Error Estimates for Nonconforming Finite Element Approximations of Drag and Lift in Channel Flows

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We propose a new method for computing drag and lift acting on bodies in channel flows by means of nonconforming finite element approximations to the incompressible Navier–Stokes equations. For the new formulas, a-priori error estimates are given which cover both the standard nonconforming finite element method and also stabilized discretizations of upwind type to handle the case of higher Reynolds numbers. For a two-dimensional DFG benchmark problem of a laminar flow around a cylinder, numerical results of the drag and the lift are presented. The numerical tests confirm our theoretical results.

Keywords. Navier–Stokes equations, drag and lift, nonconforming finite element methods

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I. INTRODUCTION

The computation of the velocity and the pressure fields in complex geometrical flow regions is an important point in computational fluid dynamics (CFD). Starting with the fundamental equations of conservation of mass and momentum, a lot of discretization techniques, including stabilized methods to handle the case of higher Reynolds numbers, have been developed to solve numerically the incompressible Navier–Stokes equations

[1, 2, 3, 4, 5]. However, what is the most interesting from an engineering point of view is not always the velocity or the pressure field itself but some functionals of them like, e.g., the drag or the lift coefficient of a body in the flow field. In this way, the numerical solution of the velocity–pressure formulation of the Navier–Stokes equations plays only an intermediate step of the overall procedure. If we use simply the traces to the boundary of the numerical solution (u_h, p_h) to calculate the corresponding surface integrals, we loose a half order of accuracy in comparison to the approximation order of (u_h, p_h) to (u, p). One of the authors has recently presented a different approach and obtained error estimates without loss of accuracy [6]. The actual computational method by this approach reveals to be the consistent flux method described in [7, 8, 9]. Note that in these early papers, no error estimates have been considered. There, the main focus was to guarantee the consistency of the numerically calculated heat flux. In [7], only standard Galerkin finite element methods have been considered whereas in [8, 9] also Petrov-Galerkin approaches have been addressed.

In this paper, we will extend the results of [6] and give formulas for the numerical calculation of drag and lift when using nonconforming finite element methods. We will thoroughly study the corresponding error estimates. We will see that these new formulas are closely connected to the numerical discretization for approximating the velocity and the pressure fields, respectively. It is remarkable how big the differences between a direct calculation of the boundary fluxes and a calculation by volume integrals can be in special situations. For example, in the scalar convection-diffusion equation, boundary layers occur which influence the accuracy of the boundary fluxes considerably. However, for a simple one-dimensional example with constant coefficient, it has been shown in [9] that, using a modified approach of the streamline-diffusion method with an optimal value of upwind parameter δ, we can get exact values for the boundary fluxes.

In this paper, the underlying technique in nonconforming finite element discretizations will be described in detail for the case of laminar channel flows around bodies. We also cover stabilized schemes of upwind type which has been analyzed in [1, 2] and numerically tested in [10, 11, 12, 13]. For other upwind or streamline-diffusion methods to solve the incompressible Navier-Stokes equation, we refer to [3, 4, 5, 14]. For conforming finite element methods, improved error estimates have just been obtained by using a duality argument [15]. We also extend these results to nonconforming finite element approximations and discuss in detail the additional assumptions which have to be made.

The plan of the paper is the following. In Section 2, we derive formulas for drag and lift which enable us to represent them by using certain volume integrals instead of surface integrals. In Section 3, we discuss how these representation formulas can be used to derive formulas for the numerical approximation of drag and lift. In particular, we study the standard nonconforming finite element approach in Sec. 3.1, where we also discuss the case of nonconforming upwind methods. In Sec. 3.2, we show how higher order a-priori estimates can be derived by using a duality argument. Finally, in Section 4, we give two-dimensional numerical test examples for the benchmark of a laminar flow around a cylinder which has been defined within the DFG Priority Research Program *Flow Simulation with High–Performance Computers* [16].

II. DRAG AND LIFT

Let us consider a laminar flow around a body in a channel with the flow region $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, with Lipschitz continuous boundary Γ. The stationary incompressible Navier–Stokes equations are formulated in primitive variables subject to Dirichlet boundary conditions:

$$\begin{aligned} \rho(u \cdot \nabla)u &= \operatorname{div} \sigma + f \quad \text{in } \Omega \,, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \,, \\ u &= g \quad \text{on } \Gamma \,. \end{aligned} \tag{2.1}$$

Here, $u = (u_1, \ldots, u_d)$ is the velocity field, p the pressure field, f an exterior force, ρ the density, μ the kinematic viscosity and σ the stress tensor given in the form

$$\sigma_{ij}(u,p) = -p\delta_{ij} + 2\mu\varepsilon_{ij}(u) \quad i,j = 1, \ldots, d$$

with the strain tensor ε defined by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i,j = 1, \ldots, d \,.$$

The boundary of Ω consists of two disjoint connected components, the boundary of the body Γ_b and the boundary of the channel $\Gamma_c = \Gamma_i \cup \Gamma_o \cup \Gamma_w$ with an inflow part Γ_i, an outflow part Γ_o and fixed walls Γ_w. We assume no–slip boundary conditions on Γ_w and Γ_b, i.e.

$$g = 0 \quad \text{on } \Gamma_w \cup \Gamma_b \,. \tag{2.2}$$

We introduce the spaces

$$V_0 := H_0^1(\Omega)^d, \qquad V_g := \{\, v \in H^1(\Omega)^d \,;\, v_{|\Gamma} = g \,\}\,, \text{and}$$

$$Q := L_0^2(\Omega) := \{\, q \in L^2(\Omega) \,;\, \int_\Omega q \, dx = 0 \,\}\,.$$

Then, a weak formulation of (2.1) reads:

Find $(u,p) \in V_g \times Q$ such that for all $(v,q) \in V_0 \times Q$

$$a(u,v) + n(u,u,v) + b(v,p) + b(u,q) = (f,v)\,, \tag{2.3}$$

where

$$\begin{aligned} a(u,v) &:= 2\mu \int_\Omega \sum_{i,j=1}^d \varepsilon_{ij}(u)\varepsilon_{ij}(v)\, dx, \\ n(w,u,v) &:= \frac{\rho}{2}\int_\Omega \sum_{i,j=1}^d \left(w_i \frac{\partial u_j}{\partial x_i} v_j - w_i \frac{\partial v_j}{\partial x_i} u_j \right) dx, \\ b(v,q) &:= -\int_\Omega q \operatorname{div} v \, dx, \end{aligned}$$

and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $L^2(\Omega)^d$. Let the given boundary condition $g \in H^{1/2}(\Gamma)^d$ satisfy the compatibility condition

$$\int_\Gamma g \cdot n \, d\gamma = 0. \tag{2.4}$$

Since $g = 0$ on Γ_b, the integral over the normal component of g on each connected component of Γ vanishes. As a conclusion, for any $\varepsilon > 0$, the Lemma of Hopf gives the existence of a divergence-free extension $u_g \in H^1(\Omega)^d$ with $u_g = g$ on Γ such that $|n(v, u_g, v)| \le \varepsilon |v|_1^2$ for all $v \in V_0$ [17]. It is well-known that under these assumptions there is at least one solution of (2.3) which is unique provided the given data are small enough.

Let the direction of the flow through the channel coincides with the x-axis and the gravity effects against the y-axis. The drag D and the lift L are given by

$$D = -\int_{\Gamma_b} \sum_{j=1}^{d} \sigma_{1j}(u,p) n_j d\gamma, \tag{2.5}$$

$$L = -\int_{\Gamma_b} \sum_{j=1}^{d} \sigma_{2j}(u,p) n_j d\gamma, \tag{2.6}$$

where $n = (n_1, \ldots, n_d)$ is the unit outer normal (from the fluid) to Γ_b. The following lemma is an easy consequence of the Gauss-Green Theorem.

Lemma 2.1. *Let $(u,p) \in (V_g \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$ be a solution to (2.3) and (2.1), respectively. Then, for any function $v \in H^1(\Omega)^d$ with $v = 0$ on Γ_c we have*

$$\int_{\Gamma_b} \sum_{i,j=1}^{d} \sigma_{ij}(u,p) n_j v_i \, d\gamma = a(u,v) + n(u,u,v) + b(v,p) - (f,v)\,. \tag{2.7}$$

Note that we get the drag and the lift from (2.7) by setting $v = w^D$ with $w_i^D = \delta_{i1}$ on Γ_b and $v = w^L$ with $w_i^L = \delta_{i2}$ on Γ_b, respectively.

Remark. *It is well-known that for a divergence–free function u and for test functions $v \in H_0^1(\Omega)^d$ the bilinear form $a : H^1(\Omega)^d \times H^1(\Omega)^d \to \mathbb{R}$ in (2.3) can be replaced by*

$$a^*(u,v) := \mu(\nabla u, \nabla v).$$

This property becomes important in the discrete case for two reasons. First, the block size of the block-diagonal matrix corresponding to a^* is smaller than that corresponding to a, and second, the bilinear form a^* is coercive even on the nonconforming finite element space but not so is the form a. Note that the trilinear form $n : H^1(\Omega)^d \times H^1(\Omega)^d \times H^1(\Omega)^d$ can be replaced also by

$$n^*(w,u,v) := \rho\left((w \cdot \nabla)u, v\right),$$

provided that div $w = 0$, *and the test function v belongs to $H_0^1(\Omega)^d$.*

Next, we show that also in (2.7) a can be replaced by a^*.

Lemma 2.2. *Let $(u,p) \in (V_g \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$ be a solution to (2.3) and (2.1), respectively. Then, for any function $v \in H^1(\Omega)^d$ with $v = 0$ on Γ_c we have*

$$\int_{\Gamma_b} \sum_{i,j=1}^{d} \sigma_{ij}(u,p) n_j v_i \, d\gamma = a^*(u,v) + n(u,u,v) + b(v,p) - (f,v) \, . \tag{2.8}$$

Proof. For sufficiently smooth functions u and v, a direct calculation gives

$$\begin{aligned} a(u,v) &= \mu(\nabla u, \nabla v) + \mu \int_\Omega \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx \\ &= a^*(u,v) + \mu(\text{div } u, \text{div } v) + \mu \int_\Gamma \sum_{i,j=1}^{d} \left(\frac{\partial u_i}{\partial x_j} n_i - \frac{\partial u_i}{\partial x_i} n_j \right) \, v_j \, d\gamma. \end{aligned}$$

Taking into consideration that $v = 0$ on Γ_c and div $u = 0$, we have to study the expression

$$\sum_{j=1}^{d} \int_{\Gamma_b} \sum_{i=1, i \neq j}^{d} \left(\frac{\partial u_i}{\partial x_j} n_i - \frac{\partial u_i}{\partial x_i} n_j \right) \, v_j \, d\gamma \, .$$

Note that

$$\frac{\partial u_i}{\partial x_j} n_i - \frac{\partial u_i}{\partial x_i} n_j$$

is a derivative of u_i in a direction perpendicular to $n = (n_1, \cdots, n_d)$ and that $u_i = 0$ on Γ_b for $i = 1, \ldots, d$. Thus,

$$\frac{\partial u_i}{\partial x_j} n_i - \frac{\partial u_i}{\partial x_i} n_j = 0 \qquad \text{on } \Gamma_b \, .$$

■

Remark. *Note that in (2.7) and (2.8), respectively, the trilinear form n can be replaced by n^*. Indeed for w, u, and $v \in H^1(\Omega)^d$, integrating by parts, we get*

$$n(w,u,v) = n^*(w,u,v) + \frac{\rho}{2}(\text{div } w, u \cdot v) - \frac{\rho}{2} \int_\Gamma w \cdot n \; v \cdot u \, d\gamma \, ,$$

and taking into consideration that div $w = 0$*, $u = 0$ on Γ_b, and $v = 0$ on Γ_c, we finally have $n(w,u,v) = n^*(w,u,v)$.*

Both the representation (2.7) and (2.8) can be used to calculate the drag and the lift provided that the solution (u,p) of (2.3) is known. For this, we choose a fixed function $v = w^D$ and $v = w^L$, respectively, from the spaces

$$\begin{aligned} V^D &:= \{ \, v \in H^2(\Omega)^d \, : \, v_i = 0 \text{ on } \Gamma_c, \; v_i = \delta_{i1} \text{ on } \Gamma_b \, \} \, , \\ V^L &:= \{ \, v \in H^2(\Omega)^d \, : \, v_i = 0 \text{ on } \Gamma_c, \; v_i = \delta_{i2} \text{ on } \Gamma_b \, \} \, . \end{aligned}$$

It is clear that the special choice of v from V^D and V^L, respectively, does not influence the left hand side of (2.7) or (2.8), because the difference of two functions $v^{(1)}$, $v^{(2)}$ from V^D and V^L, respectively, belongs to V_0 and therefore the term

$$a^*(u, v^{(1)} - v^{(2)}) + n(u,u,v^{(1)} - v^{(2)}) + b(v^{(1)} - v^{(2)}, p) - (f, v^{(1)} - v^{(2)})$$

vanishes. In the next section, we derive discrete counterparts of (2.8) which preserve this property.

III. APPROXIMATE DRAG AND LIFT

We restrict our representation to decompositions of $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ into triangles or tetrahedrons, respectively. The case of quadrilaterals or hexahedrons can be studied along the same lines. Let $\{\mathcal{T}_h\}_{h\downarrow 0}$ be a regular family of decompositions of the domain Ω into d–dimensional simplices $K \in \mathcal{T}_h$ where the mesh parameter h represents the maximum diameter of the elements $K \in \mathcal{T}_h$.

If we use directly (2.5) and (2.6) to approximate D and L, respectively, and estimate

$$|D - D_h| \leq \int_{\Gamma_b} \sum_{j=1}^{d} |(\sigma_{1j}(u,p) - \sigma_{1j}(u_h, p_h))n_j| \, d\gamma \, ,$$

we loose a half order of convergence since the approximation error in general behaves like

$$\inf_{v_h \in V_h, q_h \in Q_h} \|\sigma_{1j}(u,p) - \sigma_{1j}(v_h, q_h)\|_{0,1,\Gamma_b} \leq \text{const } h^{1/2} \left(|u|_2 + |p|_1\right) .$$

Therefore, in the following subsections, we use a different approach for calculating approximations of D and L, which admits a higher order of convergence.

A. Nonconforming finite element method

As an example we consider the nonconforming Crouzeix/Raviart finite element [18]. The technique developed here can also be applied to other nonconforming finite elements, in particular to the piecewise bilinear rotated and piecewise constant pair of quadrilateral elements proposed by Rannacher and Turek [19] and analyzed in [19, 20].

In the case of moderate or high Reynolds numbers, often a stabilization of upwind type is used to calculate the velocity and the pressure fields. We will also discuss modifications which are necessary when using the nonconforming upwind method proposed by Ohmori/Ushijima for scalar convection diffusion equations [21]. For the incompressible Navier–Stokes equation, the nonconforming upwind method has been analyzed in [2] and numerically tested in [10, 11, 12, 13]. For other upwind methods we refer to [3, 14, 20].

The approximation space V_h for the velocity consists of piecewise linear functions, where the degrees of freedoms are the mean values of the velocity components on the element faces. With V_{gh} and V_{0h} we denote spaces of all functions $v_h \in V_h$ such that the meanvalue of $v_h - g$ over any element face of the boundary is zero. The approximation space Q_h for the pressure field consists of piecewise constant functions with vanishing mean value in Ω. For this pair of finite elements, it is known that the discrete version of the Babuška–Brezzi condition is satisfied. Moreover, the bilinear form a^* introduced in Remark II. is coercive. Note that this is not the case for the original bilinear form a. In order to define a discrete problem for solving (2.3) numerically, we have to understand the forms in (2.3) in an element–wise manner, i.e., all integrals over Ω have to be calculated as the sum over all elements K of integrals over K. We will indicate this by the subscript h. In particular, we introduce the bilinear forms a_h and b_h by

$$a_h(u,v) := \sum_K \mu(\nabla u, \nabla v)_K, \quad b_h(u,v) := -\sum_K (q, \text{div } v)_K$$

and denote the mesh-depending H^1-semi norm by

$$|v|_{1,h} := \left(\sum_K |v|_{1,K}^2 \right)^{1/2} .$$

For the convective term, we will use the standard discretization

$$n_h(w,u,v) := \frac{\rho}{2}\sum_K [((w\cdot\nabla)u,v)_K - ((w\cdot\nabla)v,u)_K]$$

or an upwind discretization n_h^{upw} which has been described in detail in [2]. Then, the nonconforming method to solve (2.3) reads:

Find $(u_h,p_h)\in V_{gh}\times Q_h$ such that for all $(v_h,q_h)\in V_{0h}\times Q_h$

$$a_h(u_h,v_h)+n_h(u_h,u_h,v_h)+b_h(v_h,p_h)+b_h(u_h,q_h)=(f,v_h)\,. \tag{3.1}$$

Let us denote by i_h the interpolation operator mapping $v\in H^1(\Omega)^d$ into the nonconforming space V_h by defining the degrees of freedom as the mean values of v over the element faces . The approximate drag D_h and the approximate lift L_h are now defined by

$$D_h = -\left\{a_h(u_h,i_hw^D)+n_h(u_h,u_h,i_hw^D)+b_h(i_hw^D,p_h)-(f,i_hw^D)\right\}\,, \tag{3.2}$$

$$L_h = -\left\{a_h(u_h,i_hw^L)+n_h(u_h,u_h,i_hw^L)+b_h(i_hw^L,p_h)-(f,i_hw^L)\right\}\,. \tag{3.3}$$

Note that the nonconforming upwind method differs only in replacing n_h by n_h^{upw} in (3.1) – (3.3).

The interpolants i_hw^D and i_hw^L can be replaced by discrete functions w_h^D and w_h^L satisfying

$$w_{hi}^D = \begin{cases}\delta_{i1} & \text{at all nodes on } \Gamma_b\\ 0 & \text{at all other nodes}\end{cases}\,, \tag{3.4}$$

$$w_{hi}^L = \begin{cases}\delta_{i2} & \text{at all nodes on } \Gamma_b\\ 0 & \text{at all other nodes}\end{cases}\,, \tag{3.5}$$

respectively. This simplifies the calculation of drag and lift considerably. The replacement is justified from the fact that $i_hw^D-w_h^D$ and $i_hw^L-w_h^L$ belong to V_{0h}. Thus, neither the functions w^D, w^L, nor their interpolations have to be known explicitely.

Theorem 3.1. *Let $(u,p)\in V_g\times Q$ and $(u_h,p_h)\in V_{gh}\times Q_h$ be the solution to the problems (2.3) and (3.1), respectively, where n_h can also be replaced by the upwind discretization n_h^{upw}. Then, there is a positive constant $M=M(\|u\|_2,|p|_1,\|w\|_2)$ such that the drag and the lift, calculated by (3.2) and (3.3), satisfy the error estimates*

$$|D-D_h|,\ |L-L_h|\le Mh \tag{3.6}$$

for sufficiently small h.

Proof. Compared with the proof in [6] where the case of conforming finite element methods has been considered, we have to take into consideration the nonconformity of the method and the special upwind discretization when n_h is replaced by n_h^{upw}. We start with the standard nonconforming case.

We restrict ourselves to show the estimate of the drag since the proof for the lift is quite similar. Using the expressions (2.7) and (3.2), we have

$$\begin{aligned}D-D_h = -\{&a_h(u,w^D)-a_h(u_h,i_hw^D)+n_h(u,u,w^D)-n_h(u_h,u_h,i_hw^D)\\ &+b_h(w^D,p)-b_h(i_hw^D,p_h)-(f,w^D-i_hw^D)\}\,.\end{aligned}$$

In order to replace $(f, w^D - i_h w^D)$, we multiply the strong formulation

$$-\mu\Delta u + \rho(u \cdot \nabla)u + \nabla p = f$$

of the problem with $w^D - i_h w^D$ and integrate it in an elementwise manner over Ω. After an elementwise integration by parts, we get

$$\begin{aligned} a_h(u, w^D - i_h w^D) + n_h(u, u, w^D - i_h w^D) + b_h(w^D - i_h w^D, p) \\ = (f, w^D - i_h w^D) + e_1(u, p; w^D - i_h w^D) \end{aligned}$$

with

$$e_1(u, p; v_h) := \sum_K \int_{\partial K} \left(\mu \frac{\partial u}{\partial n} \cdot v_h - \frac{1}{2}\rho\; u \cdot n\; u \cdot v_h - p v_h \cdot n \right) d\gamma.$$

This gives for the error representation

$$\begin{aligned} D - D_h \;=\; & -\{a_h(u - u_h, i_h w^D) + n_h(u, u - u_h, i_h w^D) + n_h(u - u_h, u_h, i_h w^D) \\ & + b_h(i_h w^D, p - p_h) + e_1(u, p; w^D - i_h w^D)\}\,. \end{aligned}$$

Estimating each term separately and using the inequality

$$|i_h w^D|_{1,h} \le |i_h w^D - w^D|_{1,h} + |w^D|_1 \le (1 + ch)\|w\|_2 \le 2\|w\|_2,$$

we obtain

$$|D - D_h| \;\le\; M\left(|u - u_h|_{1,h} + |u - u_h|^2_{1,h} + \|p - p_h\|_0 + |e_1(u, p; w^D - i_h w^D)|\right).$$

Note that the consistency error can be estimated in the usual way [18]

$$\begin{aligned} |e_1(u, p; w^D - i_h w^D)| \;&\le\; c\, h\, (\|u\|_2 + \|u\|_2 |u|_1 + |p|_1) |w^D - i_h w^D|_{1,h} \\ &\le\; M\, h^2. \end{aligned}$$

Thus, the statement of the theorem follows for the standard nonconforming discretization.

Now, we consider the nonconforming upwind discretization of the convective term. The estimation starts with replacing n in (2.8) by

$$n_h^*(w, u, v) := \rho \sum_K \int_K \sum_{i,j=1}^d w_i \frac{\partial u_j}{\partial x_i} v_j \, dx\,.$$

This gives

$$\begin{aligned} D - D_h \;=\; & -\{(a_h(u, w^D) - a_h(u_h, i_h w^D)) + (n_h^*(u, u, w^D) - n_h^{upw}(u_h, u_h, i_h w^D)) \\ & + (b_h(w^D, p) - b_h(i_h w^D, p_h)) - (f, w^D - i_h w^D)\}\,. \end{aligned}$$

We estimate only a part of the second term

$$A := n_h^*(u, u, i_h w^D) - n_h^{upw}(u_h, u_h, i_h w^D),$$

as the other terms are evaluated similarly to the previous case. We split A into

$$\begin{aligned} A \;=\; & n_h^*(u, u, i_h w^D) - n_h^{upw}(i_h u, i_h u, i_h w^D) \\ & + n_h^{upw}(i_h u, i_h u - u_h, i_h w^D) + n_h^{upw}(i_h u - u_h, u_h, i_h w^D). \end{aligned}$$

Applying Lemma 4 in [2], we can estimate the difference

$$|n_h^*(u,u,i_h w^D) - n_h^{upw}(i_h u, i_h u, i_h w^D)| \leq M\ h\ |i_h w^D|_{1,h} \leq M\ h$$

and by Lemma 2 of [2] we get

$$\begin{aligned} &|n_h^{upw}(i_h u, i_h u - u_h, i_h w^D) + n_h^{upw}(i_h u - u_h, u_h, i_h w^D)| \\ &\leq\ M\ \left(|i_h u|_{1,h}\,|i_h u - u_h|_{1,h}\,|i_h w^D|_{1,h} + |i_h u - u_h|_{1,h}\,|u_h|_{1,h}\,|i_h w^D|_{1,h}\right) \\ &\leq\ M\ \left(h + |u - u_h|_{1,h}\right)\ . \end{aligned}$$

Putting all estimates together, we obtain the statement of the theorem. ∎

Remark. *From the estimates (3.6) we see that the approximate drag and the approximate lift converge to the exact values at least with the same order as the discrete velocity u_h and the discrete pressure p_h in the energy norm to the solution (u,p) of (2.3). It is important to note that for this result no additional regularity assumption is needed.*

B. Higher order convergence estimates by duality

In the previous subsection, we have seen that the special choices of the functions w^D, w^L from V^D, V^L and also w_h^D, w_h^L in (3.2), (3.3) do not influence the drag and the lift and their approximations. In this subsection we will improve the order of convergence by choosing w^D and w^L as solutions of a linearized dual problem. To do this, we need additional assumptions which guarantee the solvability and regularity of the linearized dual Navier–Stokes equation. To be more precise, we consider the following problem for given u and u_h :

Find $(\varphi, r) \in V_\psi \times Q$ such that for all $(w,q) \in V_0 \times Q$

$$a^*(w,\varphi) + n(u,w,\varphi) + n_h(w,u_h,\varphi) + b(w,r) + b(\varphi,q) = 0. \tag{3.7}$$

Here, $V_\psi := \{\, v \in H^1(\Omega)^d\,;\ v_{|\Gamma} = \psi \,\}$.

Lemma 3.2. *Let $u \in H^1(\Omega)^d$ be divergence free, $u_h \in V_h$ be bounded in $L^\infty(\Omega)^d$ with respect to the mesh size h and $\psi \in H^{1/2}(\Gamma)^d$. Then, under the assumptions*

$$\mu \geq \mu_0(\|u_h\|_\infty) > 0$$

the linearized dual problem (3.7) has a unique solution $(\varphi, r) \in V_\psi \times Q$.

Proof. The bilinear form B defined by

$$B(w,v) := a^*(w,v) + n(u,w,v) + n_h(w,u_h,v)$$

satisfies for all $w \in V_0$

$$B(w,w) = \mu|w|_1^2 + \frac{\rho}{2}\sum_K \left[((w\cdot\nabla)u_h, w)_K - ((w\cdot\nabla)w, u_h)_K\right]\ .$$

Now, integrating by parts, using the boundedness of u_h and the continuity of the trace: $H^1(K) \to L^2(\partial K)$, we get

$$
\begin{aligned}
B(w,w) &= \mu|w|_1^2 + \frac{\rho}{2}\sum_K \int_{\partial K} w\cdot n\, u_h\cdot w\, d\gamma - \frac{\rho}{2}\sum_K (\nabla\cdot w, u_h\cdot w)_K \\
&\quad -\rho\sum_K ((w\cdot\nabla)w, u_h)_K \\
&\geq (\mu - C\|u_h\|_\infty)|w|_1^2.
\end{aligned}
$$

Thus, setting $\mu_0 > C\|u_h\|_\infty$, we obtain the coerciveness of $B : V_0 \times V_0 \to \mathbb{R}$. Since the pair of spaces (V_0, Q) satisfies the inf-sup condition, we conclude the unique solvability of (3.7). ■

For getting improved error estimates, we need the solution to (3.7) satisfying $\psi = 0$ on Γ_c, $\psi = \psi_b$ on Γ_b with $\psi_{bi} = \delta_{i1}$ when considering the drag and $\psi_{bi} = \delta_{i2}$ when considering the lift. In the following, we assume that a solution to (3.7) does exist and that this solution is $H^2(\Omega)^d \times H^1(\Omega)$–regular. Thus, we assume that (φ, r) satisfy the strong form of the dual problem

$$
\begin{aligned}
-\mu\Delta\varphi_i - \rho(u\cdot\nabla)\varphi_i + \frac{\rho}{2}\left(\theta^{(i)}\cdot\varphi - u_h\cdot\frac{\partial\varphi}{\partial x_i}\right) + \frac{\partial r}{\partial x_i} &= 0 \text{ in } \Omega,\ i = 1,\dots,d \quad (3.8)\\
\operatorname{div}\varphi &= 0 \text{ in } \Omega \\
\varphi &= \psi \text{ on } \Gamma,
\end{aligned}
$$

where

$$
\theta^{(i)} = \sum_K \frac{\partial u_h}{\partial x_i}(K)\chi_K.
$$

Here, χ_K is the characteristic function of element K. Multiplying the first equation by $u - u_h$ and integrating elementwise by parts, we obtain

$$
\begin{aligned}
&a_h(u-u_h,\varphi) + n_h(u, u-u_h, \varphi) + n_h(u-u_h, u_h, \varphi) + b_h(u-u_h, r) \\
&= e_2(\varphi, r; u, u-u_h), \qquad (3.9)
\end{aligned}
$$

where

$$
e_2(\varphi, r; u, v) := \sum_K \int_{\partial K} \left\{\mu\frac{\partial\varphi}{\partial n}\cdot v + \frac{\rho}{2}u\cdot nv\cdot\varphi - rv\cdot n\right\} d\gamma.
$$

Combining (3.8) with the error representation obtained in Sect. A. where we replace $i_h w^D$ by $i_h\varphi$

$$
\begin{aligned}
D - D_h &= -\Big\{a_h(u-u_h, i_h\varphi) + n_h(u, u-u_h, i_h\varphi) \\
&\quad + n_h(u-u_h, u_h, i_h\varphi) + b_h(i_h\varphi, p-p_h) + e_1(u, p; \varphi - i_h\varphi)\Big\},
\end{aligned}
$$

we get

$$
\begin{aligned}
D - D_h &= -\Big\{a_h(u-u_h, i_h\varphi - \varphi) + n_h(u, u-u_h, i_h\varphi-\varphi) + n_h(u-u_h, u_h, i_h\varphi - \varphi) \\
&\quad - b_h(u-u_h, r) + b_h(i_h\varphi, p-p_h) \qquad (3.10)\\
&\quad + e_1(u, p; \varphi - i_h\varphi) + e_2(\varphi, r; u, u-u_h)\Big\}.
\end{aligned}
$$

Let us consider each term in (3.9) separately. We have

$$|a_h(u-u_h, i_h\varphi-\varphi)| \le ch\,|\varphi|_2\,|u-u_h|_{1,h},$$

$$|n_h(u, u-u_h, i_h\varphi-\varphi) + n_h(u-u_h, u_h, i_h\varphi-\varphi)| \le ch\,(|u|_1 + |u_h|_{1,h})|\varphi|_2|u-u_h|_{1,h}.$$

Let r_h be the interpolant of r into Q_h. Then it holds that

$$|b_h(u-u_h, r)| = |b_h(u-u_h, r-r_h)| \le ch|r|_1\,|u-u_h|_{1,h}.$$

Moreover, we have

$$|b_h(i_h\varphi, p-p_h)| = |b_h(i_h\varphi-\varphi, p-p_h)| \le ch|\varphi|_2\|p-p_h\|_0.$$

Thus, it remains to estimate $e_1(u,p;\varphi-i_h\varphi)$ and $e_2(\varphi,r;u,u-u_h)$. The first term has already considered in Sec. 3.1 and can be estimated by

$$e_1(u,p;\varphi-i_h\varphi) \le M\,h^2\,.$$

For estimating the term $e_2(\varphi,r;u,u-u_h)$, we reformulate it as a sum over all element faces E

$$e_2(\varphi,r;u,u-u_h) = \sum_E \int_E \left\{\mu\frac{\partial\varphi}{\partial n}\cdot[|u-u_h|] + \frac{\rho}{2}u\cdot n[|u-u_h|]\cdot\varphi - r[|u-u_h|]\cdot n\right\} d\gamma,$$

where $[|\cdot|]$ denotes the jump across the face E in a fixed direction and outside of Ω, u and u_h are set to be zero. Since for any element face E

$$\int_E [|u-u_h|]d\gamma = 0,$$

we can use the L^2-projection onto the space of piecewise constants [18] to get

$$|e_2(\varphi,r;u,u-u_h)| \le c\,h\,(\|\varphi\|_2\,(1+\|u\|_2) + |r|_1)\,|u-u_h|_{1,h}\,.$$

Summarizing all estimates, we get the following

Theorem 3.3. *Assume that the linearized dual problem (3.7) has a solution $(\varphi,r) \in (V_\psi \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$. Moreover, let $(u,p) \in V_\psi \times Q$ and $(u_h,p_h) \in V_{\psi h} \times Q_h$ be the solutions of the continuous and the discrete problems (2.3) and (3.1), respectively. Then, there is a positive constant $M = M(\|u\|_2, |p|_1, \|\varphi\|_2, |r|_1)$ such that the drag and the lift, calculated by (3.2) and (3.3), satisfy the error estimate*

$$|D-D_h|,\ |L-L_h| \le Mh^2 \tag{3.11}$$

for sufficiently small h.

We now study a sufficient condition for the assumptions of Lemma 3.2 and Theorem 3.3. Suppose that the Stokes problem is regular in the following sense: For any function $f \in (L^p(\Omega))^d$, $p>1$, there exists a unique solution $(u,p) \in (V_0 \cap W^{2,p}(\Omega)^d) \times (Q \cap W^{1,p}(\Omega))$ of the Stokes problem,

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \end{aligned}$$

and

$$||u||_{2,p} + ||p||_{1,p} \le c||f||_{0,p},$$

where c is a positive constant independent of f.

Theorem 3.4. *Assume that the Stokes problem is regular. Then there exists a positive constant*

$$\mu_0 = \mu_0(||g||_{H^{1/2}(\Gamma)^d}, ||f||_{L^2(\Omega)^d})$$

independent of h such that for all $\mu \ge \mu_0$ there exists unique solutions (u,p) to (2.3) and (u_h, p_h) to (3.1), respectively, and that the estimate

$$|D - D_h|,\ |L - L_h| \le M\, h^2 \tag{3.12}$$

holds, where M is a positive constant independent of h.

Proof. In the following, c_i stands for a positive constant independent of h. The first part of Theorem 3.4 is well known. We also know an error estimate

$$||u - u_h||_{1,h} + ||p - p_h||_0 \le c_1 h(|u|_{2,2} + |p|_{1,2}).$$

Hence we have

$$\begin{aligned} ||u_h||_\infty &\le ||u_h - i_h u||_\infty + ||i_h u||_\infty \\ &\le c_2\,(h^{-d/6}||u_h - i_h u||_{1,h} + ||u||_{2,2}) \\ &\le c_2\,(h^{1-d/6}\,(|u|_{2,2} + |p|_{1,2}) + ||u||_{2,2}) \\ &\le c_3\,(||u||_{2,2} + |p|_{1,2})\,. \end{aligned} \tag{3.13}$$

Here, we have used that $||v_h||_{0,6} \le c_4||v_h||_{1,h}$, see for example [1, 2]. From (3.13), we can take the constant μ_0 of Lemma 3.2 independently of h.

From (3.8) the solution (φ, r) to (3.7) satisfies the Stokes equation

$$\begin{aligned} -\mu\Delta\varphi + \nabla r &= F \quad \text{in } \Omega, \\ \operatorname{div}\varphi &= 0 \quad \text{in } \Omega, \end{aligned}$$

where

$$F_i = \rho\left\{(u\cdot\nabla)\varphi_i - \frac{1}{2}\left(\theta^{(i)}\cdot\varphi - u_h\cdot\frac{\partial\varphi}{\partial x_i}\right)\right\}.$$

Subtracting a suitable smooth function satisfying the inhomogeneous boundary condition, we can assume that $\varphi \in V_0$. We now show that $F \in L^2(\Omega)^d$. Since $\theta_j^{(i)} \in L^2(\Omega)$ and $\varphi_j \in H^1(\Omega) \subset L^6(\Omega)$, we have $\theta^{(i)}\cdot\varphi \in L^{3/2}(\Omega)$. Hence $F_i \in L^{3/2}(\Omega)$ and we obtain $\varphi_i \in W^{2,3/2}(\Omega)$ from the regularity of the Stokes problem, which implies $\varphi_i \in L^p(\Omega)$ for any $p \in [1, +\infty)$. Continuing similar arguments, we reach $F_i \in L^2(\Omega)$, which implies $(\varphi, r) \in H^2(\Omega)^d \times H^1(\Omega)$. Thus, the assumption of Theorem 3.3 is satisfied and we get the conclusion. ∎

Remark. *In order to get the higher order convergence stated in Theorem 3.4, no change in the calculation of the drag and the lift has to be made. We note that the regularity assumption of the Stokes problem holds if the boundary is smooth [17].*

IV. NUMERICAL STUDIES

Numerical tests have been performed on a benchmark problem of the DFG priority research program *Flow Simulation with High–Performance Computers*, defined in [16]. There, the Navier–Stokes equations (2.3) describe the flow in a channel around a cylinder, see Figure 1, with $\mu = 10^{-3}$, $\rho = 1$, $f = 0$, the parabolic inflow and outflow profile

$$u(0, y) = u(2.2, y) = 0.41^{-2}(1.2y(0.41 - y), 0), \quad 0 \leq y \leq 0.41,$$

and no–slip conditions on the other boundaries. The mean value of the inflow is the characteristic velocity $U = 0.2$ and the characteristic length scale is the diameter of the cylinder $L = 0.1$. The computations have been performed using the standard Galerkin discretization with the nonconforming P_1/P_0–finite element.

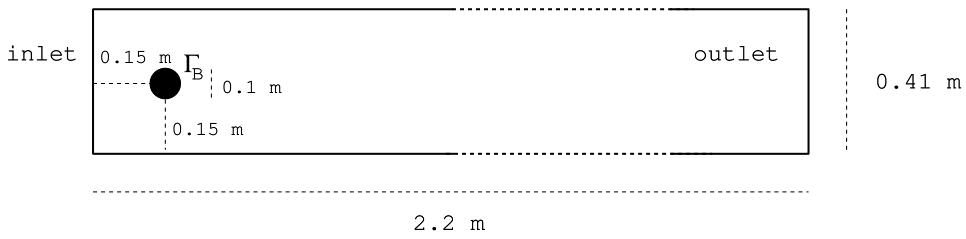

FIG. 1. Domain for the benchmark.

We have used the coarsest grids (level 0) depicted in Figure 2, Grid 1 (above) and Grid 2 (below). Uniform refinement up to level 6 results in nearly 3.2 million degrees of freedom for Grid 1 and in nearly 6.4 million degrees of freedom for Grid 2.

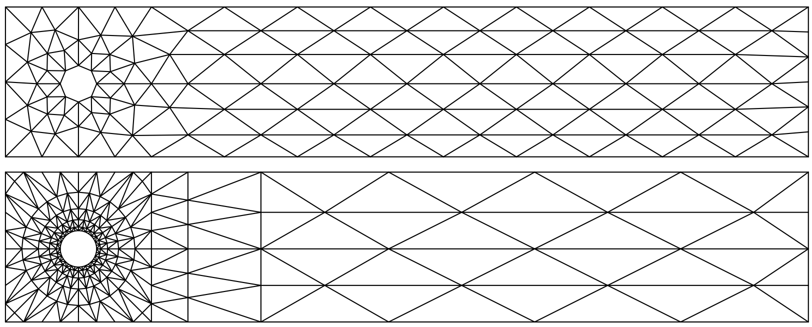

FIG. 2. Coarse grids for the computations, Grid 1 (above) and Grid 2 (below).

To have a better comparison to the benchmark results given in [16], we present instead of D_h, L_h the approximate drag coefficient $c_{D_h} := 2D_h/(\rho U^2 L)$ and the approximate lift coefficient $c_{L_h} := 2L_h/(\rho U^2 L)$. The exact values of the drag coefficient c_D and the lift coefficient c_L are not known. Starting from the assumption

$$|c_D - c_{D_h}| = Ch^\alpha,$$

we find numerically the order of convergence α by

$$\alpha_{num} := \log\left(\frac{|c_{D_{2h}} - c_{D_h}|}{|c_{D_h} - c_{D_{h/2}}|}\right) / \log(2).$$

Using (3.2) and (3.3) for the computation of c_{D_h} and c_{L_h}, we have obtained the values in Table I. The tests show an almost second order convergence that agrees well with the statements in Theorem 3.4.

TABLE I. Drag and lift coefficients with numerical order of convergence.

	Grid 1				Grid 2			
level	c_{D_h}	α_{num}	c_{L_h}	α_{num}	c_{D_h}	α_{num}	c_{L_h}	α_{num}
2	5.624571		4.223190-3		5.521853		1.276037-2	
3	5.617461		6.606430-3		5.566169		1.116959-2	
4	5.591106		9.465707-3		5.576298	2.129	1.075801-2	1.951
5	5.582555	1.624	1.032624-2	1.732	5.578733	2.056	1.065378-2	1.981
6	5.580296	1.920	1.054619-2	1.968	5.579333	2.021	1.062715-2	1.969

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