Discontinuous Galerkin Finite Element Methods for Nonconservative Hyperbolic Partial Differential Equations

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Introduction

Motivation of research:

- Many physical problems are modelled by hyperbolic partial differential equations containing nonconservative products

$$\partial_x u + A(u)\partial_x u = 0$$

- The essential feature of nonconservative products is that $A \neq Df$, hence A is **not** the Jacobian matrix of a flux function f.
- This causes problems once the solution becomes discontinuous, because the weak solution in the classical sense of distributions then does not exist.
- This also complicates the derivation of discontinous Galerkin discretizations since there is no direct link with a Riemann problem.

- **Alternative:** use the theory for nonconservative products from Dal Maso, LeFloch and Murat (DLM)

Overview of Presentation

- Overview of main results of the theory of Dal Maso, LeFloch and Murat for nonconservative products
- Space-time DG discretization of nonconservative hyperbolic partial differential equations
- Numerical examples
- Conclusions

Nonconservative Products

- Consider the function $u(x)$

$$u(x) = u_L + \mathcal{H}(x - x_d)(u_R - u_L), \quad x, x_d \in]a, b[,$$

with $\mathcal{H} : \mathbb{R} \longrightarrow \mathbb{R}$ the Heaviside function.

- For any smooth function $g : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ the product $g(u)\partial_x u$ is not defined at $x = x_d$ since here $|\partial_x u| \rightarrow \infty$.

- Introduce a smooth regularization u^ε of u. If the total variation of u^ε remains uniformly bounded with respect to ε then Dal Maso, LeFloch and Murat (DLM) showed that

$$g(u)\frac{du}{dx} \equiv \lim_{\varepsilon \to 0} g(u^\varepsilon)\frac{du^\varepsilon}{dx}$$

gives a sense to the nonconservative product as a bounded measure.

Effect of Path on Nonconservative Product

The limit of the regularized nonconservative product depends in general on the path used in the regularization.

- Introduce a Lipschitz continuous path $\phi : [0,1] \longrightarrow \mathbb{R}^m$, satisfying $\phi(0) = u_L$ and $\phi(1) = u_R$, connecting u_L and u_R in $\mathbb{R}^m$.
- The following regularization u^ε for u then emerges:

$$u^\varepsilon(x) = \begin{cases} u_L, & \text{if } x \in]a, x_d - \varepsilon[, \\ \phi(\frac{x - x_d + \varepsilon}{2\varepsilon}), & \text{if } x \in]x_d - \varepsilon, x_d + \varepsilon[, \\ u_R, & \text{if } x \in]x_d + \varepsilon, b[\end{cases} \qquad \varepsilon > 0.$$

- When ε tends to zero, then:

$$g(u^\varepsilon)\frac{du^\varepsilon}{dx} \rightharpoonup C\delta_{x_d}, \text{ with } C = \int_0^1 g(\phi(\tau))\frac{d\phi}{d\tau}(\tau)\,d\tau,$$

  weakly in the sense of measures on $]a, b[$, where δ_{x_d} is the Dirac measure at x_d.

- The limit of $g(u^\varepsilon)\partial_x u^\varepsilon$ depends on the path ϕ.

- There is one exception, namely if an $q : \mathbb{R}^m \longrightarrow \mathbb{R}$ exists with $g = \partial_u q$. In this case $C = q(u_R) - q(u_L)$.

DLM Theory

Dal Maso, LeFloch and Murat provided a general theory for nonconservative hyperbolic pde's.

- Introduce the Lipschitz continuous maps $\phi : [0,1] \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ which satisfy the following properties:

(H1) $\phi(0; u_L, u_R) = u_L,\ \phi(1; u_L, u_R) = u_R,$

(H2) $\phi(\tau; u_L, u_L) = u_L,$

(H3) $\left|\frac{\partial \phi}{\partial \tau}(\tau; u_L, u_R)\right| \leq K|u_L - u_R|$, a.e. in $[0,1]$.

- **Theorem (DLM).** Let $u :]a,b[\longrightarrow \mathbb{R}^m$ be a function of bounded variation and $g : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ a continuous function. Then, there exists a unique real-valued bounded Borel measure μ on $]a,b[$ with:

  1. If u is continuous on a Borel set $B \subset]a,b[$, then

  $$\mu(B) = \int_B g(u) \frac{du}{dx}$$

  2. If u is discontinuous at a point x_d of $]a,b[$, then

  $$\mu(\{x_d\}) = \int_0^1 g(\phi(\tau; u_L, u_R)) \frac{\partial \phi}{\partial \tau}(\tau; u_L, u_R)\, d\tau.$$

  By definition, this measure μ is the nonconservative product of $g(u)$ by $\partial_x u$ and denoted by $\mu = \left[g(u) \frac{du}{dx} \right]_\phi$.

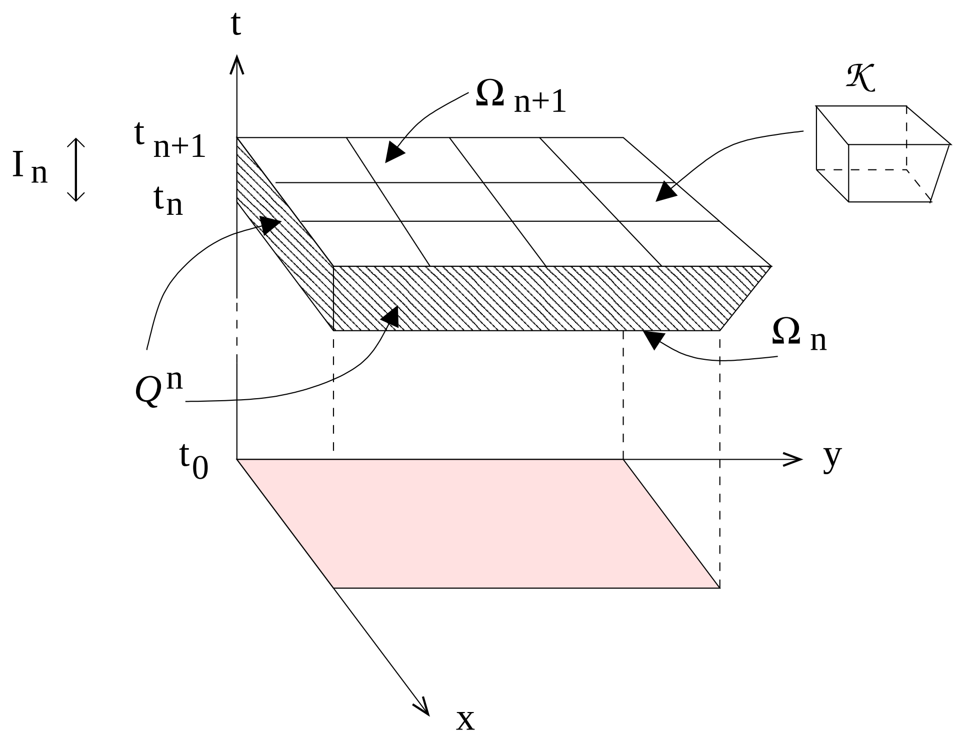
Rankine-Hugoniot Relations

- For conservative hyperbolic system of pde's, $\partial_x u + \partial_x f(u) = 0$ the Rankine-Hugoniot relations across a jump with u^L and u^R and velocity v are equal to
$$-v(u^R - u^L) + f(u^R) - f(u^L) = 0.$$

- For a nonconservative hyperbolic pde $\partial_x u + A(u)\partial_x u = 0$ the Rankine-Hugoniot relations in the DLM theory are equal to
$$-v(u^R - u^L) + \int_0^1 A(\phi_D(s, u^L, u^R))\partial_s \phi_D(s; u^L, u^R) ds = 0$$
with ϕ_D a Lipschitz continuous path satisfying $\phi_D(0; u_L, u_R) = u_L$ and $\phi_D(1; u_L, u_R) = u_R$.

- The Rankine-Hugoniot relations are essential for the definition of the NCP flux used in the DG discretization.

Space-Time Approach



- A time-dependent problem is considered directly in four dimensional space, with time as the fourth dimension

Space-Time Domain

- Consider an open domain: $\mathcal{E} \subset \mathbb{R}^d$.
- The flow domain $\Omega(t)$ at time t is defined as:

$$\Omega(t) := \{x \in \mathcal{E} \mid x_0 = t,\ t_0 < t < T\}$$

- The space-time domain boundary $\partial\mathcal{E}$ consists of the hypersurfaces:

$$\begin{aligned}\Omega(t_0) &:= \{x \in \partial\mathcal{E} \mid x_0 = t_0\},\\ \Omega(T) &:= \{x \in \partial\mathcal{E} \mid x_0 = T\},\\ \mathcal{Q} &:= \{x \in \partial\mathcal{E} \mid t_0 < x_0 < T\}.\end{aligned}$$

- The space-time domain is covered with a tessellation $\mathcal{T}_h$ consisting of space-time elements $\mathcal{K}$.

Discontinuous Finite Element Approximation

- The finite element space associated with the tessellation $\mathcal{T}_h$ is given by:

$$W_h := \{ W \in (L^2(\mathcal{E}_h))^m \;:\; W|_{\mathcal{K}} \circ G_{\mathcal{K}} \in (P^k(\hat{\mathcal{K}}))^m, \quad \forall \mathcal{K} \in \mathcal{T}_h \}$$

- The jump of f at an internal face $\mathcal{S} \in \mathcal{S}_I^n$ in the direction k of a Cartesian coordinate system is defined as:

$$[\![f]\!]_k = f^L \bar{n}_k^L + f^R \bar{n}_k^R,$$

  with $\bar{n}_k^R = -\bar{n}_k^L$.

- The average of f at $\mathcal{S} \in \mathcal{S}_I^n$ is defined as:

$$\{\!\{ f \}\!\} = \tfrac{1}{2}(f^L + f^R).$$

Space-Time Discontinuous Galerkin Discretization

Main features of a space-time DG approximation

- Basis functions are discontinuous in space and time
- Weak coupling through numerical fluxes at element faces
- Discretization results in a coupled set of nonlinear equations for the DG expansion coefficients

Benefits of Space-Time DG Discretization

Main benefits of a space-time DG approximation

- The space-time DG method results in a very local discretization, which is beneficial for:
  - hp-mesh adaptation
  - parallel computing
- The space-time DG method is well suited for problems on domains with time-dependent boundaries

Space-Time DG Formulation of Nonconservative Hyperbolic PDE's

- Consider the nonlinear hyperbolic system of partial differential equations in nonconservative form in multi-dimensions:

$$\frac{\partial U_i}{\partial t} + \frac{\partial F_{ik}}{\partial x_k} + G_{ikr}\frac{\partial U_r}{\partial x_k} = 0, \quad \bar{x} \in \Omega \subset \mathbb{R}^q,\ t > 0,$$

  with $U \in \mathbb{R}^m$, $F \in \mathbb{R}^m \times \mathbb{R}^q$, $G \in \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^m$

- These equations model for instance bubbly flows, granular flows, shallow water equations and many other physical systems.

- Weak formulation for nonconservative hyperbolic system:

Find a $U \in V_h$, such that for $V \in V_h$ the following relation is satisfied

$$
\begin{aligned}
&\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} V_i\big(U_{i,0} + F_{ik,k} + G_{ikr}U_{r,k}\big)\, d\mathcal{K} \\
&+ \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\int_{K(t_{n+1}^-)} \widehat{V}_i(U_i^R - U_i^L)\, dK - \int_{K(t_n^+)} \widehat{V}_i(U_i^R - U_i^L)\, dK \right) \\
&+ \sum_{\mathcal{S}\in\mathcal{S}_I} \int_{\mathcal{S}} \widehat{V}_i \left(\int_0^1 G_{ikr}(\phi(\tau; U^L, U^R)) \frac{\partial \phi_r}{\partial \tau}(\tau; U^L, U^R)\, d\tau\, \bar{n}_k^L \right) d\mathcal{S} \\
&- \sum_{\mathcal{S}\in\mathcal{S}_I} \int_{\mathcal{S}} \widehat{V}_i [\![F_{ik} - v_k U_i]\!]_k\, d\mathcal{S} = 0
\end{aligned}
$$

Relation with Space-Time DG Formulation of Conservative Hyperbolic PDE's

- **Theorem 2.** If the numerical flux $\hat{V}$ for the test function V is defined as:

$$\widehat{V} = \begin{cases} \{\!\{V\}\!\} & \text{at } \mathcal{S} \in \mathcal{S}_I, \\ 0 & \text{at } K(t_n) \subset \Omega_h(t_n) \quad \forall n \geq 0, \end{cases}$$

then the DG formulation will reduce to the conservative space-time DG formulation when there exists a Q, such that $G_{ikr} = \partial Q_{ik} / \partial U_r$.

- After the introduction of the numerical flux $\hat{V}$ we obtain the weak formulation:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \big(- V_{i,0}U_i - V_{i,k}F_{ik} + V_i G_{ikr} U_{r,k}\big)\, d\mathcal{K}$$

$$+ \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\int_{K(t_{n+1}^-)} V_i^L U_i^L \, dK - \int_{K(t_n^+)} V_i^L U_i^L \, dK \right)$$

$$+ \sum_{\mathcal{S}\in\mathcal{S}_I} \int_{\mathcal{S}} (V_i^L - V_i^R) \{\!\{ F_{ik} - v_k U_i \}\!\} \bar{n}_k^L \, d\mathcal{S}$$

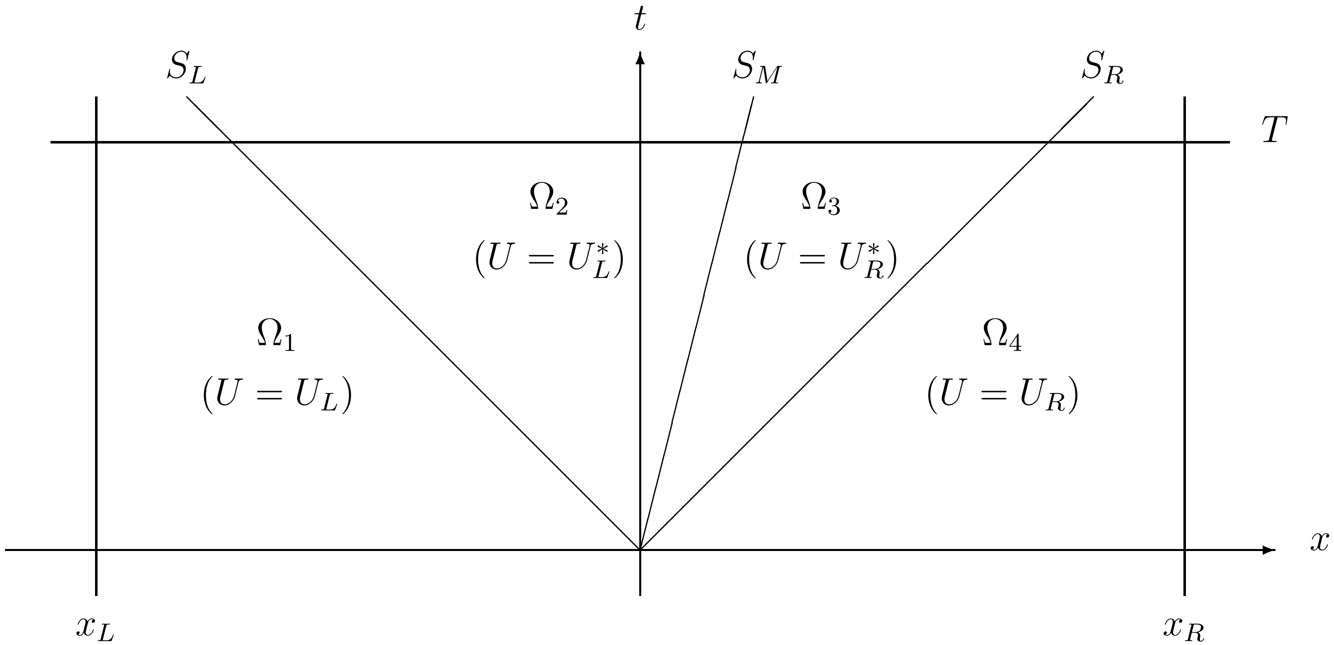
$$+ \sum_{\mathcal{S}\in\mathcal{S}_B} \int_{\mathcal{S}} V_i^L (F_{ik}^L - v_k U_i^L) \bar{n}_k^L \, d\mathcal{S}$$

$$+ \sum_{\mathcal{S}\in\mathcal{S}_I} \int_{\mathcal{S}} \{\!\{ V_i \}\!\} \left(\int_0^1 G_{ikr}(\phi(\tau; U^L, U^R)) \frac{\partial \phi_r}{\partial \tau}(\tau; U^L, U^R)\, d\tau \, \bar{n}_k^L \right) d\mathcal{S} = 0$$

Numerical Fluxes

- The fluxes at the element faces do not contain any stabilizing terms yet, both for the conservative and nonconservative part
- At the time faces, the numerical flux is selected such that causality in time is ensured
$$\widehat{U} = \begin{cases} U^L & \text{at } K(t_{n+1}^-) \\ U^R & \text{at } K(t_n^+) \end{cases} .$$
- The space-time DG formulation is stabilized using the NCP (Non-Conservative Product) flux
$$\widehat{P}_i^{nc} = \left(\{\!\{F_{ik} - v_k U_i\}\!\} + P_{ik}\right)\bar{n}_k^L$$

Nonconservative Product Flux



Wave pattern of the solution for the Riemann problem

NCP Flux

Main steps in derivation of NCP flux:

- Consider the nonconservative hyperbolic system:

$$\partial_t U + \partial_x F(U) + G(U)\partial_x U = 0,$$

- Introduce the averaged **exact** solution $\bar{U}^*_{LR}(T)$ as:

$$\bar{U}^*_{LR}(T) = \frac{1}{T(S_R - S_L)} \int_{TS_L}^{TS_R} U(x, T)\, dx.$$

- Apply the Gauss theorem over each subdomain $\Omega_1, \cdots, \Omega_4$ and connect each subdomain using the generalized Rankine-Hugoniot relations.

- The NCP-flux is then given by:

$$\widehat{P}_i^{nc}(U_L, U_R, v, \bar{n}^L) = \begin{cases} F_{ik}^L \bar{n}_k^L - \frac{1}{2}\int_0^1 G_{ikr}(\bar{\phi}(\tau; U_L, U_R))\frac{\partial \bar{\phi}_r}{\partial \tau}(\tau; U_L, U_R)\, d\tau \bar{n}_k^L \\ \qquad \text{if } S_L > v, \\ \{\!\{F_{ik}\}\!\}\bar{n}_k^L + \frac{1}{2}\big((S_R - v)\bar{U}_i^* + (S_L - v)\bar{U}_i^* - S_L U_i^L - S_R U_i^R\big) \\ \qquad \text{if } S_L < v < S_R, \\ F_{ik}^R \bar{n}_k^L + \frac{1}{2}\int_0^1 G_{ikr}(\bar{\phi}(\tau; U_L, U_R))\frac{\partial \bar{\phi}_r}{\partial \tau}(\tau; U_L, U_R)\, d\tau \bar{n}_k^L \\ \qquad \text{if } S_R < v, \end{cases}$$

- Note, if G is the Jacobian of some flux function Q, then $\widehat{P}^{nc}(U_L, U_R, v, \bar{n}^L)$ is exactly the HLL flux derived for moving grids in van der Vegt and van der Ven (2002).

Efficient Solution of Nonlinear Algebraic System

- The space-time DG discretization results in a large system of nonlinear algebraic equations:

$$\mathcal{L}(\hat{U}^n; \hat{U}^{n-1}) = 0$$

- This system is solved by marching to steady state using pseudo-time integration and multigrid techniques:

$$\frac{\partial \hat{U}}{\partial \tau} = -\frac{1}{\Delta t}\mathcal{L}(\hat{U}; \hat{U}^{n-1})$$

Test Cases

- One dimensional shallow water equations with topography

$$\partial_t U + \partial_x F + G \partial_x U = 0,$$

with:

$$U = \begin{bmatrix} b \\ h \\ hu \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ hu \\ hu^2 + \frac{1}{2}\mathsf{F}^{-2}h^2 \end{bmatrix}, \quad G(U) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathsf{F}^{-2}h & 0 & 0 \end{bmatrix}.$$

- For the space DGFEM weak formulation we can prove theoretically for linear basis functions and the path $\phi = U_L + \tau(U_R - U_L)$ that the rest flow remains at rest.

Subcritical flow over a bump

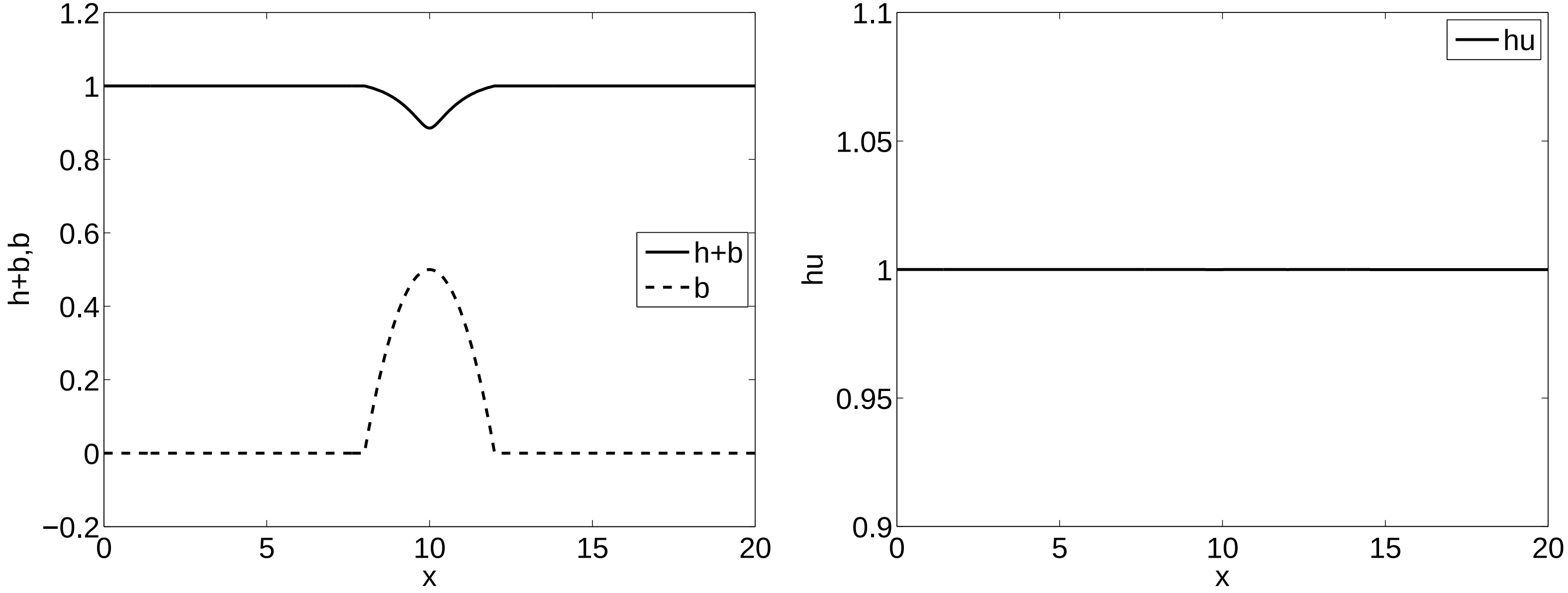
- Consider subcritical flow with a Froude number of F $= 0.2$ over a bump with shape

$$b(x) = \begin{cases} a\big(b - (x - x_p)\big)\big(b + (x - x_p)\big)\,b^{-2} & \text{for } |x - x_p| \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

  with $x_p = 10,\ a = 0.5$ and $b = 2$.

- The domain $x \in [0, 20]$ is divided into 40, 80, 160 and 320 cells

- A linear path is used $\phi = U_L + \tau(U_R - U_L)$.



(a) Water level $h(x) + b(x)$.

(b) Mass flow $hu(x)$.

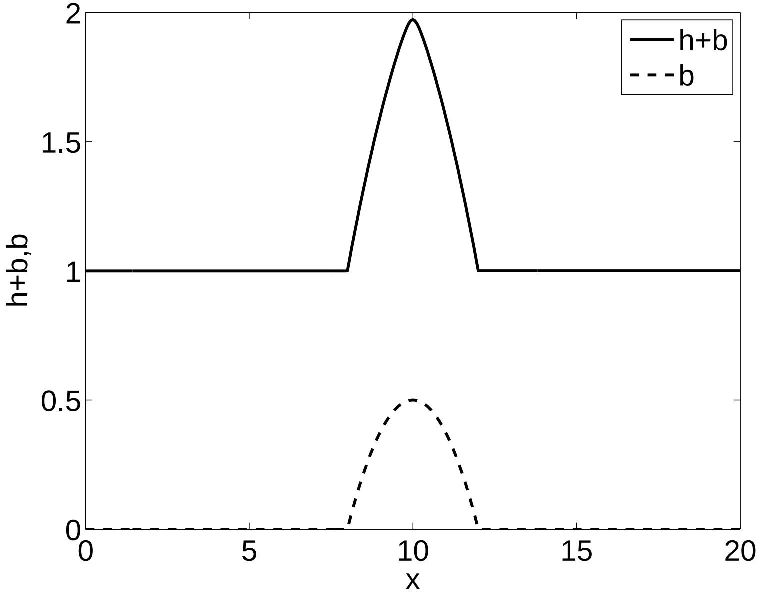
Steady-state solution for subcritical flow over a bump (F $= 0.2$ and 320 elements).

	$h+b$				hu			
N_{cells}	L^2 error	p	$L^{\max}$ error	p	L^2 error	p	$L^{\max}$ error	p
40	$0.1141 \cdot 10^{-2}$	-	$0.6559 \cdot 10^{-2}$	-	$0.1262 \cdot 10^{-2}$	-	$0.3285 \cdot 10^{-2}$	-
80	$0.3194 \cdot 10^{-3}$	1.8	$0.2387 \cdot 10^{-2}$	1.5	$0.1943 \cdot 10^{-3}$	2.7	$0.8029 \cdot 10^{-3}$	2.0
160	$0.8365 \cdot 10^{-4}$	1.9	$0.6989 \cdot 10^{-3}$	1.8	$0.2763 \cdot 10^{-4}$	2.8	$0.1369 \cdot 10^{-3}$	2.6
320	$0.2119 \cdot 10^{-4}$	2.0	$0.1847 \cdot 10^{-3}$	1.9	$0.3797 \cdot 10^{-5}$	2.9	$0.2929 \cdot 10^{-4}$	2.2

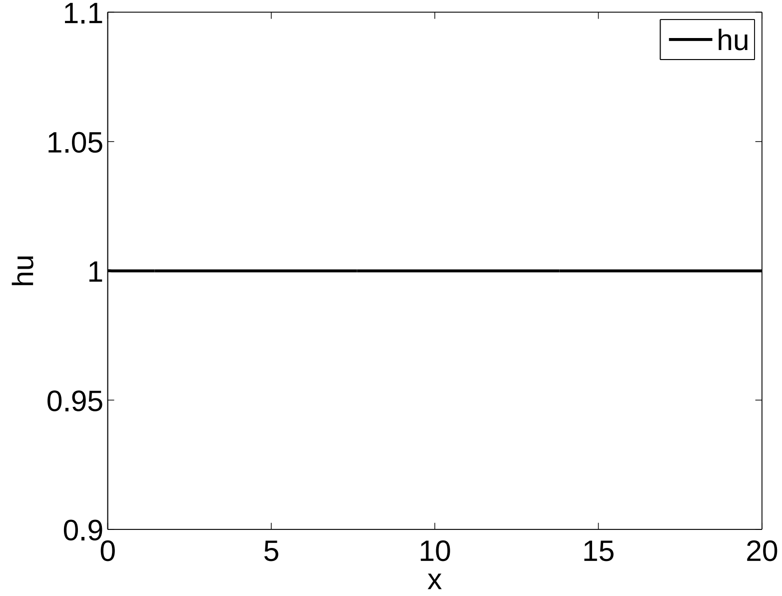
	$h+b$				hu			
N_{cells}	L^2 error	p	$L^{\max}$ error	p	L^2 error	p	$L^{\max}$ error	p
40	$0.3278 \cdot 10^{-3}$	-	$0.1836 \cdot 10^{-2}$	-	$0.2339 \cdot 10^{-3}$	-	$0.1170 \cdot 10^{-2}$	-
80	$0.4433 \cdot 10^{-4}$	2.9	$0.3195 \cdot 10^{-3}$	2.5	$0.3721 \cdot 10^{-4}$	2.7	$0.2401 \cdot 10^{-3}$	2.3
160	$0.4556 \cdot 10^{-5}$	3.3	$0.3142 \cdot 10^{-4}$	3.3	$0.5513 \cdot 10^{-5}$	2.8	$0.3596 \cdot 10^{-4}$	2.7
320	$0.5522 \cdot 10^{-6}$	3.0	$0.4407 \cdot 10^{-5}$	2.8	$0.7489 \cdot 10^{-6}$	2.9	$0.5218 \cdot 10^{-5}$	2.8

Error in $h+b$ and hu for subcritical flow over a bump at F $= 0.2$ using linear (top) and quadratic (bottom) basis functions.

Supercritical flow over a bump



(c) The water level $h(x) + b(x)$.



(d) The mass flow $hu(x)$.

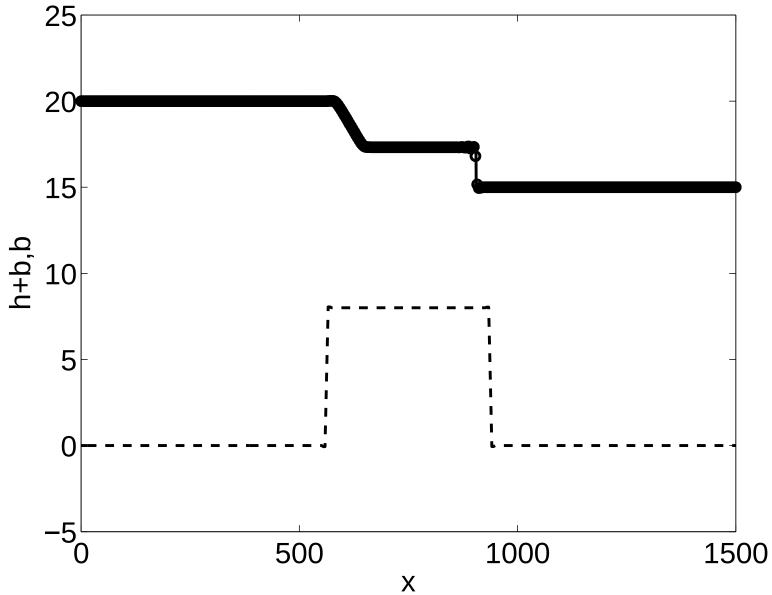
Steady-state solution for supercritical flow over a bump (F $= 1.9$ and 320 elements).

	DGFEM $h + b$				STDGFEM $h + b$			
N_{cells}	L^2 error	p	$L^{\max}$ error	p	L^2 error	p	$L^{\max}$ error	p
40	$0.7543 \cdot 10^{-2}$	-	$0.4619 \cdot 10^{-1}$	-	$0.7543 \cdot 10^{-2}$	-	$0.4619 \cdot 10^{-1}$	-
80	$0.1281 \cdot 10^{-2}$	2.6	$0.9406 \cdot 10^{-2}$	2.3	$0.1281 \cdot 10^{-2}$	2.6	$0.9406 \cdot 10^{-2}$	2.3
160	$0.3188 \cdot 10^{-3}$	2.0	$0.2615 \cdot 10^{-2}$	1.8	$0.3188 \cdot 10^{-3}$	2.0	$0.2615 \cdot 10^{-2}$	1.8
320	$0.7914 \cdot 10^{-4}$	2.0	$0.6883 \cdot 10^{-3}$	1.9	$0.7914 \cdot 10^{-4}$	2.0	$0.6883 \cdot 10^{-3}$	1.9

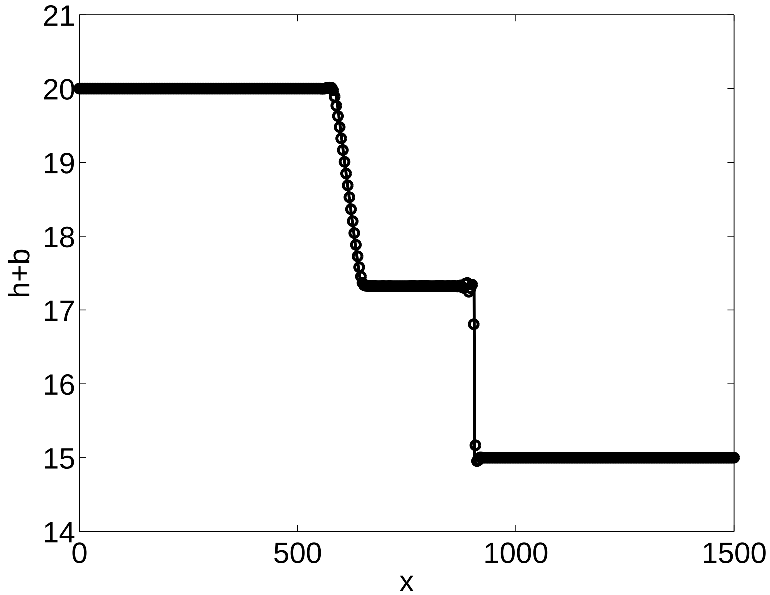
	DGFEM $h + b$				STDGFEM $h + b$			
N_{cells}	L^2 error	p	$L^{\max}$ error	p	L^2 error	p	$L^{\max}$ error	p
40	$0.1293 \cdot 10^{-2}$	-	$0.5034 \cdot 10^{-2}$	-	$0.9181 \cdot 10^{-3}$	-	$0.4946 \cdot 10^{-2}$	-
80	$0.1944 \cdot 10^{-3}$	2.7	$0.9383 \cdot 10^{-3}$	2.4	$0.1624 \cdot 10^{-3}$	2.5	$0.1127 \cdot 10^{-2}$	2.1
160	$0.2892 \cdot 10^{-4}$	2.7	$0.1545 \cdot 10^{-3}$	2.6	$0.1830 \cdot 10^{-4}$	3.1	$0.1382 \cdot 10^{-3}$	3.0
320	$0.3724 \cdot 10^{-5}$	3.0	$0.2111 \cdot 10^{-4}$	2.9	$0.2253 \cdot 10^{-5}$	3.0	$0.2002 \cdot 10^{-4}$	2.8

Error in $h + b$ for supercritical flow over a bump at F $= 1.9$ using linear (top) and quadratic (bottom) basis functions.

Dam break problem over a rectangular bump



(e) The numerical solution of the water level and the topography.



(f) The numerical solution of the water level.

Dam breaking problem at time $t = 15$. Line: 4000 cells. Dots: 400 cells.

Depth averaged two-fluid model

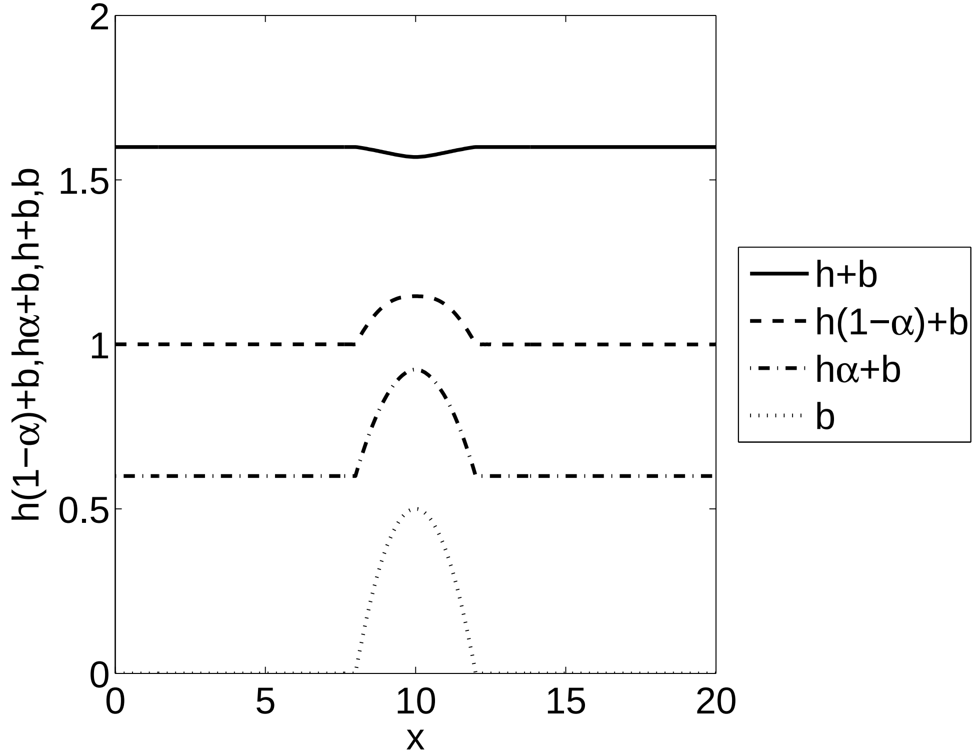
- The dimensionless depth-averaged two fluid model of Pitman and Le, ignoring source terms for simplicity, can be written as:

$$\partial_t U + \partial_x F + G\partial_x U = 0,$$

where:

$$U = \begin{bmatrix} h(1-\alpha) \\ h\alpha \\ h\alpha v \\ hu(1-\alpha) \\ b \end{bmatrix}, \quad F = \begin{bmatrix} h(1-\alpha)u \\ h\alpha v \\ h\alpha v^2 + \frac{1}{2}\varepsilon(1-\rho)\alpha_{xx} g h^2 \alpha \\ hu^2 + \frac{1}{2}\varepsilon g h^2 \\ 0 \end{bmatrix}$$

$$G(U) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \varepsilon\rho\alpha g h & \varepsilon\rho\alpha g h & 0 & 0 & \varepsilon(1-\rho)\alpha_{xx} g h\alpha + \varepsilon\rho\alpha g h \\ \frac{2u^2\alpha}{1-\alpha} - \alpha u^2 - \varepsilon g h\alpha & -\varepsilon g h\alpha - \alpha u^2 & u(\alpha-1) & u\alpha - \frac{2u\alpha}{1-\alpha} & (1-\alpha)\varepsilon g h \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$



Steady-state solution for a subcritical two-phase flow (320 cells).

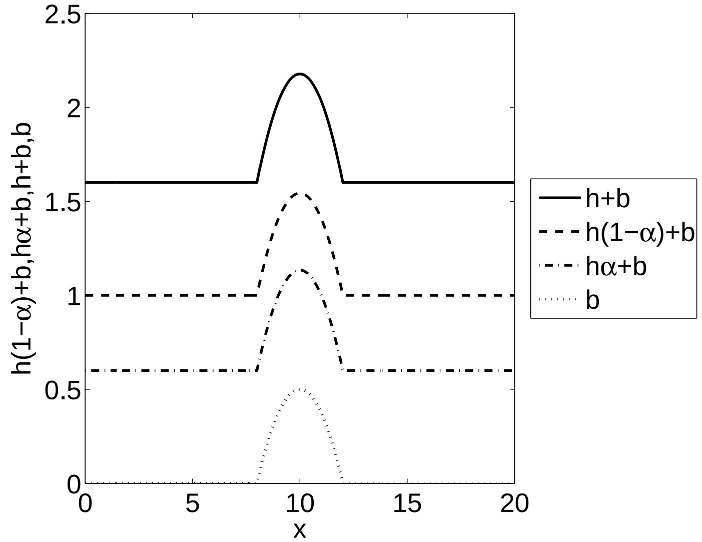
Total flow height $h + b$, flow height due to the fluid phase $h(1 - \alpha)$, flow height due to solids phase $h\alpha$ and the topography b.

STDGFEM

N_{cells}	$h(1-\alpha)+b$ L^2 error	p	L^{max} error	p	$h\alpha+b$ L^2 error	p	L^{max} error	p
40	$0.8171 \cdot 10^{-3}$	-	$0.2308 \cdot 10^{-2}$	-	$0.1404 \cdot 10^{-2}$	-	$0.4194 \cdot 10^{-2}$	-
80	$0.2025 \cdot 10^{-3}$	2.0	$0.5584 \cdot 10^{-3}$	2.0	$0.3537 \cdot 10^{-3}$	2.0	$0.9903 \cdot 10^{-3}$	2.1
160	$0.4871 \cdot 10^{-4}$	2.1	$0.1322 \cdot 10^{-3}$	2.1	$0.8511 \cdot 10^{-4}$	2.1	$0.2306 \cdot 10^{-3}$	2.1
320	$0.9789 \cdot 10^{-5}$	2.3	$0.2651 \cdot 10^{-4}$	2.3	$0.1712 \cdot 10^{-4}$	2.3	$0.4597 \cdot 10^{-4}$	2.3
N_{cells}	$hu(1-\alpha)$ L^2 error	p	L^{max} error	p	$hv(\alpha)$ L^2 error	p	L^{max} error	p
40	$0.3672 \cdot 10^{-4}$	-	$0.1442 \cdot 10^{-3}$	-	$0.1212 \cdot 10^{-4}$	-	$0.3409 \cdot 10^{-4}$	-
80	$0.5911 \cdot 10^{-5}$	2.6	$0.3448 \cdot 10^{-4}$	2.1	$0.1791 \cdot 10^{-5}$	2.8	$0.8054 \cdot 10^{-5}$	2.1
160	$0.1049 \cdot 10^{-5}$	2.5	$0.8471 \cdot 10^{-5}$	2.0	$0.3807 \cdot 10^{-6}$	2.2	$0.2048 \cdot 10^{-5}$	2.0
320	$0.1723 \cdot 10^{-6}$	2.6	$0.2078 \cdot 10^{-5}$	2.0	$0.5115 \cdot 10^{-7}$	2.9	$0.4861 \cdot 10^{-6}$	2.1

Error in $h(1-\alpha)+b$, $h\alpha+b$, $hu(1-\alpha)$ and $hv\alpha$ for subcritical flow over a bump.

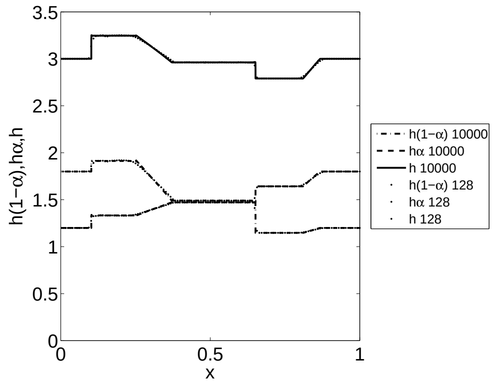
Two-phase supercritical flow



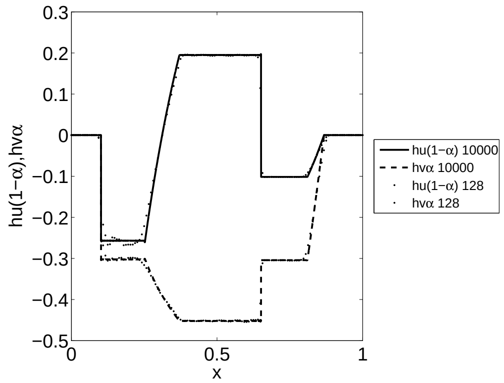
Steady-state solution for a supercritical two-phase flow (320 cells).

Total flow height $h + b$, flow height due to the fluid phase $h(1 - \alpha)$, flow height due to the solids phase $h\alpha$ and topography b.

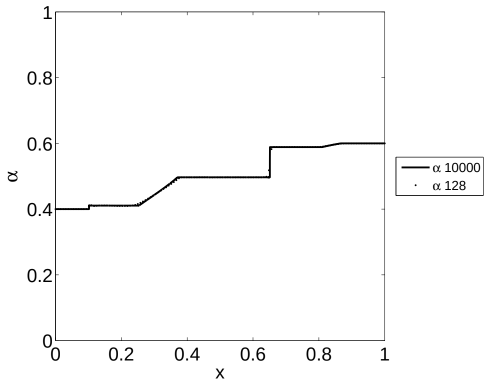
Two-phase dam break problem



(g) Solution of $h(1-\alpha)$, $h\alpha$, b and h.



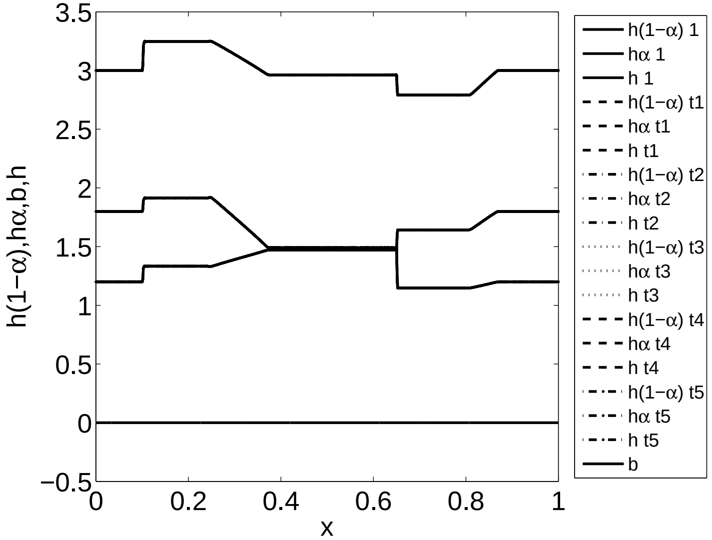
(h) Solution of $hu(1-\alpha)$ and $hv\alpha$.



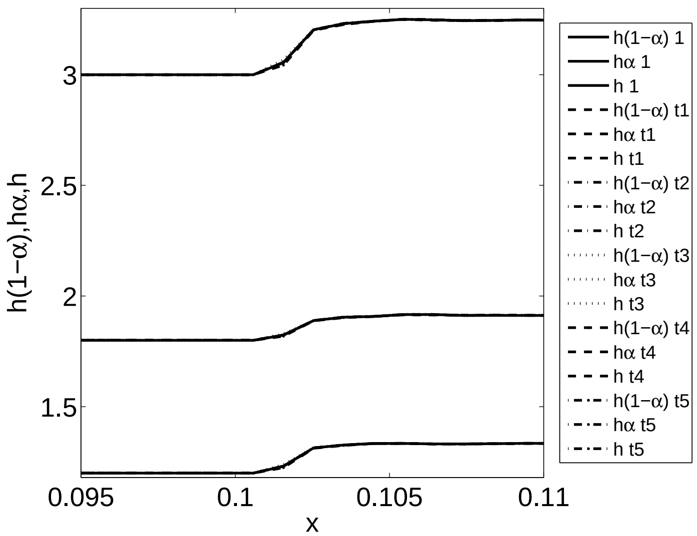
(i) Solution of α.

Two-phase dam break problem at time $t = 0.175$; mesh with 128 elements compared to mesh with 10000 elements.

Effect of Path



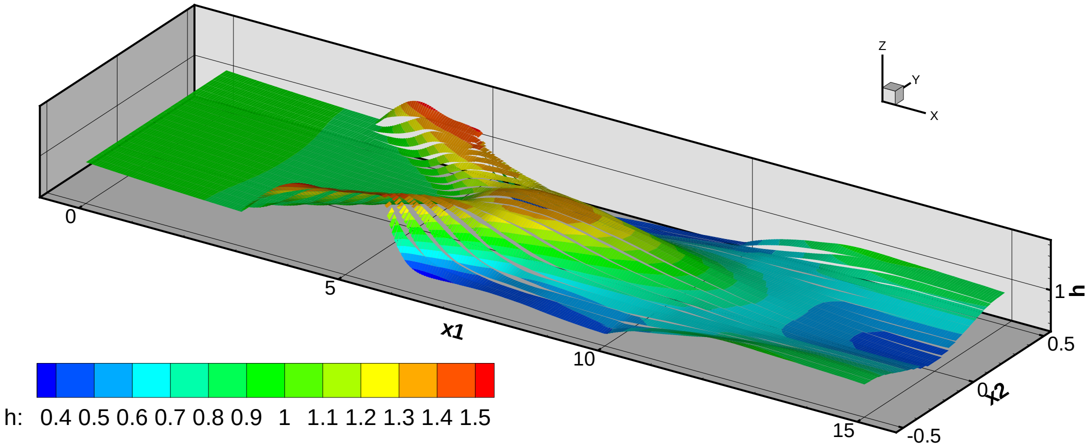
(j) The solution on the whole domain.



(k) The solution zoomed in on the left shock wave.

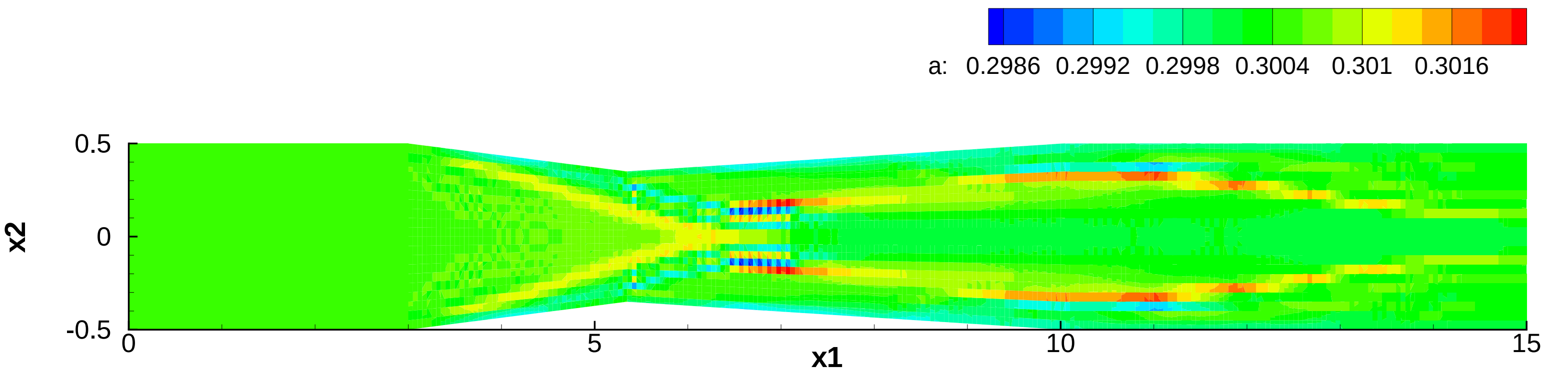
Solution of $h(1-\alpha)$, $h\alpha$, b and h at time $t = 0.175$ calculated on a mesh with 1024 elements using the paths defined by Toumi.

Flow Through a Contraction



Flow depth h of water-sand mixture in a contraction
$h/L = 0.01$, $\rho_f/\rho_s = 0.5$, slope 10°.

Flow Through a Contraction



Particle volume fraction α of water-sand mixture in a contraction
$h/L = 0.01$, $\rho_f/\rho_s = 0.5$, slope 10°.

Conclusions

- A space-time DG discretization for nonconservative hyperbolic pde's using the DLM theory has been developed.
- A new numerical flux for nonconservative hyperbolic pde's has been developed, which reduces to the HLLC flux for conservative pde's.
- The effect of the choice of the path in phase space is in practice for nearly all cases negligible.
- The algorithm has been successfully tested on the shallow water equations with non-constant topography and a depth averaged two-phase flow model.

More information:

S. Rhebergen, O. Bokhove and J.J.W. van der Vegt, Discontinuous finite element methods for hyperbolic nonconservative partial differential equations, *Journal of Computational Physics*, Vol. 227, No. 3, pp. 1887-1922, 2008

See also: wwwhome.math.utwente.nl/~vegtjjw/