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Christian-Albrechts-Universität zu Kiel
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Computational Sciences Center

Optimal control in fluid mechanics by finite elements with symmetric stabilization

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Mathematisches Seminar
Christian-Albrechts-Universität zu Kiel

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Menu

Menu

1. Motivation

Optimization problem

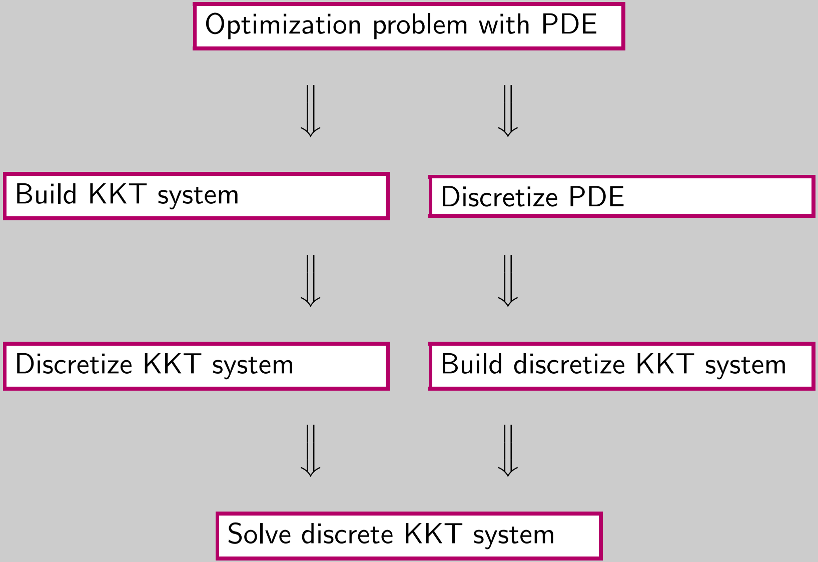
$\Downarrow$

Build Karush-Kuhn-Tucker (KKT) system

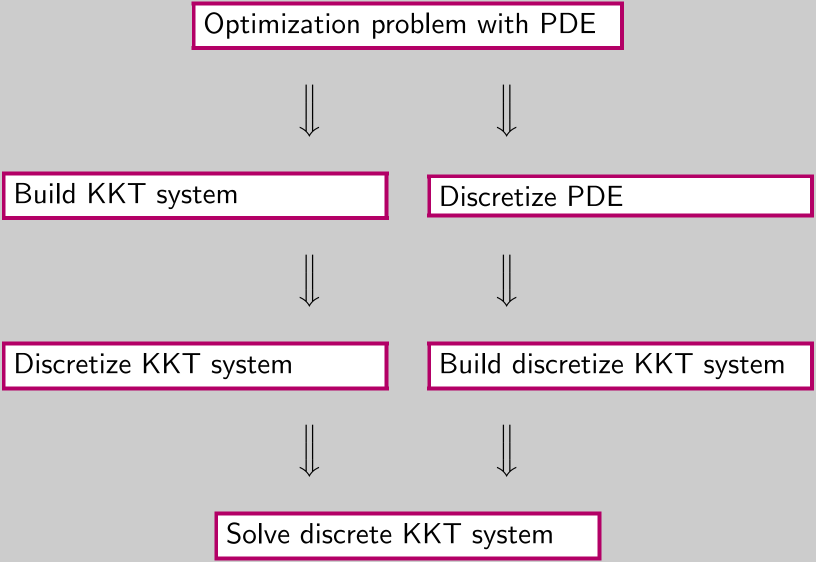
$\Downarrow$

Solve KKT system

Two possibilities for optimization with PDE



Two possibilities for optimization with PDE



Optimize-discretize **Discretize-optimize**

Model problem: Linearized Navier-Stokes with control q

$$\begin{aligned} -\mu\Delta v + (\beta\cdot\nabla)v + \sigma v + \nabla p + Bq &= f \quad \text{in } \Omega\,, \\ \operatorname{div} v &= 0 \quad \text{in } \Omega\,, \\ v &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

Objective functional:

$$J(u,q) \;:=\; \frac{1}{2}\|Cu - C\widehat{u}\|^2 + \frac{\alpha}{2}\|q\|^2 \;\rightarrow\; \min!$$

$C\widehat{u} = \widehat{v}$ observationes

Linear flow problem:

$$Au + Bq \quad = \quad f$$

state variable $u = (v, p)$, and control q

Optimal control problem:

$$\arg\min \Big\{ J(u, q) : Au + Bq = f \text{ for control } q \in Q \Big\}.$$

Augmented Lagrangian

$$L(u,q,z) \quad := \quad J(u,q) + \langle z, Au + Bq - f \rangle$$

Unrestricted minimization problem

$$min_{u,q,z} L(u,q,z)$$

Necessary conditions for saddle point of L

$$\begin{aligned} d_q L(u,q,z) = 0 \quad &\Longleftrightarrow \quad d_q J(u,q) + B^* z = 0 \\ d_u L(u,q,z) = 0 \quad &\Longleftrightarrow \quad d_u J(u,q) + A^* z = 0 \\ d_z L(u,q,\lambda) = 0 \quad &\Longleftrightarrow \quad Au + Bq \qquad\quad = f \end{aligned}$$

Continuous Karush-Kuhn-Tucker (KKT) system

$$\begin{pmatrix} \alpha I & 0 & B^* \\ 0 & C & A^* \\ B & A & 0 \end{pmatrix} \begin{pmatrix} q \\ u \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ C\widehat{u} \\ f \end{pmatrix}$$

What is an appropriate discretization of...

Primal equation

$$
\begin{aligned}
-\mu\Delta v + (\beta\cdot\nabla)v + \sigma v + \nabla p + Bq &= f \quad \text{in } \Omega\\
\operatorname{div} v &= 0 \quad \text{in } \Omega\\
v &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Adjoint equation

$$
\begin{aligned}
-\mu\Delta z_v - (\beta\cdot\nabla)z_v + \sigma z_v - \nabla z_p &= \widehat{v} - v \quad \text{in } \Omega\\
-\operatorname{div} z_v &= 0 \quad \text{in } \Omega\\
z_v &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

2. Finite element discretization

Bilinear form for $u = (v, p) \in X := [H_0^1(\Omega)]^d \times L_0^2(\Omega)$

$$a(u, \varphi) \;:=\; (\operatorname{div} v, \xi) + (\sigma v, \phi) + (\beta \cdot \nabla v, \phi) + (\mu \nabla v, \nabla \phi) - (p, \operatorname{div} \phi)$$

Influence of the control by $b : Q \times X \to \mathbb{R}$ for $q \in Q \subset L^2(\Omega)$.

Variational formulation:

$$u \in X : \qquad a(u, \varphi) + b(q, \varphi) \;=\; (f, \varphi) \quad \forall \varphi \in X$$

Galerkin formulation:

$$u_h \in X_h : \qquad a(u_h, \varphi) + b(q_h, \varphi) \;=\; (f, \varphi) \quad \forall \varphi \in X_h$$

SUPG+PSPG, Grad-div stabilization for Oseen

- Inf-sup condition not fulfilled for equal-order elements
- Dominant convective terms

$$s_h(u_h)(\varphi) = \sum_{T \in \mathcal{T}_h} \int_T \{\rho_{mom} \cdot [\delta_T(\beta \cdot \nabla)\phi + \alpha_T \nabla \xi] + (\mathrm{div}\, v)\, \gamma_T(\mathrm{div}\, \phi)\}\, dx$$

(Hughes, Johnson, Lube, Tobiska, Glowinski, Le Tallec,..)

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Discretized primal problem:

$$(A_h + S_h^u)u_h + (B_h + S_h^q)q_h \quad = \quad f_h$$

- Forget for a while the parameter dependence: S_h^u, S_h^q are linear.
- Otherwise: S_h^u, S_h^q, f_h may depend on u_h.

Adjoint equation:

$$\begin{aligned} -\mu\Delta z_v - (\beta\cdot\nabla)z_v + \sigma z_v - \nabla z_p &= \widehat{v} - v \quad \text{in } \Omega \\ -\operatorname{div} z_v &= 0 \quad \text{in } \Omega \\ z_v &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

is also of Ossen type and need to be stabilized.

Discretized adjoint problem:

$$(A_h^* + S_h^z)z_h + Cu_h = C\widehat{u}$$

For residual based stabilization: S_h^z depend on the full adjoint residual.

Discrete KKT system (optimize-discretize):

$$\begin{pmatrix} \alpha I & 0 & B_h^* \\ 0 & C_h & A_h^* + S_h^z \\ B_h + S_h^q & A_h + S_h^u & 0 \end{pmatrix} \begin{pmatrix} q_h \\ u_h \\ z_h \end{pmatrix} = \begin{pmatrix} 0 \\ C\widehat{u} \\ f_h \end{pmatrix}$$

The other way round (discretize-optimize): cf. Collis & Heinkenschloss [2002]

Build KKT system of discretized PDE:

$$(A_h + S_h^u)u + (B_h + S_h^q)q_h \quad = \quad f_h$$

$$\begin{pmatrix} \alpha I & 0 & B_h^* + (S_h^q)^* \\ 0 & C_h & A_h^* + (S_h^u)^* \\ B_h + S_h^q & A_h + S_h^u & 0 \end{pmatrix} \begin{pmatrix} q_h \\ u_h \\ z_h \end{pmatrix} = \begin{pmatrix} 0 \\ C\widehat{u} \\ f_h \end{pmatrix}$$

In general: $S_h^q \neq 0$ and $S_h^z \neq (S_h^u)^*$.

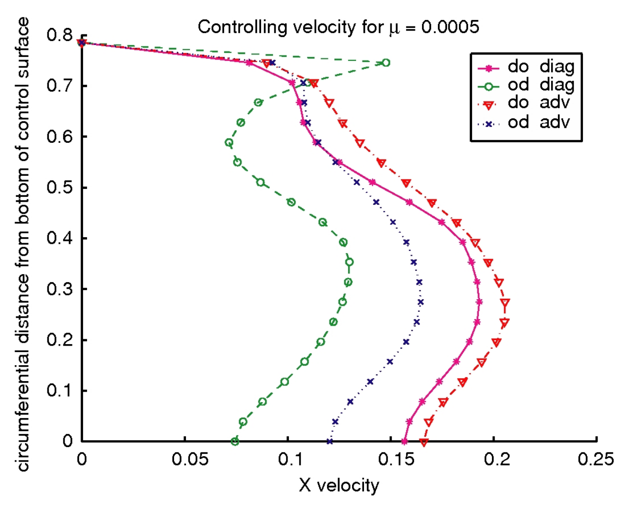
Streamline diffusion & pressure stabilized Petrov Galerkin

$$
\begin{aligned}
(S_h^u)^* - S_h^z \quad &\equiv \quad \sum_K \{(\widehat{v}_h - v_h + \sigma z^v + (\beta \cdot \nabla) z^v - \mu \Delta z^v, \delta^p \nabla \xi)_K\} \\
&+ \quad \sum_K \{(\sigma \phi + (\beta \cdot \nabla)\phi - \mu \Delta \phi, \delta^p \nabla z^p)_K\} \\
& \quad (\widehat{v}_h - v_h - \nabla z^p, \delta^v (\beta \cdot \nabla)\phi) \\
&+ \quad (\nabla \xi, \delta^v (\beta \cdot \nabla) z^v)
\end{aligned}
$$

Numerical tests by Collis & Heinkenschloss [2002]:

- D-O has better convergence properties than O-D for SUPG;
- large differences in z_h between D-O and O-D.

From Abraham, Behr, Heinkenschloss (2004): GLS



comparison of *do* and *od* with different settings of stabilization constants:

diag: $h_K :=$max. element lenght

adv: $h_K := \sum_i |(\beta_K \cdot \nabla)\phi_i|_K|/\|\beta\|_{K,\infty}$ (Tezduyar, Park (1986))

3. Finite elements with symmetric stabilization

Consider linear stabilization:

$$a(u_h, \varphi) + b(q_h, \varphi) + s_h(u_h, \varphi) \quad = \quad (f, \varphi) \qquad \forall \varphi \in X_h$$

First requirement **Symmetry:**

$$\text{(P1)} \qquad s_h(u, \varphi) \quad = \quad s_h(\varphi, u) \qquad \forall u, \varphi \in X$$

Lemma: For linear and symmetric stabilization (P1), discretization and optimization commutes.

We will show an a priori estimate in a (semi) norm:

$$|\!|\!| \cdot |\!|\!|_h : X \rightarrow \mathbb{R}_0^+$$

Second requirement **Coercivity:**

$$\text{(P2)} \qquad |\!|\!| u_h |\!|\!|_h^2 \;\lesssim\; a_h(u_h, u_h) + s_h(u_h, u_h) \qquad \forall u_h \in X_h$$

This is the case e.g. for

$$|\!|\!| u |\!|\!|_h \;:=\; (a_h(u, u) + s_h(u, u))^{1/2}$$

if $s_h(u, u) \geq 0$.

Third requirement:
$\|\!|u_h|\!\|_h$ stronger than L^2-norm of velocities:

$$\text{(P3)} \qquad \|v\| \;\lesssim\; \|\!|u|\!\|_h \qquad \forall u = (v, p) \in X$$

For example:

$$\|\!|u|\!\|_h^2 \;=\; \sigma\|v\|^2 + \mu\|\nabla v\|^2 + s_h(u, u)$$

Fourth requirement: a priori estimate for fixed control.

For $u \in [H^{r+1}(\Omega)]^{d+1}$ and finite elements of order r:

$$(\text{P4}) \qquad \|u(q) - u_h(q)\|_h \;\lesssim\; h^s \|u\|_{r+1}$$

$u(q), u_h(q) =$ solutions of continuous and discrete problems for given control $q \in Q$.

convergence order $s \leq r + 1$ (optimal $s = r + 1/2$)

Lemma: If (P4) holds for the primal problem, then it holds for the adjoint problems with given velocity field w in the rhs:

$$\|z(w) - z_h(w)\|_h \;\lesssim\; h^s \|z\|_{r+1} \qquad \text{if } z \in [H^{r+1}(\Omega)]^{d+1}$$

4. A convergence result

Theorem

Under the following conditions:

- *(P1), (P2), (P3), (P4)*
- *approximation property of the discrete control space:*

$$\|q - i_h q\| \lesssim h^s \|q\|_{r+1}$$

- *regularity of the solutions:* $u, z \in [H^{r+1}(\Omega)]^{d+1}$, $q \in H^{r+1}(\Omega)$

it holds the convergence result:

$$\|q - q_h\| \lesssim h^s(\|u\|_{r+1} + \|z\|_{r+1} + \|q\|_{r+1})$$

Principle of proof:

Since the reduced functional $j_h(q) := J(u_h(q), q)$ is at most quadratic:

$$\alpha \| \underbrace{i_h q - q_h}_{=:\delta q_h} \|^2 \quad \leq \quad j_h''(q_h)(\delta q_h) = j_h'(\underbrace{q_h + \delta q_h}_{=i_h q})(\delta q_h) - \underbrace{j_h'(q_h)(\delta q_h)}_{=0=j'(q)(\delta q_h)}$$

Expressing j' and j_h' and continuity of $b(\cdot,\cdot)$ gives ($\widehat{z}_h := z_h(u_h(i_h q))$):

$$\begin{aligned}
\alpha \|i_h q - q_h\|^2 &\leq b(i_h q - q_h, \widehat{z}_h^v - z^v) + (\alpha(i_h q - q), \delta q_h) \\
&\leq c\|\widehat{z}_h^v - z^v\| \cdot \|i_h q - q_h\| + \alpha \|i_h q - q\| \cdot \|i_h q - q_h\| \\
\|\widehat{z}_h^v - z^v\| &\leq \underbrace{\|z_h^v(u_h(i_h q)) - z_h^v(u(q))\|}_{\text{stab. disc. adjoint \& primal pb.}} + \underbrace{\|z_h^v(u(q)) - z^v(u(q))\|}_{\text{prev. Lemma}}
\end{aligned}$$

Theorem

Under the same conditions as the previous theorem with $s = r + \frac{1}{2}$*:*

$$\|u - u_h\|_h^2 \lesssim h^{r+\frac{1}{2}}(\|u\|_{r+1} + \|z\|_{r+1} + \|q\|_{r+1})$$

Proof.

$$\|u - u_h\|_h \leq \underbrace{\|u(q) - u_h(q)\|_h}_{h^{r+\frac{1}{2}}\|u\|_{r+1} \text{ due to (P4)}} + \|u_h(q) - u_h(q_h)\|_h$$

Coercivity (P2) for $w_h := u_h(q) - u_h(q_h)$:

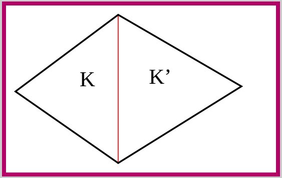
$$\|w_h\|_h^2 \lesssim a(w_h, w_h) + s_h(w_h, w_h) = -(B(q - q_h), w_h^v)$$

Cauchy-Schwarz, (P3) and continuity of B:

$$\|w_h\|_h \lesssim \|B(q - q_h)\| \lesssim \|q - q_h\| \qquad \square$$

5. Examples of symmetric stabilization techniques

Edge oriented stabilization (EOS) [Burman, Hansbo]



Jumps across edges:

$$[\![u(x)]\!] \quad := \quad u(x)|_K - u(x)|_{K'} \,.$$

Stabilization terms:

$$\begin{aligned} s_h^{es}(u,\varphi) &:= s_h^{es,p}(p,\xi) + s_h^{es,v}(v,\phi) \\ s_h^{es,p}(p,\xi) &:= \sum_{K\in\mathcal{T}_h} \int_{\partial K} \alpha_K [\![\nabla p]\!] \cdot [\![\nabla \xi]\!] \, ds \\ s_h^{es,v}(v,\phi) &:= \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left\{ \delta_K [\![n\cdot\nabla v]\!] \cdot [\![n\cdot\nabla\phi]\!] + \gamma_K [\![\operatorname{div} v]\!] \cdot [\![\operatorname{div}\phi]\!] \right\} ds \end{aligned}$$

Fulfill (P1), (P2), (P3) and (P4).

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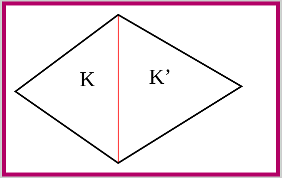

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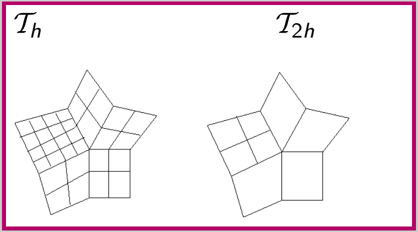
Fulfill (P1), (P2), (P3) and (P4). Hence: optimal order of convergence.

Local projection stabilization (LPS) [Becker, Br., Burman, Tobiska, Matthies, Lube, Rapin]

Step 1 - Definition of fluctuation operator:

- D_{2h}^{r-1} = discontinuous, patchwise polynomial order $r-1$.



- Patchwise L^2-projection

$$\pi_h : L^2(\Omega) \to D_{2h}^{r-1}$$

- Fluctuation operator

$$\kappa_h = i - \pi_h$$

Example $r = 1$**:** Patch-wise projection on constants:

$$\kappa_h \nabla p|_K = \nabla p - \frac{1}{|K|} \int_K \nabla p \, dx \,, \quad K \in \mathcal{T}_{2h}$$

Step 2 - Definition of stabilization terms

- Pressure stabilization (Br. & Becker '00)

$$S_h(u,\varphi) \;=\; \Big(\kappa_h(\nabla p), \alpha\kappa_h(\nabla\xi)\Big)$$

- stabilization of convective terms by the full gradient

$$\ldots + \Big(\kappa_h(\nabla v), \delta\kappa_h(\nabla\phi)\Big)$$

- or streamline derivatives + stabilization of divergence-free condition

$$\ldots + \Big(\kappa_h((\beta\cdot\nabla)v), \delta\kappa((\beta\cdot\nabla)\phi)\Big) + \Big(\kappa_h(\operatorname{div} v), \gamma\kappa((\operatorname{div}\phi)\Big)$$

But: nonlinear for Navier-Stokes.

Fulfill (P1), (P2), (P3) and (P4).

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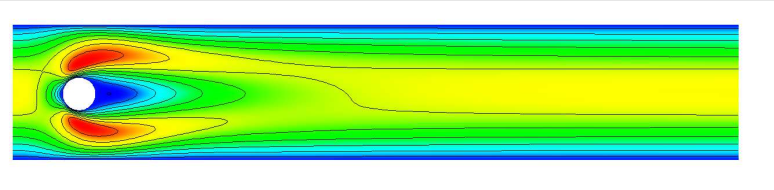
6. Numerical validation

Navier-Stokes:

$$\begin{aligned} -\mu\Delta v + (v\cdot\nabla)v + \nabla p + Bq &= f \quad \text{in } \Omega\,, \\ \operatorname{div} v &= 0 \quad \text{in } \Omega\,, \\ v &= v_0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

Discretized with local projection stabilization.

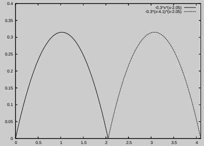
DFG benchmark: (uncontroled solution at $Re = 100$)



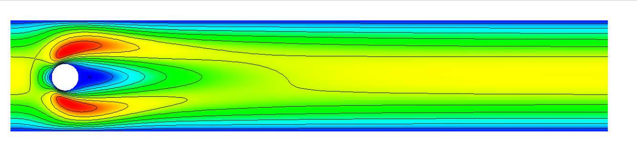
Objective functional:

$$J(v,q) \quad := \quad \frac{1}{2}\|v-\widehat{v}\|^2 \;\rightarrow\; \text{min!}$$

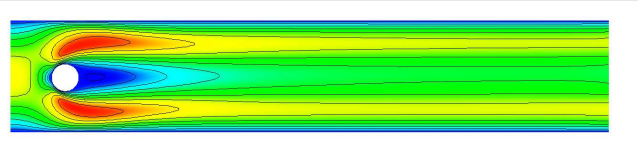
$\widehat{v}(x,y)$ =double-Poiseulle flow (parabolic)



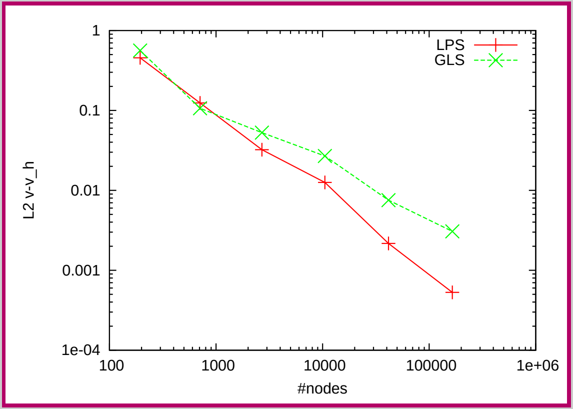
Initial sol.



optimiz. sol.



Comparison of convergence:



LPS = local projection stabilization (symmetric)
GLS = PSPG / SUPG optimize-discretize

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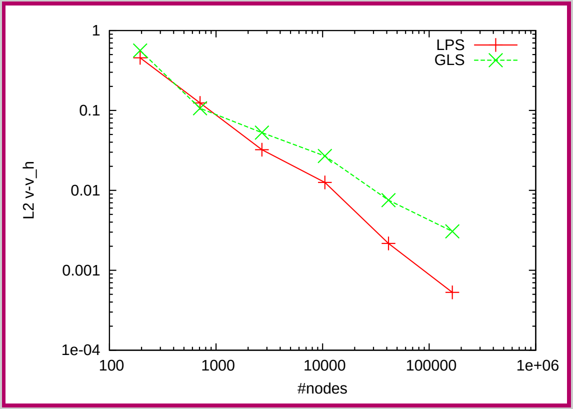

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Further optimization results with LPS: Becker, Meidner, Vexler

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- Finite element schemes may provide consistent KKT systems when symmetric stabilization is used.
- Convergence proof for Oseen with general symmetric stabilization (LPS,EOS,...)
- First numerical test problem indicate the benefit of symmetric stabilization.

Thanks a lot!