Computing Turbulence with General Galerkin Methods

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Outline

- Computational challenges of turbulent flow
- General Galerkin (G2) methods
- Blowup, d'Alembert paradox, and turbulent separation
- Applications

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Challenges of high Re flow

- Mathematical model: existence/uniqueness: regularity/blowup, well-posedness/chaos,…

  - Only regularized (Euler) solutions available!?

- Computational resolution: turbulence, shocks, boundary layers, complex geometry,…

  - Only underresolved (no DNS) computations available!

- What information is available from underresolved computations of regularized mathematical models?

High Re flow: Euler equations

Euler residual $R(\hat{u}) = 0$, for $\hat{u} = (\rho, m, \varepsilon)$

$$\partial\rho/\partial t + \nabla\cdot(u\rho) = 0 \qquad \text{(Mass conservation)}$$

$$\partial m/\partial t + \nabla\cdot(um) + \nabla p = 0 \qquad \text{(Newton 2}^{\text{nd}}\text{ law)}$$

$$\partial\varepsilon/\partial t + \nabla\cdot(u\varepsilon) + \nabla\cdot(pu) = 0 \qquad \text{(1}^{\text{st}}\text{ Law: total energy } \varepsilon\text{)}$$

with momentum $m=u\rho$, density ρ, velocity u, pressure p

- Ideal gas law of state: $p = (\gamma-1)e$ (e.g. air)
- Incompressibility: $\nabla\cdot u = 0$ (e.g. water)

General Galerkin G2

V_h = {piecewise linear FE in space and time}

W_h = {piecewise linear FE in space, piecewise constant in time}

cG(1)cG(1): For all $(v_\rho, v_m, v_\varepsilon)$ in W_h : find (ρ, m, ε) in V_h :

$$(\rho_t, v_\rho) - (u\rho, \nabla v_\rho) + (\delta u\cdot\nabla\rho, u\cdot\nabla v_\rho) + (\nu_\rho\nabla\rho, \nabla v_\rho) = 0$$

$$(m_t, v_m) - (um, \nabla v_m) + (\delta u\cdot\nabla m, u\cdot\nabla v_m) + (\nu_m\nabla m, \nabla v_m) - (p, \nabla\cdot v_m) = 0$$

$$(\varepsilon_t, v_\varepsilon) - (u\varepsilon, \nabla v_\varepsilon) + (\delta u\cdot\nabla\varepsilon, u\cdot\nabla v_\varepsilon) + (\nu_\varepsilon\nabla\varepsilon, \nabla v_\varepsilon) - (pu, \nabla v_\varepsilon) = 0$$

- Streamline diffusion: $\delta = C_1 \left((1/k)^2 + (u/h)^2 \right)^{-1/2}$
- Shock capt: $\nu_\alpha = \max(C_\alpha h^2 R_\alpha/|\alpha|, C_2 h^{3/2})$ $\quad \nu_m = \nu_{m_i}\delta_{ij}$, $C_2 \sim uL^{-1/2}$
- Exact global conservation of mass, momentum and energy!

General Galerkin G2

- Incompressible flow: Euler residual $R(\hat{u}) = 0$, for $\hat{u} = (u,p)$

$$u_t + u\cdot\nabla u + \nabla p = 0$$
$$\nabla\cdot u = 0$$

- G2: For all test functions (v,q) in W_h : find (u,p) in V_h :

$$(u_t + u\cdot\nabla u,v) + (\delta u\cdot\nabla u,u\cdot\nabla v) - (p,\nabla\cdot v) = 0$$
$$(\delta\nabla p,\nabla q) = (q,\nabla\cdot u)$$

General Galerkin G2

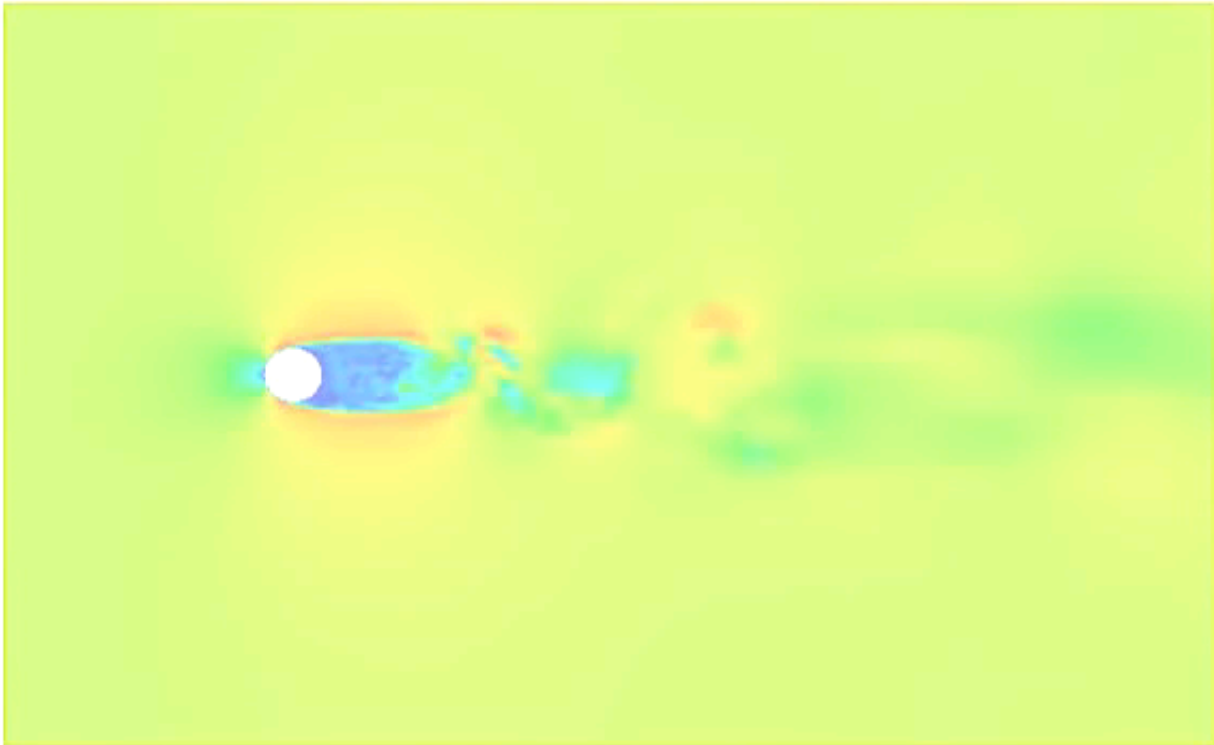
- G2 approximate solution $\hat{u}$: Euler residual $R(\hat{u}) \neq 0$
- G2 as weak Euler solutions: $||R(\hat{u})||_{-1} \leq ||hR(\hat{u})|| \leq Ch^{1/2}$
  Proof: a priori (incompr flow) alt. a posteriori (compr flow)
- G2 solutions are well-posed with respect to output M:

  $|M(\hat{u}) - M(\hat{U})| \leq S\,(\,||R(\hat{u})||_{-1} + ||R(\hat{U})||_{-1}\,) \leq S\,Ch^{1/2}$

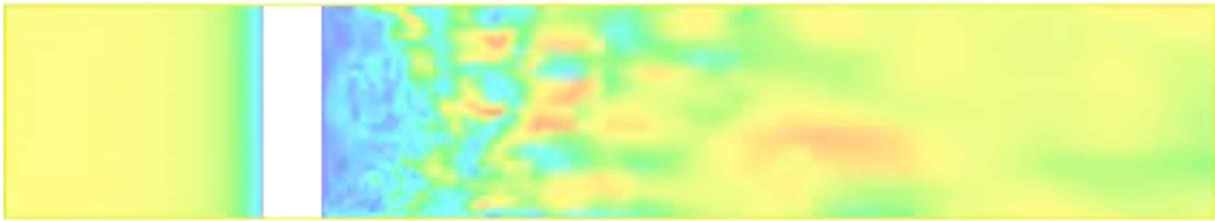
  S = stability factor (a posteriori from computed dual solution)

- Weak solution is not enough! Need to consider perturbations!
- No explicit turbulence model! (G2 regular. with error control)
- A posteriori error estimate: $|M(\hat{u}) - M(\hat{U})| \leq \sum_K E_K$ (cells K)

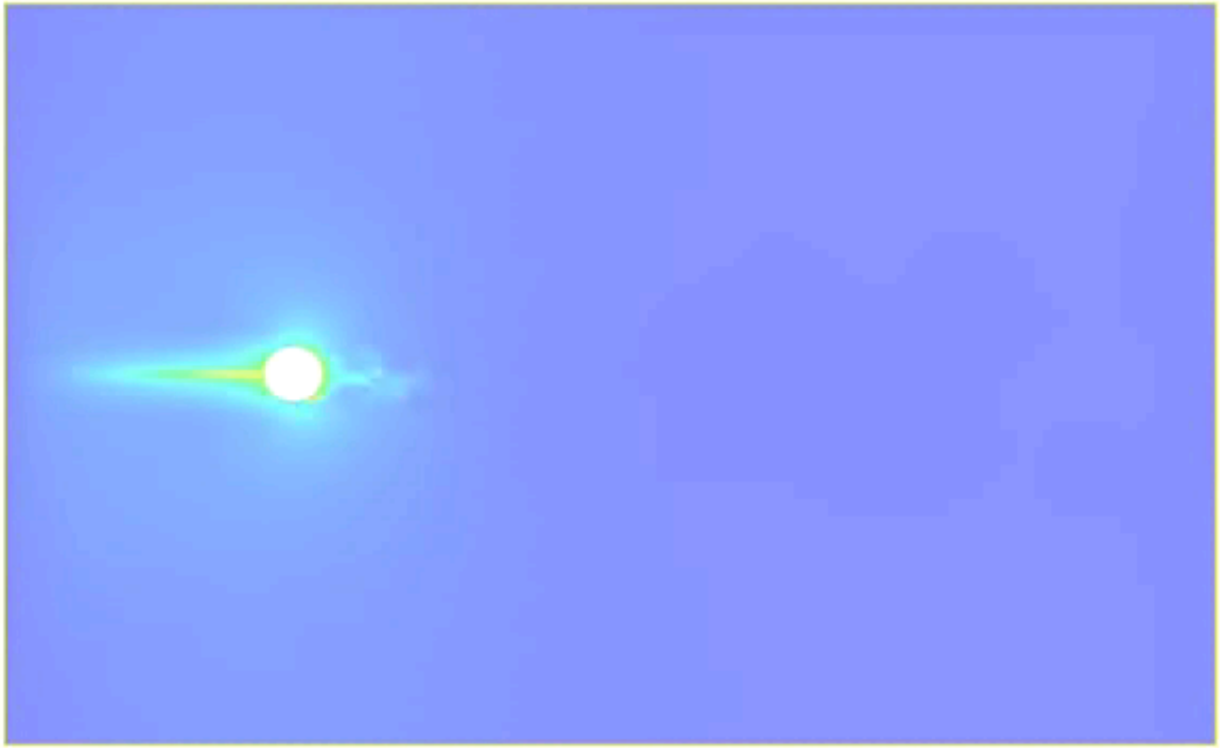
3d circular cylinder Re=3900



3d circular cylinder Re=3900

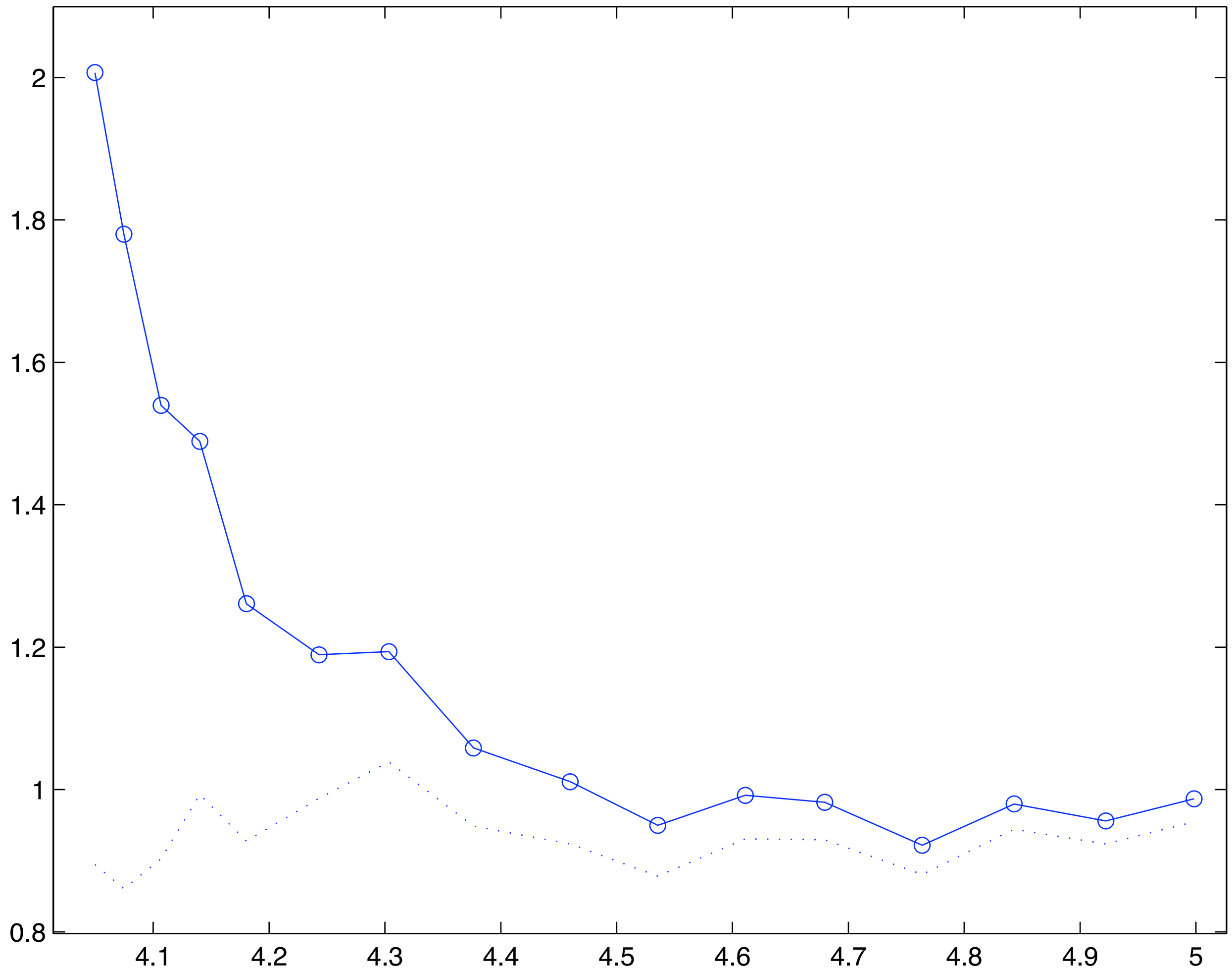


Dual solution for drag

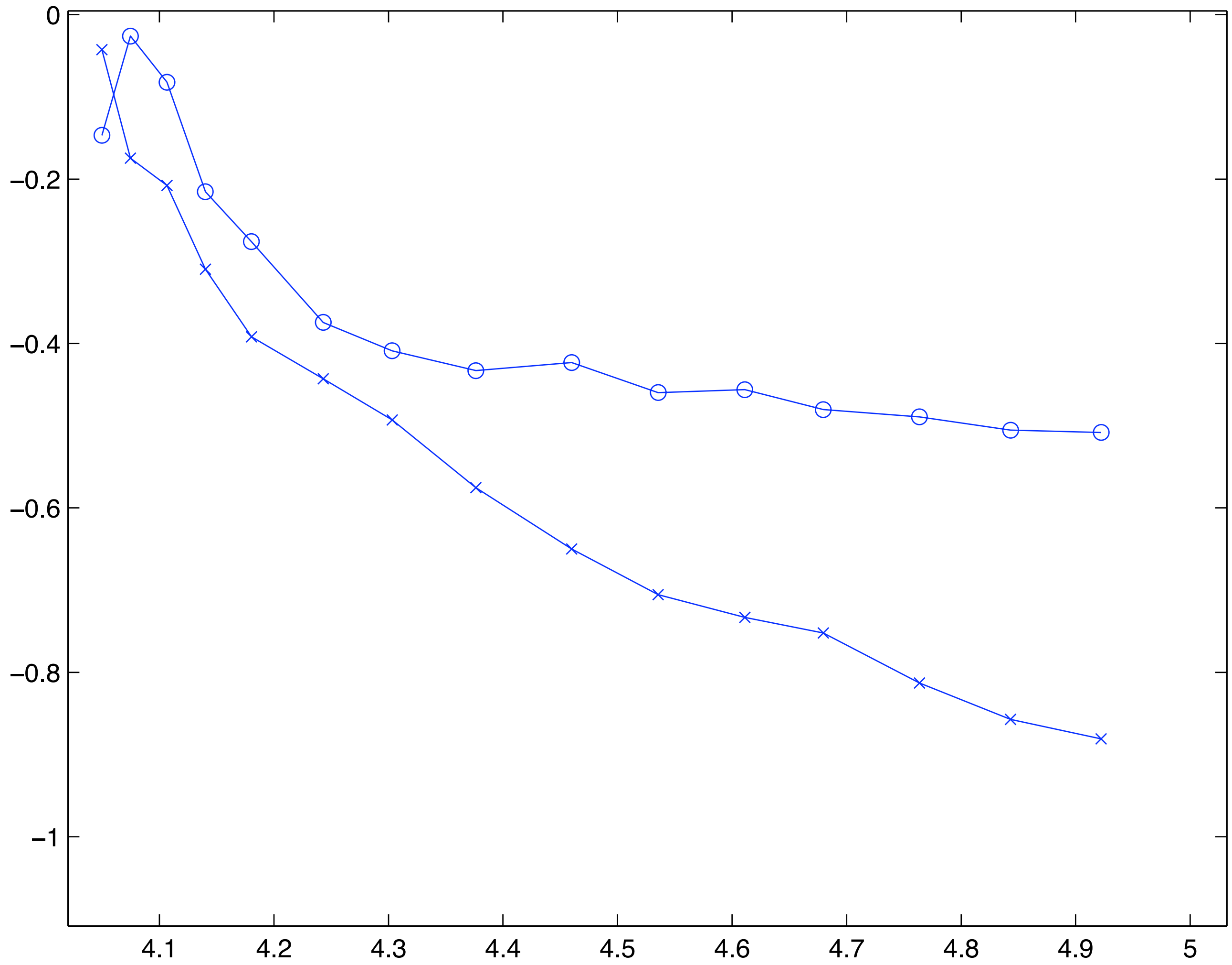


Dual solution for drag

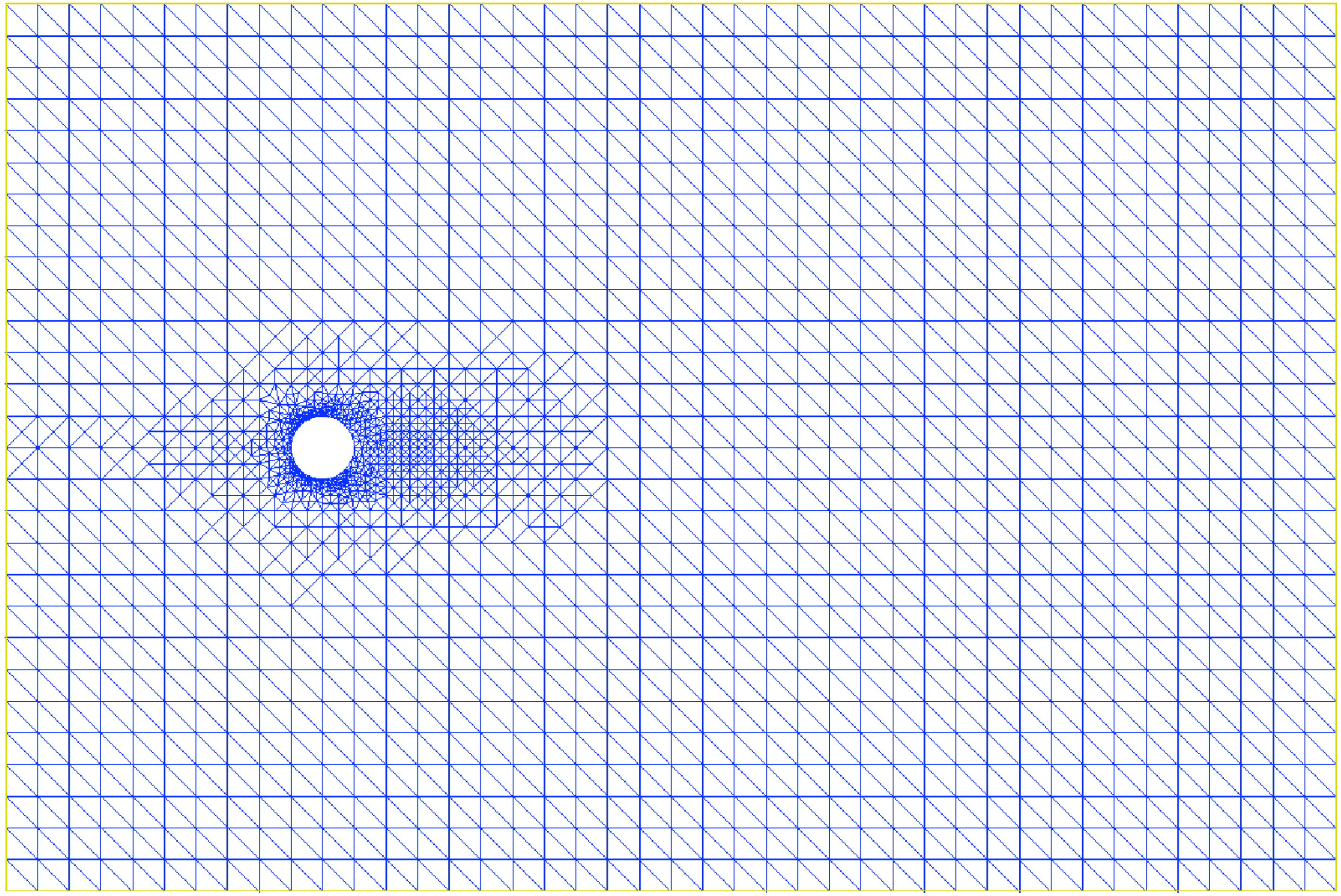
Drag coeff under mesh refinement



Error indicator (in drag)



Mesh after 15 refinements (about 100 000 vertices)



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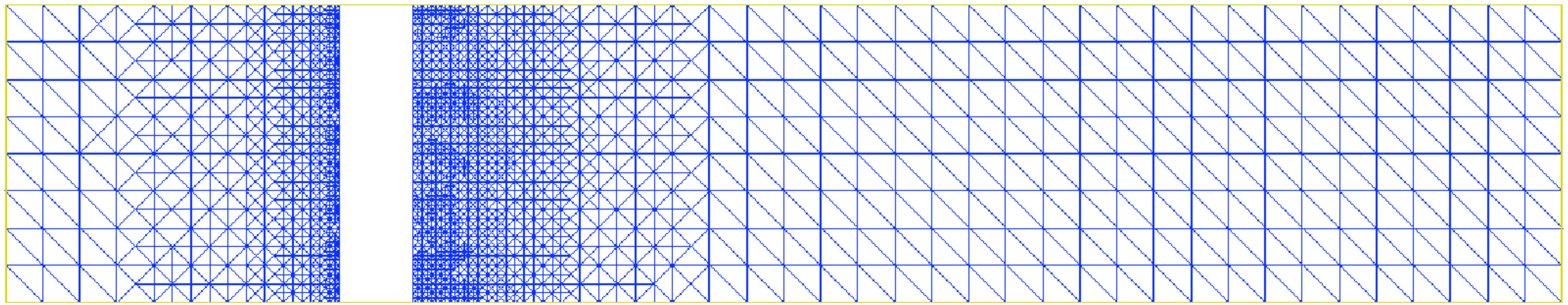
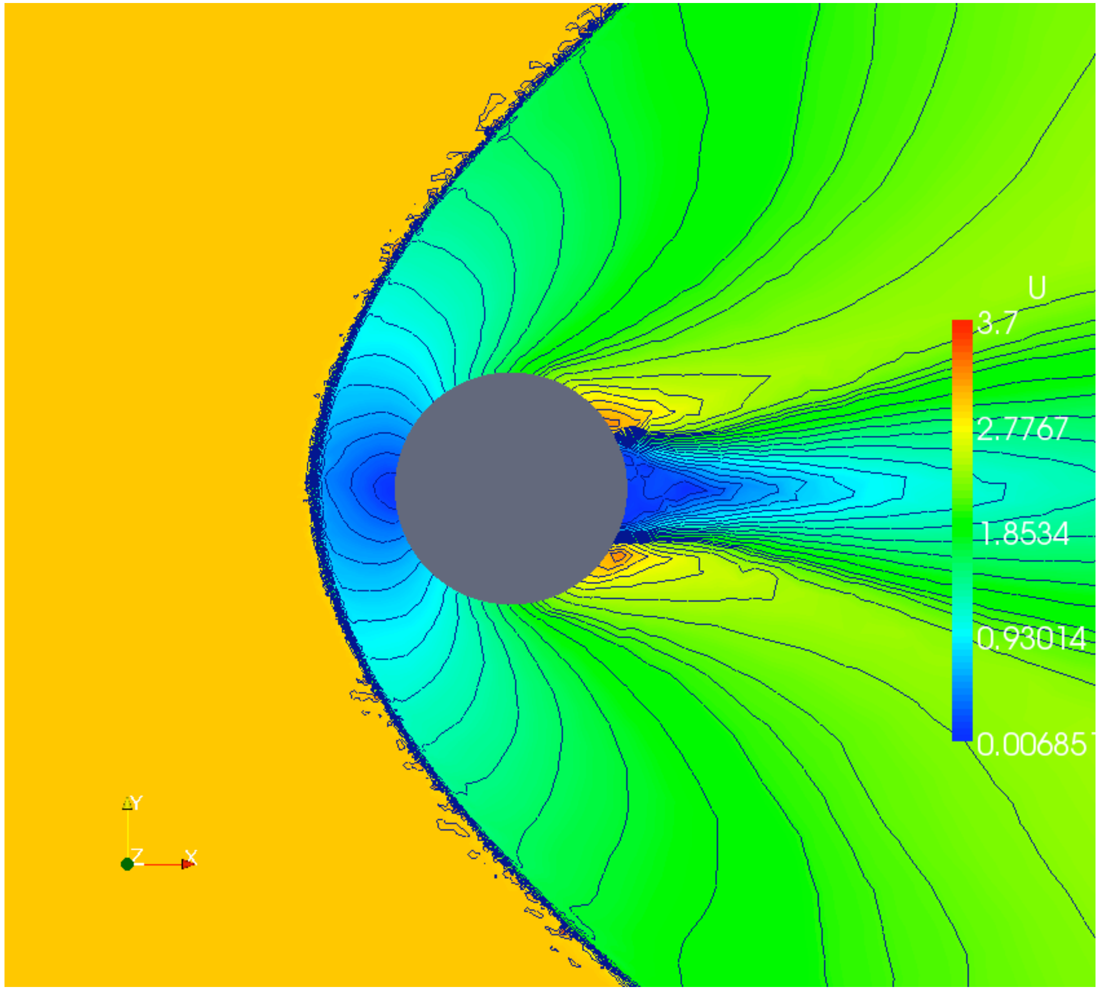
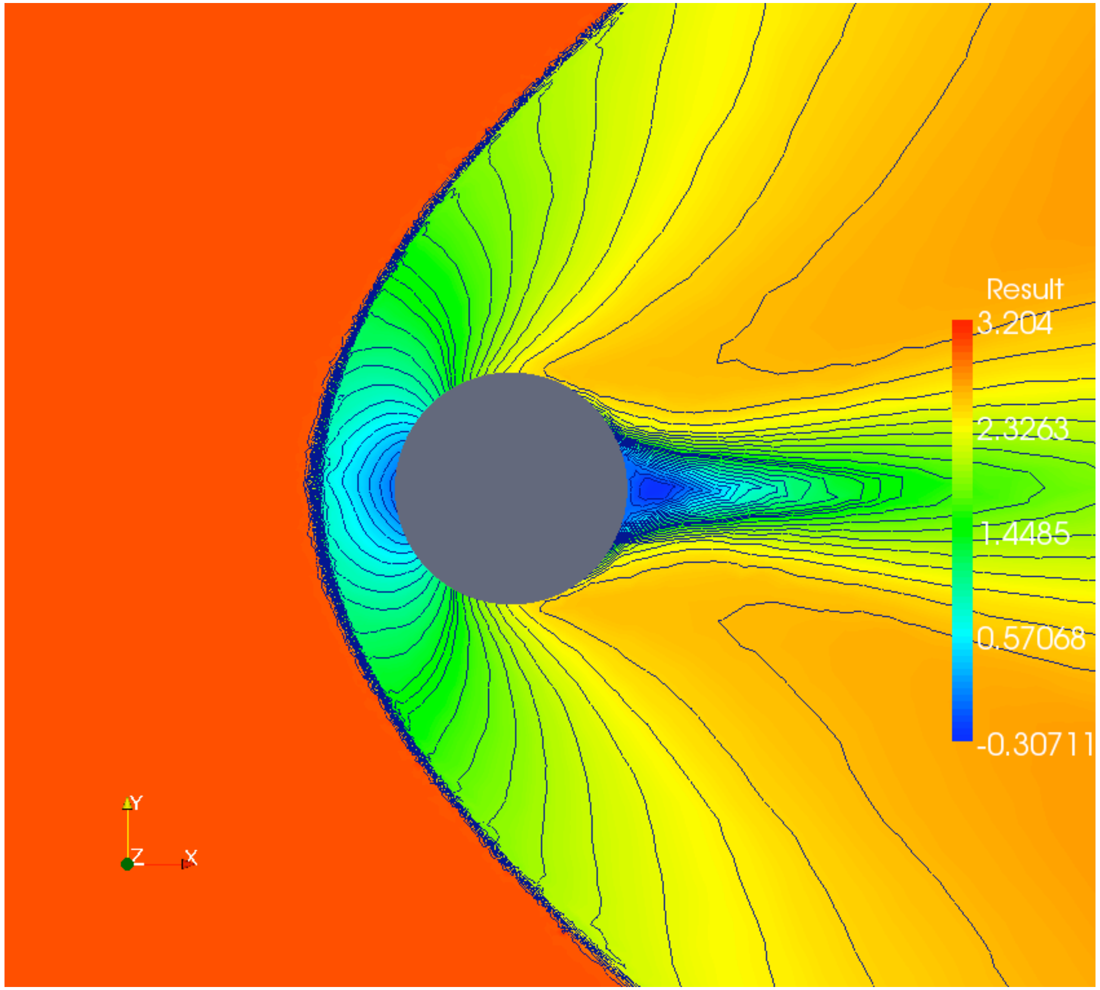


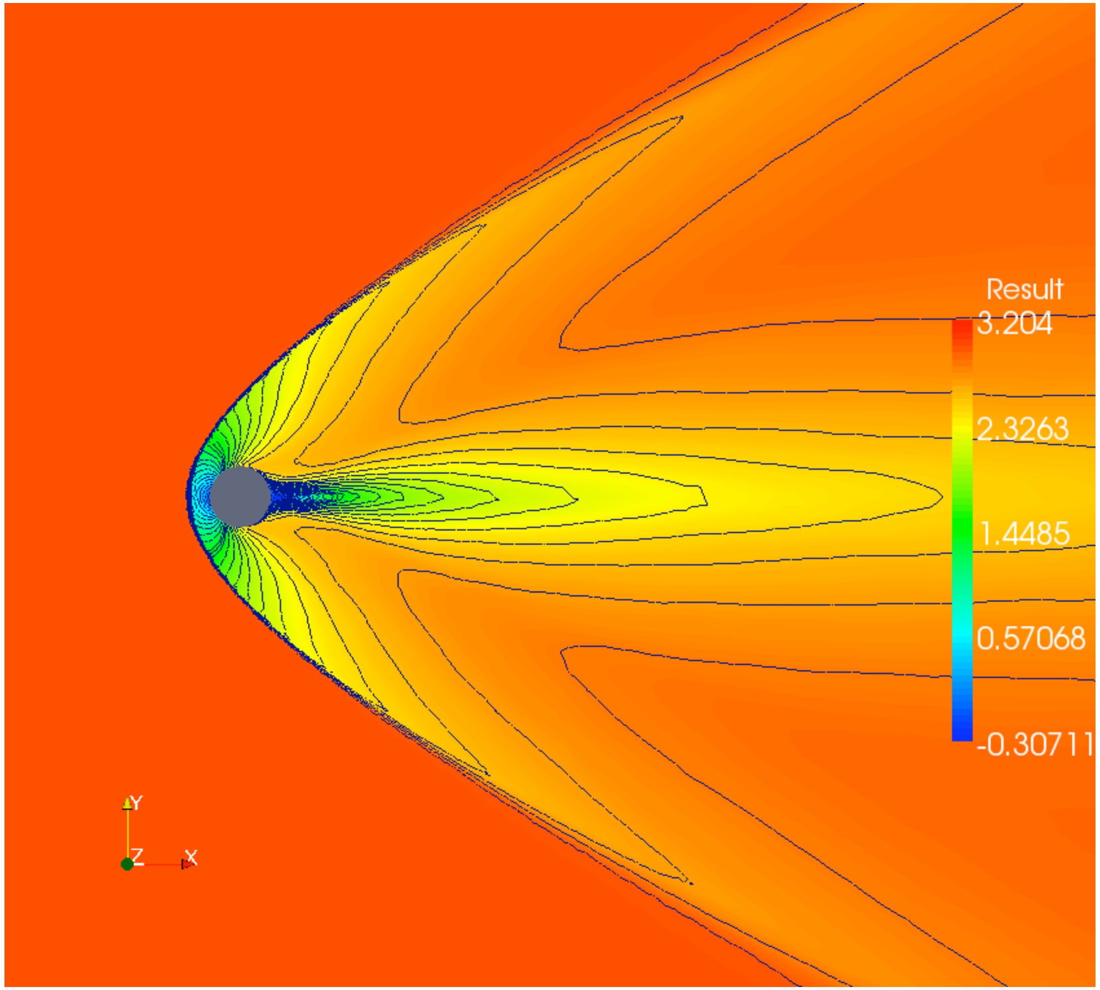
2d compressible flow: Mach number
14 refinements (about 15 000 vertices)



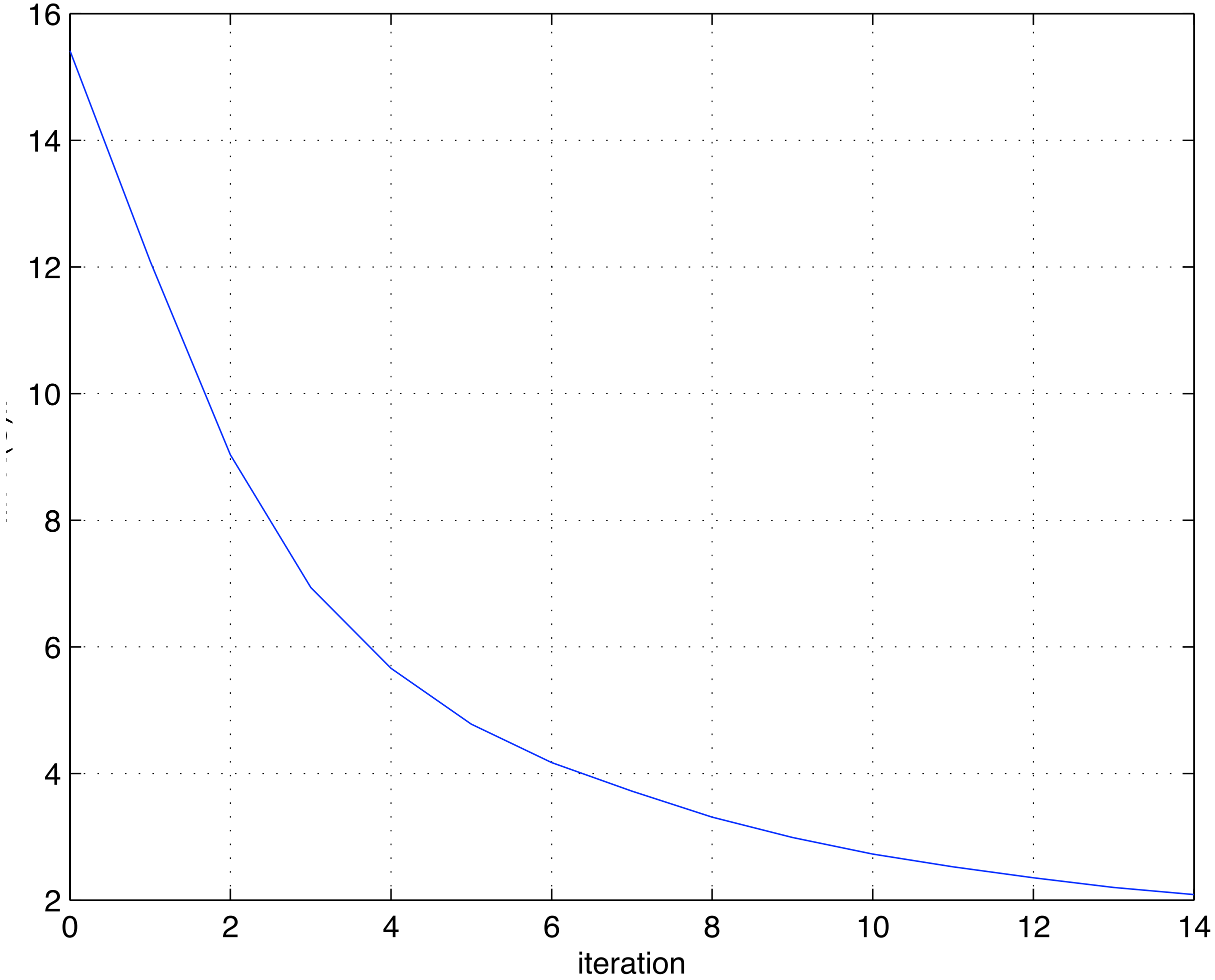
2d compressible flow: velocity
14 refinements (about 15 000 vertices)



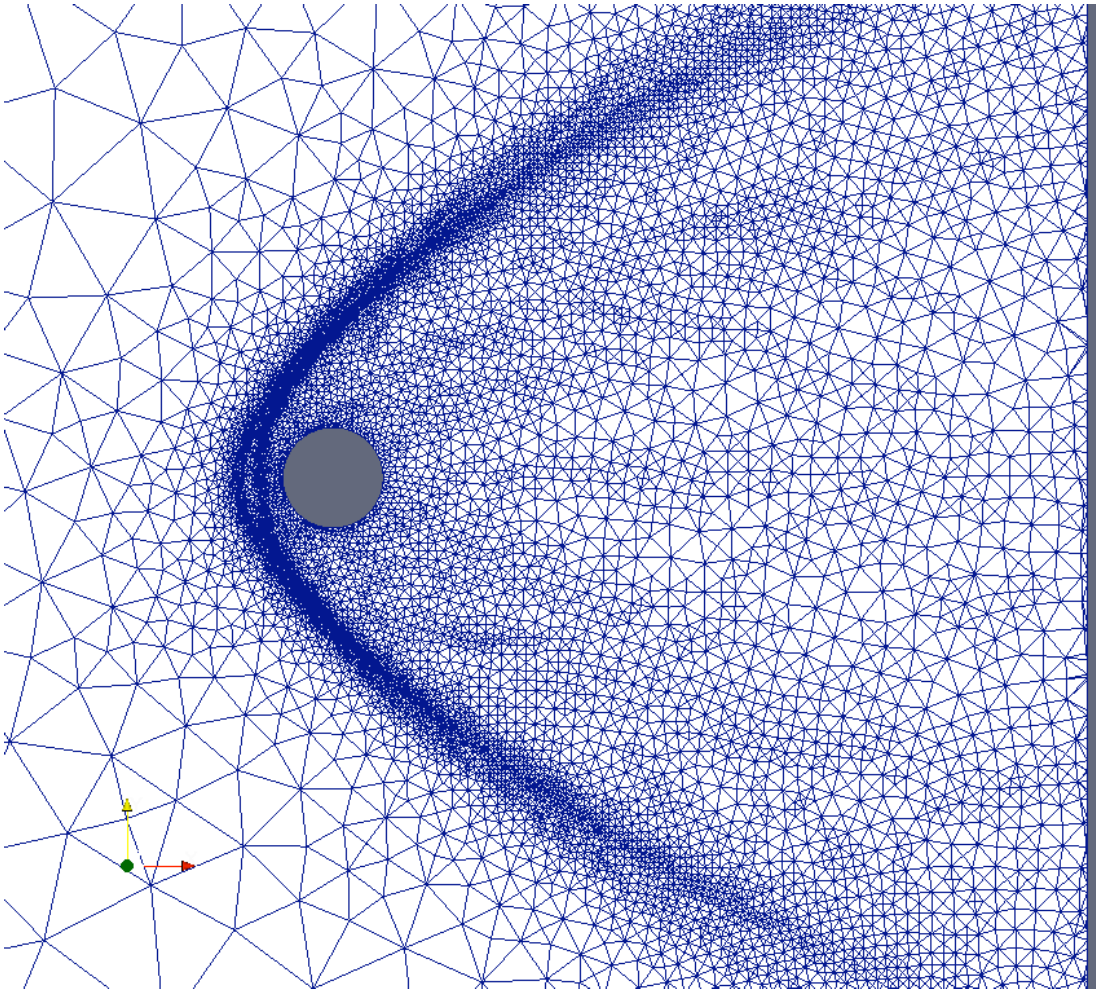
2d compressible flow: velocity
14 refinements (about 15 000 vertices)



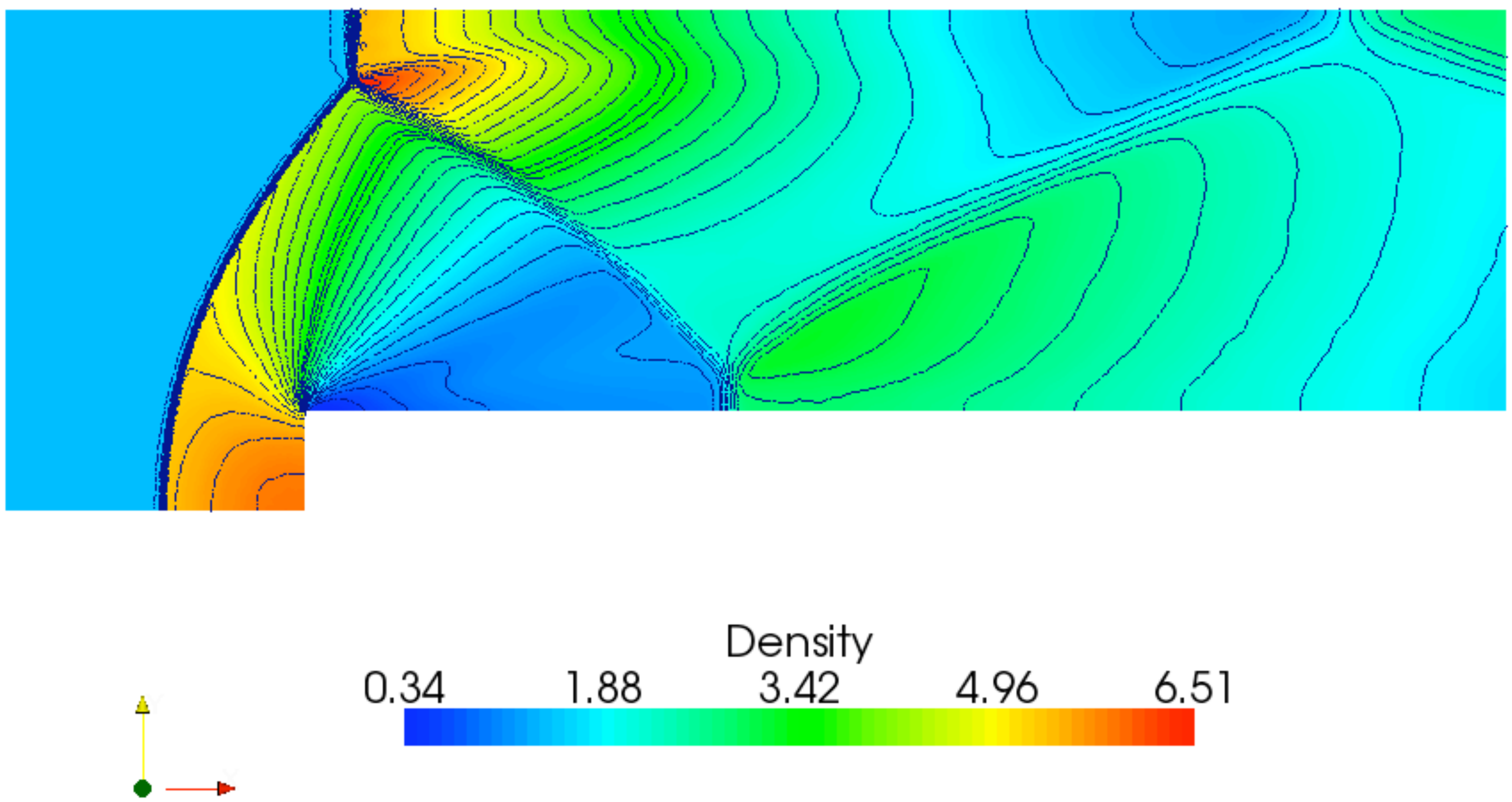
Error indicator: ||hR|| (no dual)



Adaptive mesh refinement
43 refinements (about 20 000 vertices)



2d compressible flow: shock reflection
11 refinements (about 30 000 vertices)



G2 Thermodynamics

- Kinetic energy $k = \rho|u|^2/2$, internal energy $e = \rho T$, temperature T

1st Law : $d/dt(K+E) = 0$ ($K = \int k\,dx$, $E = \int e\,dx$, $W = \int w\,dx$)

2nd Law: $dK/dt - W + D = 0$, $dE/dt + W - D = 0$, with $D \geq 0$

- Energy transfer: reversible (work $w = p\nabla\cdot u$), irreversible ($D > 0$)

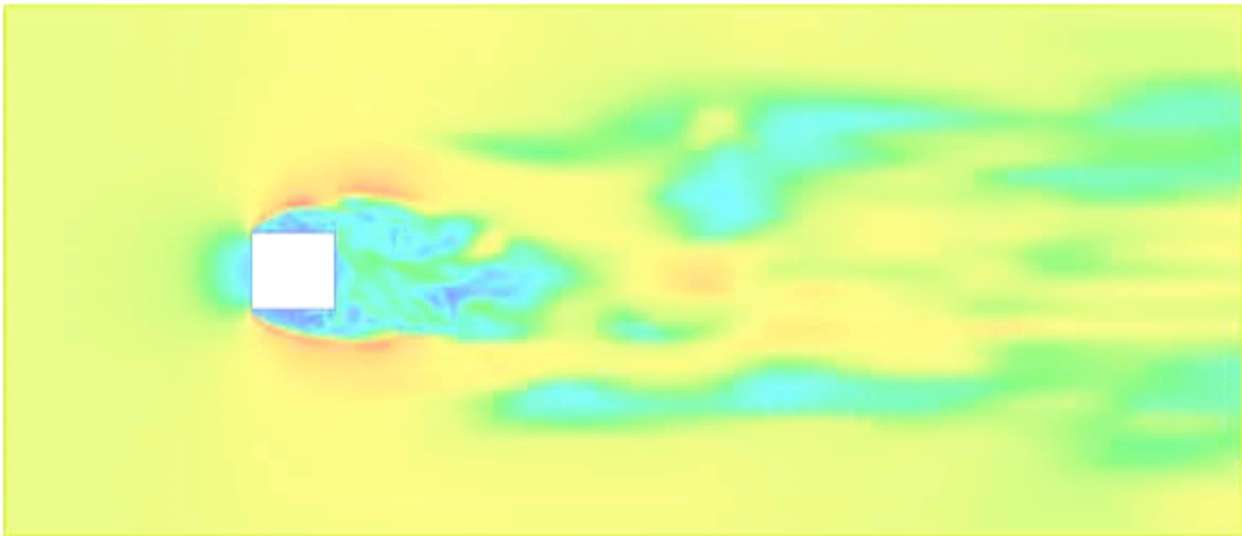
G2: If $||hR(\hat{u})|| \leq Ch^{1/2}$ then we have (also local/suitable variant):

$$|dK/dt - W + D| \leq Ch^{1/2} \quad |dE/dt + W - D| \leq Ch^{1/2}$$

$D = D_h$: dissipative G2 terms $D_h = \int \delta\rho|u\cdot\nabla u|^2 + \nu_\alpha\rho|\nabla u|^2\,dx \geq 0$

Law of finite dissipation $D_h \sim 1$ (indep of h) for turbulence/shocks

Surface mounted cube Re = 40 000
Incompressible flow: W=0, D>0



Surface mounted cube Re = 40 000
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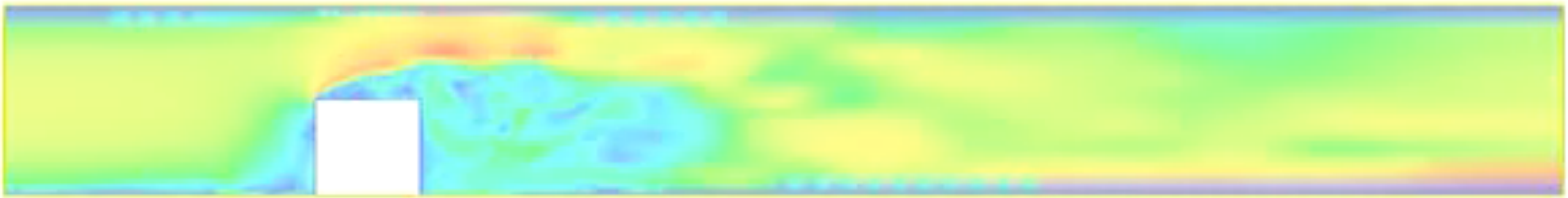
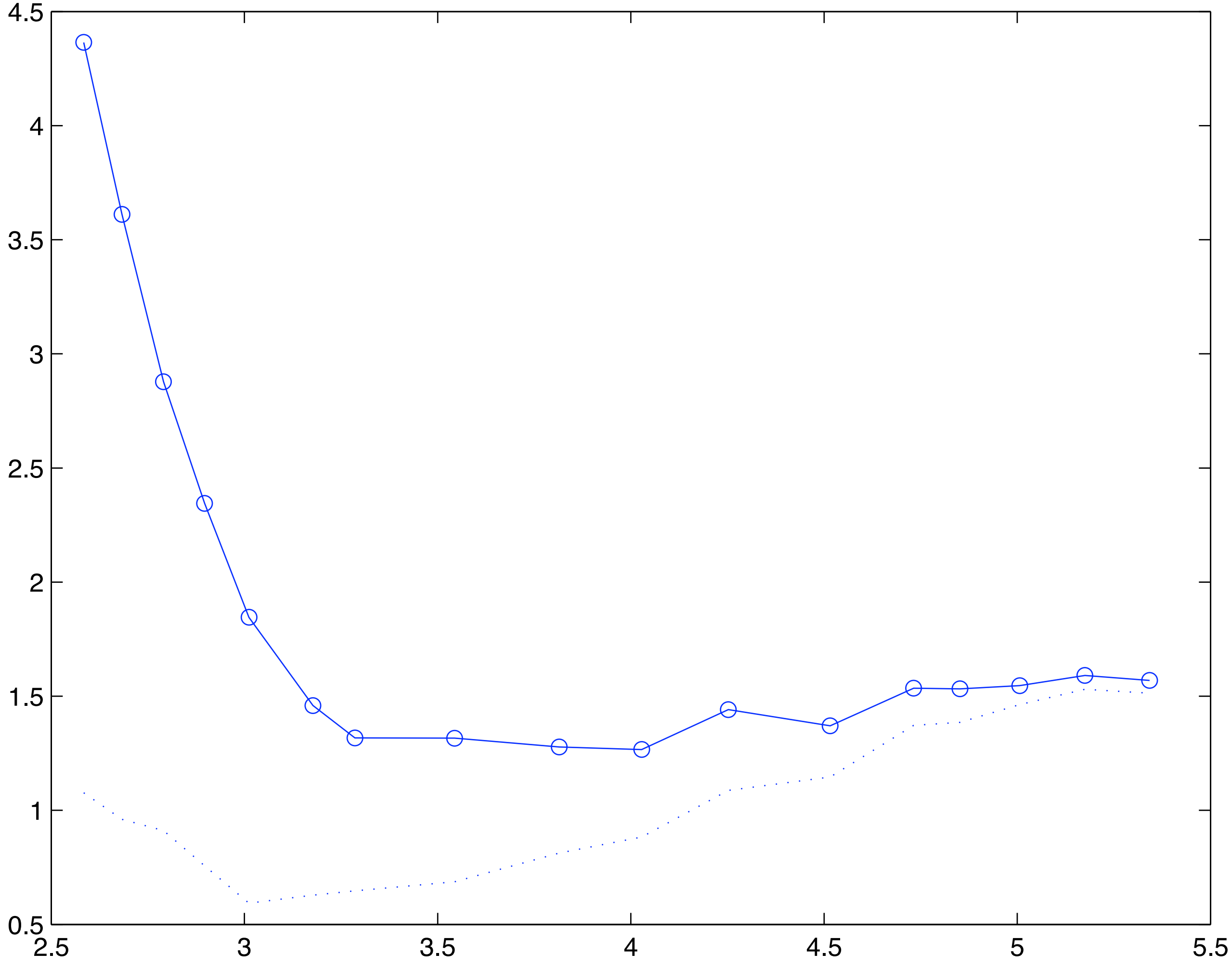
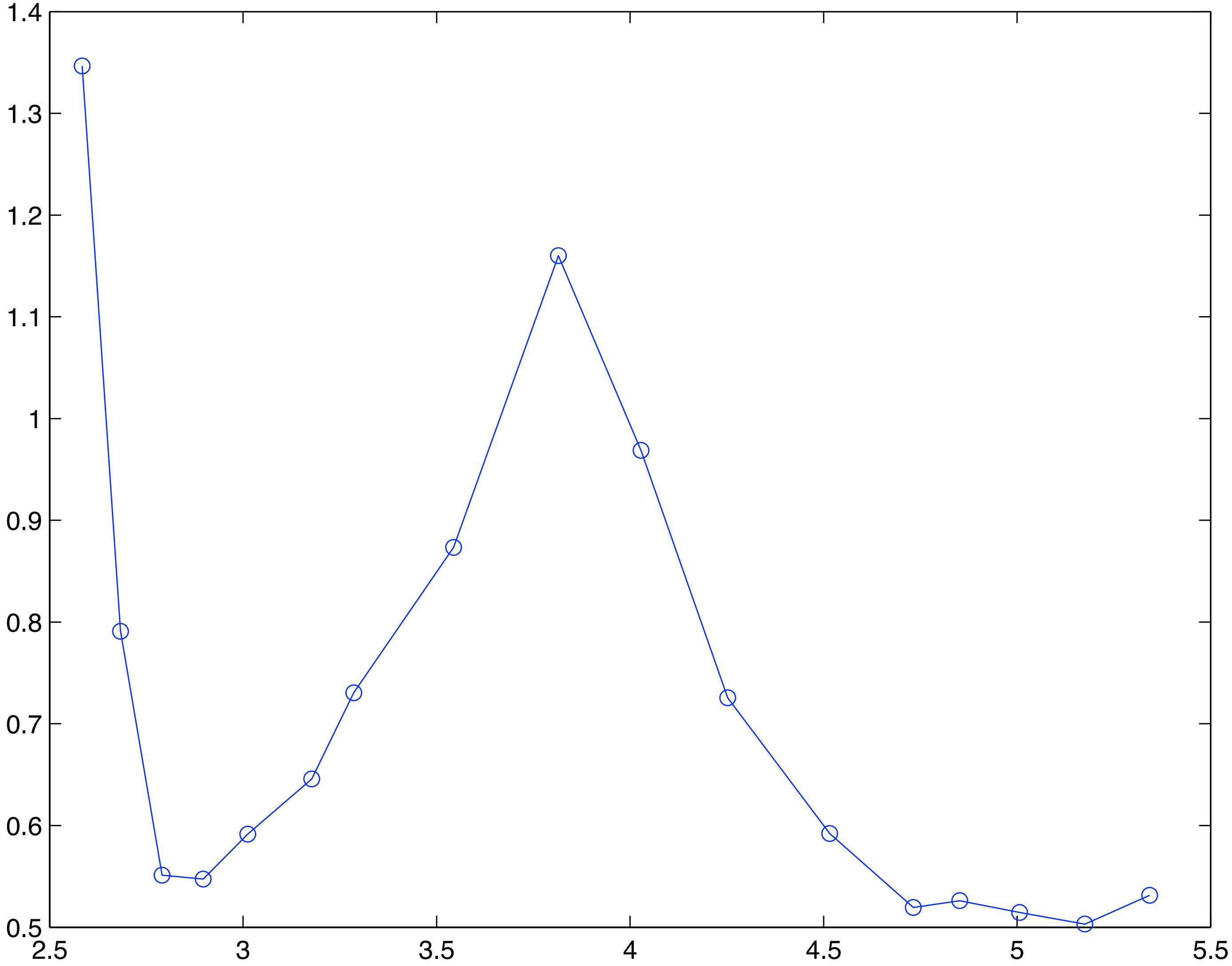


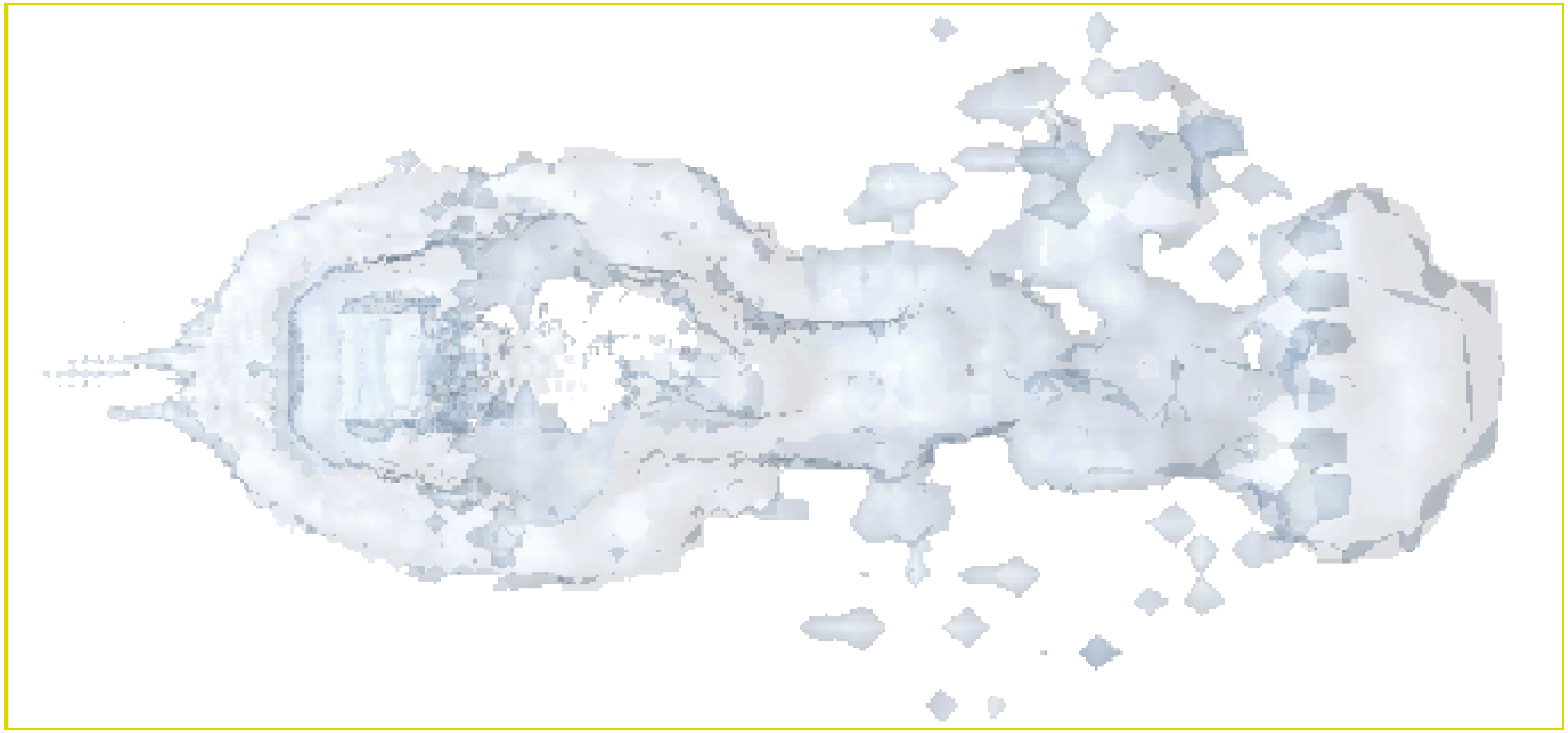
Drag coefficient (~ D_h) under mesh refinement



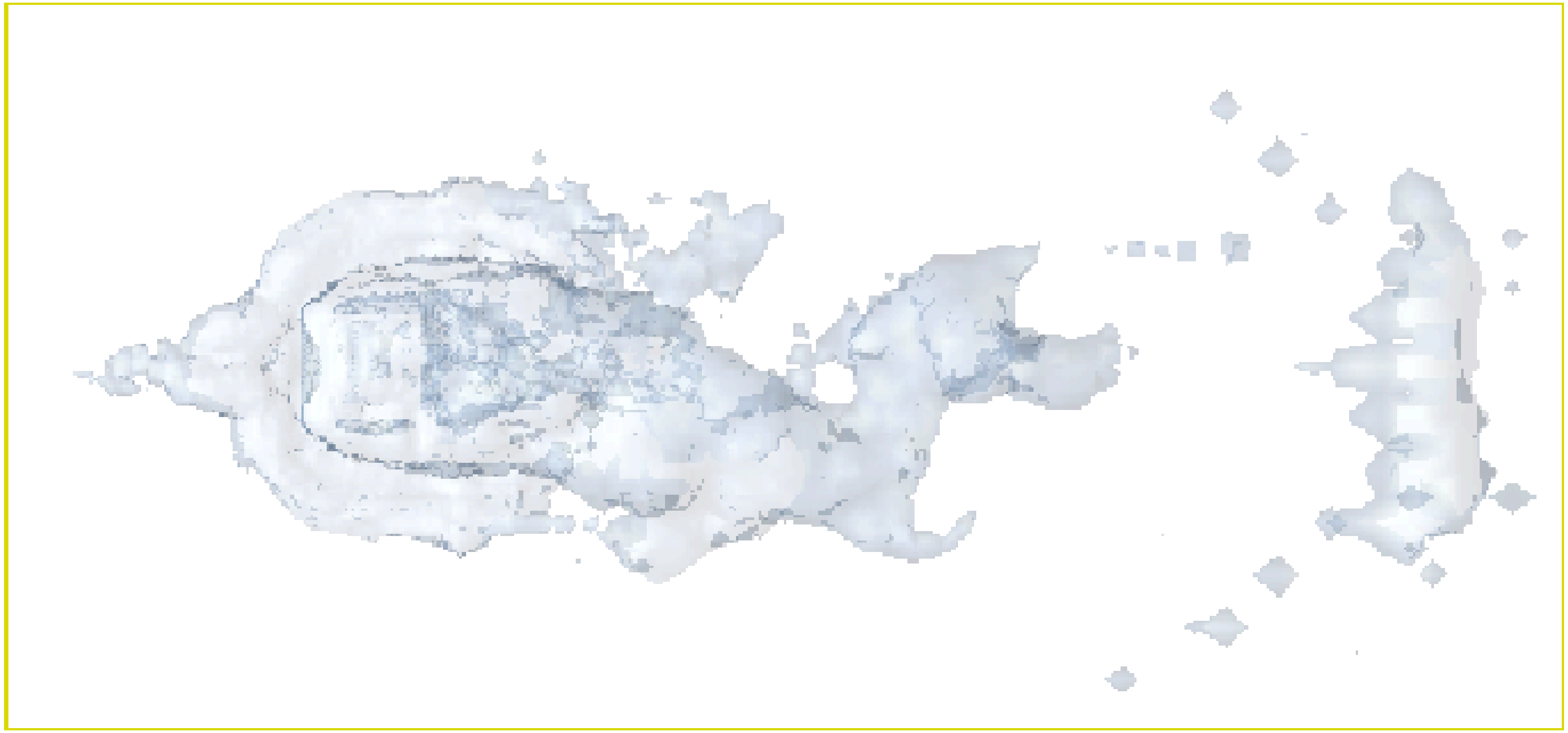
Law of finite dissipation: $D_h \sim 1$ (D_h independent of mesh size h)



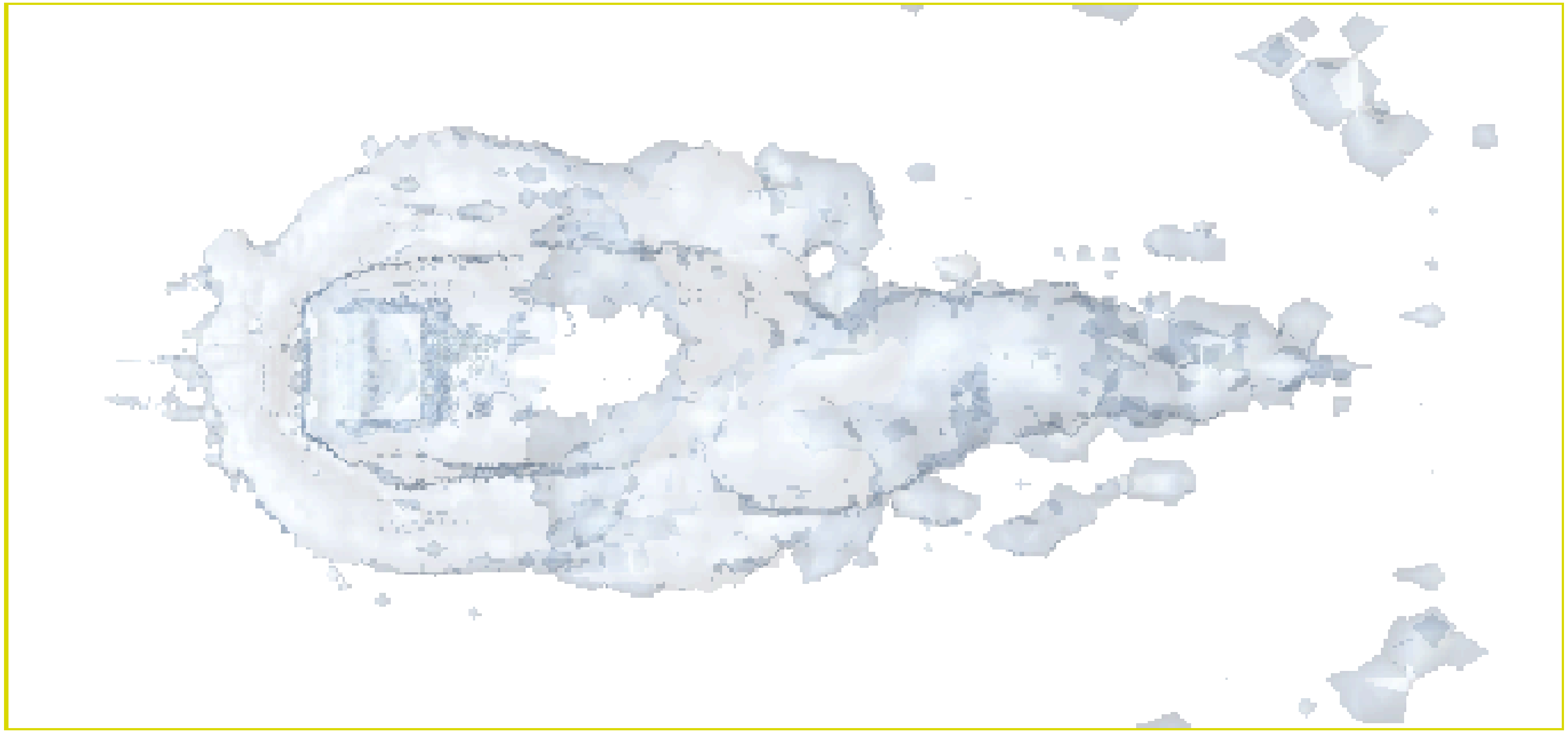
Isosurface of D_h after 13 refinements (D_h independent of mesh size h)



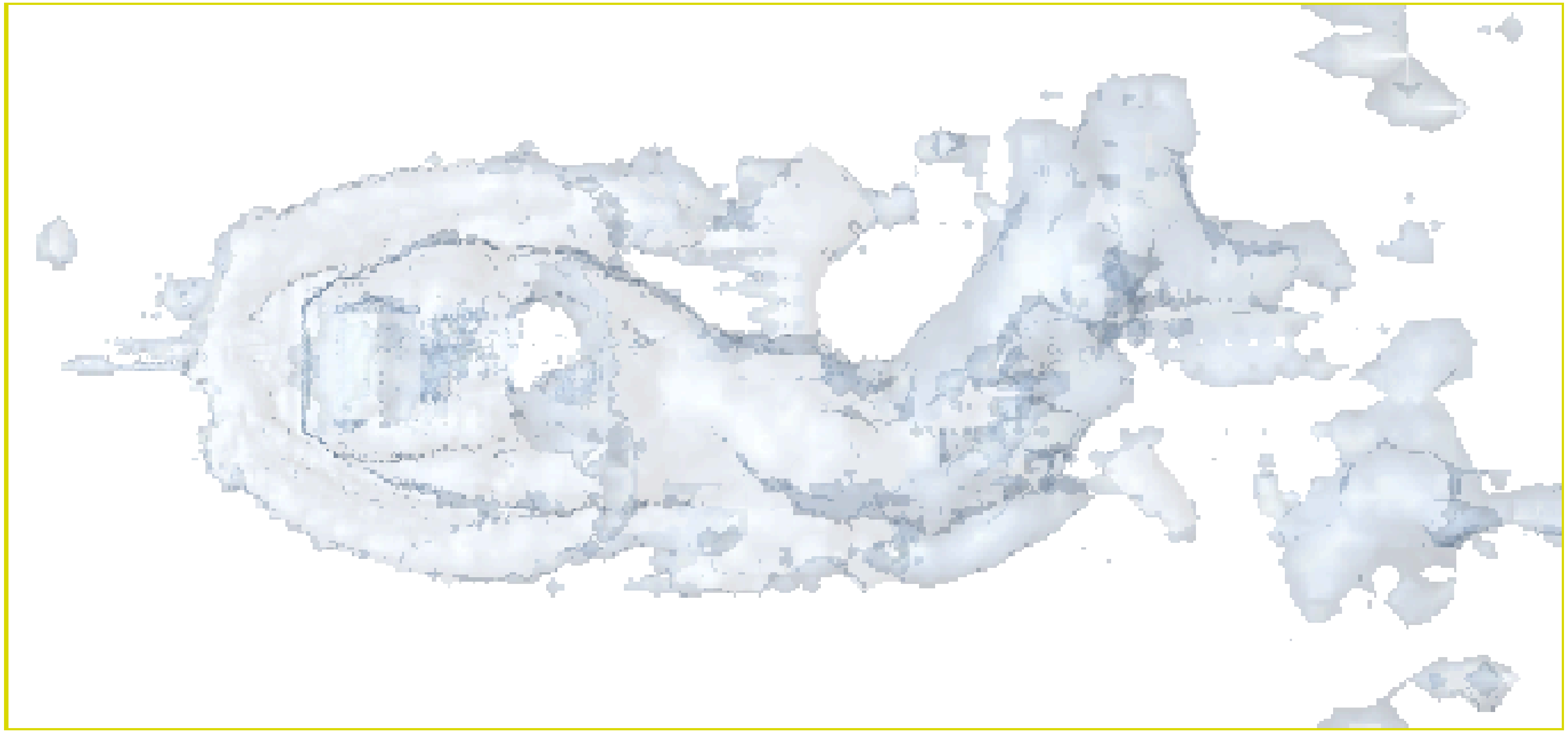
Isosurface of D_h after 14 refinements (D_h independent of mesh size h)



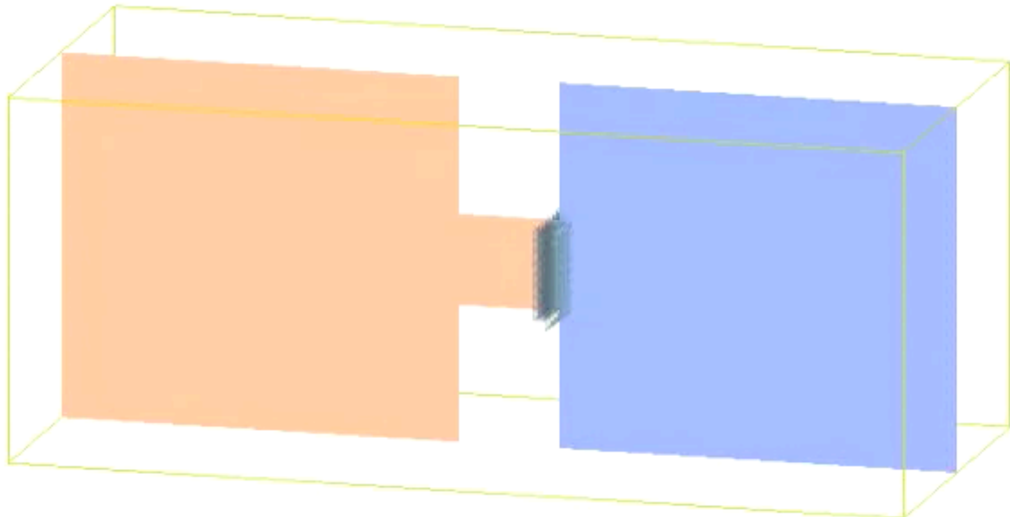
Isosurface of D_h after 15 refinements (D_h independent of mesh size h)



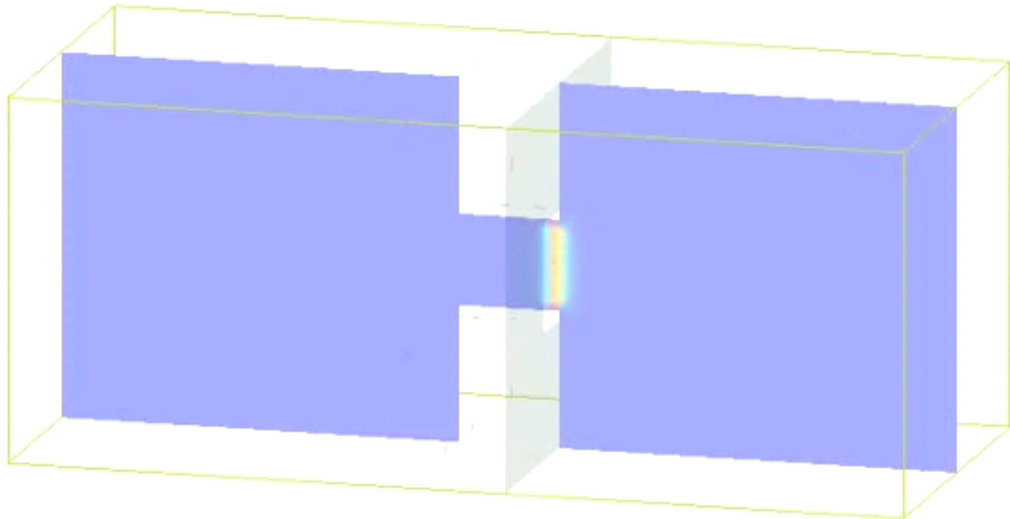
Isosurface of D_h after 16 refinements (D_h independent of mesh size h)



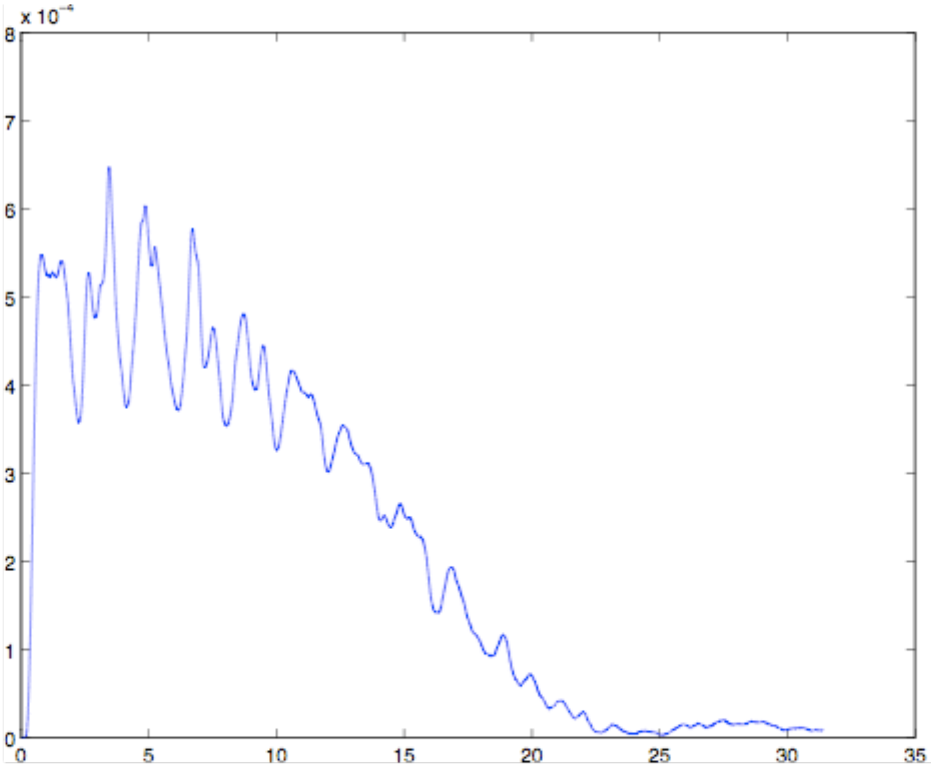
3d compressible turbulent flow
Expanding gas (W > 0): density

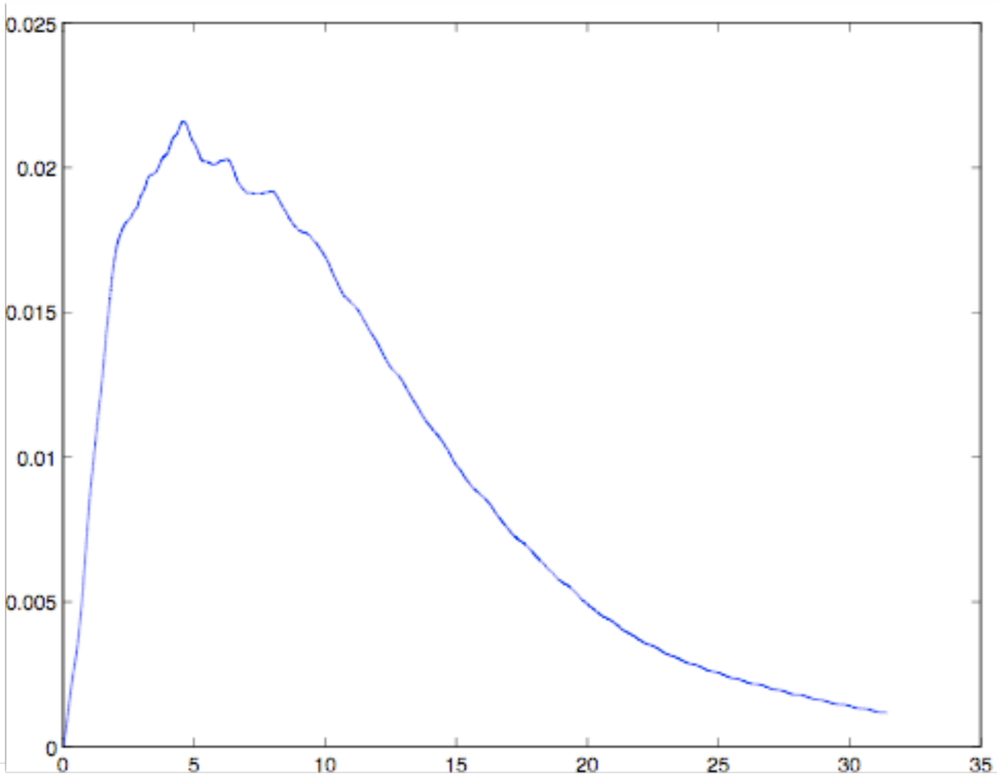


3d compressible turbulent flow
Expanding gas (W > 0): momentum



3d compressible turbulent flow
Kinetic energy left/right chamber: D>0





References

- Stab. FEM (80's)*:* Hughes, Tezduyar, Johnson, Szepessy, Hansbo,…
- A posteriori error control (duality) for fluids (90's): Johnson, Eriksson, Hansbo, Burman, Rannacher, Becker, Braack, Hartmann, Süli, Houston, Barth,…

- G2 incompressible turbulent flow*:* Hoffman *(SISC 04,05, JFM 06, CM 06…),* Hoffman/Johnson *(CMAME 06, Springer 07,…)*
- G2 compressible turbulent flow*:* Hoffman/Nazarov/Jansson *(SISC 08),* Hoffman/Johnson *(Springer 08)*

Blowup of smooth Euler solutions

Q: Does a smooth Euler solution blow up (in shock/turbulence)?

Compressible Euler: well-known to exhibit blowup in shocks

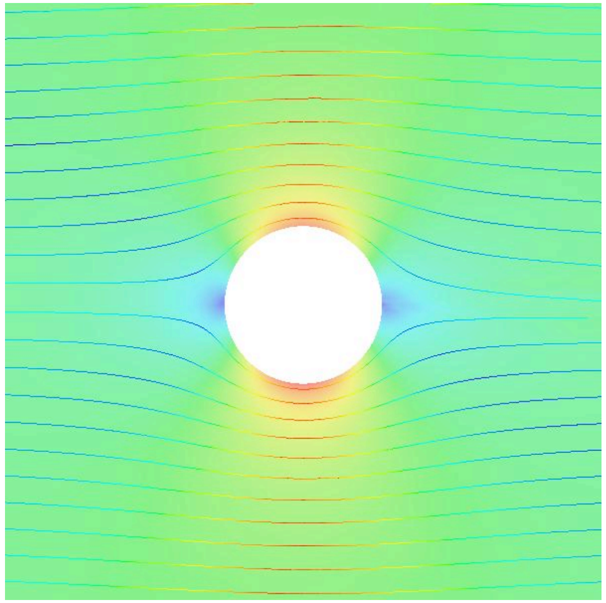
- No exact solutions exist! Only regularized (weak) solutions!

Incompressible Euler: blowup or not? ($1 million Clay Prize)[2]

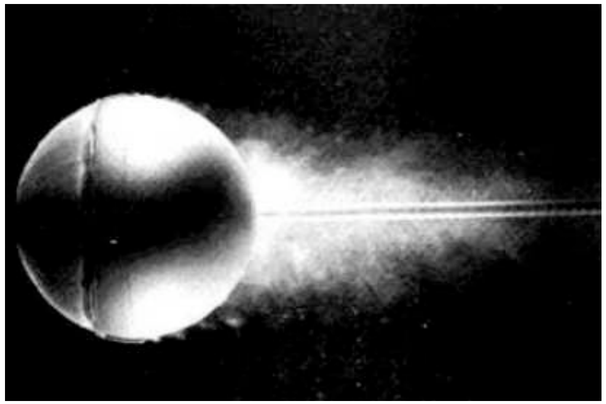
- Local blowup: find T such that û is non-smooth ($\hat{u} \notin C^\infty$) for $t \geq T$: detection requires infinite resolution: vorticity $\omega \sim \exp(\exp(t))$ No perturbation analysis (cont dep on data)! *(Hou/Li, Physica D 08)*
- Global blowup: $||R(\hat{u})|| \sim h^{-1/2}$ under mesh refinement: detection on finite meshes: $D_h \sim ||h^{1/2}R(\hat{u})|| \sim 1$ (finite dissipation) Stability analysis wrt mesh perturburbations! Well-posedness in M! *(Hoffman/Johnson, BIT 08)*

Blowup of Potential Flow
(The d'Alembert paradox)

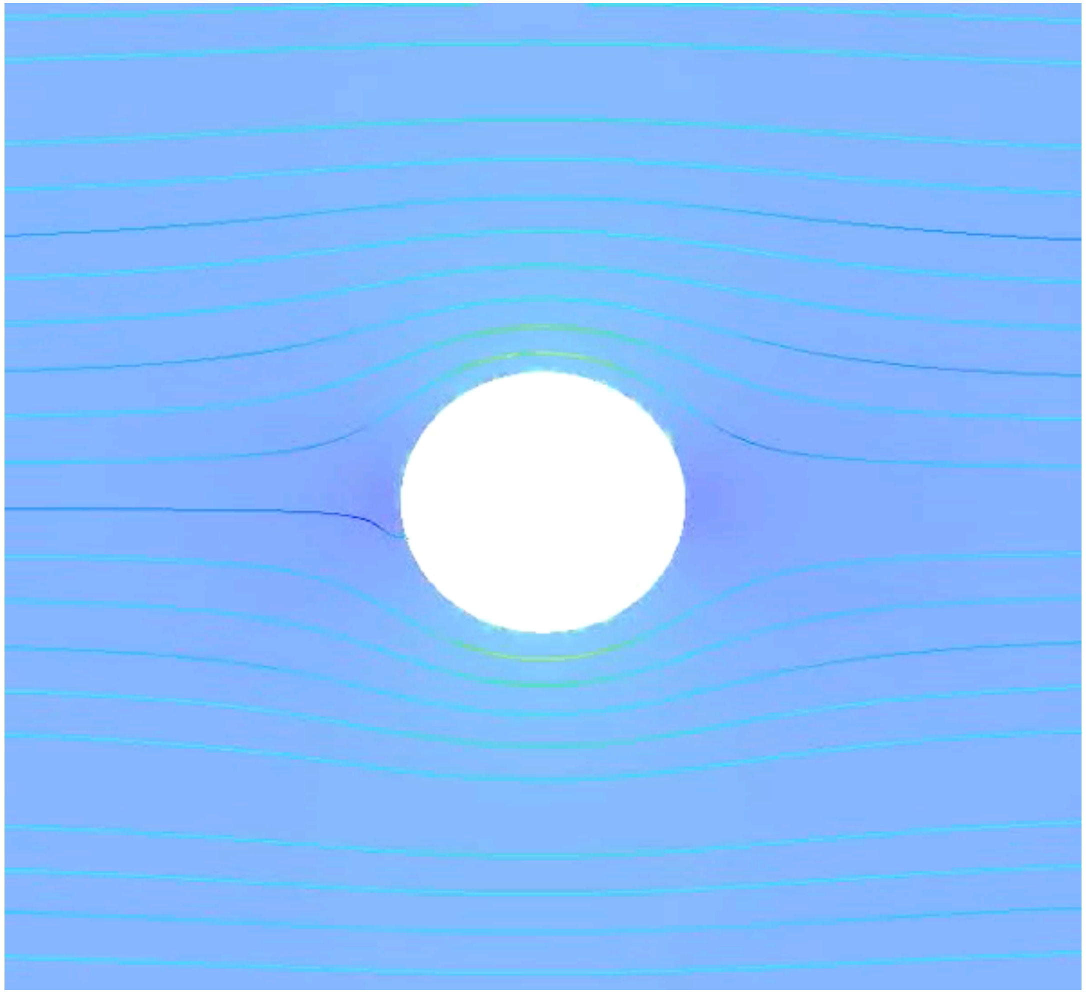
- d'Alembert paradox (1752):
  Potential flow has no drag or lift!
  Not compatible with experiments!
  (high drag turbulent solution)
- Prandtl (1904): boundary layer
  gives separation into turbulence:
  Euler with slip bc invalid!
- Must resolve boundary layer!
- High Re computation impossible!



- Does the potential solution blowup?!
- Not without perturbations (Kelvin thm)!

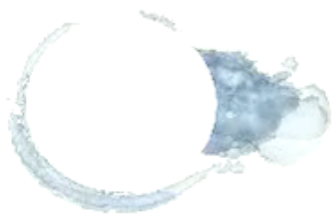


G2 Blowup of Potential Flow: velocity (The d'Alembert paradox)

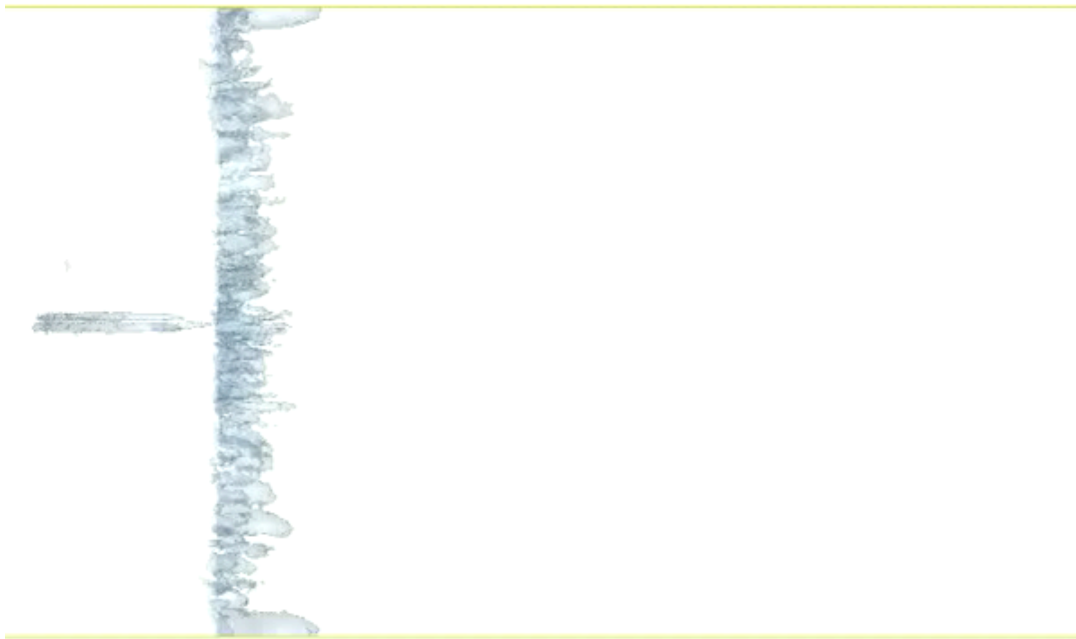


G2 Blowup of Potential Flow: pressure (The d'Alembert paradox)

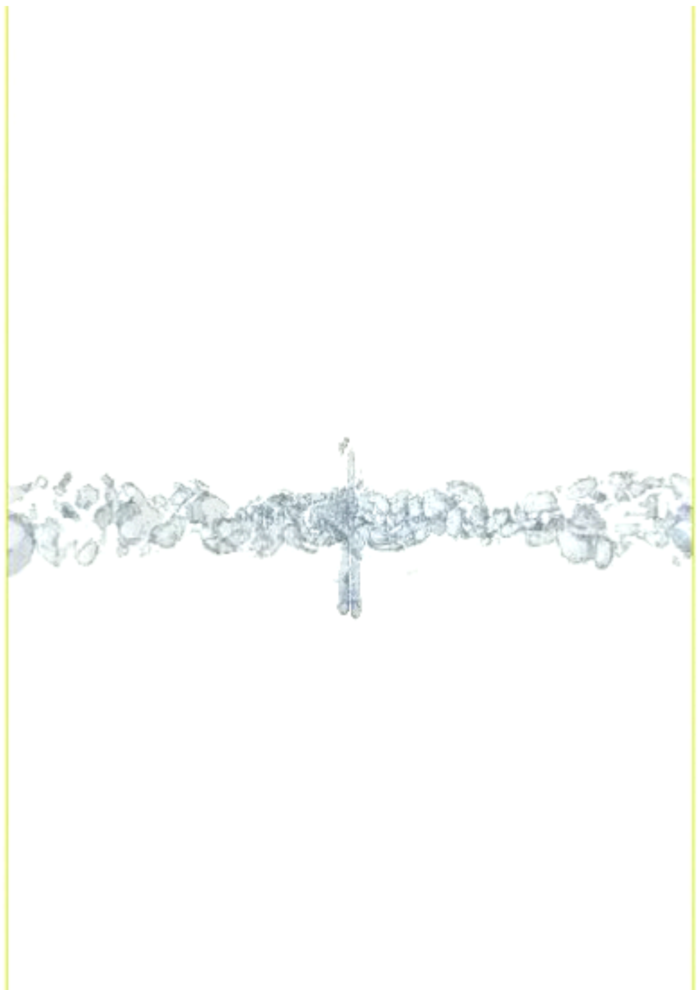
Blowup of Potential Flow: vorticity ω_1
(The d'Alembert paradox)



Blowup of Potential Flow: vorticity ω_1 (The d'Alembert paradox)

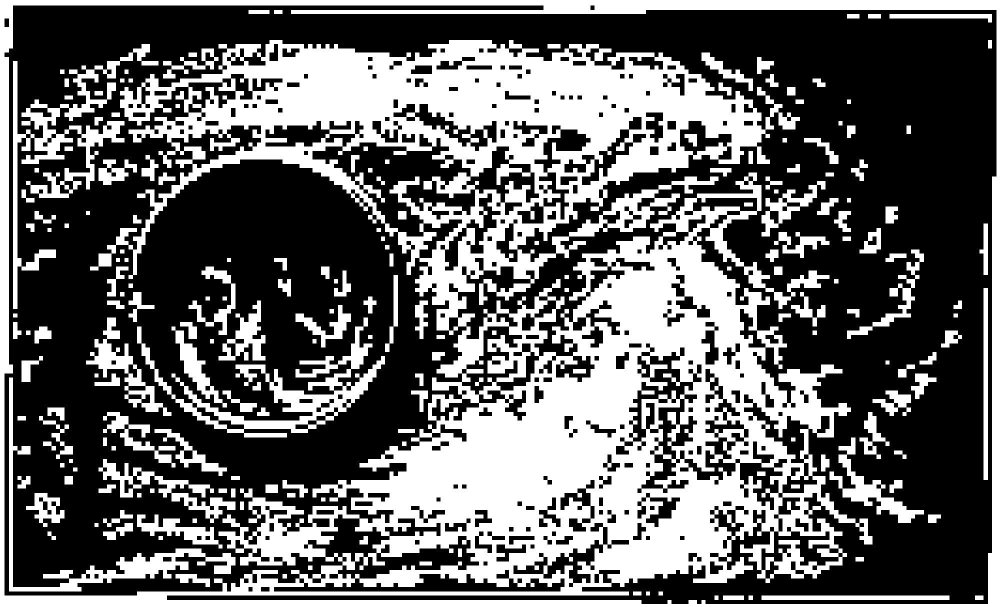


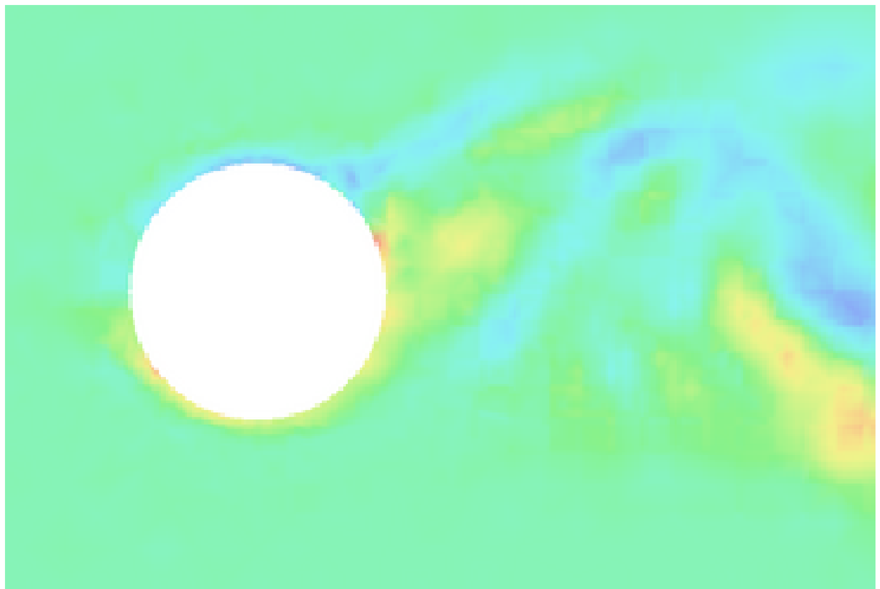
Blowup of Potential Flow: vorticity ω_1
(The d'Alembert paradox)



New resolution of d'Alembert paradox (Hoffman/Johnson, JMFM 08)

- Standard linear stability analysis: Euler solutions (u,p) exponentially unstable! (sum of eigenvalues zero for ∇u)
- Streamwise vorticity ω_1 grows exponentially in time at rear separation of cylinder (vortex stretching)! Turbulent wake!
- Global blowup! Potential solution not well-posed (for any M)!



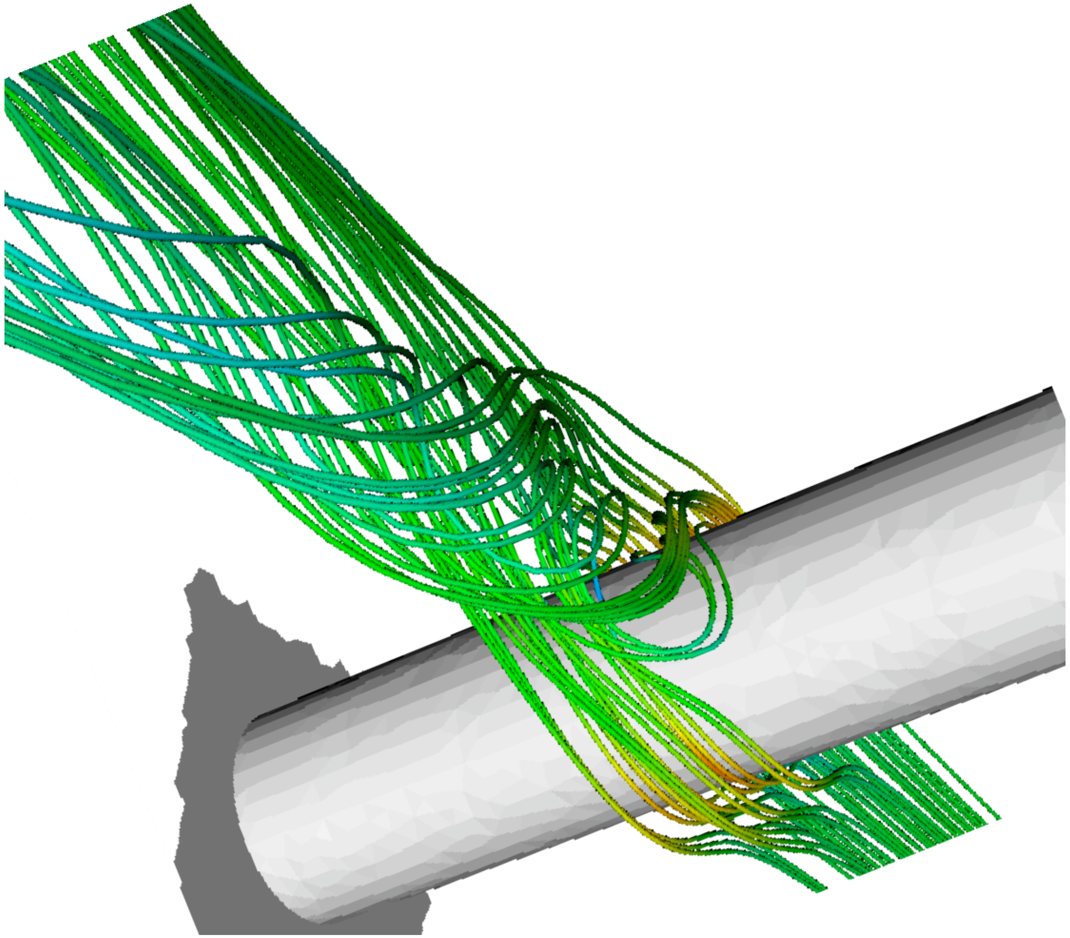


Turbulent boundary layer separation

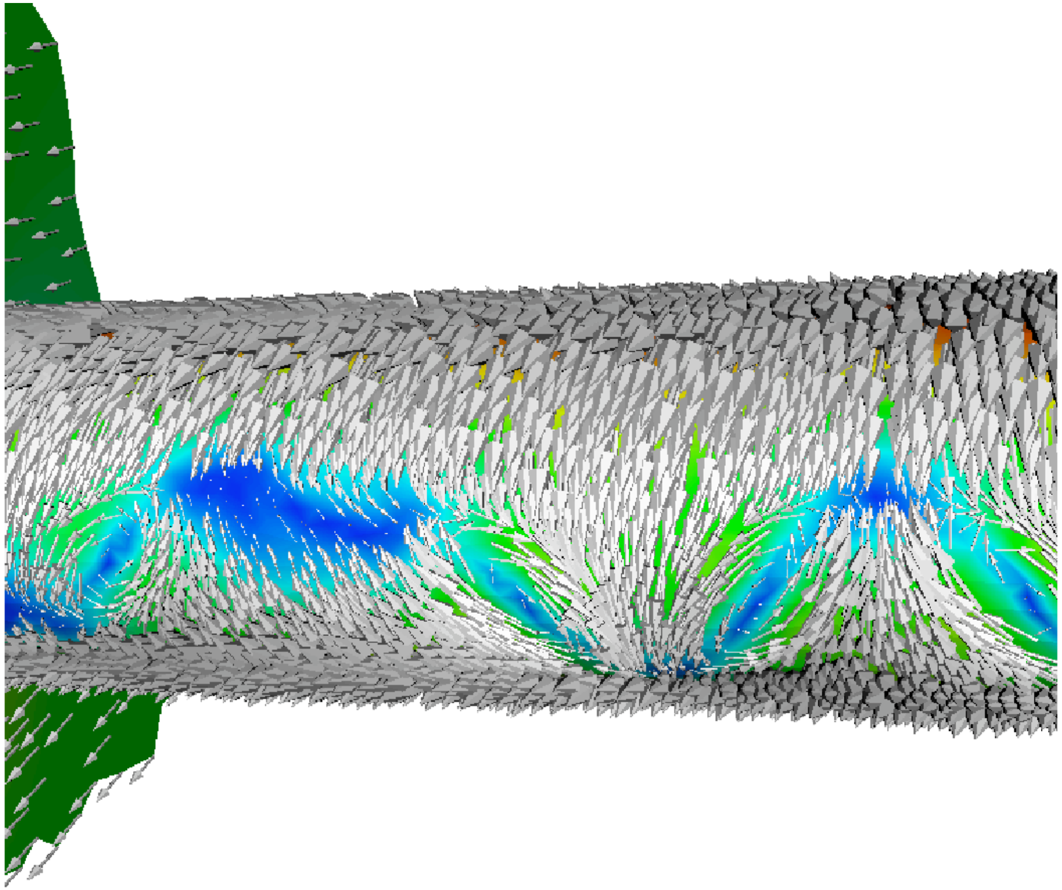
- Prandtl (1904): boundary layer separation by adverse pressure gradient $\nabla p \cdot t > 0$ (pressure increase in tangent direction): velocity reduced to zero (separation by recirculation)

- Hoffman/Johnson (2008): slip (no boundary layer) separation by insufficient normal pressure gradient $\nabla p \cdot n$
- Wall normal vorticity reduces pressure (and wall normal pressure gradient) to allow separation!
- Boundary layer not needed for separation!
- Use instead skin friction boundary conditions: cheap! (*John/Liakos IJNMF 06, Hoffman CM 06)*
- (Very) high Re: skin friction (coeff) = 0: computation possible!

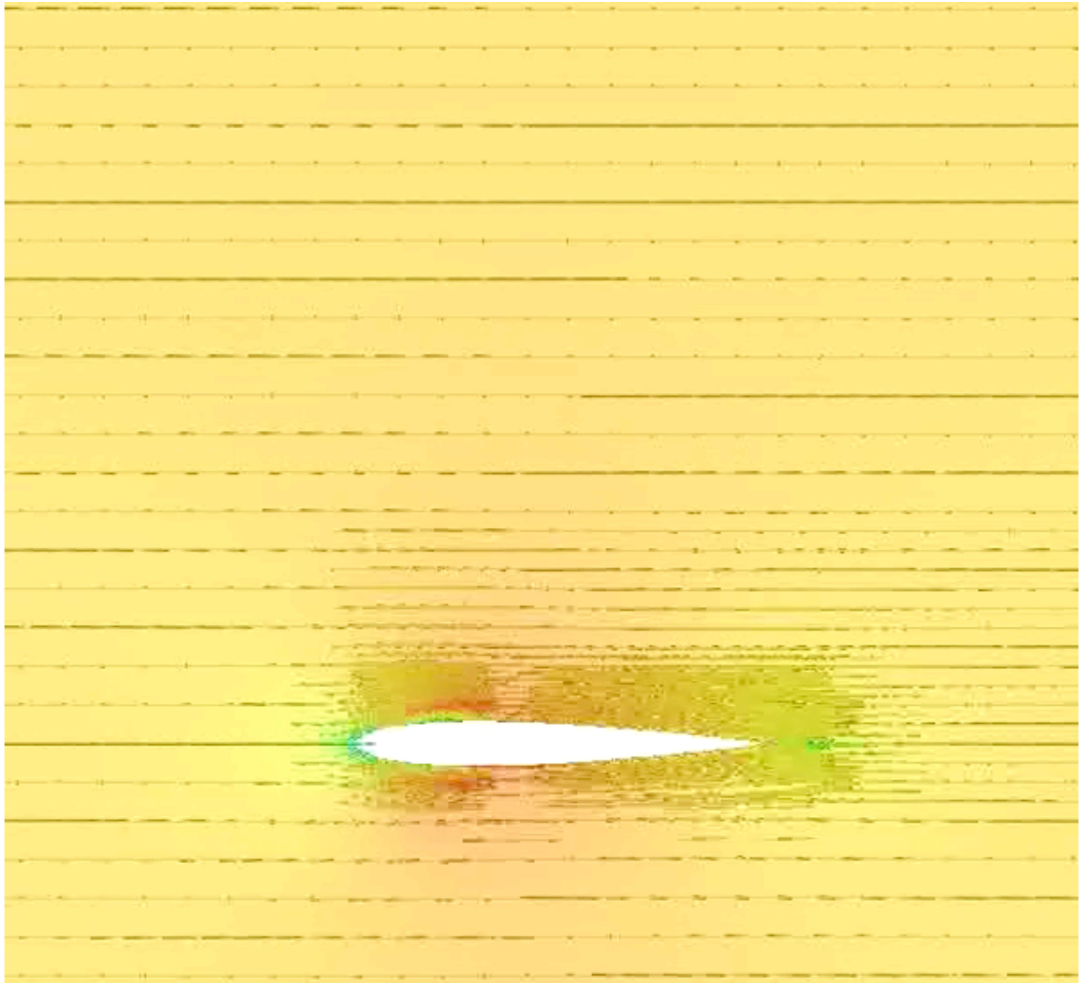
Turbulent boundary layer separation (separation with slip bc)



Turbulent boundary layer separation (separation with slip bc)

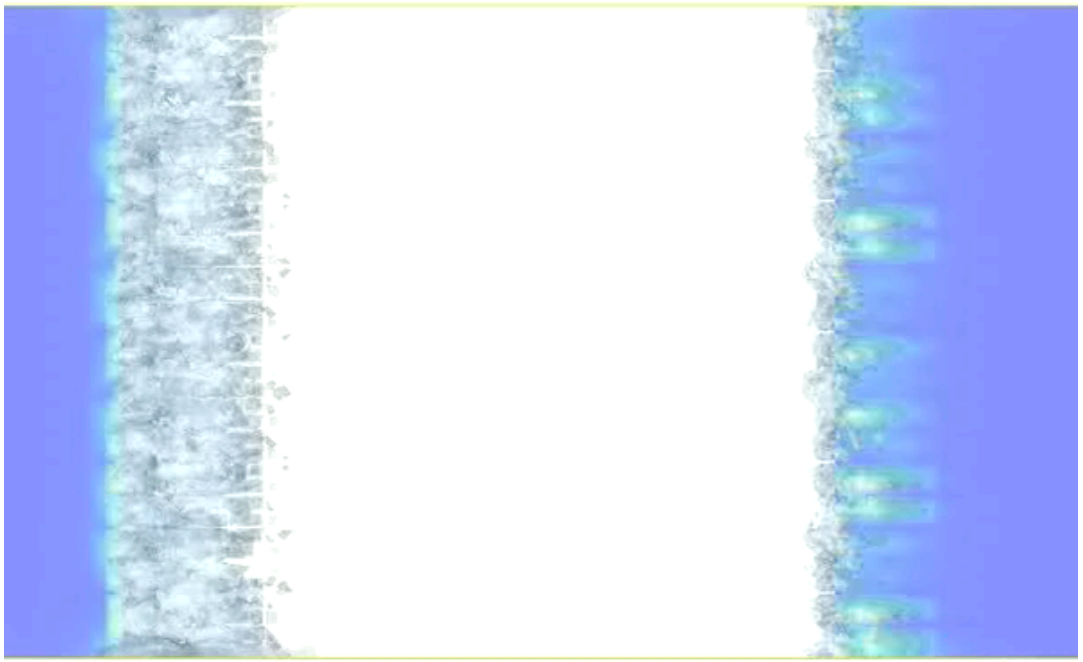


NACA 0012 take-off: velocity

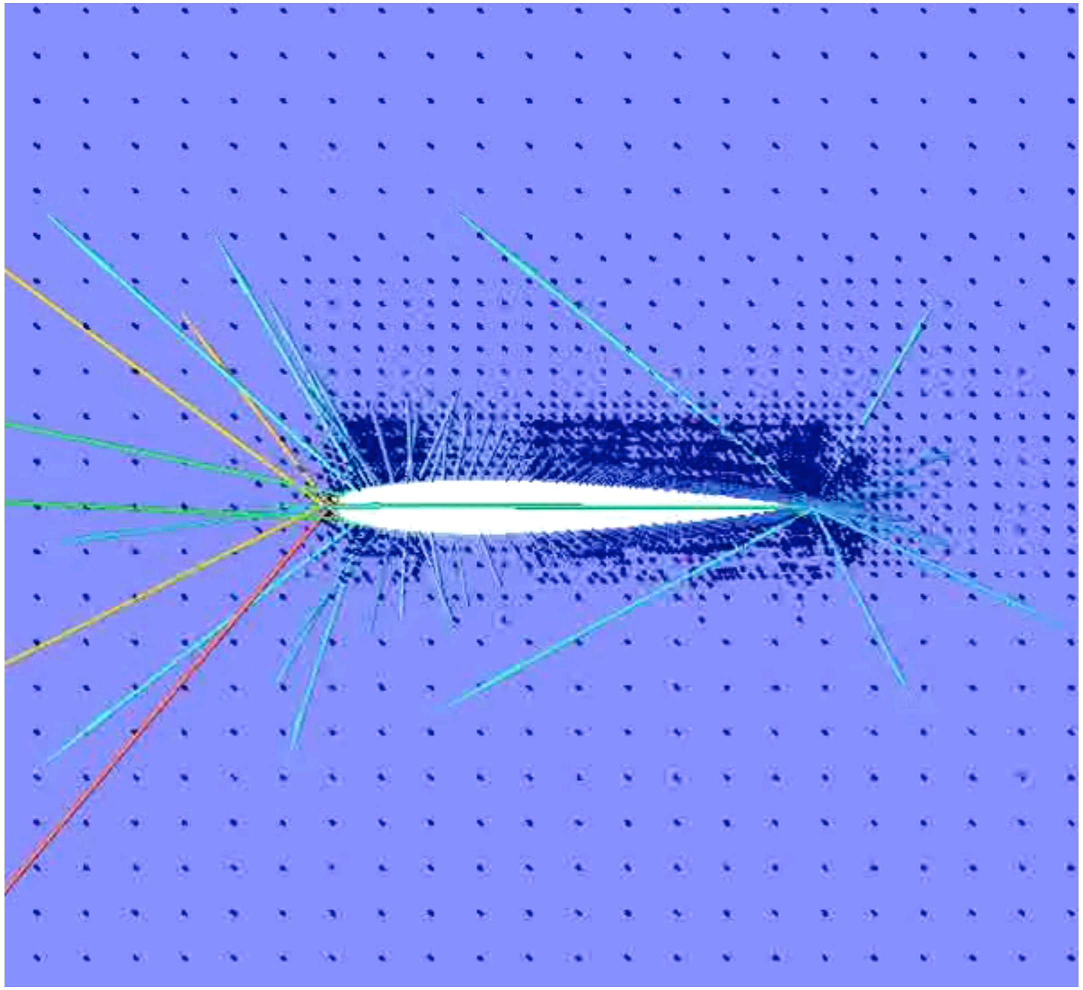


NACA 0012 take-off: pressure

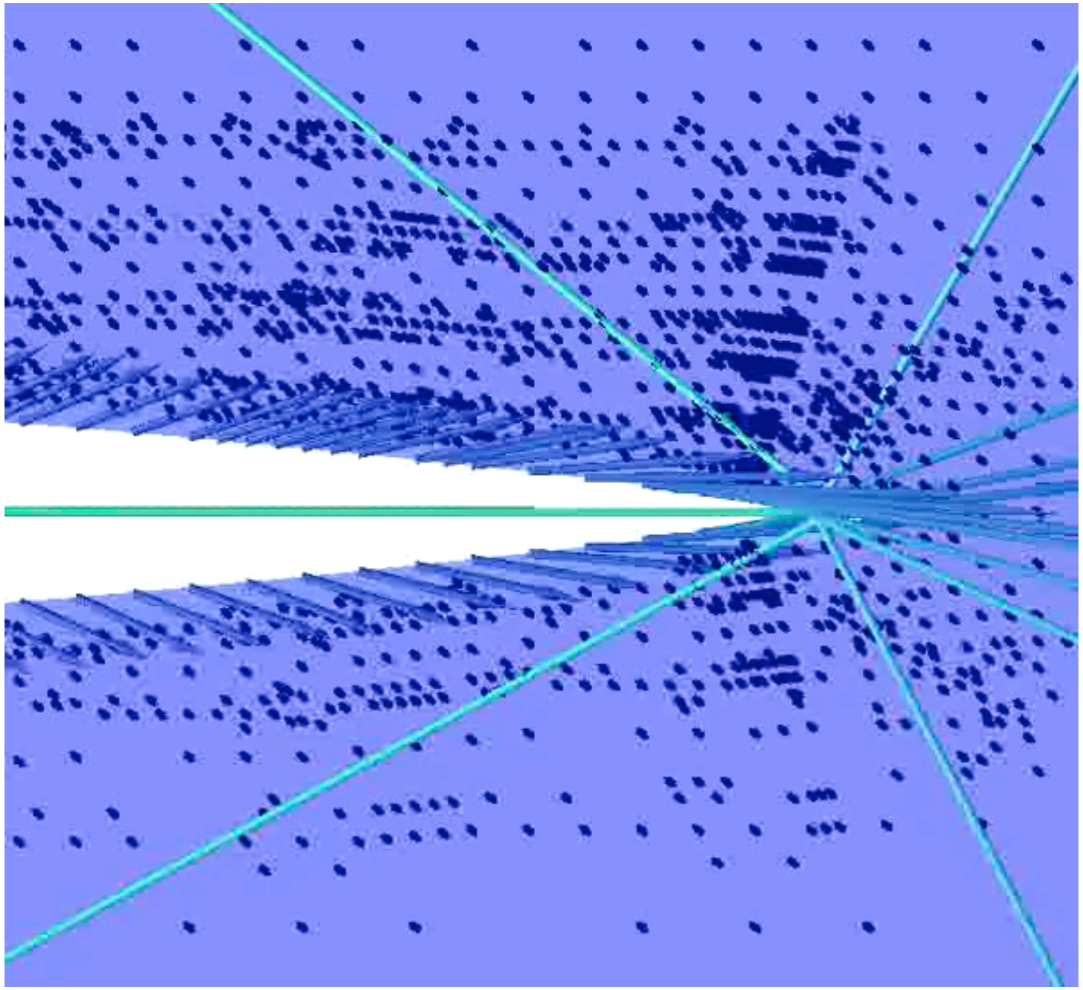
NACA 0012 take-off: vorticity



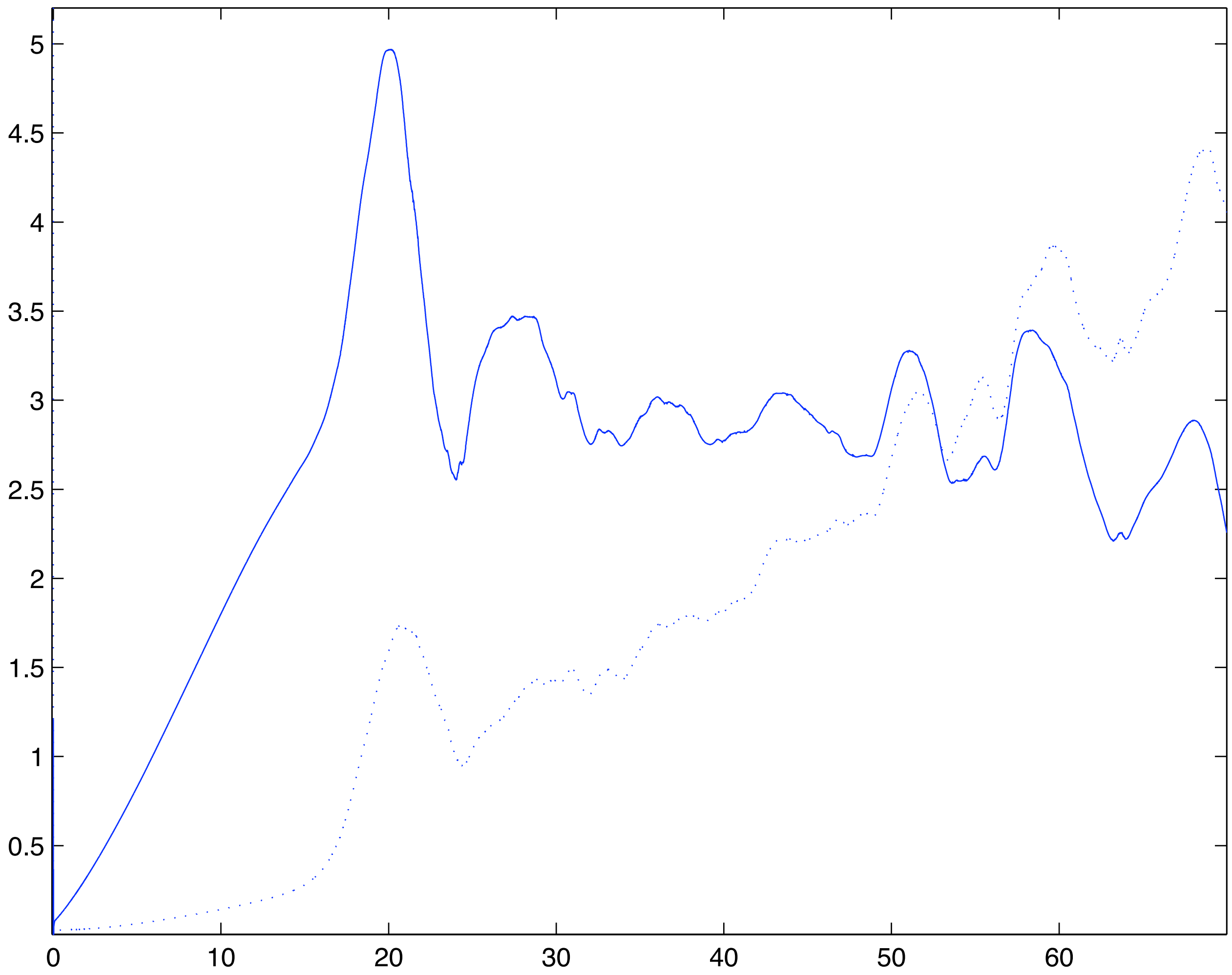
NACA 0012 take-off: $\nabla p \cdot n$



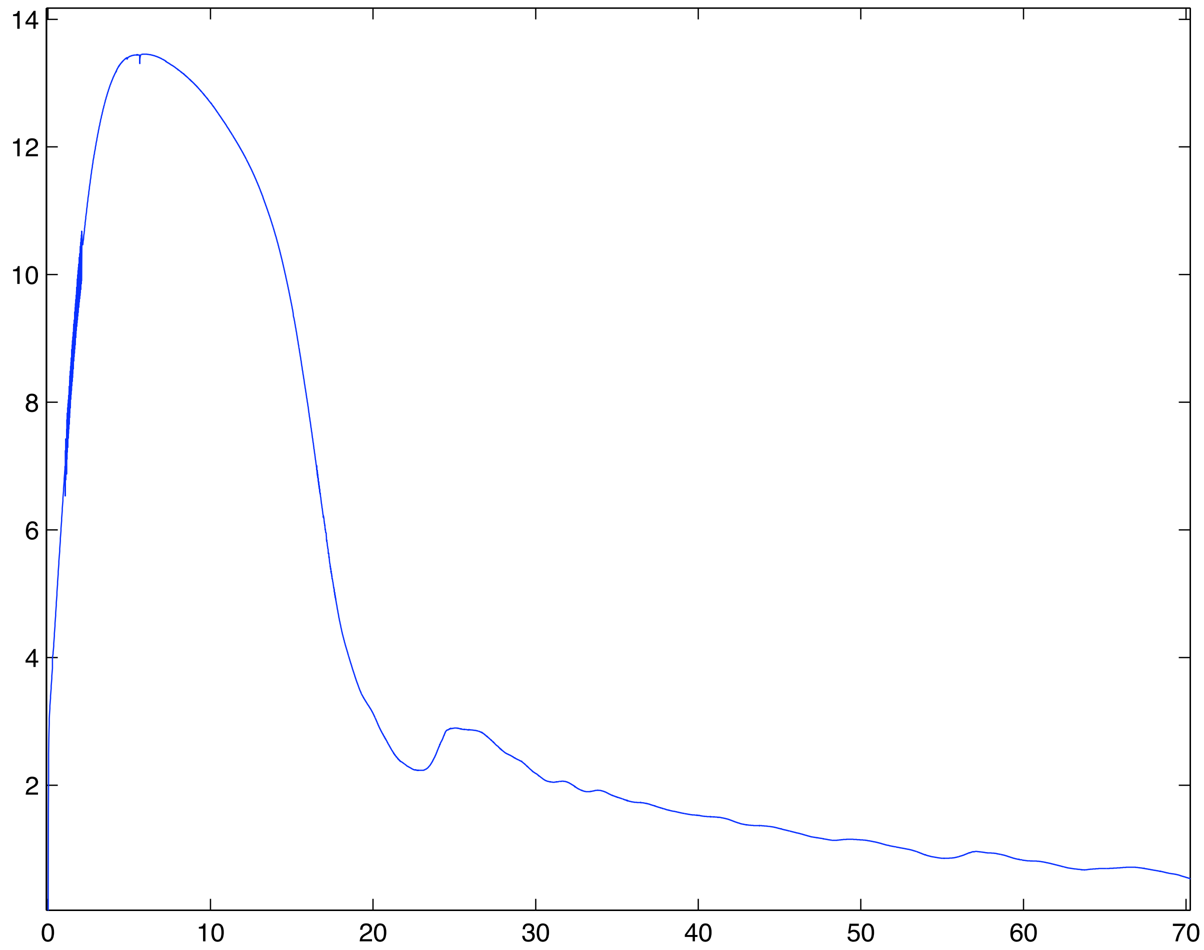
NACA 0012 take-off: $\nabla p \cdot n$



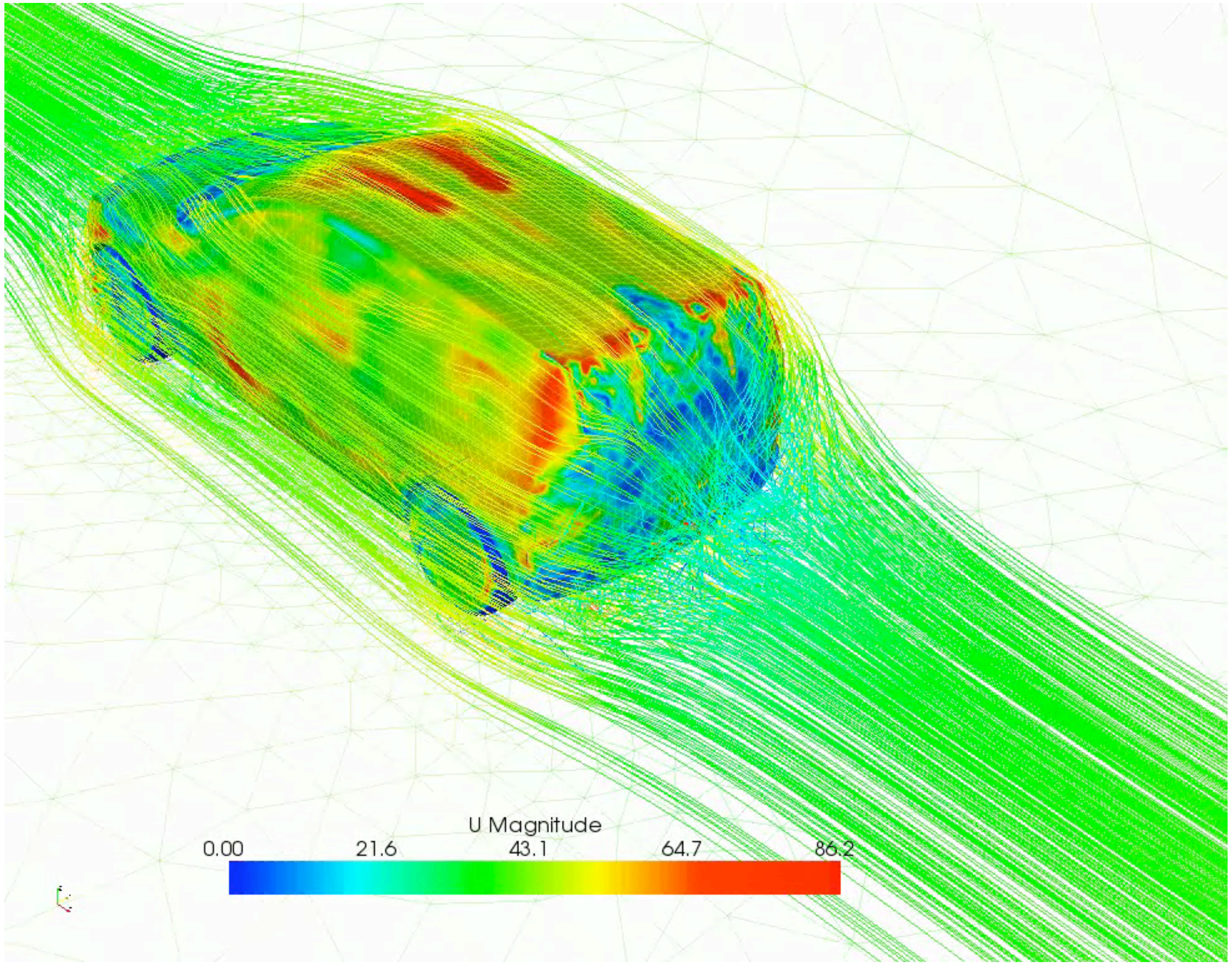
c_L and c_D vs angle of attack (consistent with experimental results)



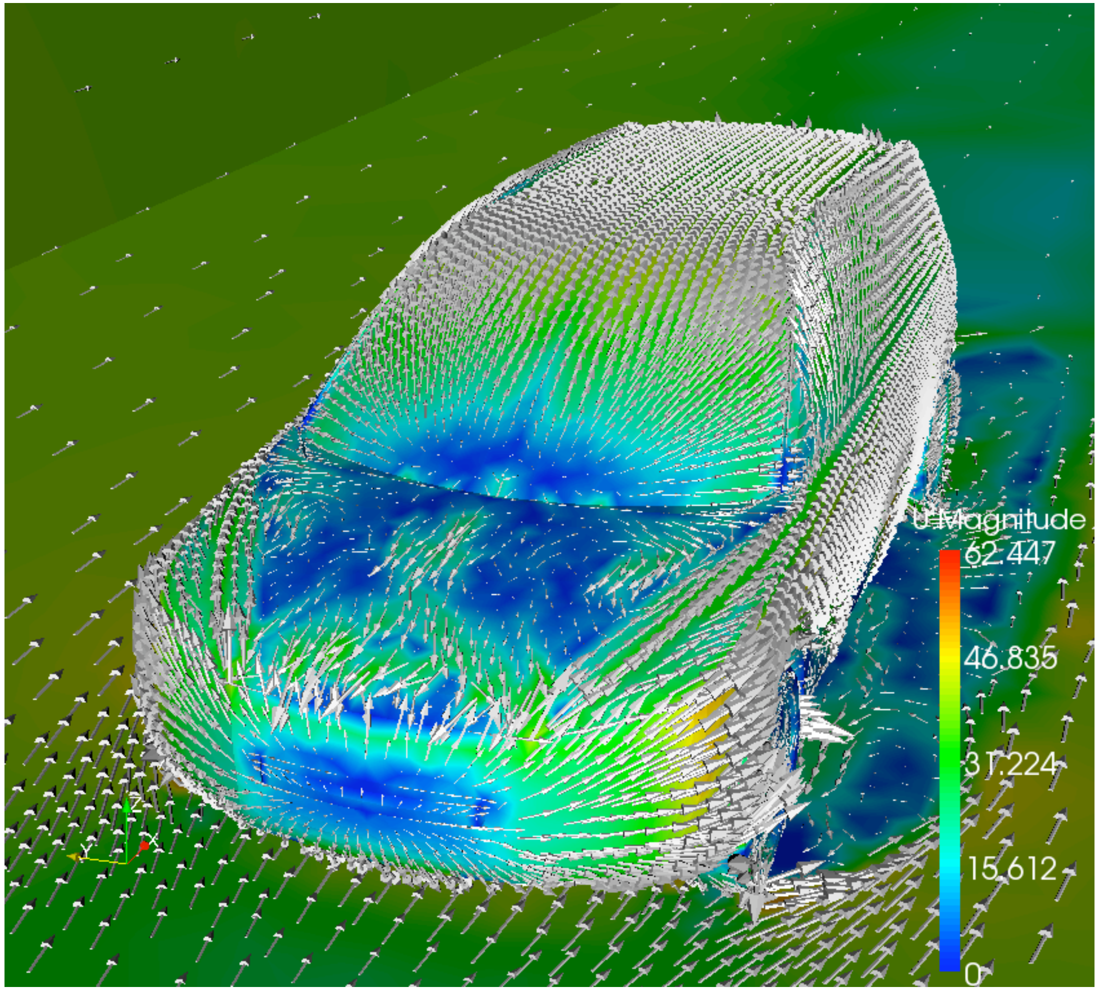
c_L/c_D vs angle of attack (consistent with experimental results)



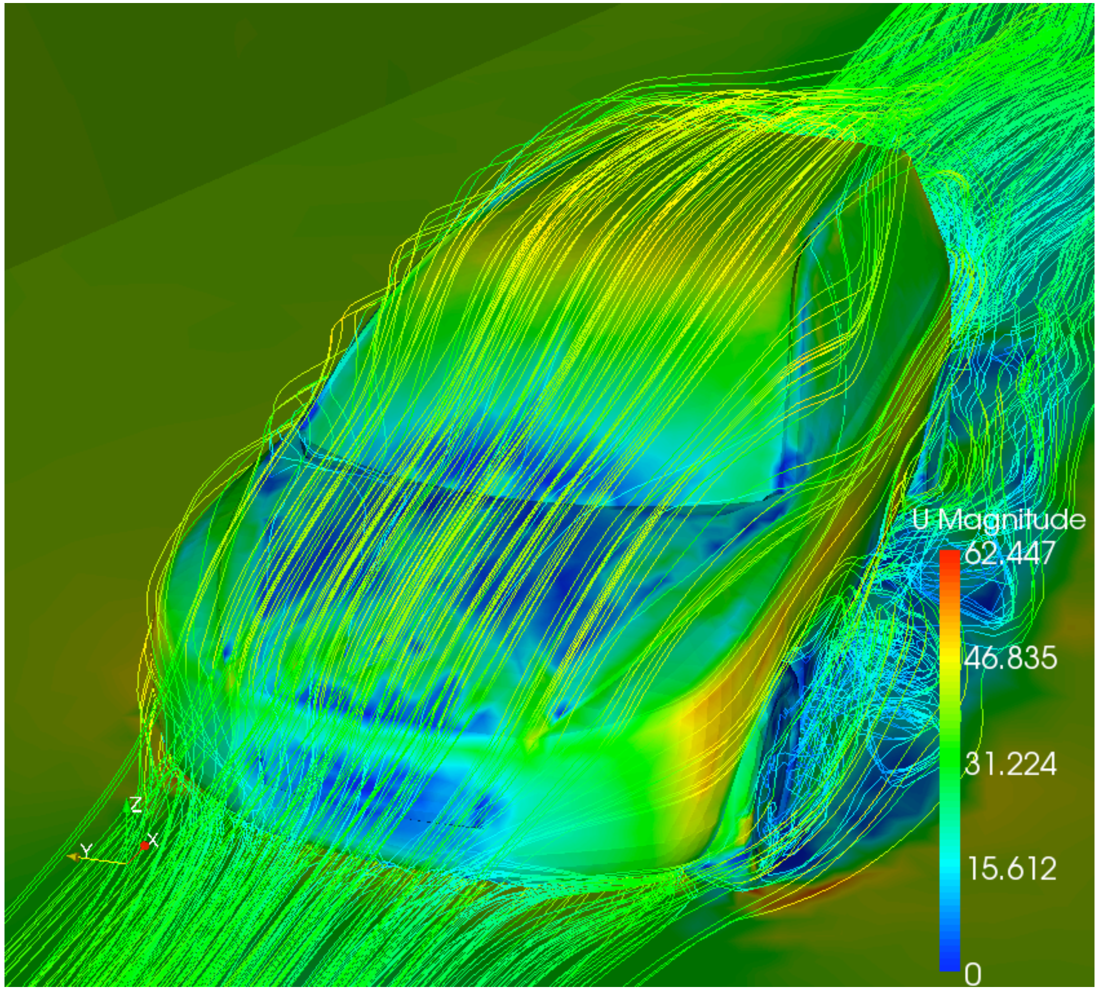
Turbulent flow past car (geometry by Volvo Cars)



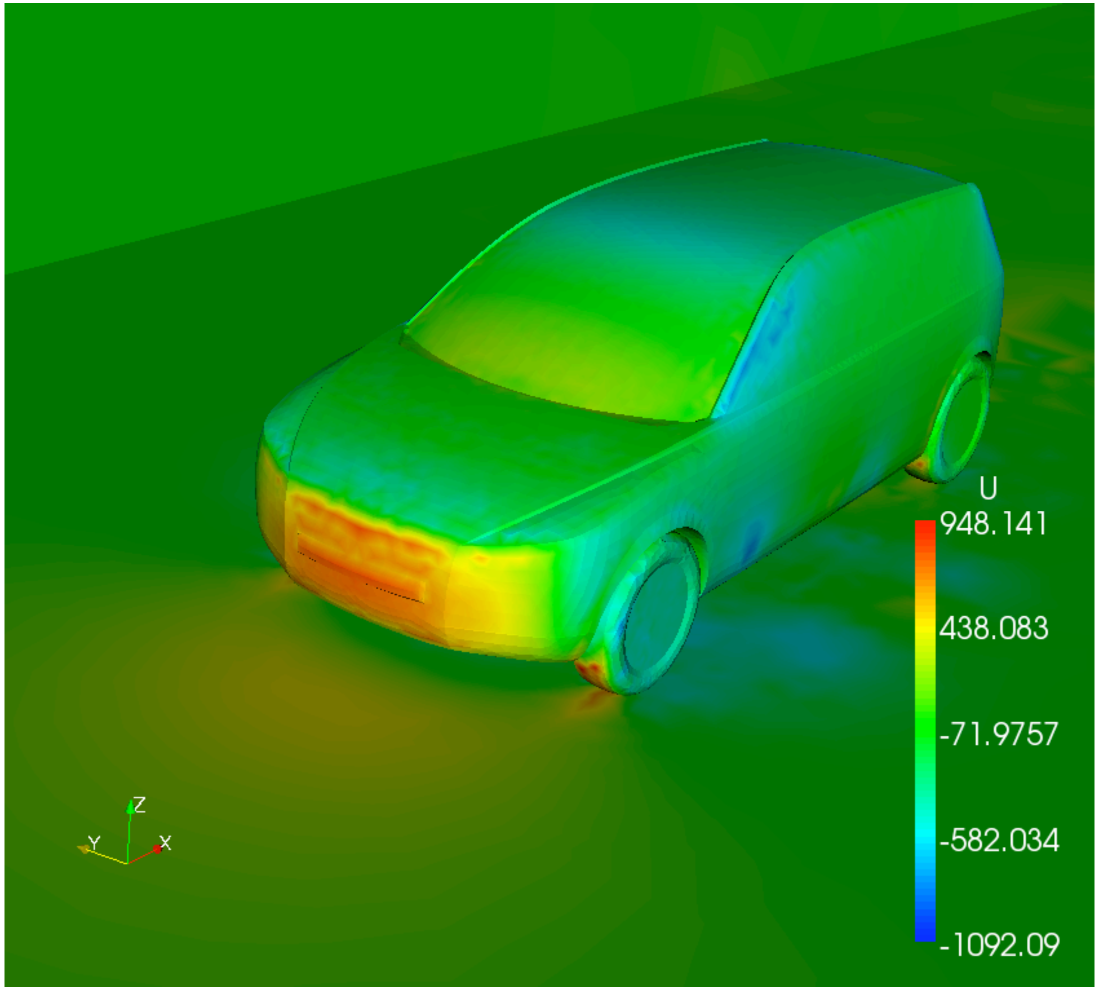
Velocity



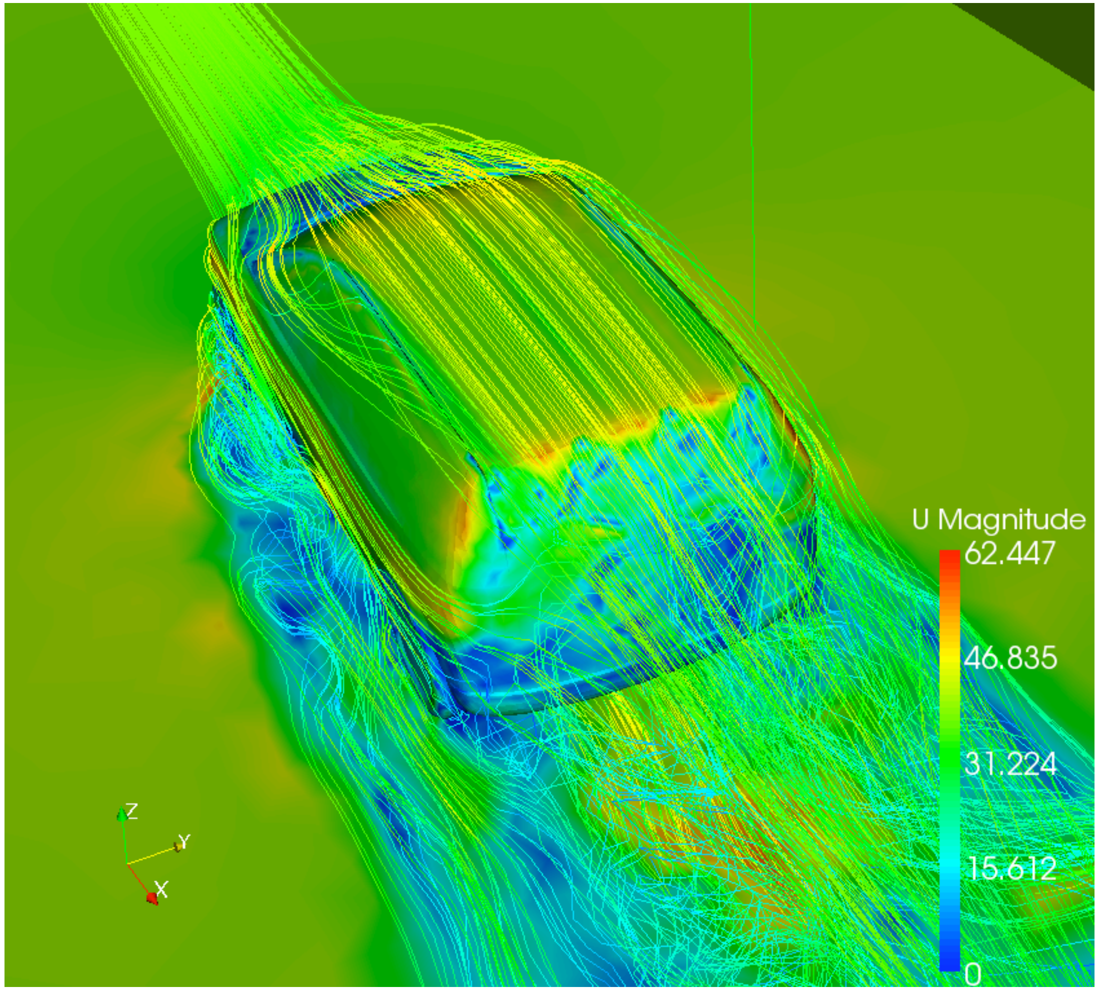
Streamlines



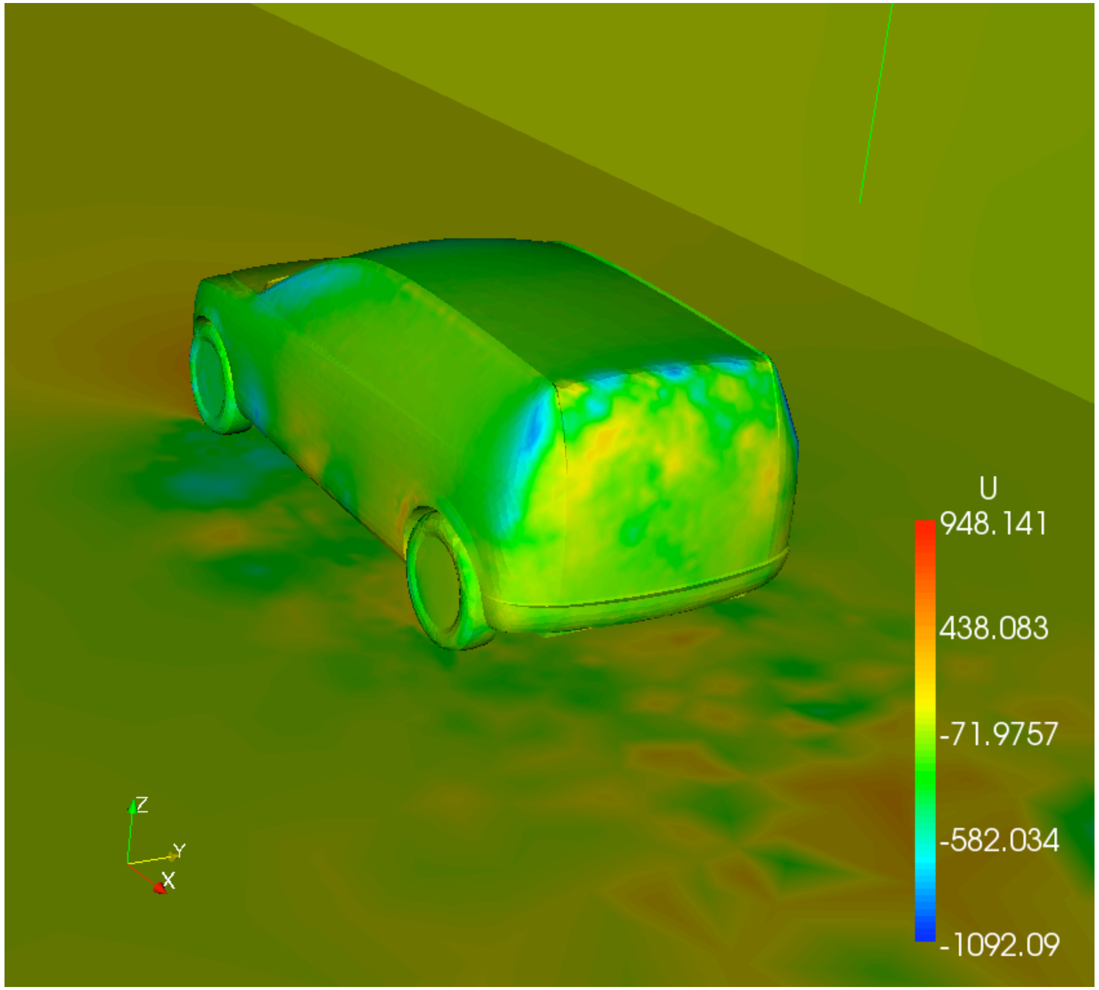
Pressure



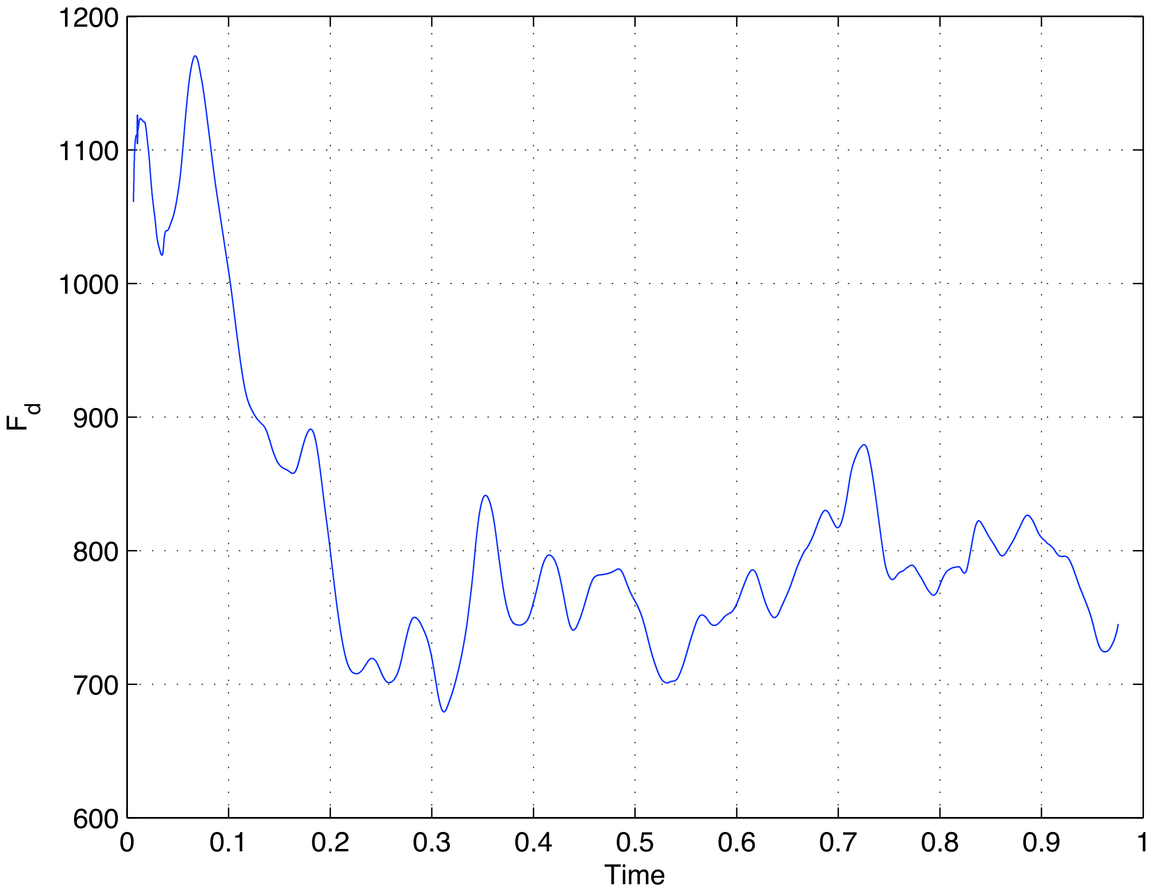
Streamlines



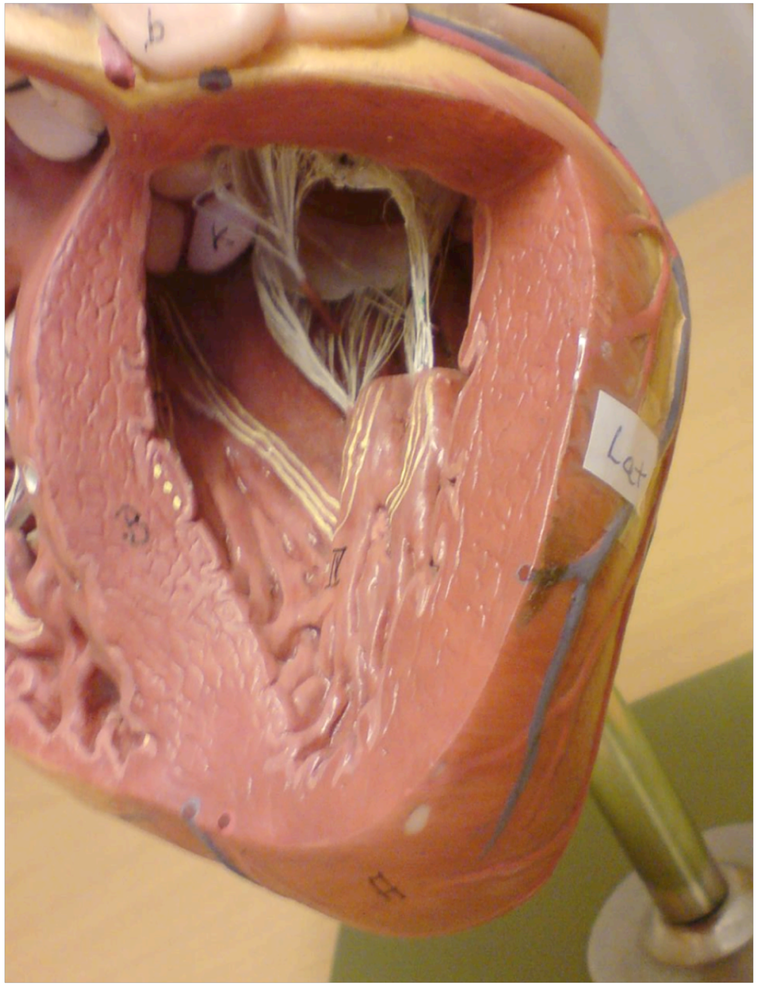
Pressure

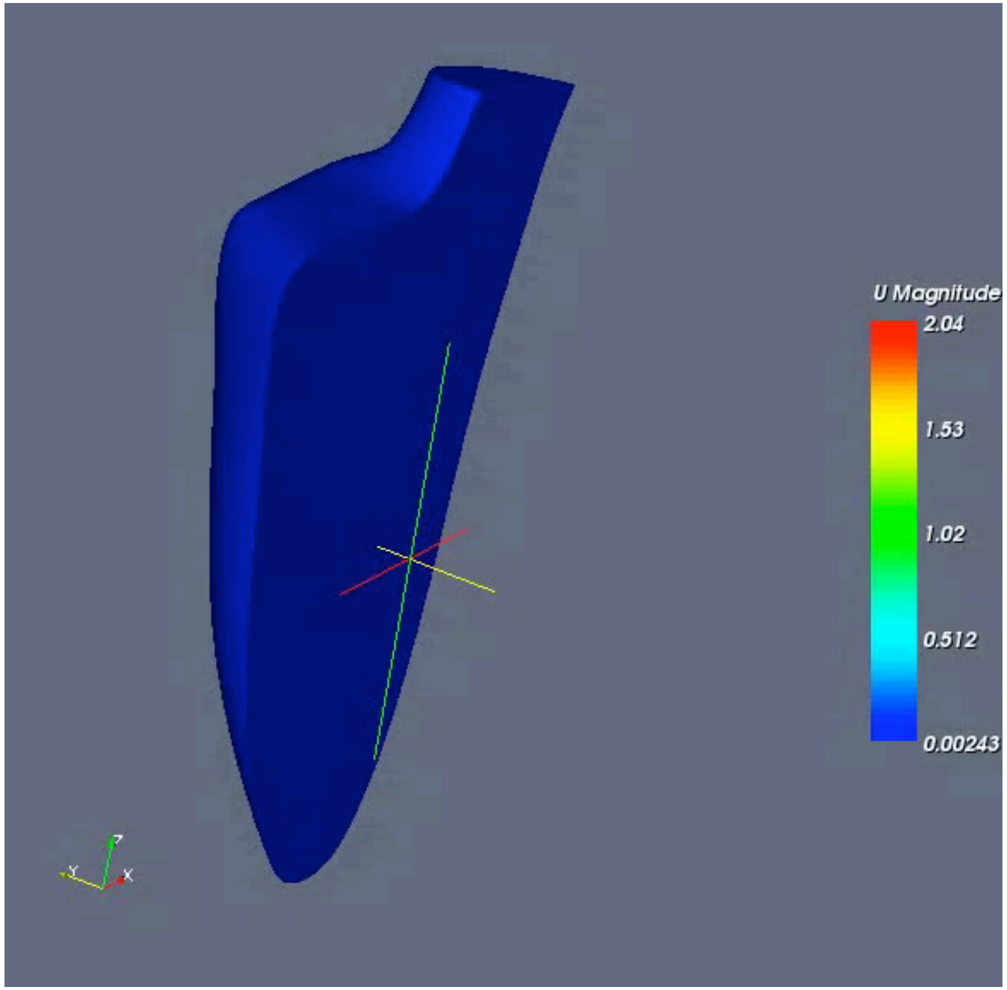


Drag force car
(consistent with experimental results)



Biomedical computing - moving mesh ALE
(with LiU/UMU – Mats Larson, Per Vesterlund)





Summary

- o General Galerkin (G2) computational method for turbulent flow, with adaptive mesh refinement and a posteriori error control
- No explicit turbulence modeling
- Detection of (global) blowup on finite meshes
- Resolution of d'Alembert Paradox
- Turbulent separation with skin friction bc

Ongoing work

- Adaptive turbulent compressible flow
- Adaptive fluid-structure interaction
- Improved efficiency: parallel adaptivity
- Acoustics

Mixer: engine exhaust system (with Swenox AB and KTH MWL)

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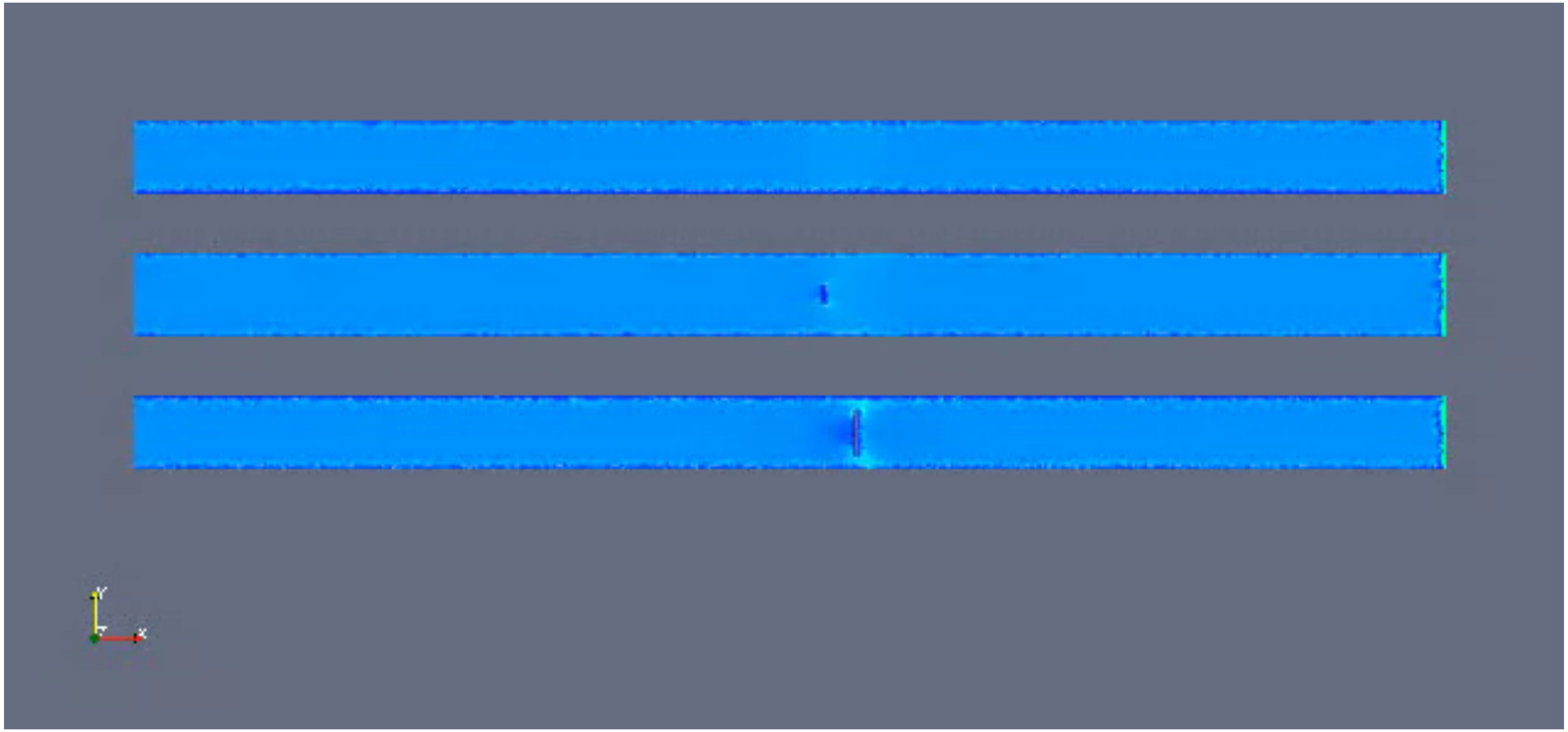
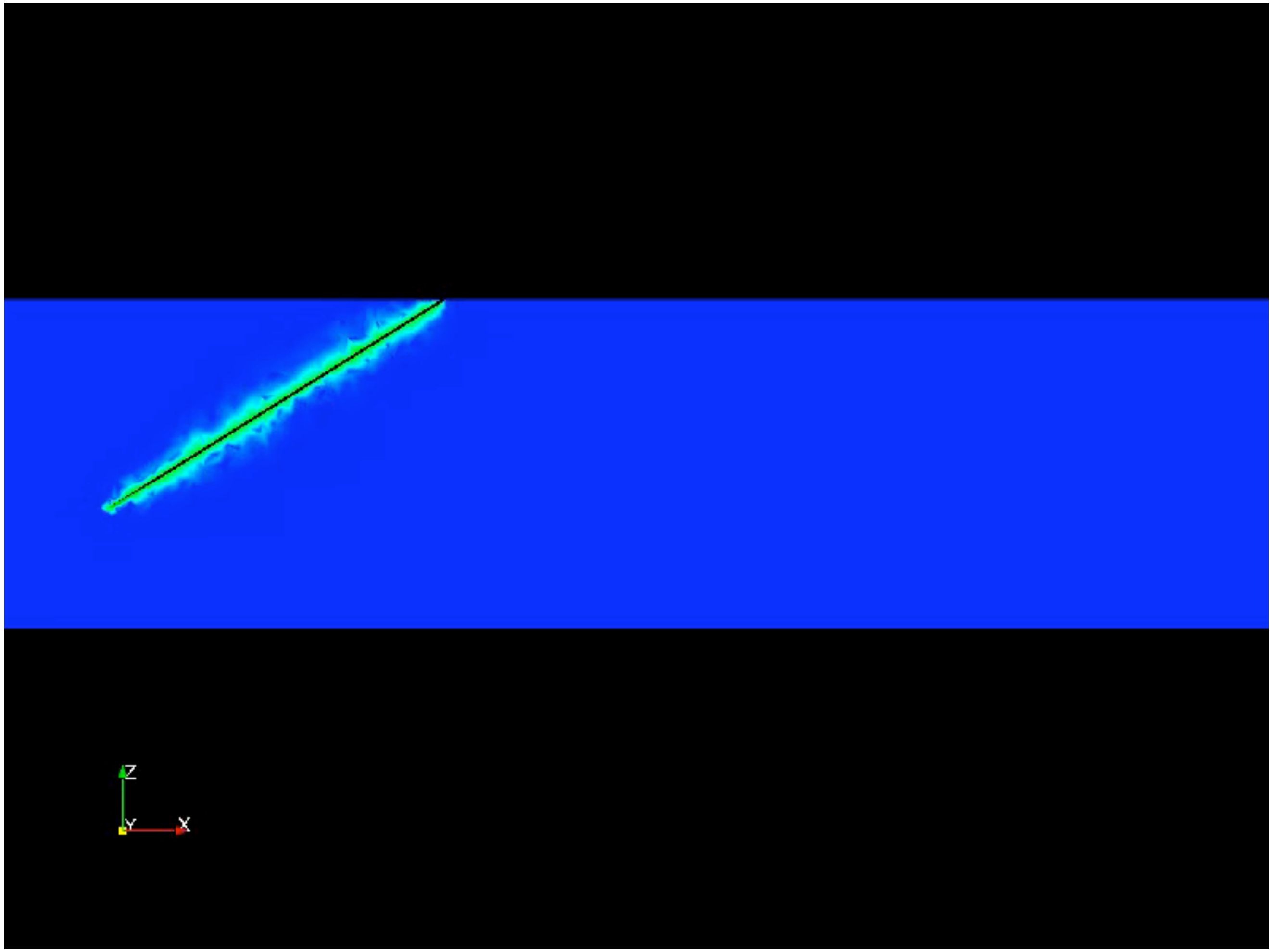
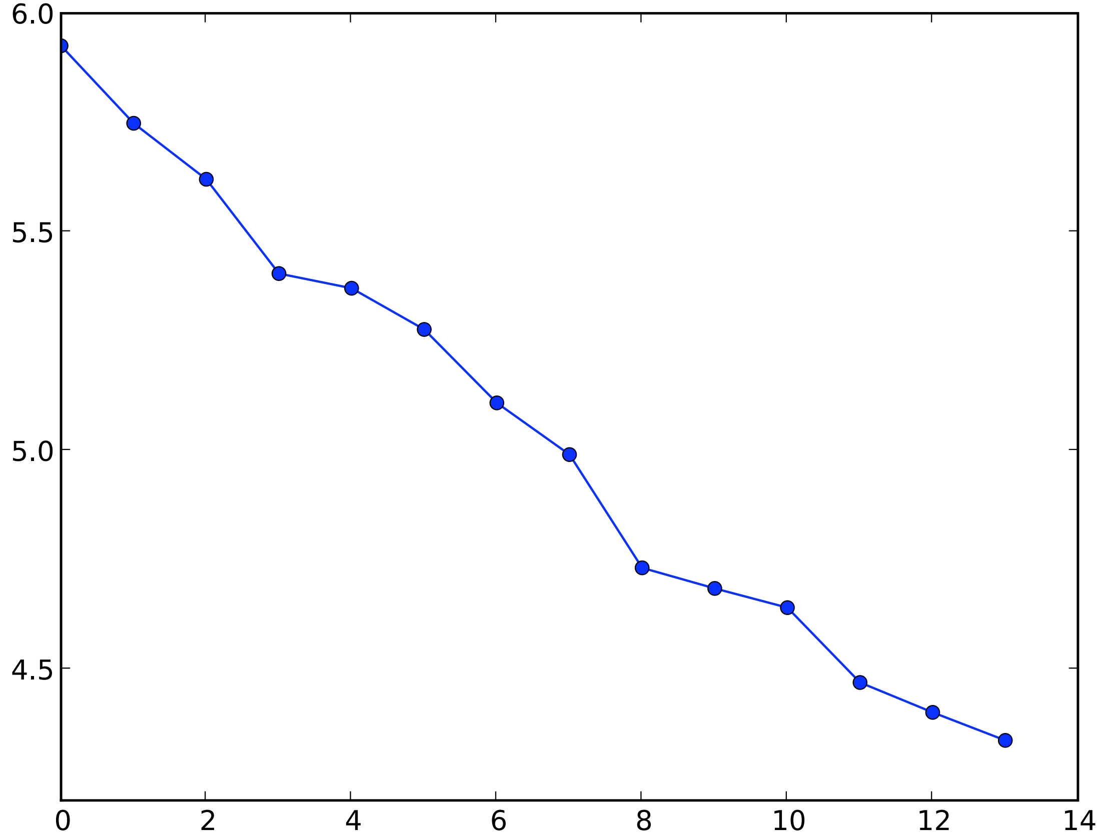


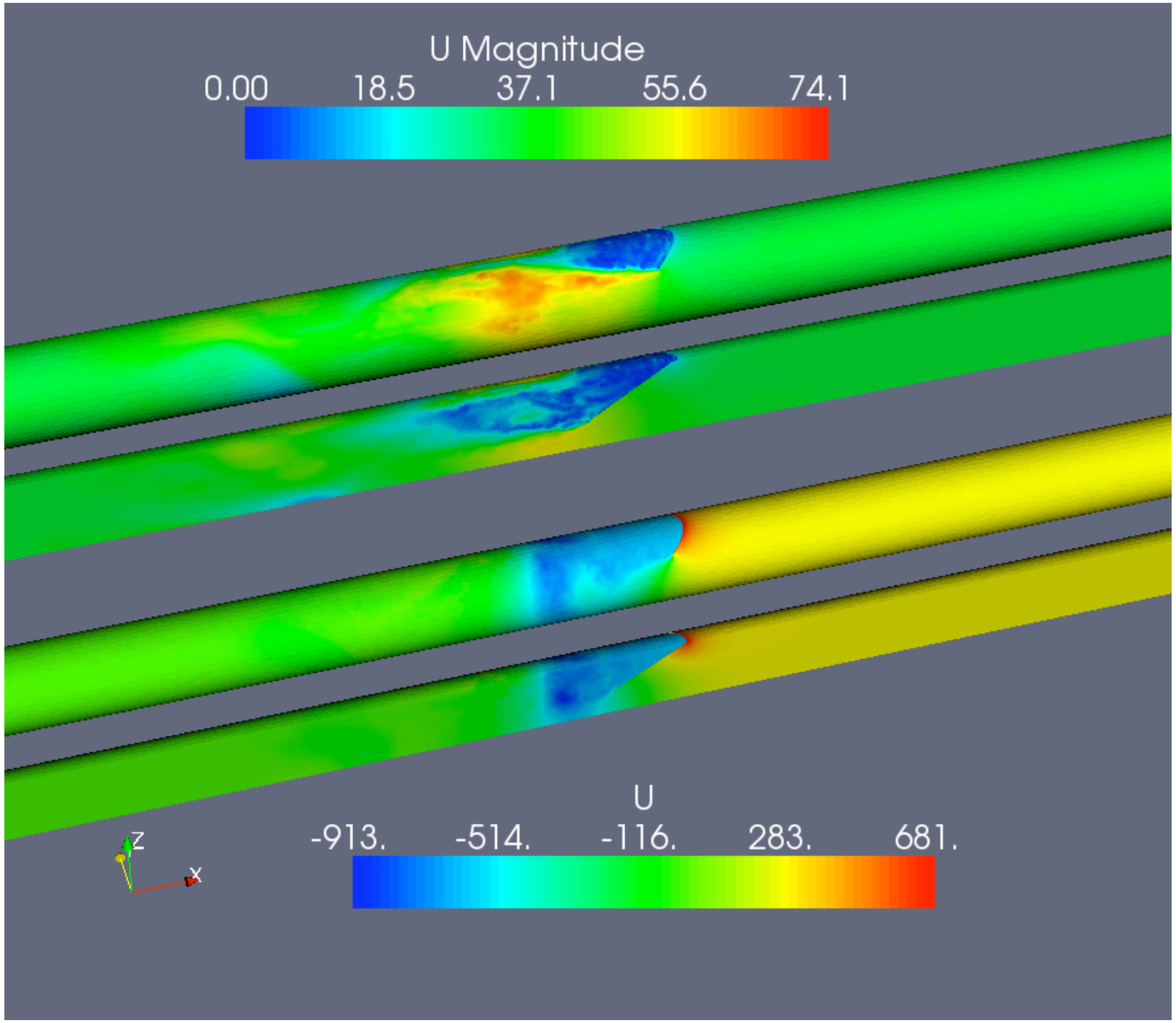
Dual solution for drag



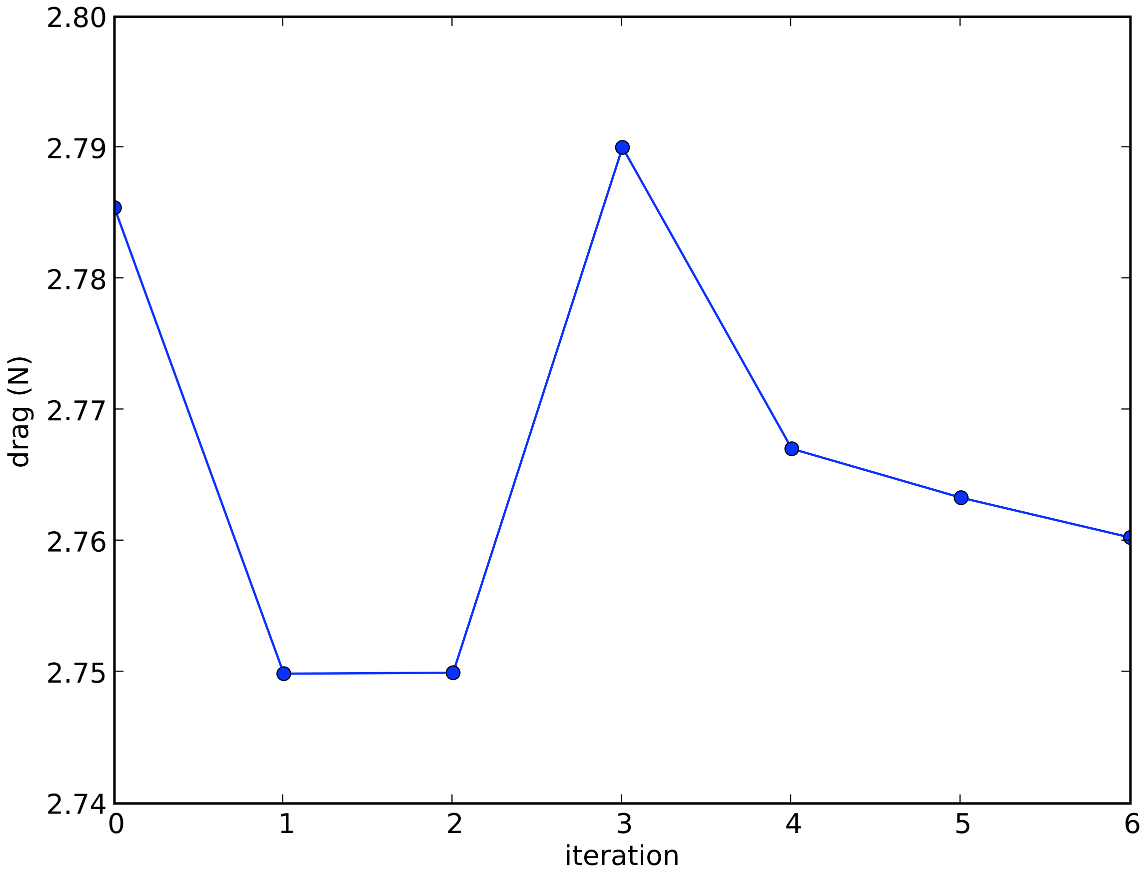
Drag convergence: no slip b.c.
(slow due to turbulent boundary layers)



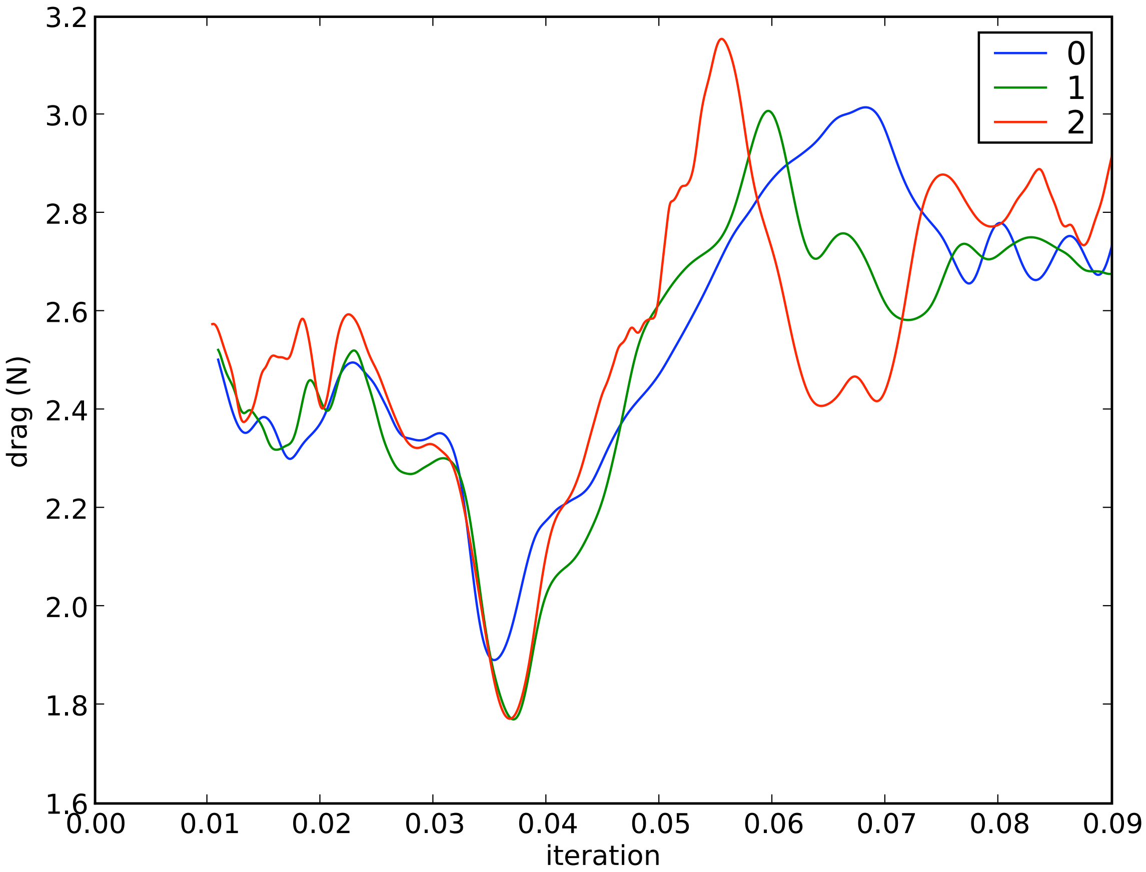
Mixer: slip b.c.



Drag convergence: slip b.c. (35 399 vertices: no boundary layers)



Time series for drag: 3 meshes (consistent with experimental results)



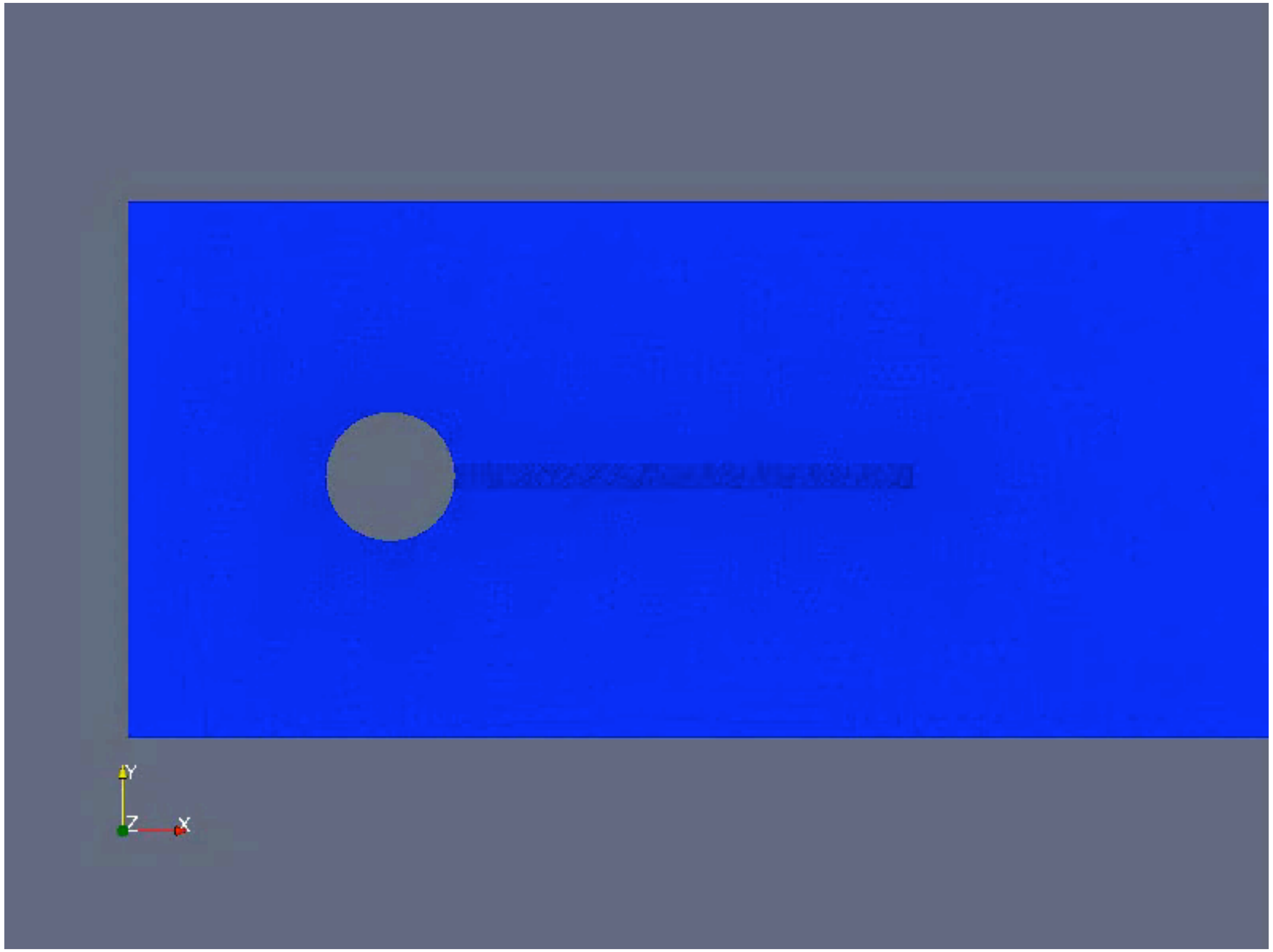
Unified continuum FSI

- Unified continuum ALE fluid-structure interaction
- High stability compared to domain decomposition: global coupling by conservation of momentum (consistent results by Hron/Turek's Monolitic method)

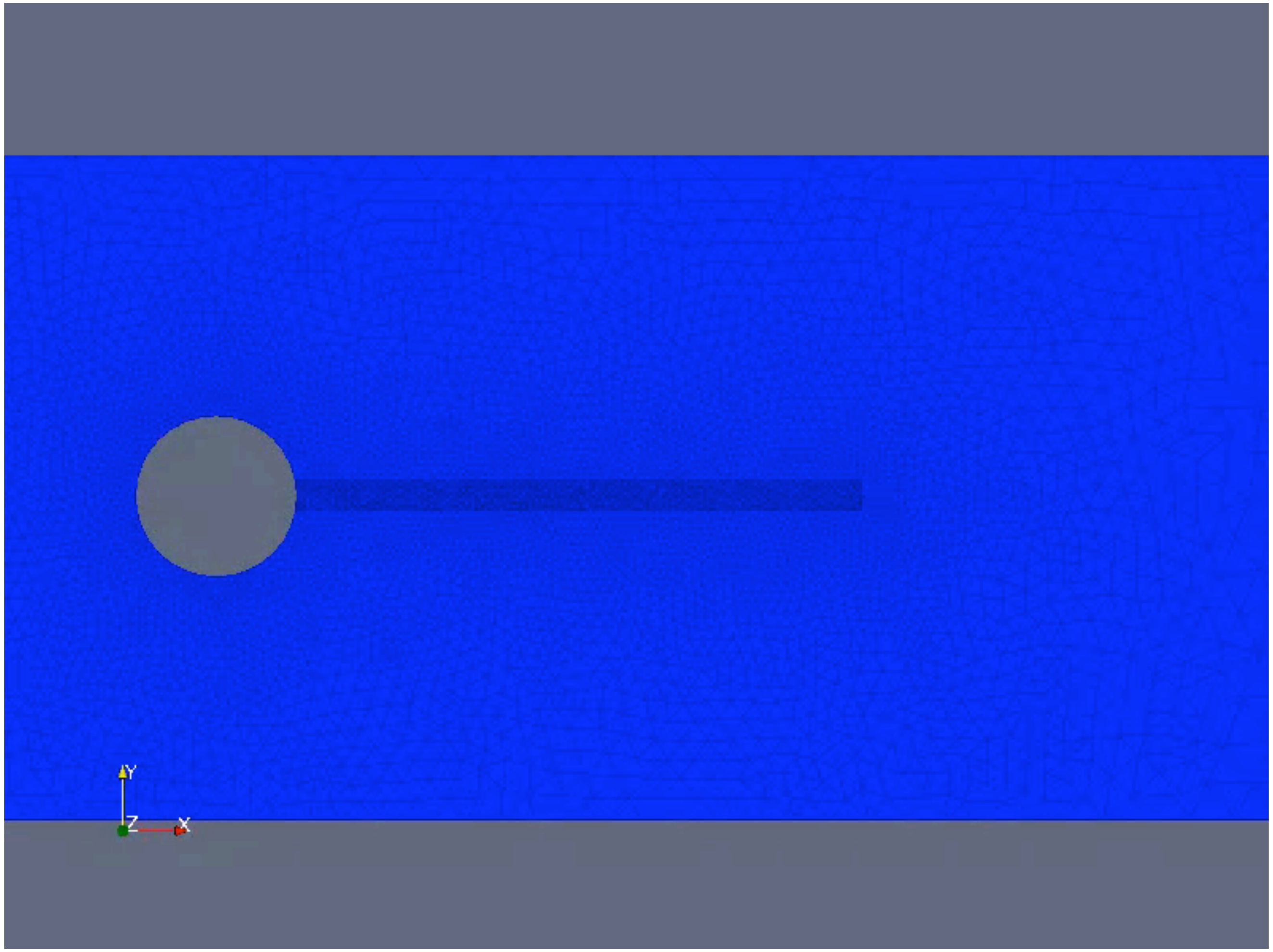
Ex: incompressible fluid - elastic incompressible structure

- Newtonian fluid: $\sigma = 2\mu\varepsilon(u) - pI$
- Solid: $\sigma = \sigma_D - pI$, $\sigma_D = G(F)$ (Hookes law: $D_t\sigma_D = E\ \varepsilon(u)$)
- Incompressibility by pressure law: $(\delta\nabla p,\nabla q) = (\nabla\cdot u,q)$

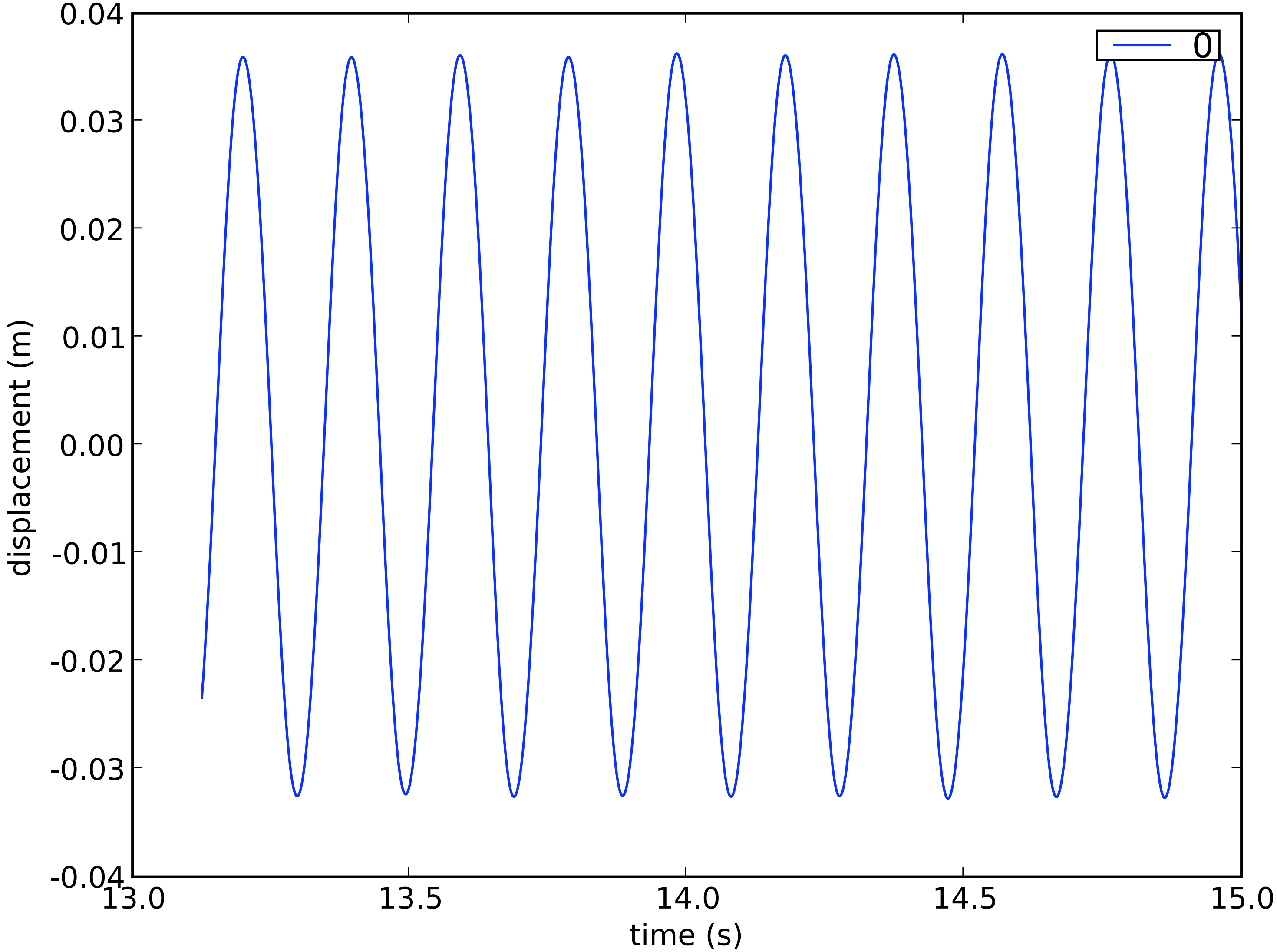
Hron/Turek benchmark 2
(5415 vertices)



Hron/Turek benchmark 3
(5415 vertices)



Bmk 3: consistent displacement



3d FSI: elastic mixer (Swenox/MWL)

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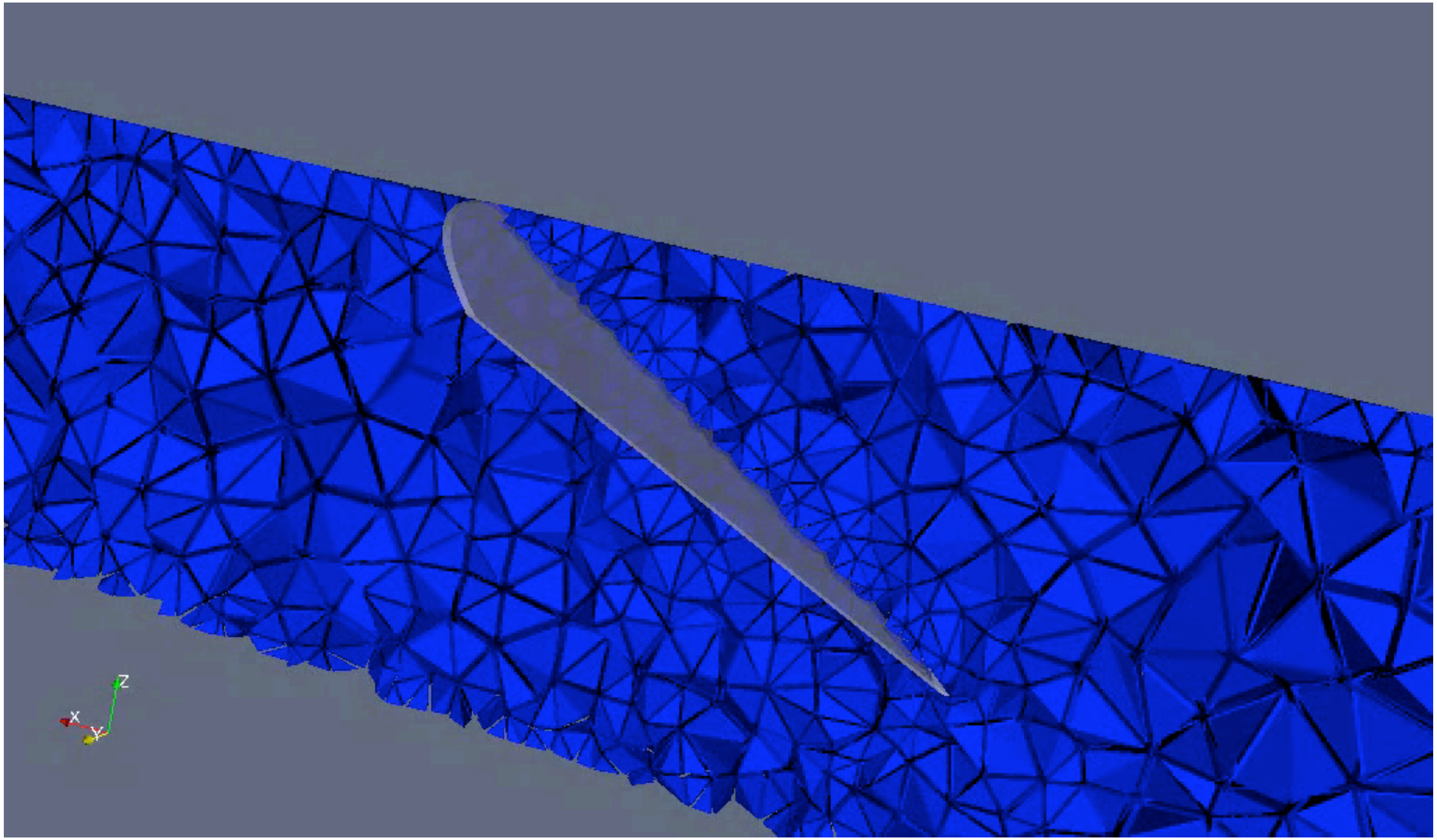
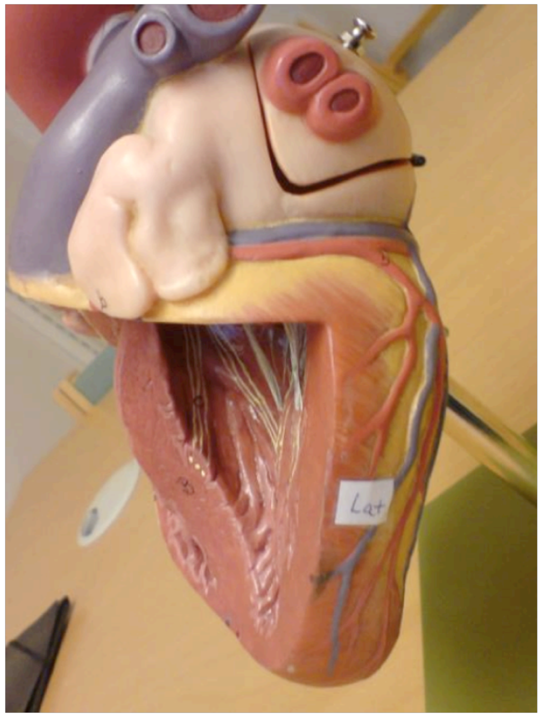


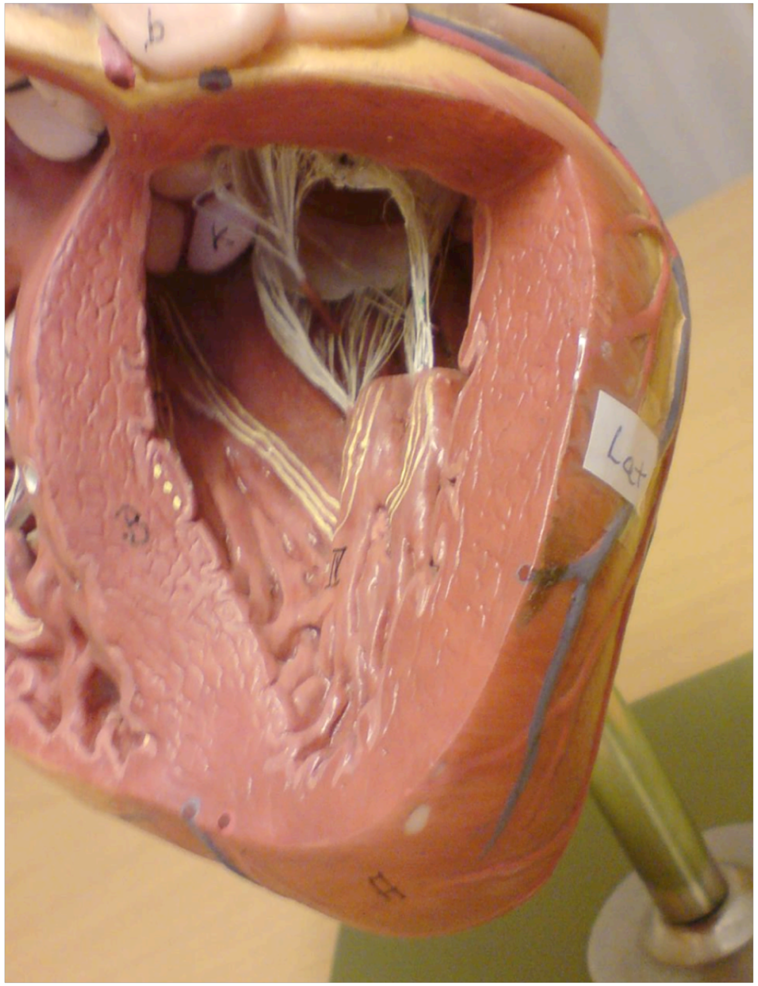
Biomedical computing

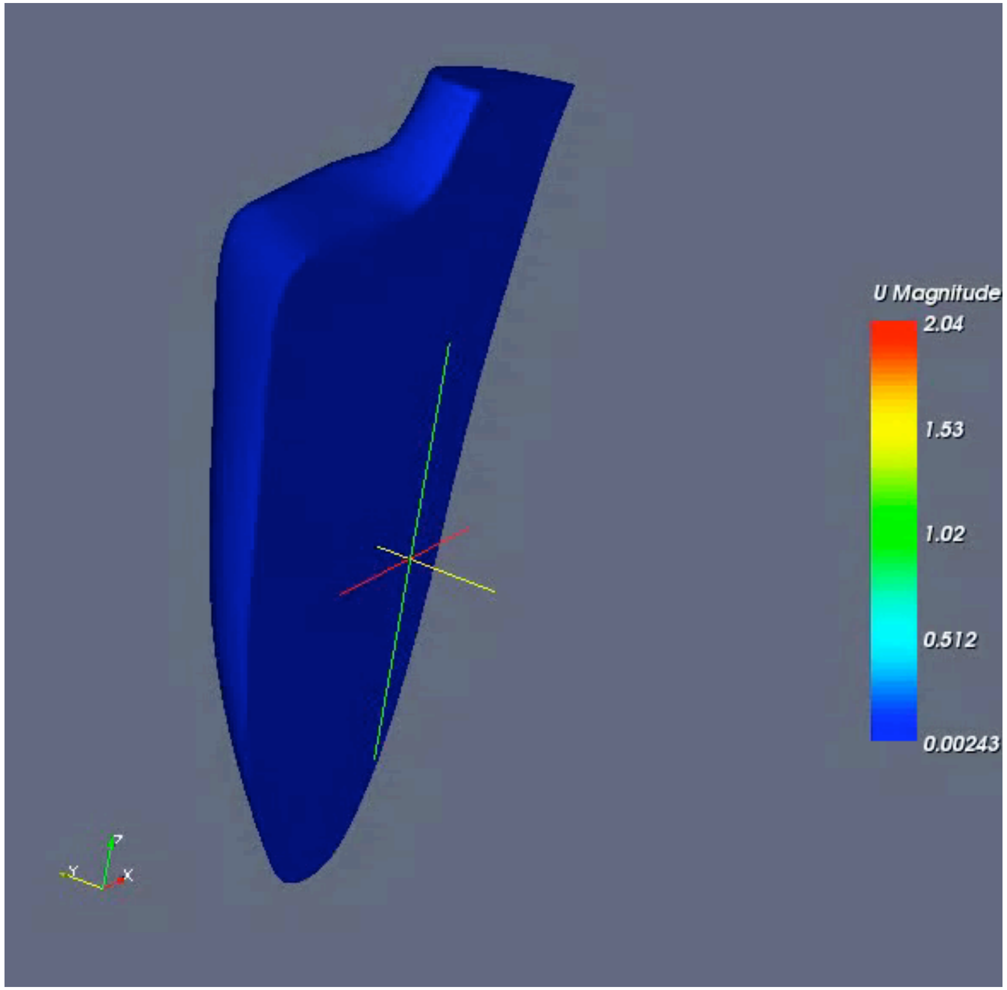
- Matthias Aechtner, msc (2008)
- Joint project: KTH, UmU, LiU (Mats Larson, Per Vesterlund,…)



- Scanned patient specific geometry
- Modeling of human heart fluid mechanics by moving mesh simulation

Biomedical computing





Implementation: Unicorn (2007)

- Open source software at fenics.org (dep. FIAT/FFC/Dolfin)
- Incompressible/compressible, laminar/turbulent, high/low Mach number, multiphase/fluid-structure interaction,...
- Tetrahedral meshes
- General friction/penetration type boundary conditions
- ALE G2 discretization with mesh smoothing (elastic analogy)
- Adaptivity based on a posteriori error estimation with duality
- Distributed parallel computing (MPI, PARMETIS/SCOTCH)
- Discrete solver: Newton + GMRES for conservation laws (PETSc), AMG for incompressible pressure law (Hypre)

Development process: different solvers being merged into one