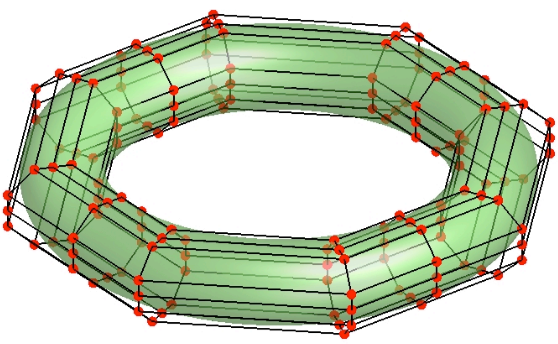


Control net

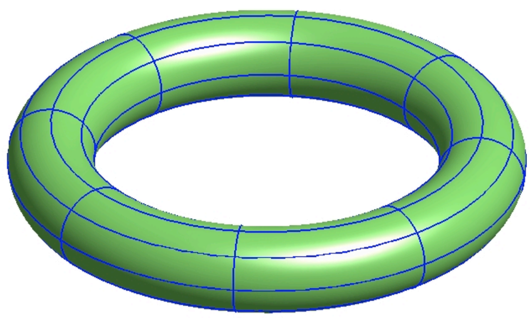


Mesh

h-refined Surface

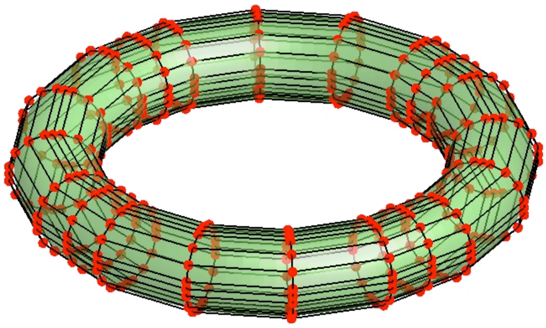


Control net

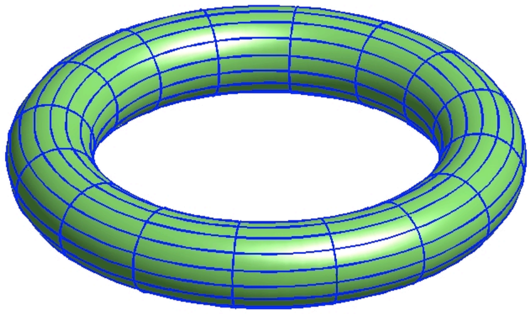


Mesh

Further *h*-refined Surface

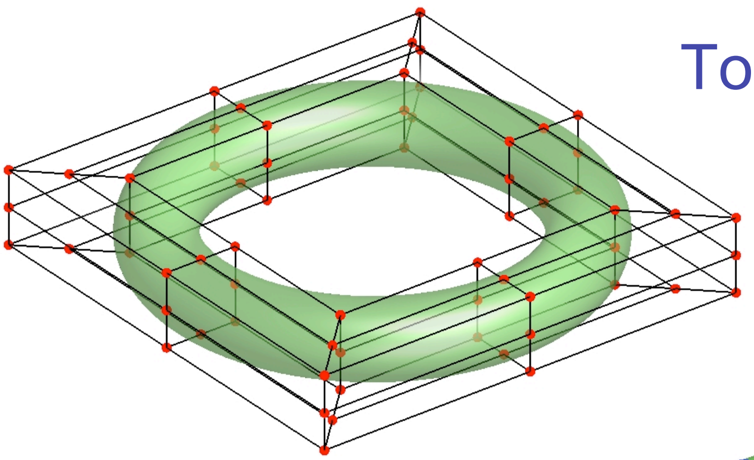


Control net

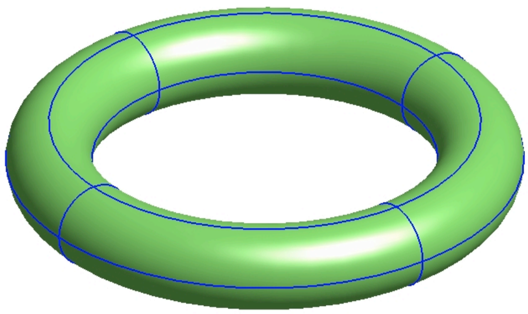


Mesh

Toroidal Surface

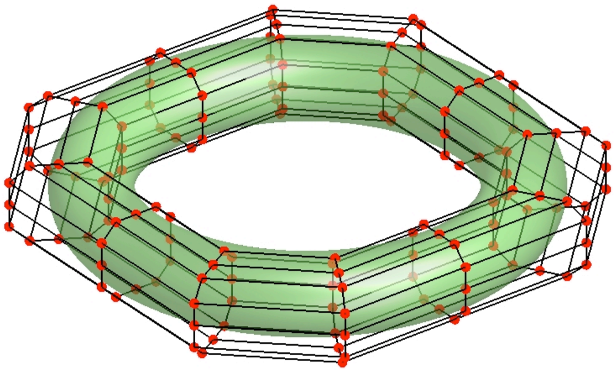


Control net

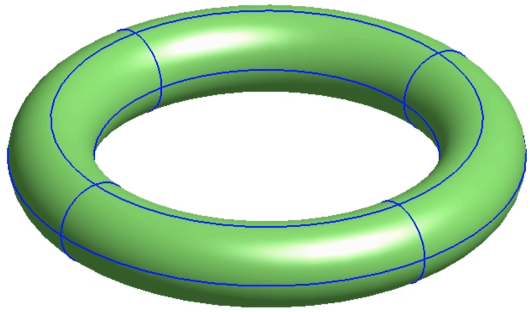


Mesh

Cubic *p*-refined Surface

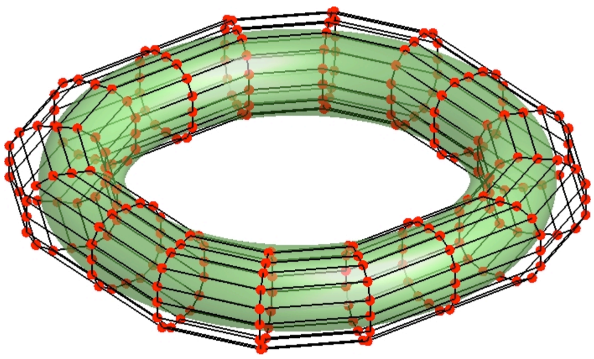


Control net

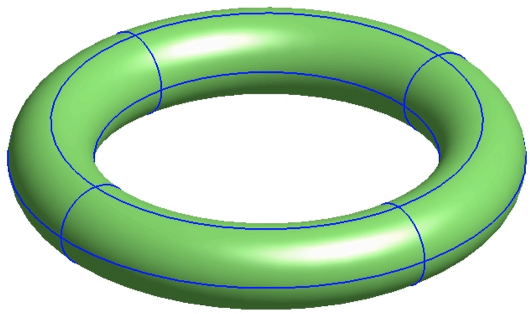


Mesh

Quartic *p*-refined Surface

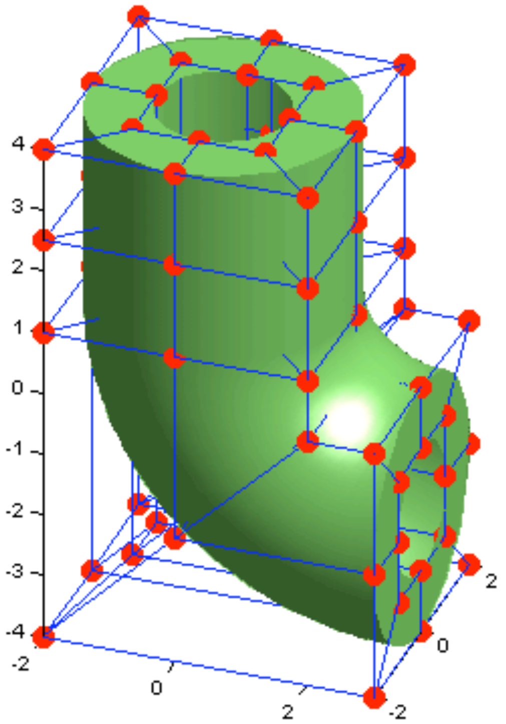


Control net

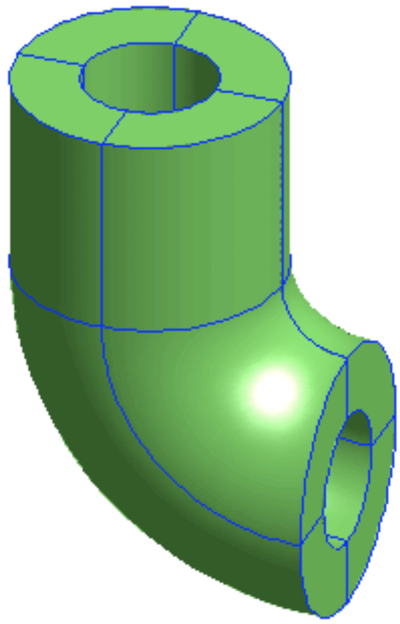


Mesh

Control Net

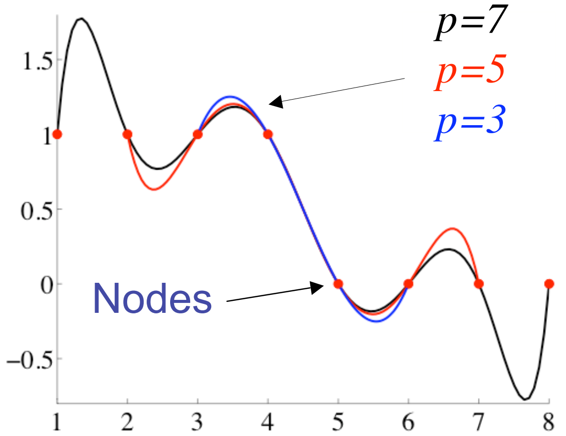


Mesh

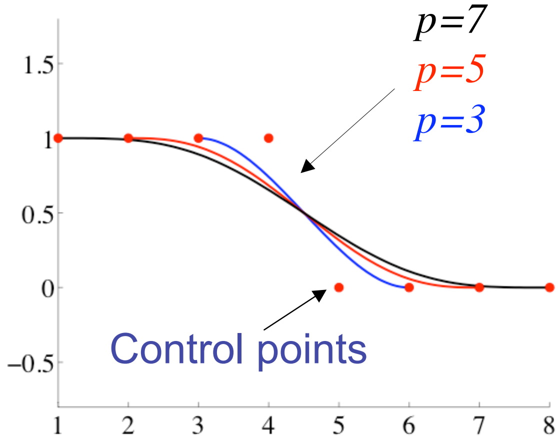


Variation Diminishing Property

Lagrange polynomials



NURBS



Finite Element Analysis and Isogeometric Analysis

▪ Compact support
▪ Partition of unity
▪ Affine covariance
▪ Isoparametric concept
▪ Patch tests satisfied

Pure Advection

Problem:

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + u\dfrac{\partial \varphi}{\partial x} = 0 \quad \text{for} \quad x \in (0, L) & (1) \\ \varphi(0,t) = \varphi(L,t) & (2) \end{cases}$$

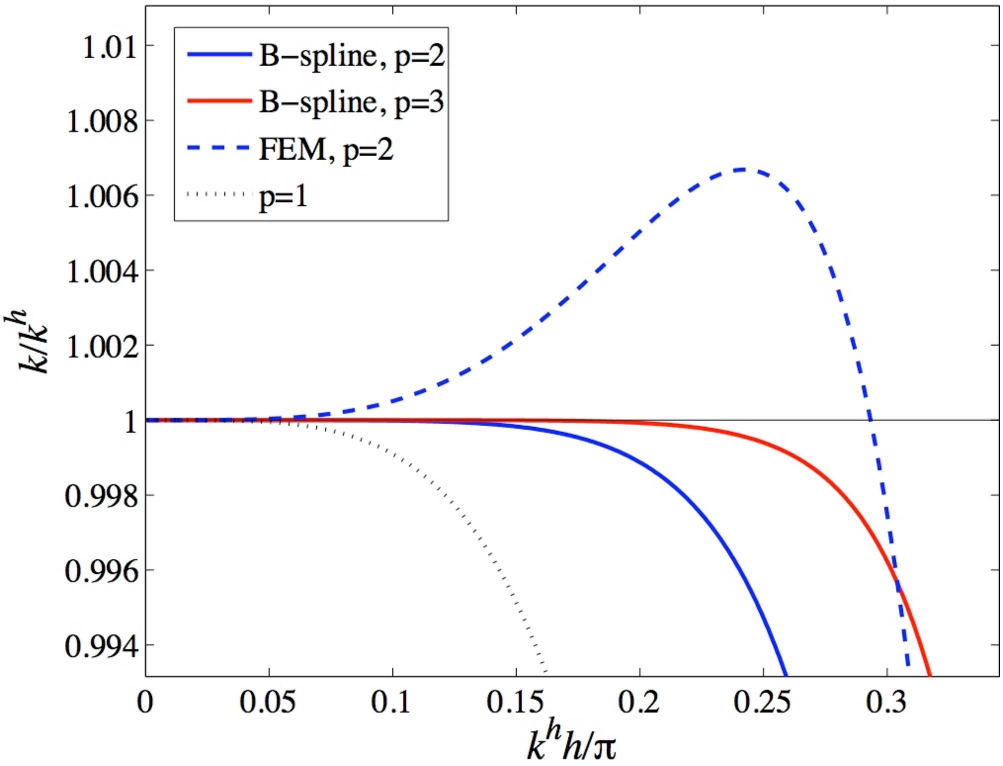
Solutions take the form:

$$\varphi_n(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t)$$

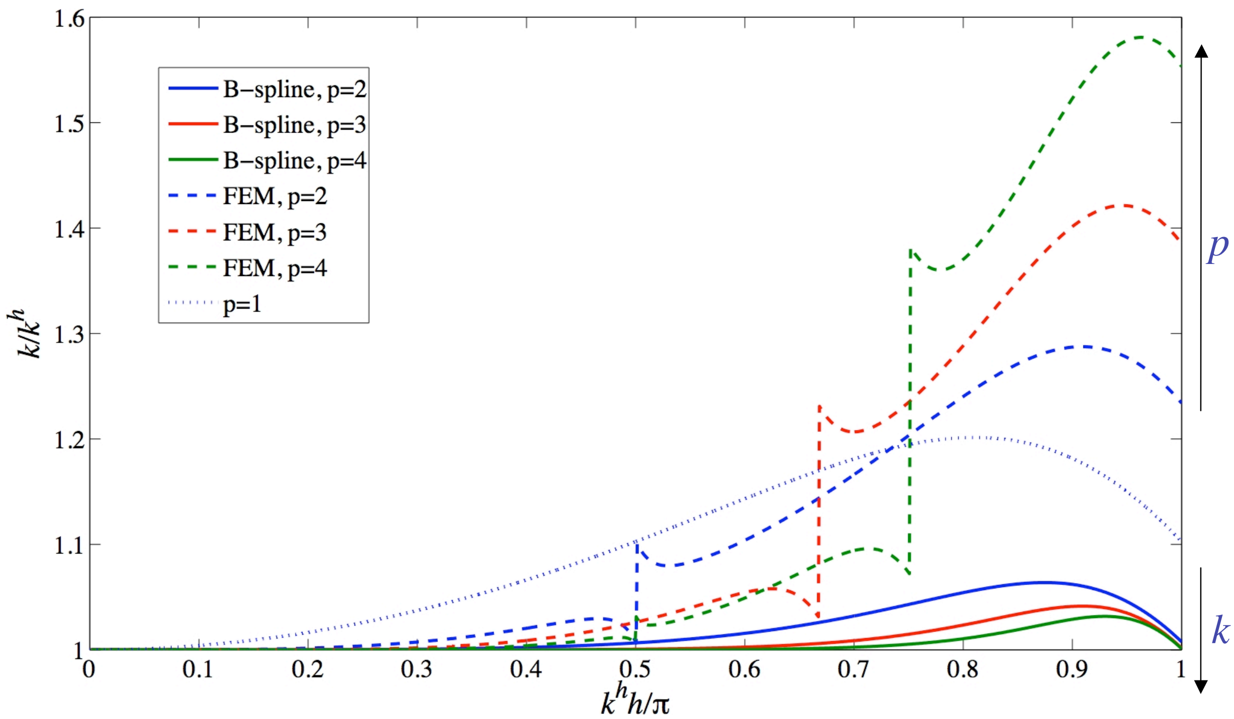
$$(1) \quad \Rightarrow \quad \omega / k = u, \quad \boxed{\text{phase speed is a constant}}$$

$$(2) \quad \Rightarrow \quad k = 2n\pi / L, \quad n = 0,1,2,\ldots$$

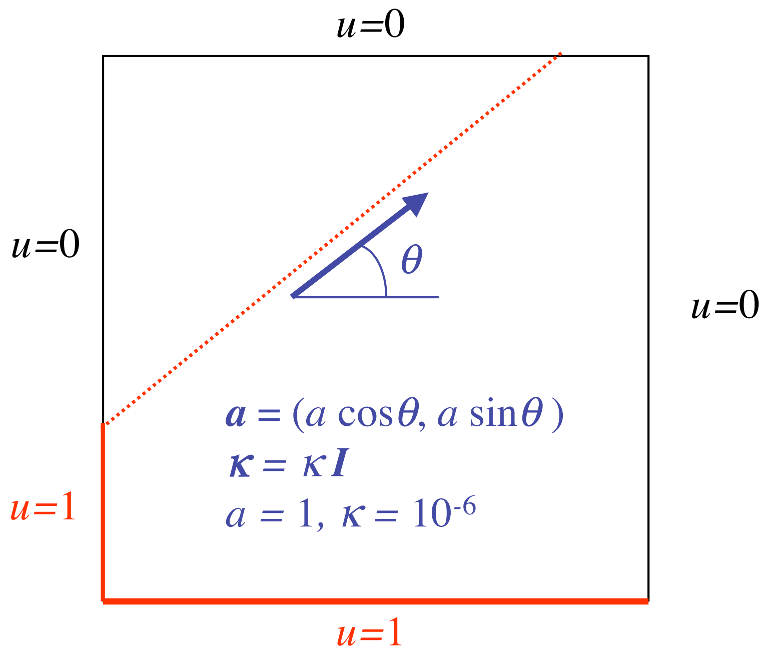
Pure Advection Phase Error



Pure Diffusion Phase Error

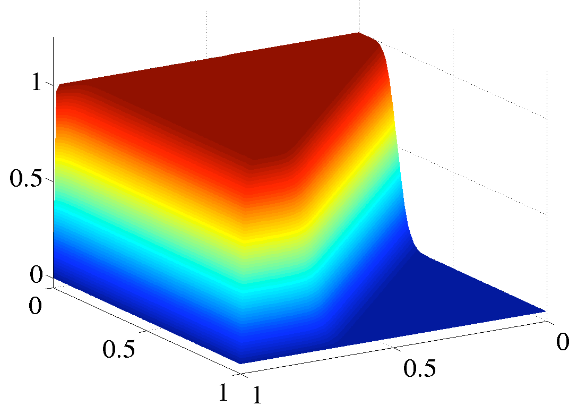


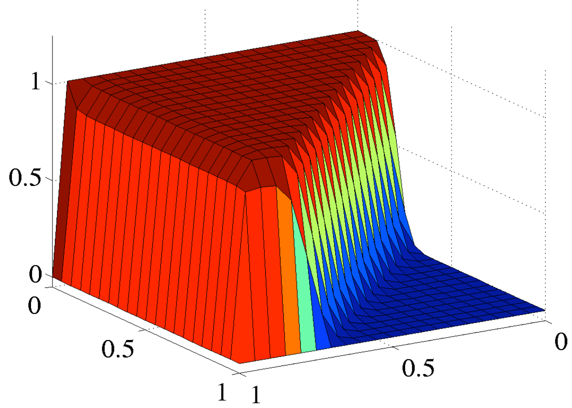
Advection Skew to Mesh: Problem Setup



Advection Skew to Mesh at 45º

- SUPG formulation
- NURBS: p =15 (14 continuous derivatives)





20 x 20 element mesh
33 x 33 control variables (dof)

Cahn-Hilliard Equation (1957)

- Applications:
  - Phase segregation of binary alloys
  - Image processing
  - Planet formation
  - Growth of tumors
  - Etc.
- Spatial derivatives of order four

Cahn-Hilliard Equation

$$E = \int_{\Omega} \left(f_0(c) + \kappa \|\nabla c\|^2 \right) \mathrm{d}x, \qquad \dot{E} \le 0$$

Conservation of mass:

$$\frac{\partial c}{\partial t} + \nabla \cdot J = 0$$

Dynamics driven by variational derivative of energy:

$$J = -M_c \nabla \left(\frac{\delta E}{\delta c} \right), \qquad \frac{\delta E}{\delta c} = f_0'(c) - 2\kappa \Delta c = \mu_c - \lambda \Delta c$$

Cahn-Hilliard equation:

$$\frac{\partial c}{\partial t} = \nabla \cdot \left(M_c \nabla \left(\mu_c - \lambda \Delta c \right) \right), \; M_c = Dc(1-c), \; \mu_c = f_0', \; \lambda \text{ const.}$$

Cahn-Hilliard Equation

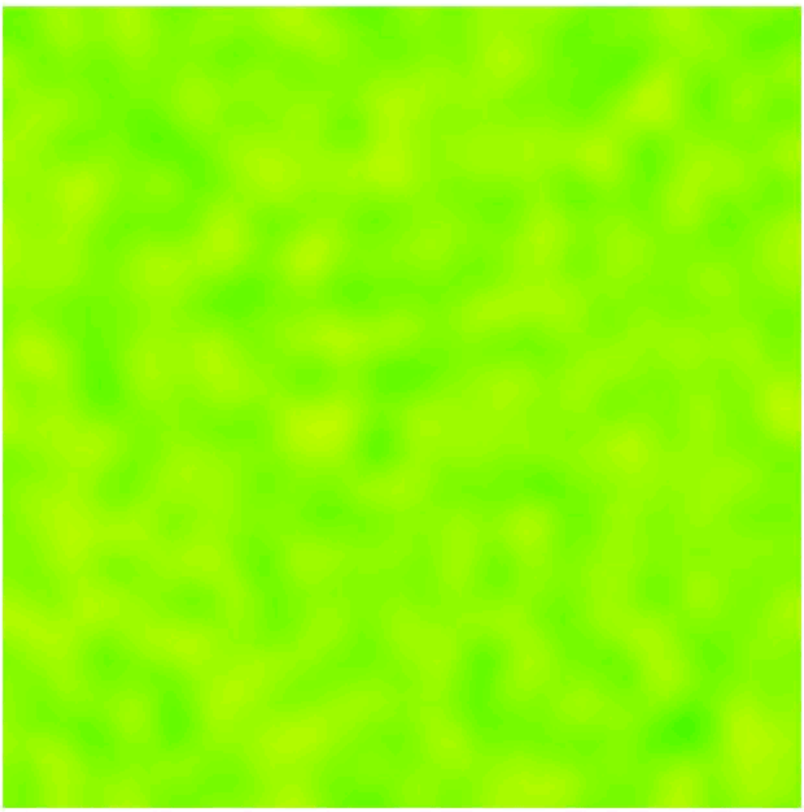
Variational Formulation

Find $c \in V$ such that $\forall w \in V,\ B(w,c) = 0$ where

$$B(w,c) = \left(w, \frac{\partial c}{\partial t} \right)_\Omega + \left(\nabla w, M_c \nabla \mu_c + \lambda \nabla M_c \Delta c \right)_\Omega + (\Delta w, \lambda M_c \Delta c)_\Omega$$

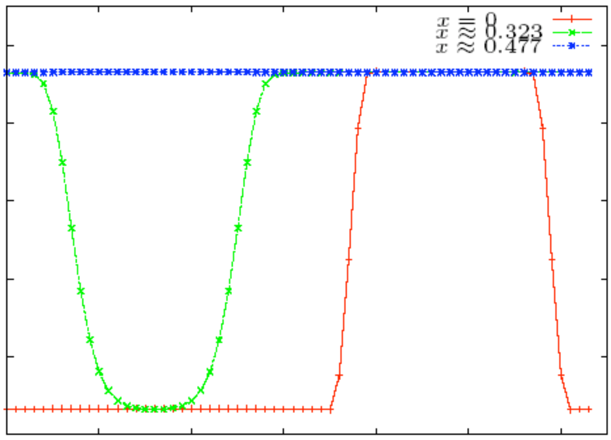
Remarks

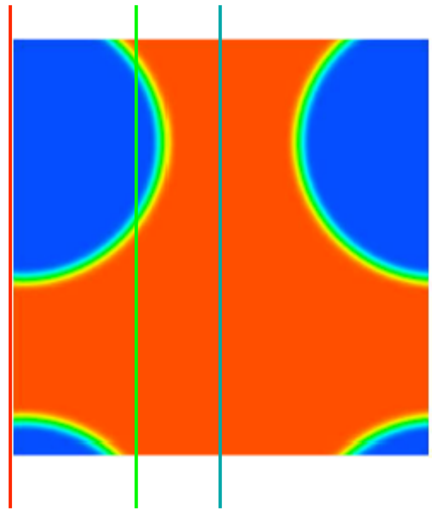
1. Periodic boundary conditions assumed
2. Galerkin method
3. C^k-continuous B-splines, $k \geq 1$



Numerical Results: $\alpha = 3000$, $\bar{c} = 0.63$

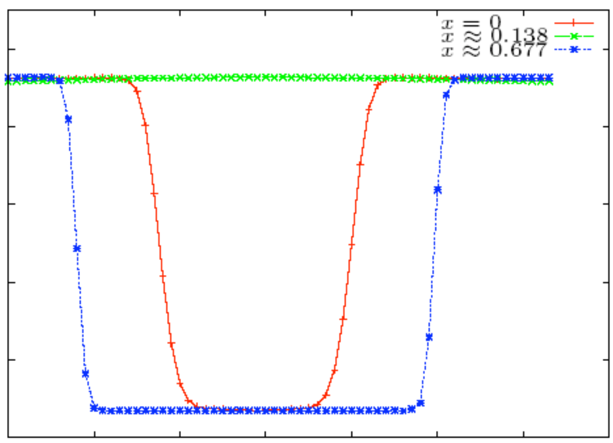
C^1-continuous quadratics (64^2)

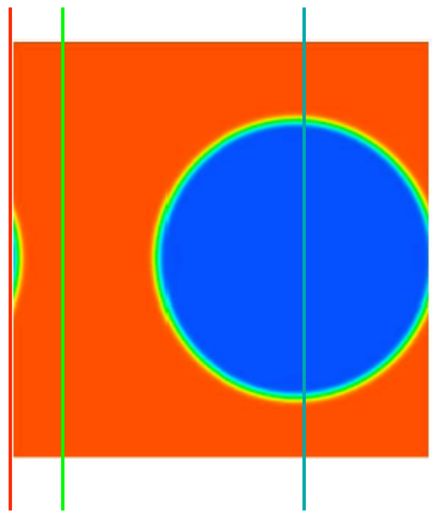




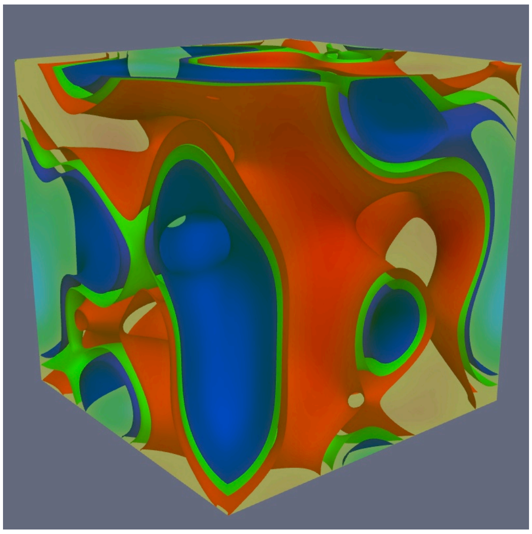
Numerical Results: $\alpha = 6000$, $\bar{c} = 0.63$

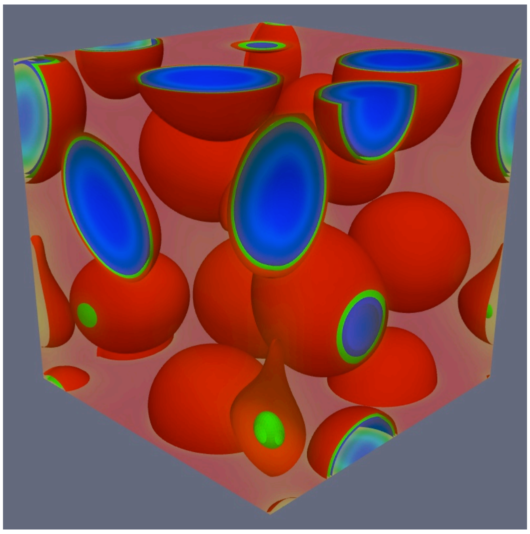
C^3-continuous quartics (64^2)



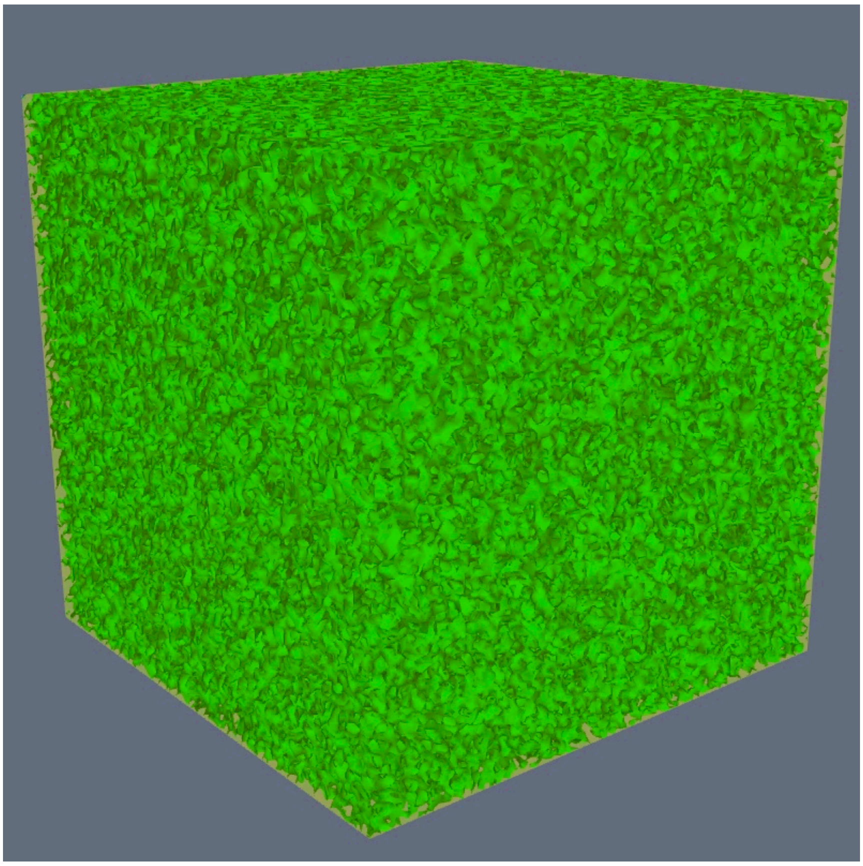


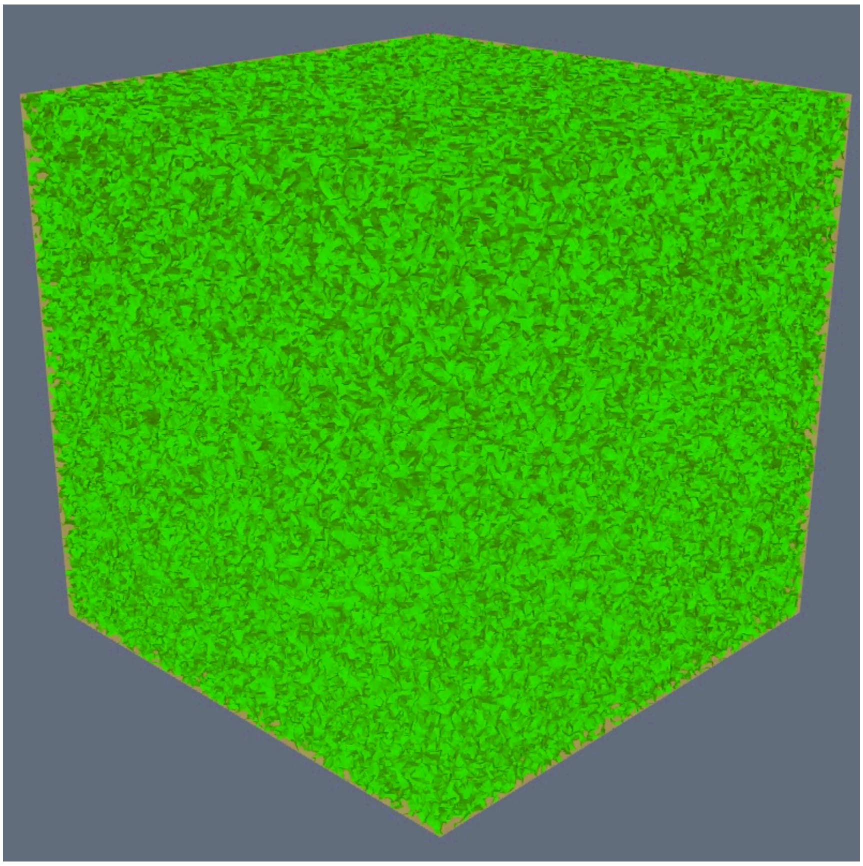
3D Numerical Results (128^3)





$\bar{c} = 0.63$ $\bar{c} = 0.75$





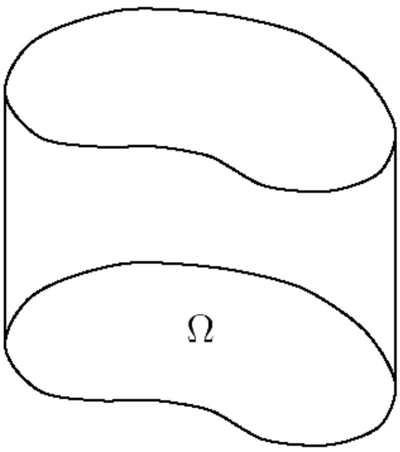
Incompressible Navier-Stokes Equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \nu\, \Delta \boldsymbol{u} + \boldsymbol{f} \quad \text{in Q}$$

$$\nabla \cdot \boldsymbol{u} = 0 \qquad \text{in Q}$$

$$\boldsymbol{u} = \mathbf{0} \qquad \text{on P}$$

$$\boldsymbol{u}(t = 0^+) = \boldsymbol{u}(t = 0^-) \quad \text{on } \Omega$$



$$\mathsf{Q} = \Omega \times (0, T)$$
$$\mathsf{P} = \Gamma \times (0, T)$$
$$\Gamma = \partial\Omega$$

Variational Space-Time Formulation

Find $\boldsymbol{U} = \{\boldsymbol{u}, p\} \in \boldsymbol{V}$ such that $\forall \boldsymbol{W} = \{\boldsymbol{w}, q\} \in \boldsymbol{V}$

$$B(\boldsymbol{W},\boldsymbol{U}) = B_1(\boldsymbol{W},\boldsymbol{U}) + B_2(\boldsymbol{W},\boldsymbol{U},\boldsymbol{U}) = (\boldsymbol{W},\boldsymbol{F})$$

where

$$B_1(\boldsymbol{W},\boldsymbol{U}) = \left(\boldsymbol{w}(T^-),\boldsymbol{u}(T^-)\right)_\Omega - \left(\frac{\partial \boldsymbol{w}}{\partial t},\boldsymbol{u}\right)_Q + \left(\nabla^s \boldsymbol{w}, 2\nu\, \nabla^s \boldsymbol{u}\right)_Q$$
$$+ \left(q, \nabla\cdot\boldsymbol{u}\right)_Q - \left(\nabla\cdot\boldsymbol{w}, p\right)_Q$$

$$B_2(\boldsymbol{W},\boldsymbol{U},\boldsymbol{U}) = -\left(\nabla \boldsymbol{w}, \boldsymbol{u}\otimes\boldsymbol{u}\right)_Q$$

$$(\boldsymbol{W},\boldsymbol{F}) = (\boldsymbol{w},\boldsymbol{f})_Q + \left(\boldsymbol{w}(0^+),\boldsymbol{u}(0^-)\right)_\Omega$$

Variational Multiscale Formulation

Split

$$\boldsymbol{V} = \bar{\boldsymbol{V}} \oplus \boldsymbol{V}'$$

Finite dimensional subspace
Standard FE, spectral or NURBS space, etc.

$$\boldsymbol{V} \setminus \bar{\boldsymbol{V}}$$

Original problem becomes:

Find $\boldsymbol{U} = \bar{\boldsymbol{U}} + \boldsymbol{U}'$, $\bar{\boldsymbol{U}} \in \bar{\boldsymbol{V}}$, $\boldsymbol{U}' \in \boldsymbol{V}'$, such that

$$(\bar{\boldsymbol{U}}) \qquad B(\bar{\boldsymbol{W}},\bar{\boldsymbol{U}}+\boldsymbol{U}') = (\bar{\boldsymbol{W}},\boldsymbol{F}) \qquad \forall \bar{\boldsymbol{W}} \in \bar{\boldsymbol{V}}$$

$$(\boldsymbol{U}') \qquad B(\boldsymbol{W}',\bar{\boldsymbol{U}}+\boldsymbol{U}') = (\boldsymbol{W}',\boldsymbol{F}) \qquad \forall \boldsymbol{W}' \in \boldsymbol{V}'$$

Variational Multiscale Method

Exact Theory

$$(\bar{\boldsymbol{U}}) \qquad B(\bar{\boldsymbol{W}}, \bar{\boldsymbol{U}} + \boldsymbol{U}') = (\bar{\boldsymbol{W}}, \boldsymbol{F})$$

$$(\boldsymbol{U}') \qquad \begin{cases} B(\boldsymbol{W}', \bar{\boldsymbol{U}} + \boldsymbol{U}') = (\boldsymbol{W}', \boldsymbol{F}) \\ \boxed{\boldsymbol{U}' = \mathfrak{F}'(\bar{\boldsymbol{u}}, \boldsymbol{R}'(\bar{\boldsymbol{U}}))} \\ (\boldsymbol{W}', \boldsymbol{R}'(\bar{\boldsymbol{U}})) = B(\boldsymbol{W}', \bar{\boldsymbol{U}}) - (\boldsymbol{W}', \boldsymbol{F}) \\ \boldsymbol{R}'(\bar{\boldsymbol{U}}) = \text{Residual of coarse scales} \end{cases}$$

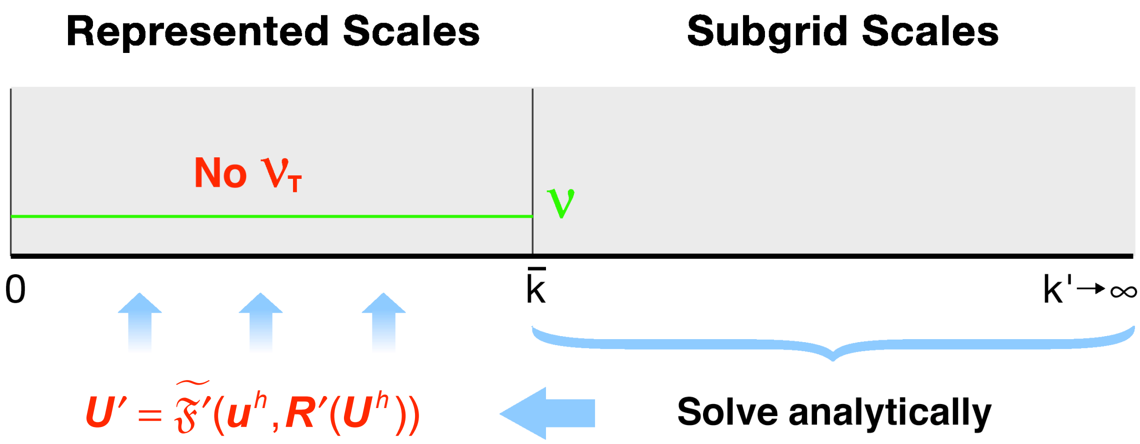
Turbulence Modeling Theory

Set $\bar{\boldsymbol{U}} \equiv \boldsymbol{U}^h$ and $\bar{\boldsymbol{W}} \equiv \boldsymbol{W}^h$ in $(\bar{\boldsymbol{U}})$,

$$(\boldsymbol{U}^h) \qquad B(\boldsymbol{W}^h, \boldsymbol{U}^h + \widetilde{\mathfrak{F}'}(\boldsymbol{u}^h, \boldsymbol{R}'(\boldsymbol{U}^h))) = (\boldsymbol{W}^h, \boldsymbol{F})$$

where $\widetilde{\mathfrak{F}'} \approx \mathfrak{F}'$ is the only approximation. The projector $\bar{\boldsymbol{P}} : \boldsymbol{U} \to \bar{\boldsymbol{U}}$ is fundamental; see H.-Sangalli, *SIAM J. Num. Anal.*, 2007.

Variational Multiscale Method



Fine-Scale Equations

$$
\left.\begin{aligned}
&\left(\boldsymbol{w}', \frac{\partial \boldsymbol{u}'}{\partial t} + \nabla\cdot\left(\boldsymbol{u}'\otimes\boldsymbol{u}^h + \boldsymbol{u}^h\otimes\boldsymbol{u}' + \boldsymbol{u}'\otimes\boldsymbol{u}' - 2\nu\,\nabla^s\boldsymbol{u}'\right)\right)_{\tilde{Q}_n} \\
&+\left(\boldsymbol{w}', \nabla p'\right)_{\tilde{Q}_n} + \left(q', \nabla\cdot\boldsymbol{u}'\right)_{\tilde{Q}_n} \\
&+\left(\boldsymbol{w}', \left[\!\left[2\nu\nabla^s\boldsymbol{u}'\cdot\boldsymbol{n}\right]\!\right]\right)_{\tilde{P}_n\setminus P_n} \\
&+\left(\boldsymbol{w}'(t_n^+), \boldsymbol{u}'(t_n^+)\right)_{\Omega} =
\end{aligned}\right\}\quad \textit{Fine-scale operator}
$$

$$
\left.\begin{aligned}
&-\left(\boldsymbol{w}', \frac{\partial \boldsymbol{u}^h}{\partial t} + \nabla\cdot\left(\boldsymbol{u}^h\otimes\boldsymbol{u}^h - p^h\mathbf{1} - 2\nu\,\nabla^s\boldsymbol{u}\right) - \boldsymbol{f}\right)_{\tilde{Q}_n} \\
&-\left(q', \nabla\cdot\boldsymbol{u}^h\right)_{\tilde{Q}_n}
\end{aligned}\right\}\quad \textit{PDE system residual}
$$

$$
-\left(\boldsymbol{w}', \left[\!\left[2\nu\nabla^s\boldsymbol{u}^h\cdot\boldsymbol{n}\right]\!\right]\right)_{\tilde{P}_n\setminus P_n} \longleftarrow \quad \textit{Viscous flux residual}
$$

$$
-\left(\boldsymbol{w}'(t_n^+), \boldsymbol{u}^h(t_n^+) - \boldsymbol{u}^h(t_n^-)\right)_{\Omega} \longleftarrow \quad \textit{IC residual}
$$

Exact Solution for the Fine Scales

Assumption (inspired by LES)

$\mathfrak{F}'(\boldsymbol{u}^h, \mathbf{0}) = \mathbf{0}$, and if $\boldsymbol{R}'(\boldsymbol{U}^h)$ is "small", $\mathfrak{F}'(\boldsymbol{u}^h, \boldsymbol{R}'(\boldsymbol{U}^h))$ is "small"

Perturbation series

$$
\boldsymbol{U}' = \sum_{k=1}^{\infty} \varepsilon^k\,\boldsymbol{U}'_k = \varepsilon\,\boldsymbol{U}'_1 + \varepsilon^2\,\boldsymbol{U}'_2 + \ldots, \qquad \varepsilon = \left\|\boldsymbol{R}'(\boldsymbol{U}^h)\right\|
$$

$$
\varepsilon\,\boldsymbol{U}'_1 = G'_{\boldsymbol{u}^h}(\boldsymbol{R}'(\boldsymbol{U}^h)) \qquad \left(G'_{\boldsymbol{u}^h} = \text{Green's operator}\right)
$$

$$
\varepsilon^2\,\boldsymbol{U}'_2 = G'_{\boldsymbol{u}^h}(\nabla\cdot(\varepsilon\,\boldsymbol{u}'_1\otimes\varepsilon\,\boldsymbol{u}'_1))
$$

$$
\vdots
$$

$$
\varepsilon^j\,\boldsymbol{U}'_j = G'_{\boldsymbol{u}^h}(\nabla\cdot(\sum_{k=1}^{j-1}\varepsilon^k\boldsymbol{u}'_k \otimes \varepsilon^{j-k}\boldsymbol{u}'_{j-k}))
$$

Residual-Driven Turbulence Modeling

Green's operator approximation

$$\varepsilon\, \boldsymbol{U}'_1 \approx \tilde{G}'_{\boldsymbol{u}^h}(\boldsymbol{R}'(\boldsymbol{U}^h)) \qquad \left(\tilde{G}'_{\boldsymbol{u}^h} \approx \boldsymbol{M}'_{\boldsymbol{u}^h}\right)$$

$$\varepsilon^2\, \boldsymbol{U}'_2 \approx \tilde{G}'_{\boldsymbol{u}^h}(\nabla\cdot(\varepsilon\, \boldsymbol{u}'_1 \otimes \varepsilon\, \boldsymbol{u}'_1))$$

$$\vdots$$

$$\varepsilon^j\, \boldsymbol{U}'_j \approx \tilde{G}'_{\boldsymbol{u}^h}(\nabla\cdot(\sum_{k=1}^{j-1} \varepsilon^k \boldsymbol{u}'_k \otimes \varepsilon^{j-k} \boldsymbol{u}'_{j-k}))$$

Inspiration from Stabilized Methods

$\tilde{G}'_{\boldsymbol{u}^h} = \boldsymbol{\tau}$ (selected by scaling arguments)

Residual-Driven Turbulence Modeling

Fine-scale approximation:

$$\boldsymbol{u}' = -\boldsymbol{\tau}_M \boldsymbol{r}_M \quad \text{and} \quad p' = -\tau_C\, r_C$$

$$\boldsymbol{\tau}_M = \left(\left(\frac{2}{\Delta t}\right)^2 + \boldsymbol{u}^h \cdot \boldsymbol{G}\boldsymbol{u}^h + C_I \nu^2 \operatorname{tr}\left(\boldsymbol{G}\cdot\boldsymbol{G}\right)\right)^{-1/2}$$

$$\boldsymbol{\tau}_C = \left(\boldsymbol{g}\cdot\boldsymbol{\tau}_M \boldsymbol{g}\right)^{-1}$$

where

$$\boldsymbol{G} = \frac{\partial \xi}{\partial \boldsymbol{x}}^T \frac{\partial \xi}{\partial \boldsymbol{x}} \quad \text{and} \quad g_i = \sum_{j=1}^{3} G_{ij}$$

Discrete "Conservative" Form

$$\left.\begin{aligned}
&\left(\boldsymbol{w}^h(T^-),\boldsymbol{u}^h(T^-)\right)_\Omega-\left(\frac{\partial \boldsymbol{w}^h}{\partial t},\boldsymbol{u}^h\right)_Q\\
&+\left(\nabla^s\boldsymbol{w}^h,2\nu\,\nabla^s\boldsymbol{u}^h\right)_Q+\left(q^h,\nabla\cdot\boldsymbol{u}^h\right)_Q-\left(\nabla\cdot\boldsymbol{w}^h,p^h\right)_Q\\
&-\left(\nabla\boldsymbol{w}^h,\boldsymbol{u}^h\otimes\boldsymbol{u}^h\right)_Q\\
&-\left(\boldsymbol{w}^h(0^+),\boldsymbol{u}^h(0^-)\right)_\Omega-\left(\boldsymbol{w}^h,\boldsymbol{f}\right)_Q
\end{aligned}\right\}\ \textit{Galerkin terms}$$

$$\left.\begin{aligned}
&-\left(\frac{\partial \boldsymbol{w}^h}{\partial t}+\nabla\cdot\left(2\nu\,\nabla^s\boldsymbol{w}^h\right)+\nabla q^h,\boldsymbol{u}'\right)_{\tilde{Q}}-\left(\nabla\cdot\boldsymbol{w}^h,p'\right)_{\tilde{Q}}\\
&-\left(\nabla\boldsymbol{w},\boldsymbol{u}'\otimes\boldsymbol{u}^h+\boldsymbol{u}^h\otimes\boldsymbol{u}'+\boldsymbol{u}'\otimes\boldsymbol{u}'\right)_{\tilde{Q}}
\end{aligned}\right\}\ \textit{Multiscale Terms}$$

Fine-scale approximation: $\boldsymbol{u}'=-\boldsymbol{\tau}_M\,\boldsymbol{r}_M$ and $p'=-\tau_C\,r_C$

Comparison with Classical Stabilized Methods

Coarse-scale equation

$$\begin{aligned}
0&=B(\boldsymbol{W}^h,\boldsymbol{U}^h)-\left(\boldsymbol{W}^h,\boldsymbol{F}\right) &&\leftarrow \text{Galerkin terms}\\
&+B_1(\boldsymbol{W}^h,\boldsymbol{U}')+B_2(\boldsymbol{W}^h,\boldsymbol{U}',\boldsymbol{U}^h) &&\leftarrow \text{Classical stabilization}\\
&+B_2(\boldsymbol{W}^h,\boldsymbol{U}^h,\boldsymbol{U}')+B_2(\boldsymbol{W}^h,\boldsymbol{U}',\boldsymbol{U}') &&\leftarrow \text{Non-classical stabilization}
\end{aligned}$$

Remarks

- These terms omitted in classical stabilized methods
- Extension of the ideas of SUPG, GLS and MS
- ***No eddy viscosity***

Forced Isotropic Turbulence

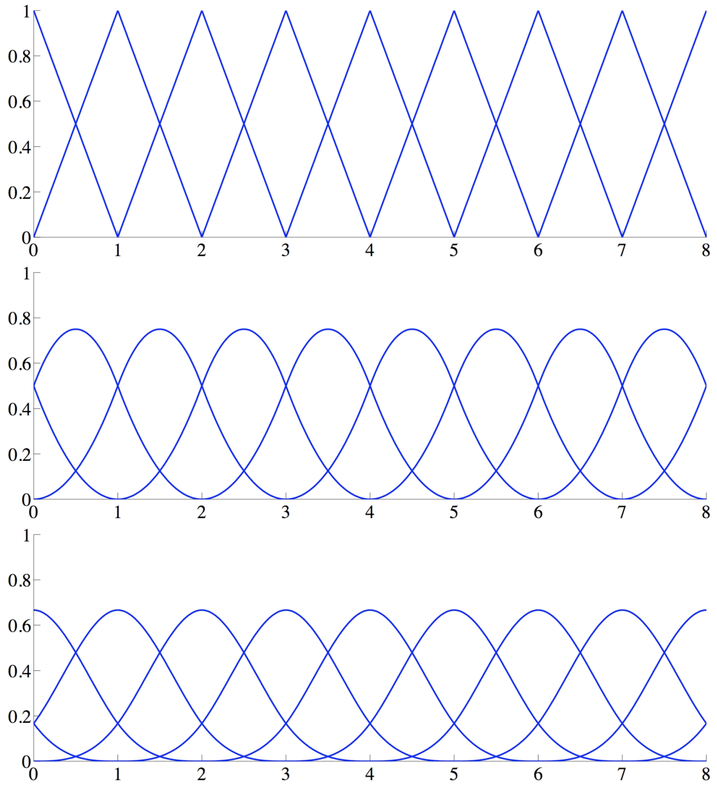
- NURBS (B-splines)
  - Linears (standard hexahedral finite elements), C^0
    - Meshes: 32^3, 64^3, 128^3 , 256^3
  - Quadratics, C^1
    - Meshes: 32^3, 64^3, 128^3
  - Cubics, C^2
    - Meshes: 32^3, 64^3
- Uniform mesh in all three directions
- Statistics:
  - Energy spectra
  - Third-order structure functions $S_3 = \left\langle \left(u(x+r) - u(x) \right)^3 \right\rangle$
  - Forcing of three lowest modes of the velocity field at each instant
- Power input kept constant (P_{input} = 62.8)
- Samples taken ~0.4 T_{eddy} *apart*

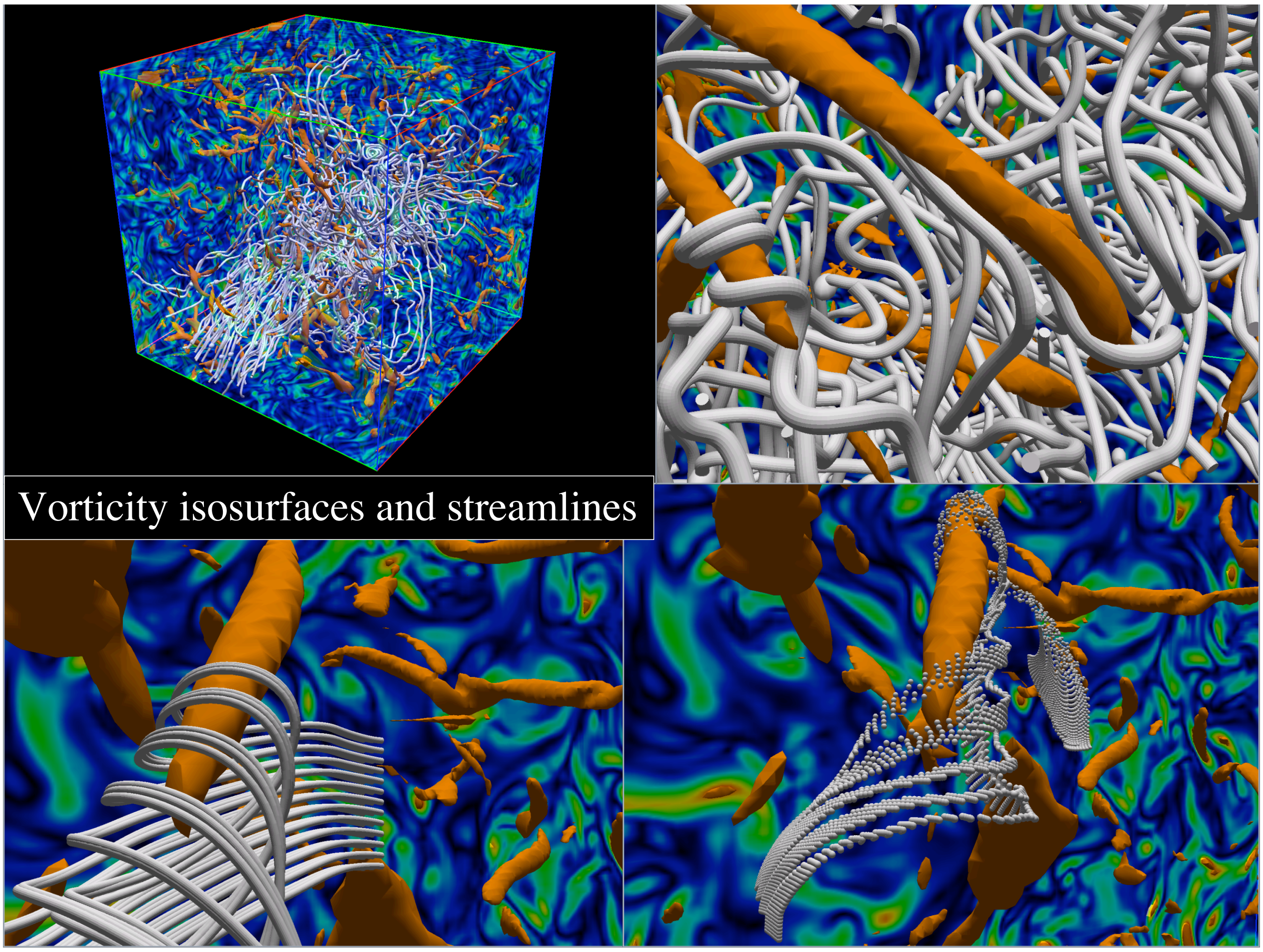
Periodic NURBS (B-spline) Basis Functions

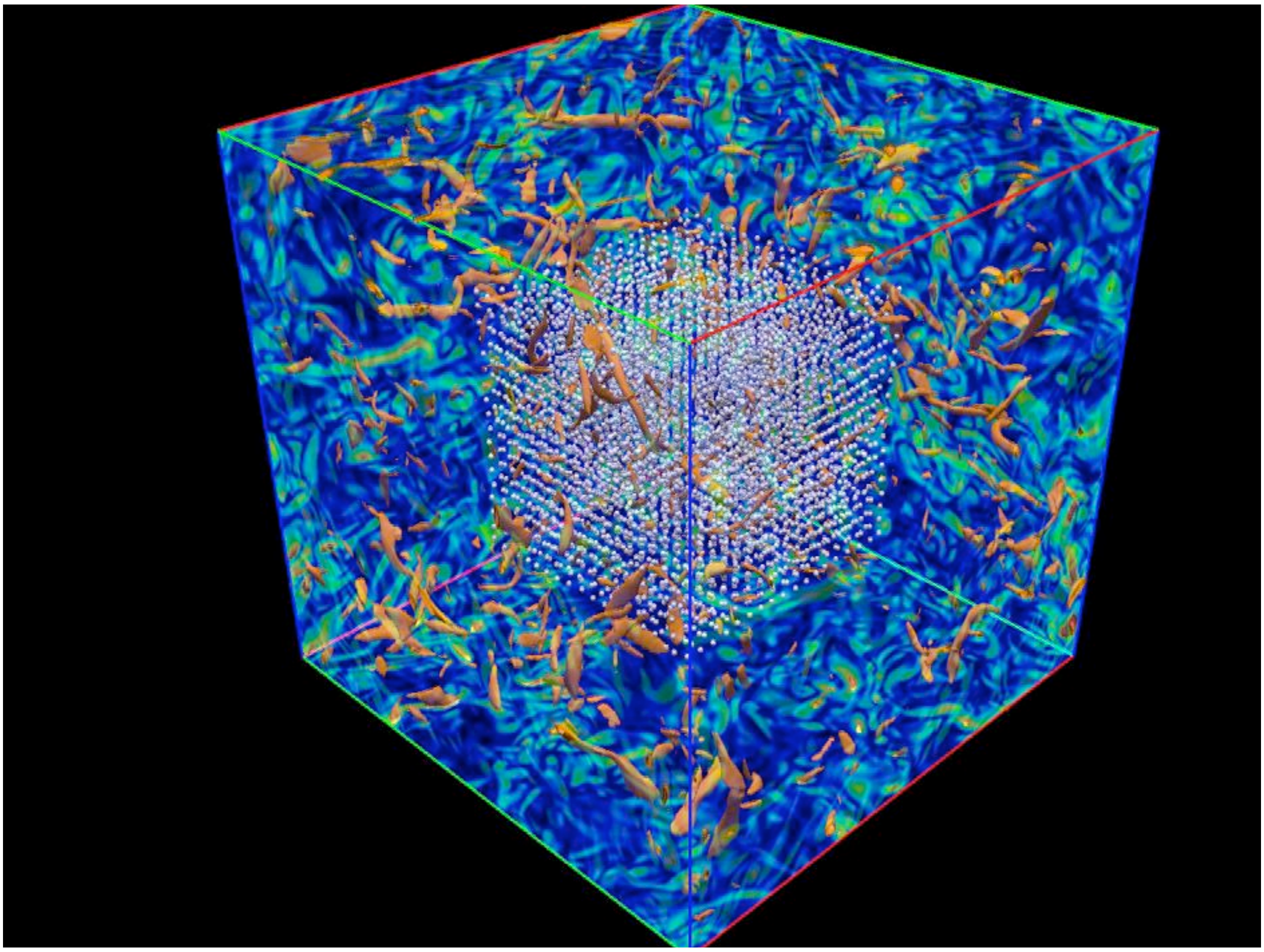
Linears

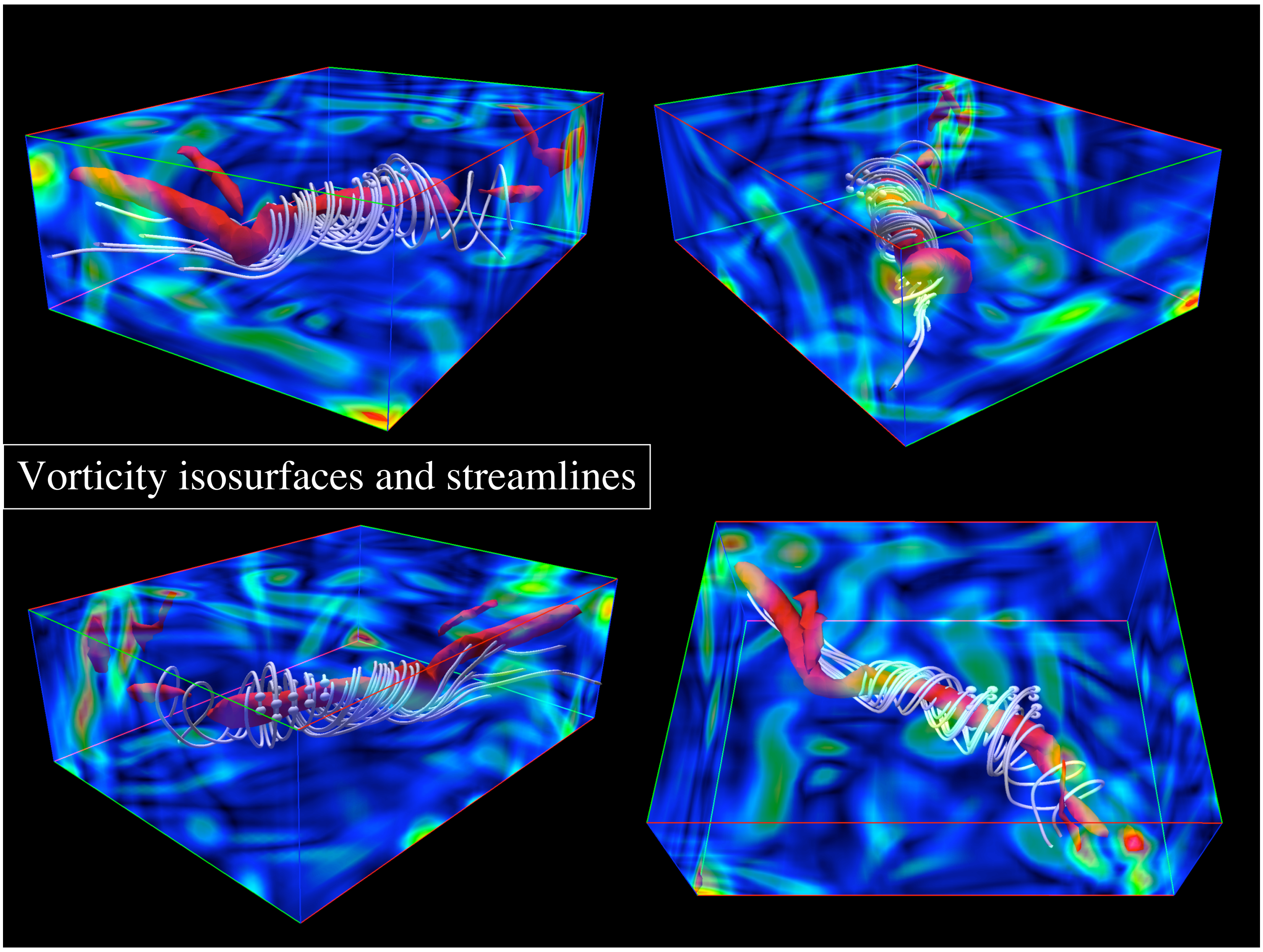
Quadratics

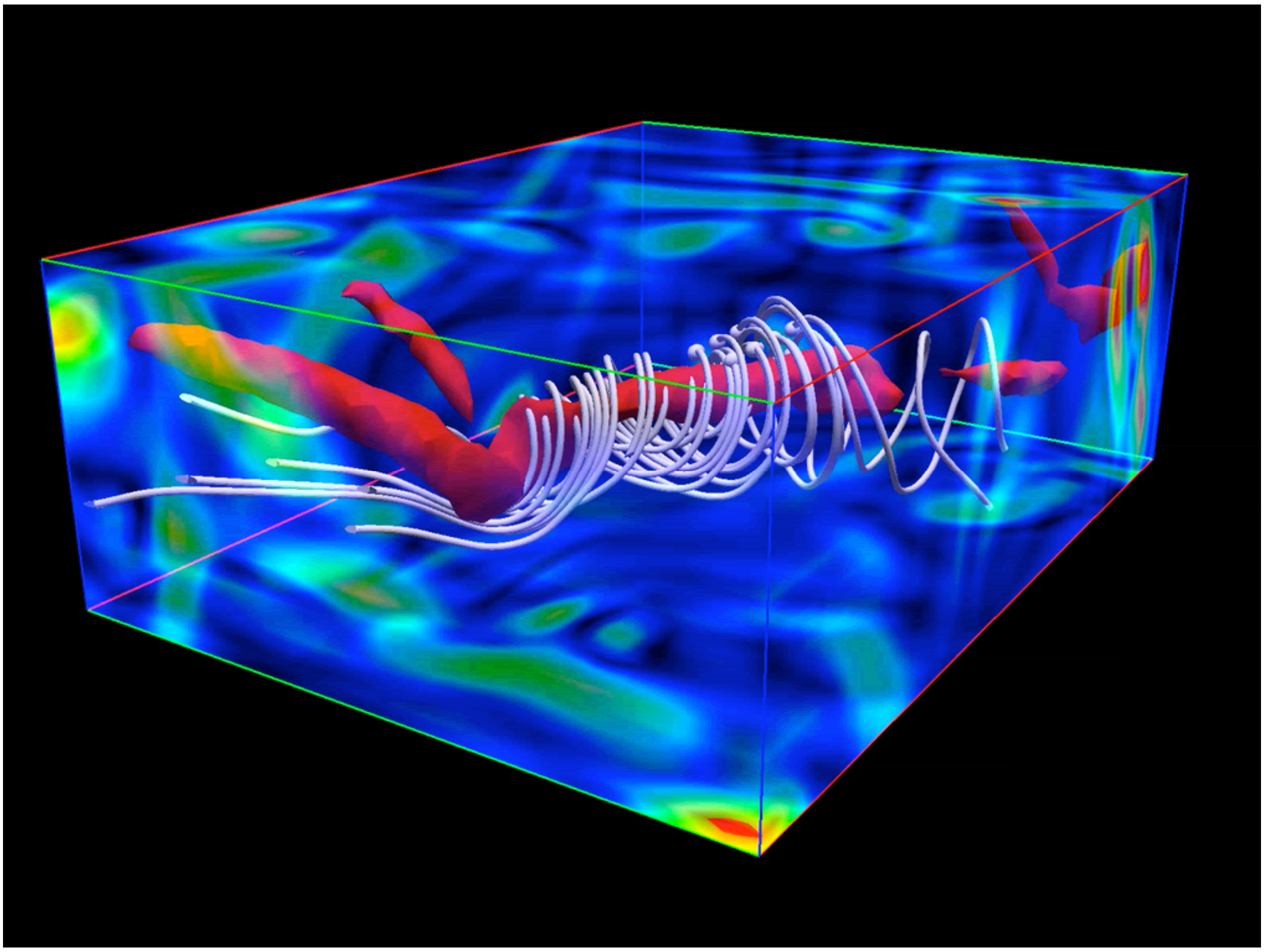
Cubics

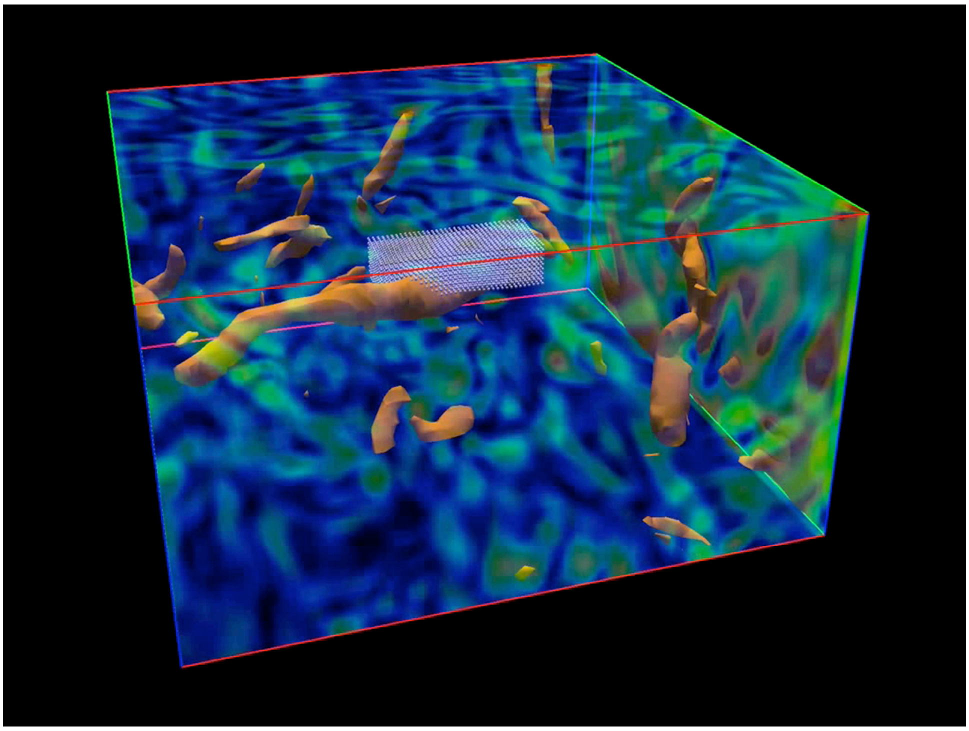



Vorticity isosurfaces and streamlines

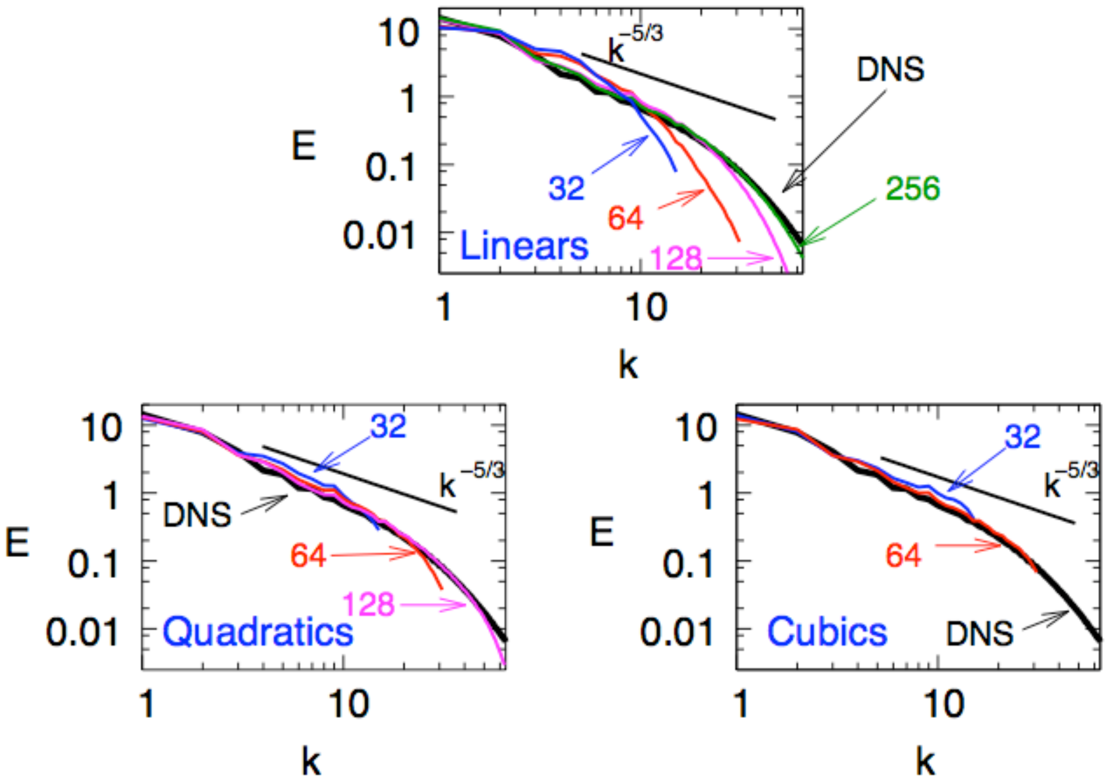



Vorticity isosurfaces and streamlines

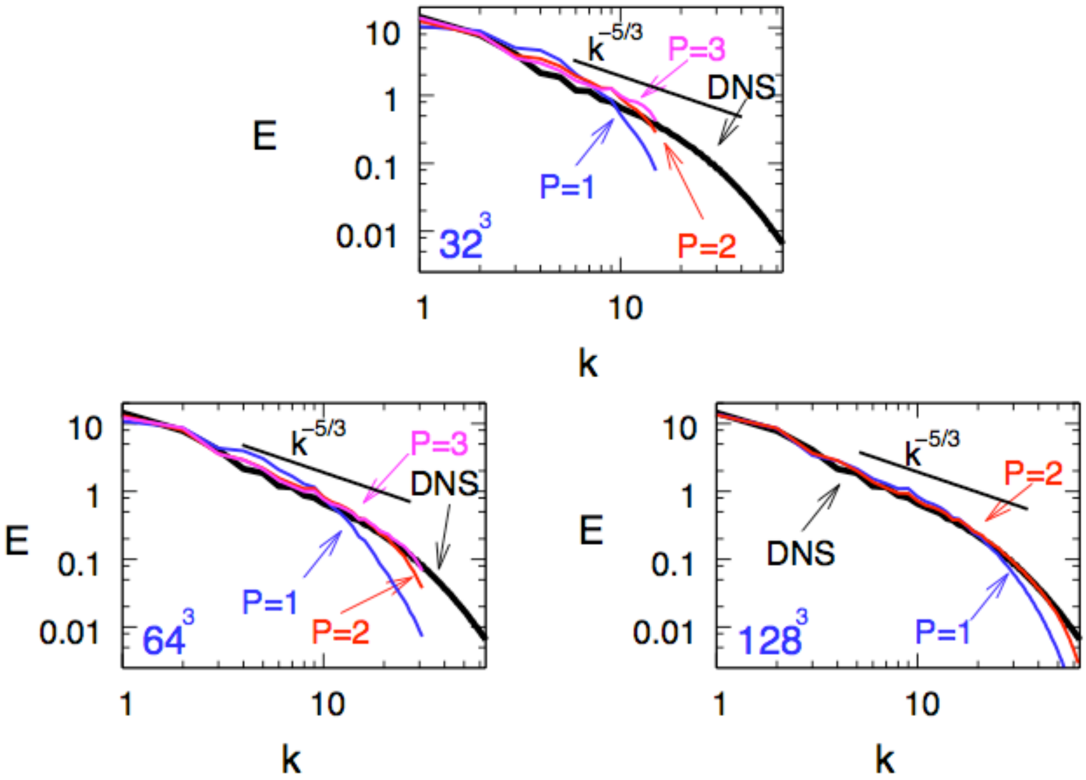




Energy Spectra Re_λ=164 (*h*-refinement)

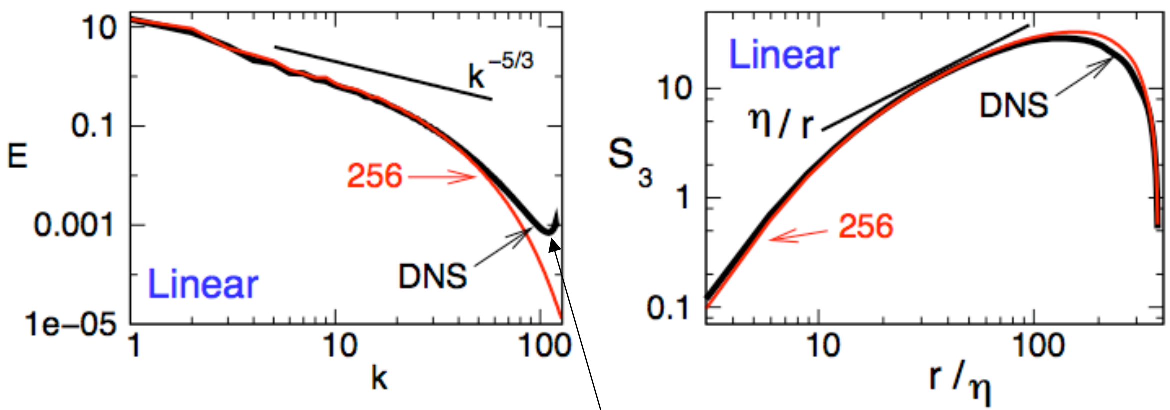


Energy Spectra Re_λ=164 (k-refinement)



Energy Spectra and Third-order Structure Function

Re_λ=164

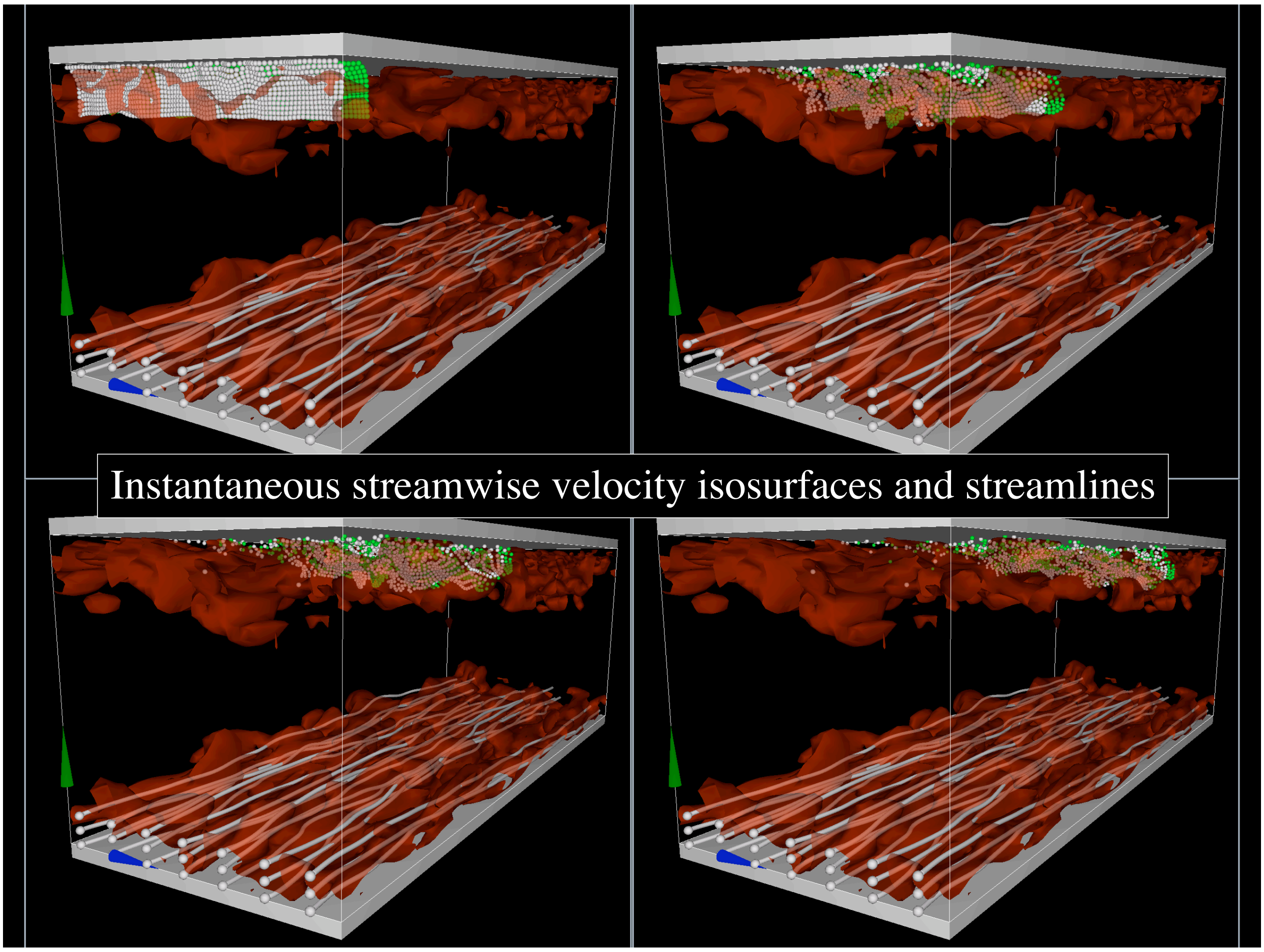


$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}$$

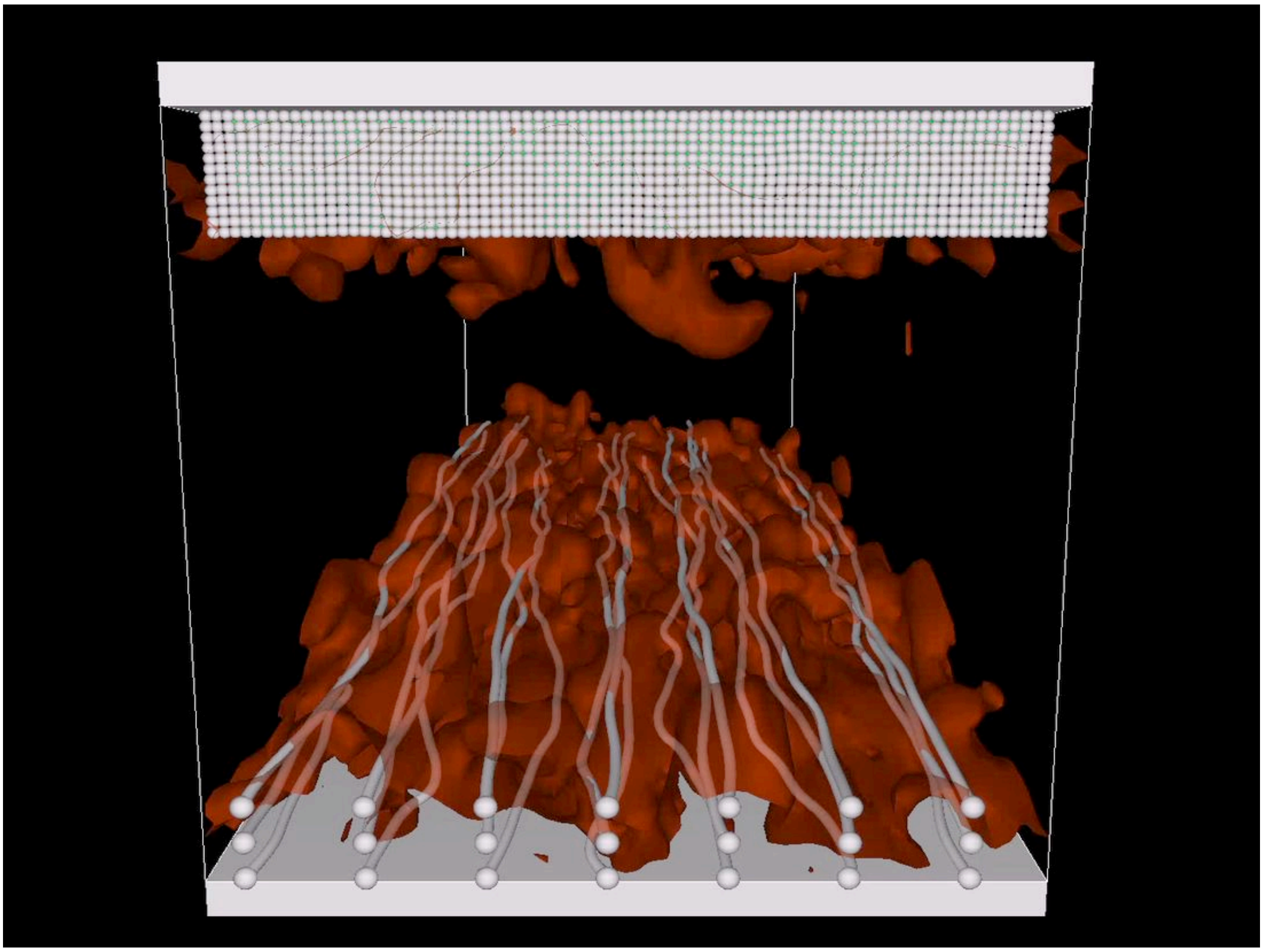
Energy pile-up for spectral method (cf. Guermond)

Turbulent Channel Flow at Re=395

- Meshes of 32^3 and 64^3 elements
- Hyperbolic tangent stretching
- NURBS
- *h*-and *k*-refinement
- Domain size: $2\pi \times 2 \times 2\pi/3$
- Comparison with DNS of Moser, Kim, and Mansour (1999)
  - Used 256 x 193 x 192 spectral functions

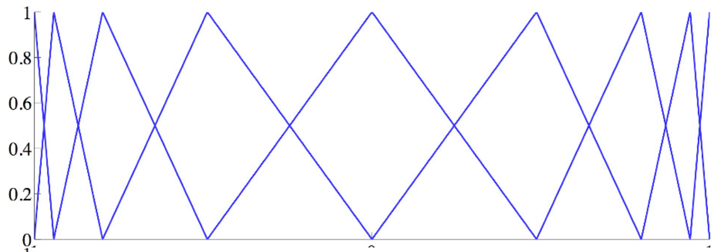


Instantaneous streamwise velocity isosurfaces and streamlines

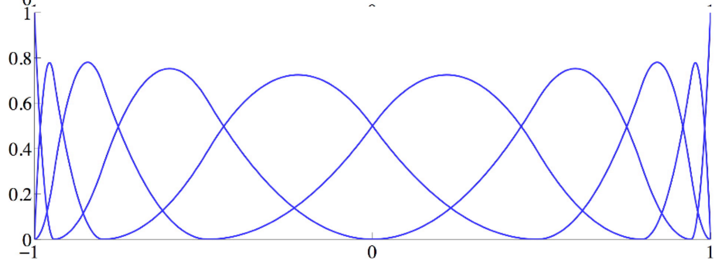


NURBS (B-spline) Basis Functions Normal to Wall

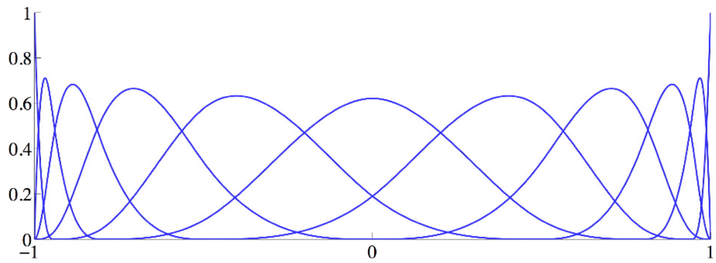
Linear

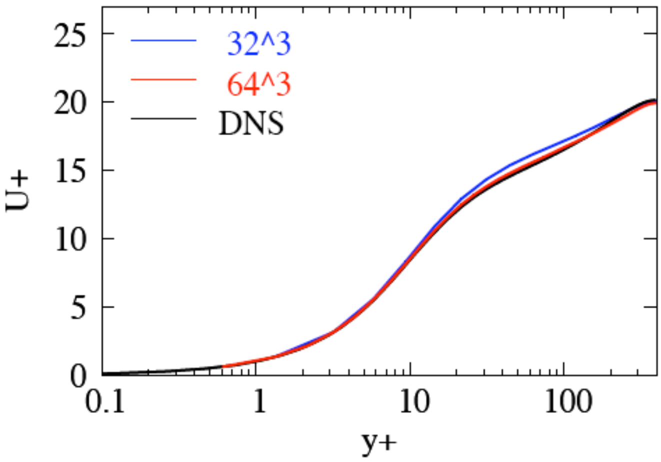


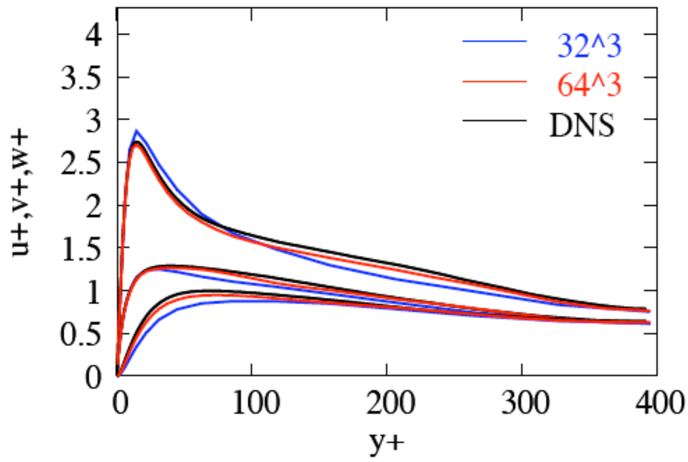
Quadratic



Cubic



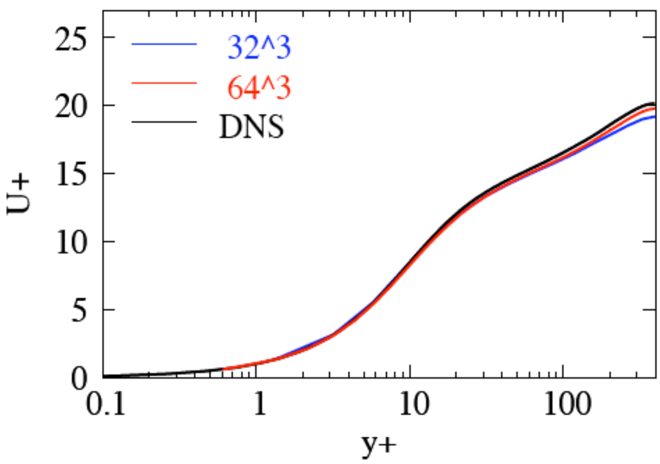


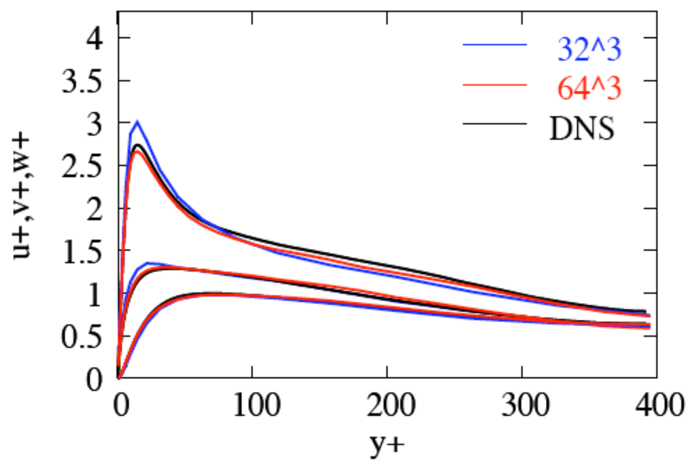


Turbulent channel flow at Re=395

Quadratic NURBS

h-refinement

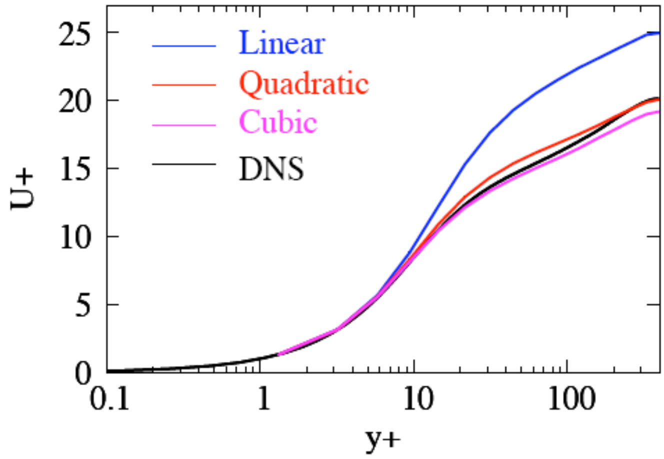




Turbulent channel flow at Re=395

Cubic NURBS

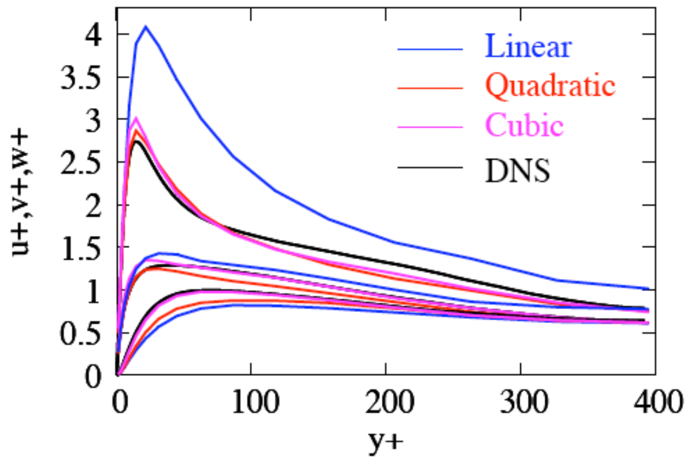
h-refinement



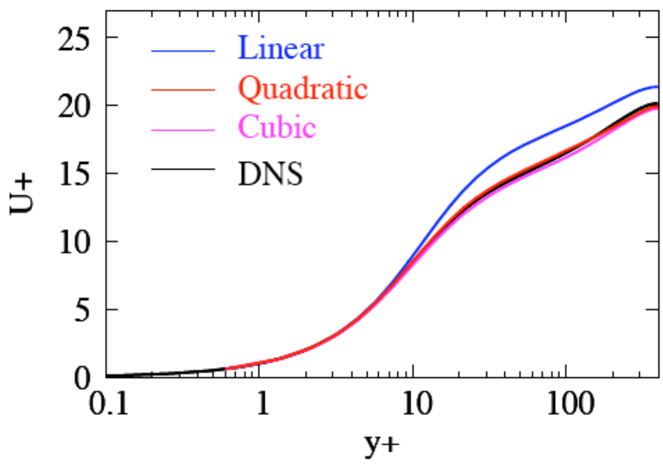
Turbulent channel flow at Re=395

32^3 mesh

k-refinement



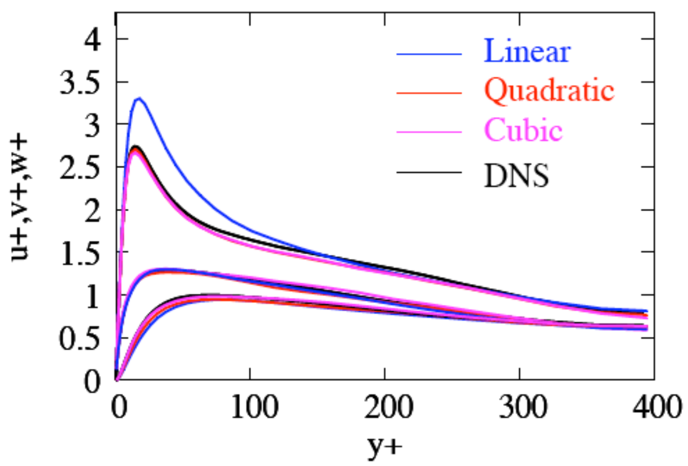
Note the increase in accuracy from linear to quadratic



Turbulent channel flow at Re=395

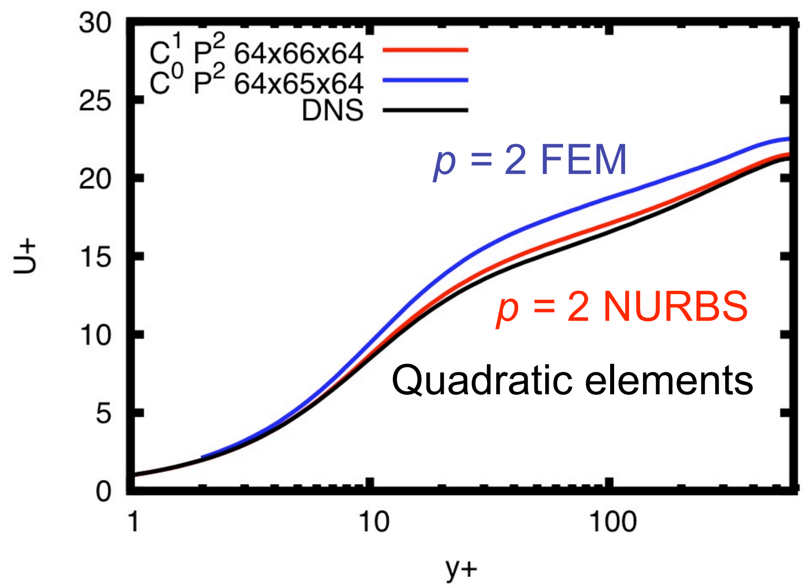
64^3 mesh

k-refinement

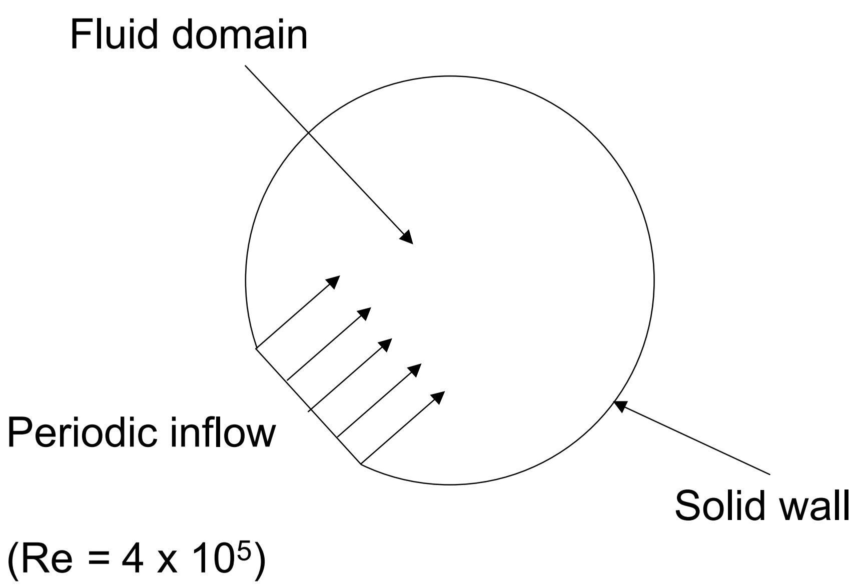


Both quadratics and cubics are very accurate

Turbulent channel flow at Re=590, 64^3 mesh



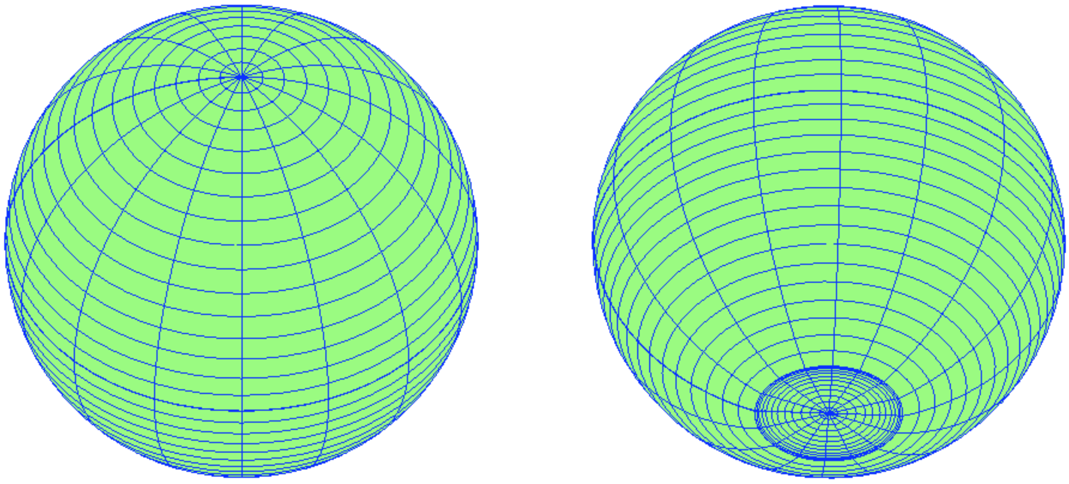
Balloon Containing an Incompressible Fluid



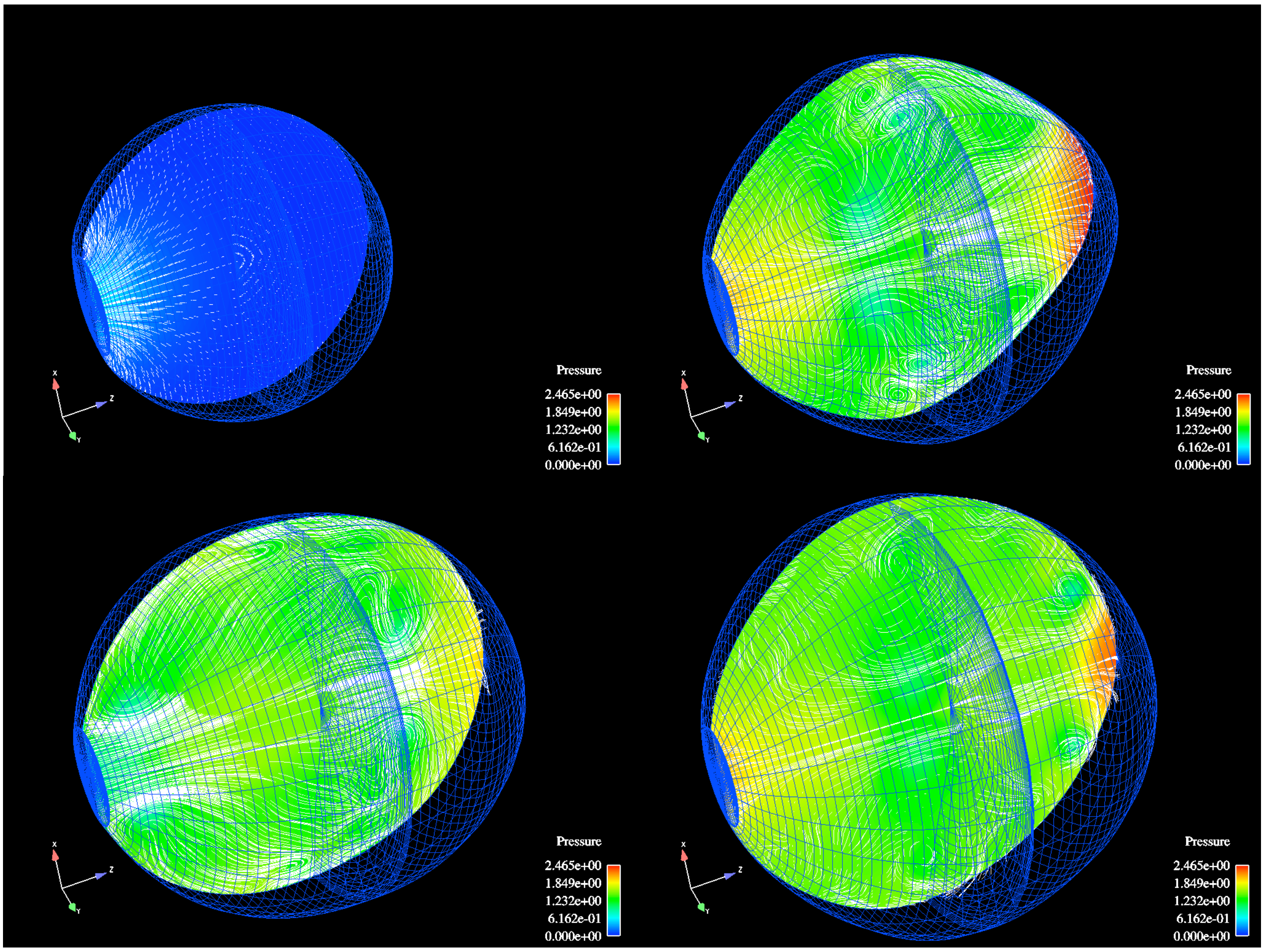
From Wall ’06, Tezduyar ’07

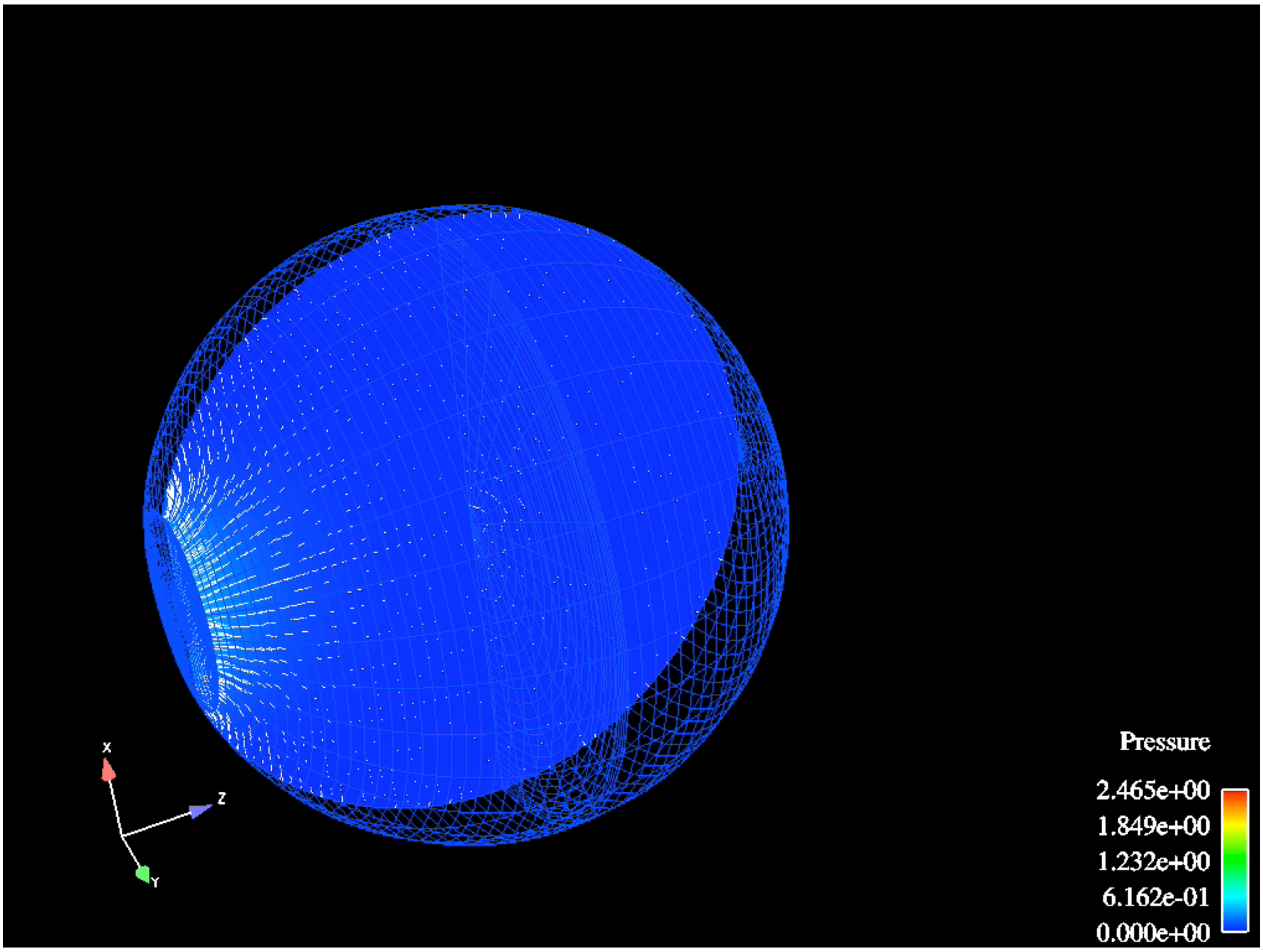
Balloon Containing an Incompressible Fluid

- Quadratic NURBS for both solid and fluid
- Boundary layer meshing



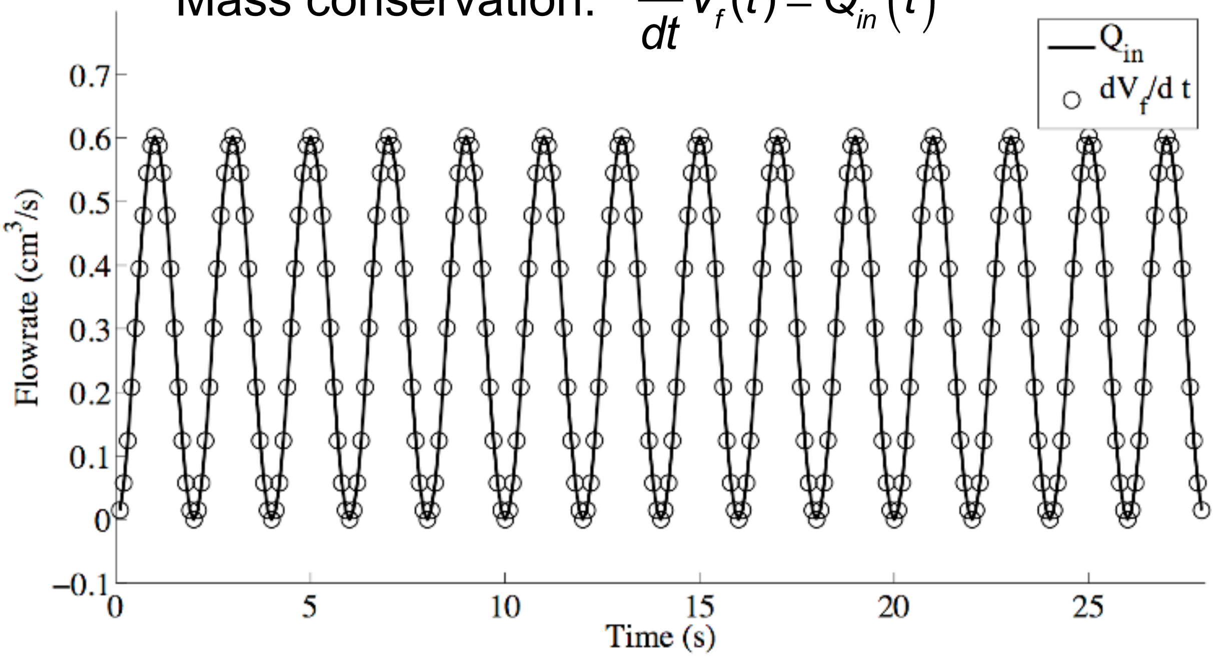
(a) Top view (b) Bottom view





Balloon Containing an Incompressible Fluid

Mass conservation: $\frac{d}{dt}V_f(t) = Q_{in}(t)$



Weak Boundary Conditions

- Wall-bounded turbulent flows
- Discontinuous Galerkin methodology used to enforce no-slip Dirichlet conditions weakly
- Turbulent channel flow
  - Quadratic NURBS
  - Re = 395
    - 32^3 mesh, stretched and uniform, 2π x 2 x $2\pi/3$
    - DNS: 256 x 193 x 192, spectral, 2π x 2 x π
  - Re = 950
    - 64^3 mesh, stretched and uniform, 4π x 2 x $4\pi/3$
    - DNS: 3072 x 385 x 2304, spectral, 8π x 2 x 3π

Weak Dirichlet BC Formulation

Find $\{\boldsymbol{u}^h, p^h\}$, $\boldsymbol{u}^h \cdot \boldsymbol{n} = 0$ on Γ, s.t. $\forall \{\boldsymbol{w}^h, q^h\}$, $\boldsymbol{w}^h \cdot \boldsymbol{n} = 0$ on Γ,

$$
\begin{aligned}
B^{MS}\left(\{\boldsymbol{w}^h, q^h\}, \{\boldsymbol{u}^h, p^h\}\right) - F^{MS}\left(\{\boldsymbol{w}^h, q^h\}\right) \\
-\left(\boldsymbol{w}^h, 2\nu\nabla^s \boldsymbol{u}^h \cdot \boldsymbol{n}\right)_{\tilde{\Gamma}} \\
-\left(2\nu\nabla^s \boldsymbol{w}^h \cdot \boldsymbol{n}, \boldsymbol{u}^h - \boldsymbol{0}\right)_{\tilde{\Gamma}} \\
-\left(\boldsymbol{w}^h \tau_B, \boldsymbol{u}^h - \boldsymbol{0}\right)_{\tilde{\Gamma}}
\end{aligned}
$$

Multiscale formulation terms

Consistency term

Adjoint-consistency term

Penalty term

Penalty Term

Original formulation (purely *numerical*):

$$\tau_B = \frac{C_B^I \nu}{h}$$

$C_B^I \geq C > 0$ (stability)

h – element size in the wall-normal direction

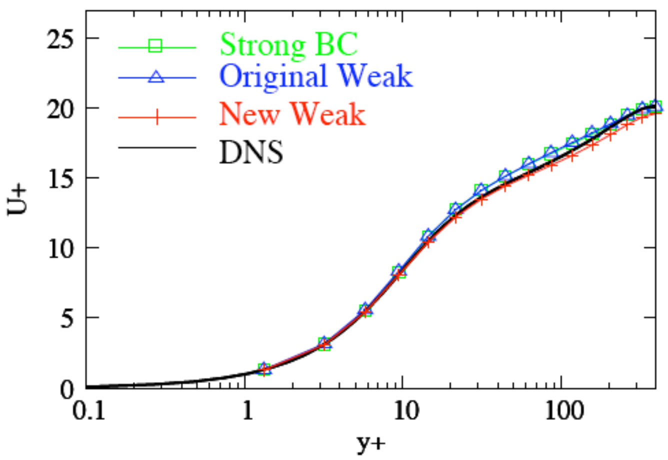
New formulation (based on *wall function* ideas):

$$\tau_B = \frac{u^{*2}}{\left\| \boldsymbol{u}^h \right\|_{l_2}}$$

u^* = wall friction velocity, satisfying

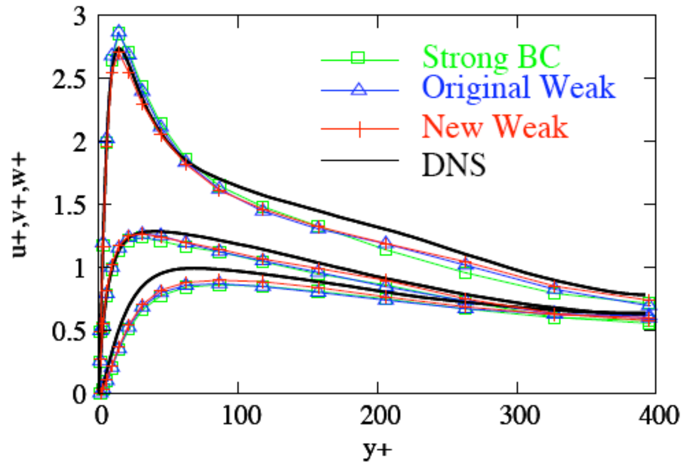
"law of the wall," $y^+ = f(u^+)$, where

$$y^+ = \frac{yu^*}{\nu}, \; u^+ = \frac{\left\| \boldsymbol{u}^h \right\|_{l_2}}{u^*}, \; y = \text{normal distance from the wall}$$

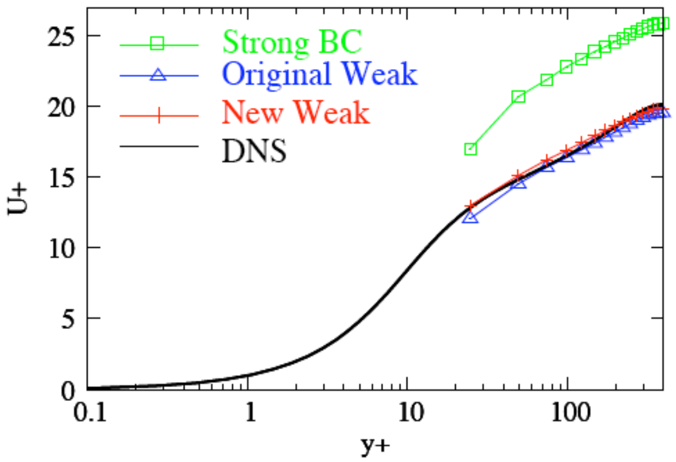


Turbulent channel flow at Re=395

Stretched mesh of 32^3 elements

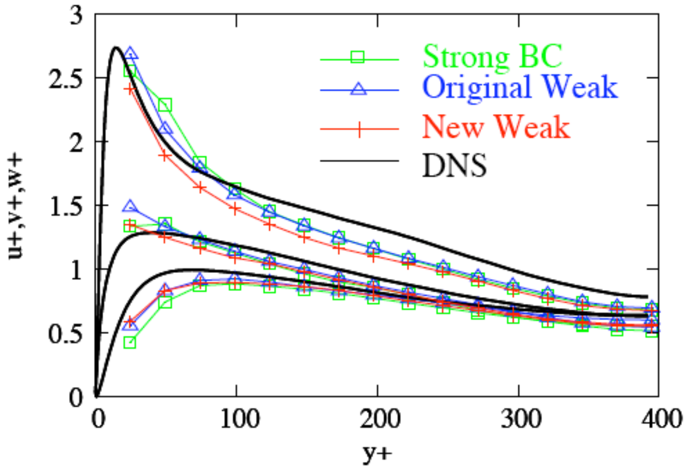


Weak approaches strong in the limit of vanishing wall-normal mesh size, thus results are similar

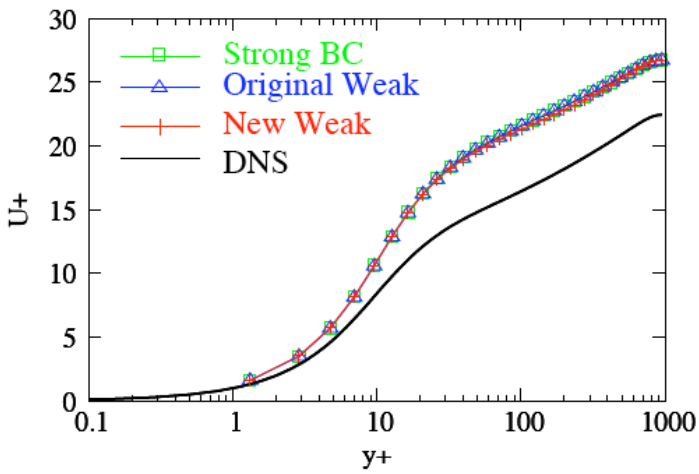


Turbulent channel flow at Re=395

Uniform mesh of 32^3 elements

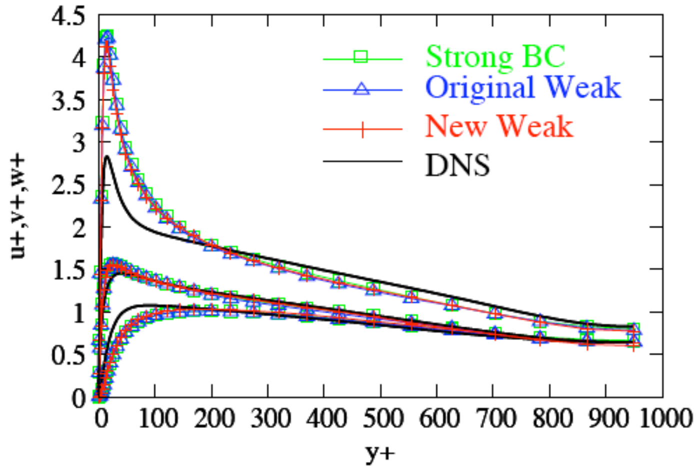


Weak attains accuracy in the core of the flow despite the coarseness of the boundary layer mesh

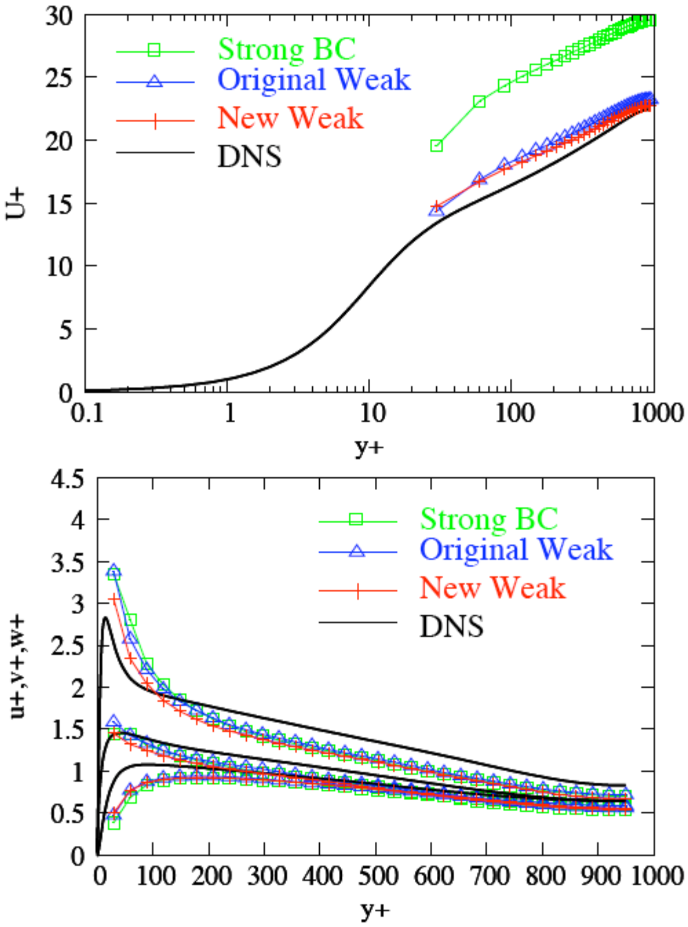


Turbulent channel flow at Re=950

Stretched mesh of 64^3 elements



Weak approaches strong in the limit of vanishing wall-normal mesh size, thus results are similar



Turbulent channel flow at Re=950

Uniform mesh of 64^3 elements

Weak is a considerable improvement over strong when the boundary layer is not resolved

Conclusions: Weak BC

- Weak BC
  - At least as accurate as strong in all cases
  - Much more accurate mean flow velocity than strong when the boundary layer is under-resolved
  - Should be the default strategy
- The *law of the wall* version is slightly more accurate than the *purely numerical* version
- It may be possible to improve the weak BC further

Conclusions: VMS

- VMS is a *rational framework* for LES turbulence modeling
  - Fine-scale fields are approximated
  - *No eddy viscosities*
  - Numerics and modeling are inextricably interlinked
- Very good results in all cases
- Approximation properties of NURBS appear to be partially responsible for accuracy of results
  - C^1 quadratic NURBS are a good combination of accuracy and efficiency

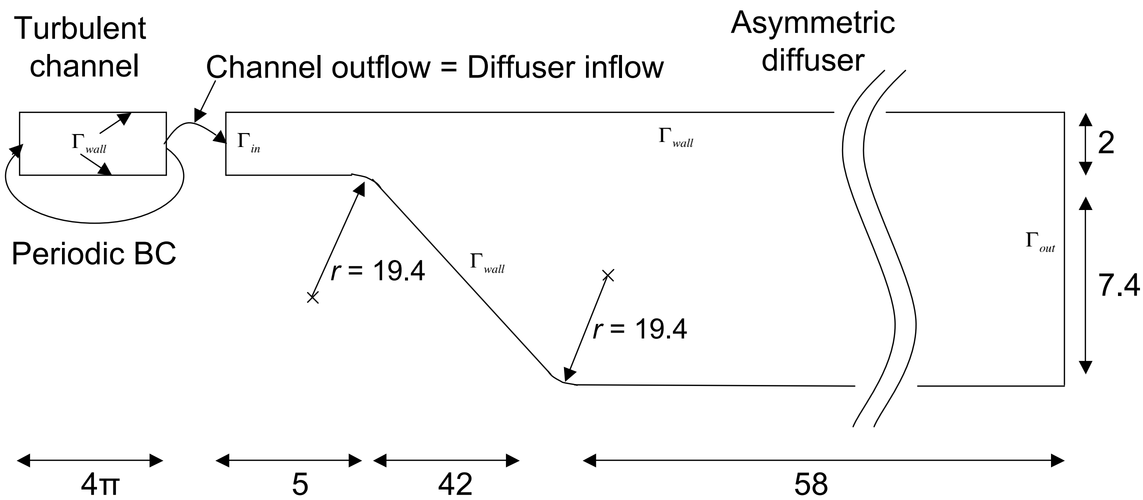
VMS: The Future

- Stabilized methods may be viewed as historical *stepping stones* leading to VMS
- A *mathematical theory* of fine-scale approximations is needed
- Each fine-scale approximation is a *new and different* turbulence model
  - Even the *simplest* seems effective
  - It *should* get better
- *Extending* the VMS formulation of turbulence modeling to other situations, such as RANS, etc.
- How far can it go?

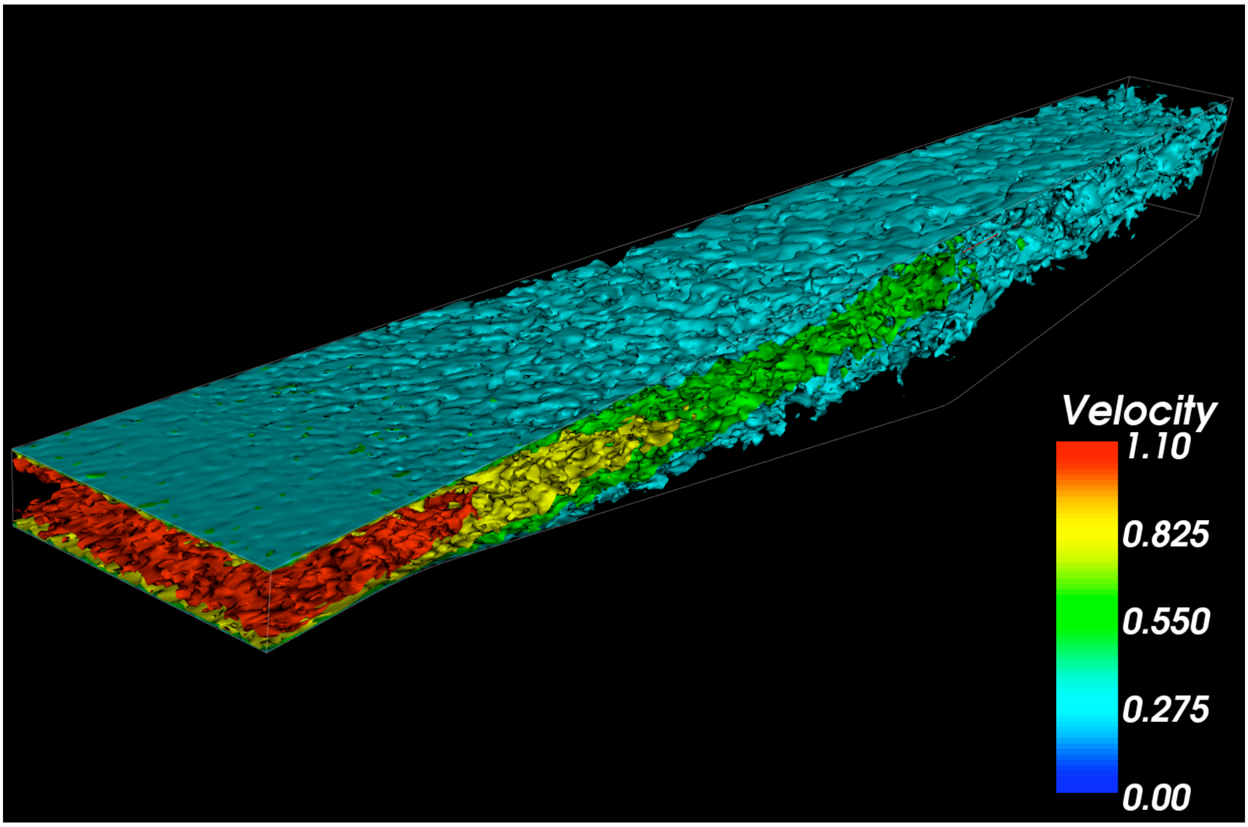
Latest Research

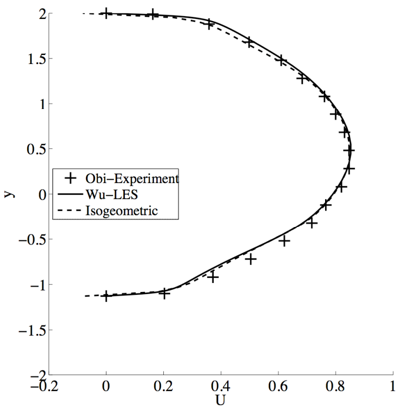
- Google “UT Austin ICES” and click on “Research” to find all my preprints.

Turbulent Flow in an Asymmetric Diffuser

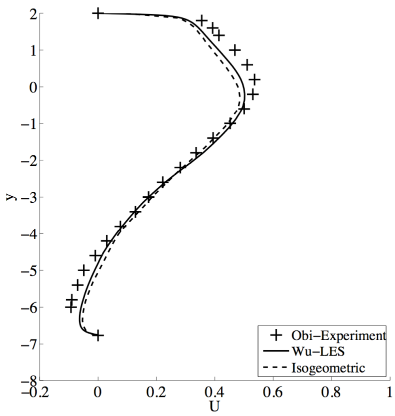


- C^1-continous quadratic NURBS elements
- *Uniform* 302 x 64 x 64 mesh (i.e., *no* stretching)
- *Weakly* enforced no-slip boundary condition

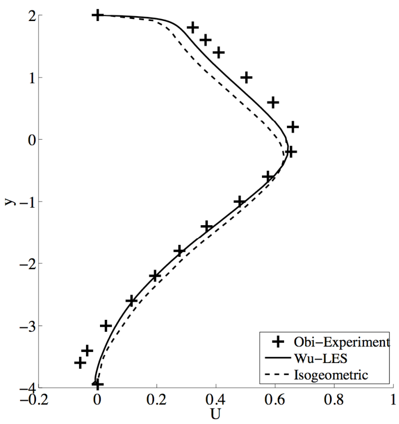




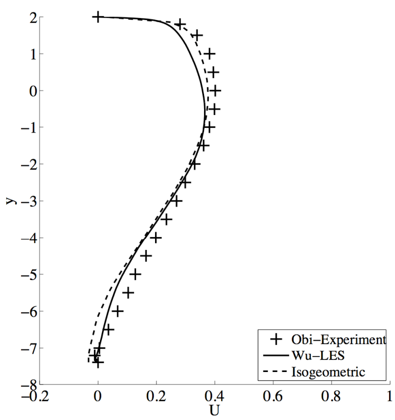
(a) Mean stream-wise velocity at $x = 6.4$



(c) Mean stream-wise velocity at $x = 38.4$



(b) Mean stream-wise velocity at $x = 22.4$



(d) Mean stream-wise velocity at $x = 58.4$