A Posteriori Error Estimators for the Nonconforming P_1–Finite Element Discretization of Convection–Diffusion Equations

Volker John
Otto–von–Guericke–Universität Magdeburg
Institut für Analysis und Numerik
Postfach 4120
D-39016 Magdeburg
Germany
email: volker.john@mathematik.uni-magdeburg.de

Abstract

In this paper we focus on adaptive methods for discretizing convection–diffusion equations by means of the nonconforming P_1–finite element. Local lower estimates for residual based a posteriori error estimators in a discrete H^1–norm and in the L^2–norm are proven. Moreover, a residual based a posteriori error estimator in the L^2–norm is derived. The behaviour of local error estimators is illustrated by numerical examples. Finally, we compare the global error estimate with the real error numerically and describe a way of approximating the size of a stability constant which is essential for the reliability of the error estimate.

Keywords. Convection–diffusion equations, nonconforming finite element methods, residual based a posteriori error estimators, adaptive grid refinement.

AMS(MOS) subject classifications. 65N15, 65N30, 65N50

1 Introduction

The importance of adaptive methods for the accurate and fast numerical solution of partial differential equations arising in applications, e.g. in computational fluid dynamics, has increased enormously during the last two decades. Adaptive methods aim at obtaining the numerical solution of an equation within a prescribed tolerance and with a minimal amount of numerical work. A posteriori error estimators are the most essential tool in adaptive methods. From practical point of view two–sided estimates are desirable, a local lower estimate and a global upper estimate. The local lower estimate controls the adaptive grid generation and the global upper estimate serves as a measure for the quality of the numerical solution and therefore as a stopping criterion of the solution process.

Lets given a partial differential equation with solution u, a discrete problem belonging to that equation with solution u_h, a norm $\|\cdot\|$, and an element–wise computed local a posteriori error estimator η_T. The usual practice consists in spending an increased amount of numerical work in elements which have a large estimated local error η_T. For this purpose we use adaptive mesh refinement, the so-called h–method. Let η be a global a posteriori error estimator.

Definition 1.1 *The efficiency index of an a posteriori error estimator is the ratio between the estimated and the real error*

$$I_{eff} = \frac{\eta}{\|u - u_h\|_\Omega}.$$

An error estimator is called efficient if I_{eff} and I_{eff}^{-1} are bounded for all triangulations.

An efficient error estimator is called robust for a class of problems if I_{eff} and I_{eff}^{-1} are bounded independently of the concrete problem.

The robustness of error estimators with respect to the diffusion parameter ν is a very severe problem for convection–diffusion equations. It turns out that this question is closely connected with the stability constant of the dual problem. Up to now only Angermann [Ang95] could develop a robust a posteriori error estimator for a rather complicated operator norm. Closely related are investigations on robust error estimators for the Stokes and Navier–Stokes equations. First results has been obtained by Rannacher and Johnson, see [JR94], [JRB95] and [Joh95].

Nonconforming finite element methods are attractive for problems in fluid dynamics, such as the Stokes or the Navier–Stokes equations. Often low order elements are used, e.g. the nonconforming P_1/P_0–finite element by Crouzeix and Raviart [CR73] and the nonconforming rotated Q_1/P_0–finite element by Rannacher and Turek [RT92]. These pairs of elements guarantee the Babuška–Brezzi stability uniformly with respect to the mesh size. In addition, they show an advantageous behaviour on parallel computers since all unknowns are stored on at most two processors, see e.g. Schieweck [Sch96] or John [Joh97]. Thus, one communication is sufficient to exchange information.

A comprehensive survey of different types of a posteriori error estimators has been given by Verfürth [Ver96b]. We consider in this paper so–called residual based a posteriori error estimators. These estimators contain the residual of the strong formulation of the partial differential equation to solve and terms which are the connection of the weak and the strong formulation. In general these terms can be expressed by jumps of derivatives of discrete functions across edges. For details of techniques of proving such error estimates for conforming discretizations see Verfürth [Ver96b] or Johnson et. all [EEHJ95].

In order to extent the results to nonconforming discretizations an additional control of the consistency error is needed. Dari et. all [DDPV96] have shown the necessity of additional terms in global a posteriori error estimators. They derived an a posteriori error estimator for the Laplace equation in the energy norm. The additional term is a sum of the jumps of the tangential derivative of the nonconforming function across the edges. The results had been extended to the Stokes equations [DDP95]. Residual based a posteriori error estimators for the energy norm and the nonconforming P_1–finite element discretization of the Laplace equation have been considered also by Wohlmuth in [Woh95]. There the additional term is a sum of the jumps of the nonconforming function across the edges. An essential rôle plays a saturation assumption with respect to the solution of the continuous problem and the solution of a conforming discrete problem with piecewise quadratics.

In this paper we consider the nonconforming P_1–finite element discretization of convection–diffusion problems. We present local a posteriori error estimators in a discrete H^1–norm and in the L^2–norm as well as a global error estimator in the L^2–norm. The local a posteriori error estimators in the discrete H^1–norm correspond in the special case of the Laplace equation with that of Dari et. all [DDPV96] and Wohlmuth [Woh95]. However, in these papers error estimators in the L^2–norm have not been considered. We present a comprehensive numerical study which demonstrates the mode of working of one of the local error estimators in the discrete H^1–norm and one of the local error estimators in the L^2–norm. The proof of the global a posteriori error estimator in the L^2–norm follows a technique of Johnson [EEHJ95]. It results in an estimate with a problem dependent stability constant as factor. Information about the size of this constant are crucial for obtaining a reliable global error estimate. A heuristic way of approximating this constant is presented.

The paper is organized as follows. In Section 2 we introduce the equations and notations. Section 3 deals with cut–off functions which are an essential tool of the investigations of the local error estimators in Section 4 and 5. In Section 6 the global error estimator is studied. Section 7 presents numerical results which illustrate the behaviour of the local error estimators. In the numerical tests of Section 8 the global error estimate is compared to real errors and a way for the approximation of the stability constant is described.

2 The Problem and Its Discretization, Notations

Let $\Omega \subset \mathbb{R}^2$ be bounded polygonal domain with boundary $\partial\Omega$. We consider the convection–diffusion equation

$$\begin{array}{rcll} -\nabla \cdot \nu \nabla u + b \cdot \nabla u + cu &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{2.1}$$

where $\nu \in \mathbb{R}^+$, $b \in \left(C^1(\overline{\Omega})\right)^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ are given functions and

$$c - 1/2(\nabla \cdot b) \geq c_0 > 0 \quad \text{in } \Omega$$

is fulfilled. For the solution u of (2.1) we assume

$$u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.2}$$

With $(\cdot,\cdot)$ we denote the inner product in $L^2(\Omega)$, with $(\cdot,\cdot)_\omega$ the inner product in $L^2(\omega)$ with $\omega \subset \Omega$. A weak formulation of (2.1) reads as follows

Find $u \in H_0^1(\Omega)$ with

$$a(u,v) + (b \cdot \nabla u, v) + (cu, v) \quad = \quad (f, v) \quad \forall v \in H_0^1(\Omega) \tag{2.3}$$

with

$$a(u,v) = (\nu \nabla u, \nabla v).$$

Problem (2.3) is called the convective form of the weak formulation of (2.1).

In this paper we focus on the nonconforming P_1–finite element discretization of (2.3). The nonconforming P_1–finite element has been introduced by Crouzeix and Raviart [CR73]. Let $\mathcal{T}_h$ be a shape regular decomposition of Ω into triangles. For a given triangulation we denote with $\{T_i\}_{i=1}^M$ the set of triangles, with $\{E_i\}_{i=1}^N$ the set of the edges of the triangles which do not belong to $\partial\Omega$, with $\{E_i\}_{i=N+1}^{N+N_D}$ the set of edges which belong to $\partial\Omega$, and with B_i the midpoint of the edge E_i, $i = 1,\ldots,N+N_D$. The space $P_k(T)$ is the space of all polynomials defined on T with degree not greater than k. Then the space of the nonconforming P_1–finite elements is defined as

$$V_h := \left\{ \begin{array}{l} v_h \in L^2(\Omega) \mid v_h|_T \in P_1(T)\ \forall T \in \mathcal{T}_h,\ v_h \text{ is continuous in } B_i, \\ i = 1,\ldots,N \text{ and } v_h(B_i) = 0,\ i = N+1,\ldots,N+N_D \end{array} \right\}.$$

The functions of V_h are linear on each triangle T_i, in general discontinuous across the edges, but continuous in the midpoints of the edges $\{B_i\}_{i=1}^N$. In V_h we introduce the norm

$$|v_h|_h = \left(\sum_{T \in \mathcal{T}_h} |v_h|_{1,T}^2 \right)^{1/2},$$

which is a norm due to the homogeneous boundary data.

Stabilisation techniques for the convective term $(b \cdot \nabla u, v)$ are necessary in order to obtain a stable discrete problem of (2.3). On the one hand upwind techniques are used, see Roos, Stynes and Tobiska [RST96], and on the other hand nonconforming streamline–diffusion methods, see John, Maubach and Tobiska [JMT96].

For $v \in H_0^1(\Omega)$ we set

$$\mathrm{curl} v = \begin{pmatrix} -\frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} \end{pmatrix}.$$

Every edge E which separates two neighbouring triangles T_1 and T_2 is associated with a uniquely oriented normal $n_E = (n_1, n_2)$ (for definiteness from T_1 to T_2). The jump of a function $v_h \in V_h$ across an edge E depends on this orientation and is denoted by

$$[|v_h|]_E = v_h|_{T_1} - v_h|_{T_2}.$$

The functions belonging to V_h fulfill

$$\int_E q\,[|v_h|]_E\,ds = 0 \quad \forall q \in P_0(E), v_h \in V_h. \tag{2.4}$$

With $t_E = (-n_2, n_1)$ we denote the tangential vector on E.

The symbol h_T stands for the diameter of a triangle (longest edge), the symbol h_E for the length of an edge, and the symbol ρ_T for the diameter of the largest inscribed ball. We shall consider only shape regular triangulations. Thus we assume the existence of a constant C such that

$$\frac{h_T}{\rho_T} \le C \quad \text{or} \quad \frac{h_T}{h_E} \le C \quad \forall T \in \mathcal{T}_h, \forall E \subset \partial T. \tag{2.5}$$

Let ω be the union of a number of triangles of $\mathcal{T}_h$. Then we define the discrete semi norm

$$|v_h|_{h,\omega} = \left(\sum_{T \subset \omega} |v_h|_{1,T}^2 \right)^{1/2}.$$

The symbol ω_E stands for the union of triangles with have the common edge E and the symbol ω_T for the union of all triangles which have a common edge with the triangle T, for typical situations away from the boundary $\partial\Omega$ see Figure 1.

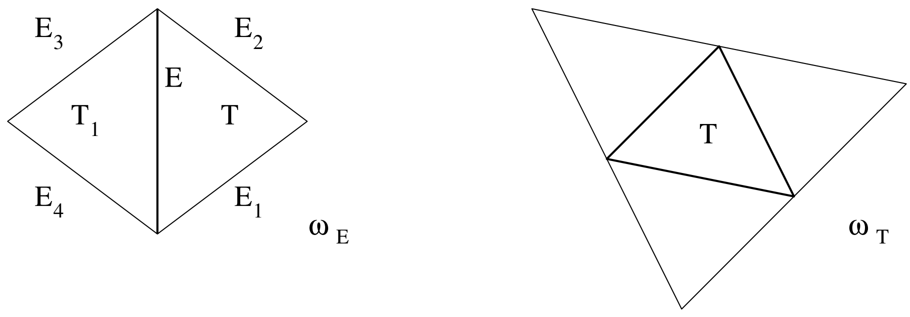

Figure 1: The local domains ω_E and ω_T.

With X_h we denote the space of conforming piecewise linear functions which vanish on $\partial\Omega$. The inclusion $X_h \subset V_h$ holds. Let $R_{X_h} : H^2(\Omega) \mapsto X_h$ be the standard nodal interpolation operator, satisfying

$$\|v - R_{X_h} v\|_{0,T} \le C_1 h_T^2 \|v\|_{2,T} \; \forall v \in H^2(T), \tag{2.6}$$

$$\|v - R_{X_h} v\|_{0,E} \le C_2 h_E^{3/2} \|v\|_{2,T} \; \forall v \in H^2(T), E \subset \partial T. \tag{2.7}$$

The constants C_1 and C_2 depend only on the smallest angle of the triangles. We set $C_I = \max\{C_1, C_2\}$.

With $\mathcal{M}_E^k : H^1(T) \mapsto P_k(E)$ we denote the $L^2(E)$–projection onto the space of the restriction to E of all polynomials of degree k. The inequality

$$\left| \int\limits_E u(v - \mathcal{M}_E^k v)\, ds \right| \le C h_T^{m+1/2} \|u\|_{0,E} |v|_{m+1,T} \tag{2.8}$$

with $0 \le m \le k$, for all elements $T \subset \omega_E$, $u \in L^2(E)$, and $v \in H^{m+1}(T)$ is proven in [JMT96].

Constants which are independent of h_T, h_E and ν are denoted by C.

3 Cut–off Functions and Auxiliary Results

We start with inverse estimates of norms and semi norms of polynomials in Sobolev spaces.

Lemma 3.1 *Let $\mathcal{T}_h$ be a shape regular triangulation and $\omega = T$ or $\omega = \omega_E$ or $\omega = \omega_T$. If $\omega = \omega_E$ then it is $\mathsf{h} = h_E$ else $\mathsf{h} = h_T$.*

i) For $m \ge 1, p \in [1, \infty]$ holds

$$\sum_{T \subset \omega} |v|_{m,p,T} \le \frac{C}{\mathsf{h}} \sum_{T \subset \omega} |v|_{m-1,p,T} \quad \forall v \text{ with } v|_T \in P_k(T),\ \forall T \subset \omega. \tag{3.1}$$

ii) For all elements $T \in \mathcal{T}_h$ and all edges $E \subset \partial T$ we have

$$\|v|_T\|_{0,E} \le C h_E^{-1/2} \|v\|_{0,T} \quad \forall v \in P_k(T). \tag{3.2}$$

Proof: The proof of i) follows the lines of Ciarlet [Cia91], Theorem 17.2. Each element is mapped by an affine transformation onto a reference element and the equivalence of norms in finite dimensional spaces is used. The choices $\mathsf{h} = h_E$ for $\omega = \omega_E$ and $\mathsf{h} = h_T$ for $\omega = \omega_T$ are possible due to the shape regularity of the triangulation. The proof of ii) requires in addition a standard estimation of the trace of v. □

Cut-off functions are a standard tool for proving local a posteriori error estimates, see e.g Verfürth [Ver96a] and [Ver96b]. Let $T \in \mathcal{T}_h$ be an arbitrary triangle and $E \subset \partial T$. Then we denote with λ_E the linear function which is zero on E and 1 in the corner of T opposite to E. With the notations of Figure 1 we define the so–called bubble functions

$$\mathrm{B}_T = \begin{cases} 27\lambda_E \lambda_{E_1} \lambda_{E_2} & \text{in } T \\ 0 & \text{otherwise,} \end{cases} \qquad \mathrm{B}_E = \begin{cases} 4\lambda_{E_1}\lambda_{E_2} & \text{in } T \\ 4\lambda_{E_3}\lambda_{E_4} & \text{in } T_1 \\ 0 & \text{otherwise.} \end{cases}$$

The form of these bubble functions is presented in Figure 2. Bubble functions which will be used for the local a posteriori error estimates in the L^2–norm have to be $C^1(\bar{\Omega})$. First we take

$$\mathrm{B}_T^* = \mathrm{B}_T^2.$$

Next we construct an edge bubble function B_E^*, see Figure 3. Let E be the common edge of the triangles T_1 and T_2. The other edges of T_1 are E_1 and E_2. The triangle T_1 is reflected on E with the images E_1^* and E_2^* of E_1 and E_2, respectively. In the same way we obtain E_3^* and E_4^* by reflecting T_2 on E. Then we define

$$\mathrm{B}_E^* = \begin{cases} 256\lambda_{E_1}^2 \lambda_{E_2}^2 \lambda_{E_1^*}^2 \lambda_{E_2^*}^2 & \text{on } T_1 \\ 256\lambda_{E_3}^2 \lambda_{E_4}^2 \lambda_{E_3^*}^2 \lambda_{E_4^*}^2 & \text{on } T_2 \\ 0 & \text{otherwise.} \end{cases}$$

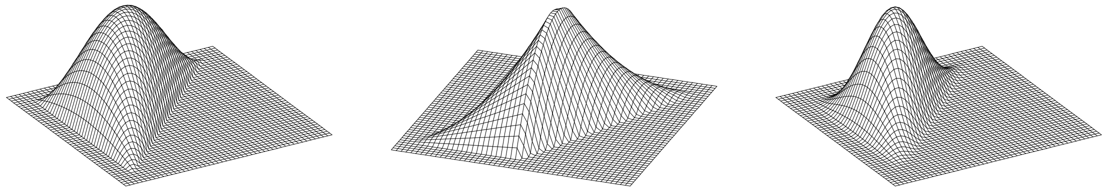

Figure 2: Cut–off functions $\mathrm{B}_T, \mathrm{B}_E, \mathrm{B}_T^*$.

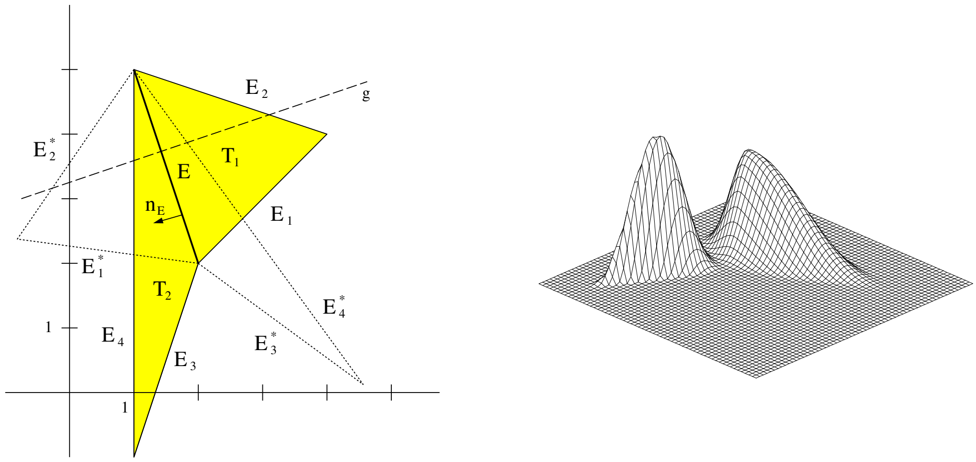

Figure 3: Cut–off function B_E^*.

Lemma 3.2 *The cut–off functions have the following properties*

i) $$\mathrm{B}_T, \mathrm{B}_E \in C^0(\bar{\Omega}), \quad \mathrm{B}_T|_T \in P_3(T), \mathrm{B}_E|_T \in P_2(T)\ \forall T \subset \omega_E,$$

ii) $$\mathrm{B}_T^*, \mathrm{B}_E^* \in C^1(\bar{\Omega}), \quad \mathrm{B}_T^*|_T \in P_6(T), \mathrm{B}_E^*|_T \in P_8(T)\ \forall T \subset \omega_E,$$

iii) $$0 \leq \mathrm{B}_T, \mathrm{B}_E, \mathrm{B}_T^*, \mathrm{B}_E^* \leq 1,$$

iv) $$\mathrm{B}_T = 0, \mathrm{B}_T^* = 0 \text{ on } \partial T, \quad \mathrm{B}_E = 0, \mathrm{B}_E^* = 0 \text{ on } \partial\omega_E, \tag{3.3}$$

v) $$\nabla \mathrm{B}_T^* = \mathbf{0} \text{ on } \partial T, \quad \nabla \mathrm{B}_E^* = \mathbf{0} \text{ on } \partial\omega_E, \tag{3.4}$$

vi) $$\nabla \mathrm{B}_E^* \cdot n_E = 0 \text{ on } E. \tag{3.5}$$

vii) B_E^* *is continuous on* E.

viii) $\mathrm{B}_E^*|_E$ *is an even function on* E *with respect to the midpoint of* E.

Proof: The properties i) – v) follow directly from the definitions of the cut–off functions.

vi) We consider a line g in the direction of n_E across E, see Figure 3. The definition of B_E^* on T_1 is continued on the mirror image of T_1. Then it follows from the definition of B_E^* that this function is continuous differentiable on g and symmetric to E. Therefore $\mathrm{B}_E^*|_g$ has a local extremum on E, that means $\nabla \mathrm{B}_E^* \cdot n_E = 0$ in the cross point of g and E. This is true for all points on E. Analogously, $\nabla \mathrm{B}_E^* \cdot n_E = 0$ on E is proven considering T_2.

vii) We start with T_1. On E it is $\lambda_{E_1} = \lambda_{E_1^*}$ and $\lambda_{E_2} = \lambda_{E_2^*}$. Hence, $\mathrm{B}_E^*|_E = 256\lambda_{E_1}^4\lambda_{E_2}^4$. With respect to T_2 holds $\lambda_{E_3} = \lambda_{E_3^*} = \lambda_{E_1}$ and $\lambda_{E_4} = \lambda_{E_4^*} = \lambda_{E_2}$ on E and therefore $\mathrm{B}_E^*|_E = 256\lambda_{E_1}^4\lambda_{E_2}^4$.

viii) From vii) we know $\mathrm{B}_E^*|_E = 256\lambda_{E_1}^4\lambda_{E_2}^4$. This function is an even function with respect to the midpoint of E. □

In the following sections we need uniform norm equivalences of weighted L^2–norms in finite dimensional spaces.

Lemma 3.3 *The following inequalities hold*

$$\|v\|_{0,T}^2 \le C\int_T v^2 \mathrm{B}_T dx \quad \forall v \in P_k(T), \tag{3.6}$$

$$\|v\|_{0,T}^2 \le C\int_T v^2 \mathrm{B}_T^* dx \quad \forall v \in P_k(T), \tag{3.7}$$

$$\|v\mathrm{B}_T\|_{0,T} \le \|v\|_{0,T} \quad \forall v \in P_k(T), \tag{3.8}$$

$$\|v\mathrm{B}_T^*\|_{0,T} \le \|v\|_{0,T} \quad \forall v \in P_k(T), \tag{3.9}$$

$$\|v\|_{0,E}^2 \le C\int_E v^2 \mathrm{B}_E dx \quad \forall v \in P_k(E), \tag{3.10}$$

$$\|v\|_{0,E}^2 \le C\int_E v^2 \mathrm{B}_E^* dx \quad \forall v \in P_k(E), \tag{3.11}$$

$$\|v\mathrm{B}_E\|_{0,\omega_E} \le \|v\|_{0,\omega_E} \quad \forall v \text{ with } v|_{T_i} \in P_k(T_i), T_i \subset \omega_E, \tag{3.12}$$

$$\|v\mathrm{B}_E^*\|_{0,\omega_E} \le \|v\|_{0,\omega_E} \quad \forall v \text{ with } v|_{T_i} \in P_k(T_i), T_i \subset \omega_E, \tag{3.13}$$

where C is a constant independent of T and h_T.

Proof: The proofs are based on techniques developed in Verfürth [Ver96b], Lemma 3.3. (3.6), (3.7), (3.10), and (3.11) follow by transformation on a reference element, the positivity of the bubble functions in the local domains with the exception of a set of measure zero and the equivalence of norms in finite dimensional spaces.

For proving (3.8), (3.9), (3.12), and (3.13) we use that the bubble functions are bounded on the considered domain. □

4 A Local A Posteriori Error Estimate in the discrete H^1–norm

Let T be an arbitrary element of the shape regular triangulation $\mathcal{T}_h$ and $u_h \in V_h$ be an arbitrary function. We define

$$\eta_{T,1}^2 = h_T^2\|f_h + \nu\Delta u_h - b_h\cdot\nabla u_h - c_h u_h\|_{0,T}^2 + \sum_{E\subset\partial T} h_E\|\,[|\nu\nabla u_h\cdot n_E|]_E\,\|_{0,E}^2, \tag{4.1}$$

$$\begin{aligned}\eta_{T,2}^2 &= h_T^2\|f_h + \nu\Delta u_h - b_h\cdot\nabla u_h - c_h u_h\|_{0,T}^2 \\ &\quad + \sum_{E\subset\partial T}\left(h_E\|\,[|\nu\nabla u_h\cdot n_E|]_E\,\|_{0,E}^2 + h_E^{-1}\|\nu\,[|u_h|]_E\,\|_{0,E}^2\right),\end{aligned} \tag{4.2}$$

where f_h, b_h and c_h are approximations on f, b, c such that the restriction to each element T of $f_h + \nu\Delta u_h - b_h\cdot\nabla u_h - c_h u_h$ belongs to $P_k(T)$ for some k. The first term of $\eta_{T,i}$ is the norm of the residual of the strong formulation (2.1) of the convection–diffusion equation. The second term involves jumps of the gradient of u_h across edges. The third term in $\eta_{T,2}$ measures the nonconformity of the function u_h. In this section we are going to show that under certain approximation properties on f_h, b_h and c_h a local inequality of the type

$$\eta_{T,i} \le C(\nu + h_T)|u - u_h|_{h,\omega_T} + \mathcal{O}(h_T^2 |u - u_h|_{h,\omega_T}) \quad \forall T \in \mathcal{T}_h, \; i = 1, 2$$

is valid.

In the next lemmata each term of (4.1) and (4.2) is estimated separately. The techniques which we use are standard ones, see e.g. Verfürth [Ver96b]. They are based on multiplying the terms of (4.1) and (4.2) with appropriate bubble functions and integrating by parts, where the occurring line integrals are absorbed by the bubble functions.

Lemma 4.1 *Let u be the solution of (2.1) and $u_h \in V_h$ an arbitrary function. Then the inequality*

$$\begin{aligned} & h_T \|f_h + \nu\Delta u_h - b_h \cdot \nabla u_h - c_h u_h\|_{0,T} \\ & \le \; C\, [\nu|u - u_h|_{1,T} + h_T\left(\|b\|_{\infty,T}|u - u_h|_{1,T} + \|c\|_{\infty,T}\|u - u_h\|_{0,T} + \|f - f_h\|_{0,T}\right) \\ & \quad + (\|b - b_h\|_{\infty,T} + h_T\|c - c_h\|_{\infty,T})\|u_h\|_{0,T}] \end{aligned} \tag{4.3}$$

holds $\forall T \in \mathcal{T}_h$.

Proof: We set $w_h = -\nu\Delta u_h + b_h \cdot \nabla u_h + c_h u_h$. Using (3.6), (2.1), integration by parts, and (3.3) we have

$$\begin{aligned} & \|f_h - w_h\|_{0,T}^2 \\ & \le \; C \left[\int_T \nu\nabla(u - u_h) \cdot \nabla\left((f_h - w_h)\mathrm{B}_T\right) dx + \int_T (b \cdot \nabla(u - u_h) + c(u - u_h))(f_h - w_h)\mathrm{B}_T dx \right. \\ & \quad \left. + \int_T ((b - b_h) \cdot \nabla u_h + (c - c_h)u_h)(f_h - w_h)\mathrm{B}_T dx + \int_T (f_h - f)(f_h - w_h)\mathrm{B}_T dx \right]. \end{aligned}$$

Applying the Cauchy–Schwarz inequality, (3.1), and (3.8) we obtain

$$\begin{aligned} & \|f_h - w_h\|_{0,T}^2 \\ & \le \; C\left[\nu h_T^{-1}|u - u_h|_{1,T} + \|b\|_{\infty,T}|u - u_h|_{1,T} + \|c\|_{\infty,T}\|u - u_h\|_{0,T} + \|f - f_h\|_{0,T} \right. \\ & \quad \left. + \left(h_T^{-1}\|b - b_h\|_{\infty,T} + \|c - c_h\|_{\infty,T}\right)\|u_h\|_{0,T}\right] \|f_h - w_h\|_{0,T}. \end{aligned}$$

From this the statement of the lemma follows. □

Before estimating the jumps of the normal derivatives across the edges we note that $[|\nu\nabla u_h \cdot n_E|]_E$ is a constant function on each edge E, therefore

$$\| \, [|\nu\nabla u_h \cdot n_E|]_E \, \|_{0,E} = h_E^{1/2} | \, [|\nu\nabla u_h \cdot n_E|]_E \, |. \tag{4.4}$$

Let us denote with $[|\nu\nabla u_h \cdot n_E|]_E^C$ the constant continuation of $[|\nu\nabla u_h \cdot n_E|]_E$ to ω_E. Then, taking into consideration the regularity of the triangulation we have the estimate

$$\left\| [|\nu\nabla u_h \cdot n_E|]_E^C \right\|_{0,\omega_E} \le C h_E^{1/2} \| \, [|\nu\nabla u_h \cdot n_E|]_E \, \|_{0,E}. \tag{4.5}$$

Lemma 4.2 *Assume the shape regularity of the triangulation, the regularity of the exact solution given by (2.2), and $u_h \in V_h$ an arbitrary function. Then, for each edge E we have*

$$
\begin{aligned}
& h_E^{1/2} \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E \,\|_{0,E} \\
& \quad \leq \; C \left[\nu |u-u_h|_{h,\omega_E} + h_E \left(\|b\|_{\infty,\omega_E} |u-u_h|_{h,\omega_E} + \|c\|_{\infty,\omega_E} \|u-u_h\|_{0,\omega_E} + \|f-f_h\|_{0,\omega_E} \right) \right. \\
& \qquad \left. + \left(\|b-b_h\|_{\infty,\omega_E} + h_E \|c-c_h\|_{\infty,\omega_E} \right) \|u_h\|_{0,\omega_E} \right]. \qquad (4.6)
\end{aligned}
$$

Proof: The proof follows the arguments of Lemma 4.1, now with B_E instead of B_T. Taking into account that ∇u is continuous a.e. we obtain with the help of (3.10), element–wise integration by parts, and (3.3)

$$
\begin{aligned}
& \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E \,\|_{0,E}^2 \\
& \quad \leq \; C \left\| \sum_{T \subset \omega_E} \int_T \nu \nabla (u_h - u) \cdot \nabla \left([\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E \right) dx + \int_T \nu \Delta (u_h - u) \, [\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E dx \right\| .
\end{aligned}
$$

Both terms are estimated separately. For the first one we get using the Cauchy–Schwarz inequality, (3.1), (3.12), and (4.4)

$$
\sum_{T \subset \omega_E} \int_T \nu \nabla (u_h - u) \cdot \nabla ([\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E) dx \quad \leq \quad C h_E^{-1/2} \nu |u - u_h|_{h,\omega_E} \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E \,\|_{0,E}.
$$

The second term is split into three parts

$$
\begin{aligned}
& \sum_{T \subset \omega_E} \int_T \nu \Delta (u_h - u) \, [\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E dx \\
& \quad \leq \; \|\nu \Delta (u_h - u)\|_{0,\omega_E} \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E \|_{0,\omega_E} \\
& \quad \leq \; \Big(\| - f_h + b \cdot \nabla u_h + c u_h + f - b \cdot \nabla u - c u \|_{0,\omega_E} + \|f_h + \nu \Delta u_h - b_h \cdot \nabla u_h - c_h u_h\|_{0,\omega_E} \\
& \qquad + \|(b - b_h) \cdot \nabla u_h - (c - c_h) u_h \|_{0,\omega_E} \Big) \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E \|_{0,\omega_E}.
\end{aligned}
$$

The first part is estimated using the Cauchy–Schwarz inequality and (3.12)

$$
\begin{aligned}
& \| - f_h + b \cdot \nabla u_h + c u_h + f - b \cdot \nabla u - c u \|_{0,\omega_E} \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E \|_{0,\omega_E} \\
& \quad \leq \; C h_E^{1/2} \left[\|f - f_h\|_{0,\omega_E} + \|b\|_{\infty,\omega_E} |u - u_h|_{h,\omega_E} + \|c\|_{\infty,\omega_E} \|u - u_h\|_{0,\omega_E} \right] \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E \,\|_{0,E}.
\end{aligned}
$$

For the second part we apply (4.3) for all triangles which belong to ω_E and use $h_E \leq h_T \leq C h_E$

$$
\begin{aligned}
& \|f_h + \nu \Delta u_h - b_h \cdot \nabla u_h - c_h u_h\|_{0,\omega_E} \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E \|_{0,\omega_E} \\
& \quad \leq \; C h_E^{1/2} \Big[\nu h_E^{-1} |u - u_h|_{h,\omega_E} + (\|b\|_{\infty,\omega_E} |u - u_h|_{h,\omega_E} + \|c\|_{\infty,\omega_E} \|u - u_h\|_{0,\omega_E} \\
& \qquad + \|f_h - f\|_{0,\omega_E}) + (h_E^{-1} \|b - b_h\|_{\infty,\omega_E} + \|c - c_h\|_{\infty,\omega_E}) \|u_h\|_{0,\omega_E} \Big] \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E \,\|_{0,E}.
\end{aligned}
$$

The third part is estimated in the proof of Lemma 4.1

$$
\begin{aligned}
& \|(b - b_h) \cdot \nabla u_h - (c - c_h) u_h \|_{0,\omega_E} \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E^C \,\mathrm{B}_E \|_{0,\omega_E} \\
& \quad \leq \; C h_E^{1/2} \|u_h\|_{0,\omega_E} \left(h_E^{-1} \|b - b_h\|_{\infty,\omega_E} + \|c - c_h\|_{\infty,\omega_E} \right) \| \,[\![\nu \nabla u_h \cdot n_E]\!]_E \,\|_{0,E}.
\end{aligned}
$$

Summarizing the individual estimates we obtain inequality (4.6). $\square$

The jump $\left[\left|\frac{\partial u_h}{\partial t_E}\right|\right]_E$ of the tangential derivative of u_h across E is a constant function on each edge. We denote by $\left[\left|\frac{\partial u_h}{\partial t_E}\right|\right]_E^C$ its continuation on ω_E. For shape regular triangulations we have

$$\left\| \left[\left|\frac{\partial u_h}{\partial t_E}\right|\right]_E^C \right\|_{0,\omega_E} \leq C h_E^{1/2} \left\| \left[\left|\frac{\partial u_h}{\partial t_E}\right|\right]_E \right\|_{0,E}. \tag{4.7}$$

Lemma 4.3 *Assume the shape regularity of the triangulation, the regularity of the exact solution given by (2.2), and $u_h \in V_h$ an arbitrary function. Then, for each edge E we have*

$$h_E^{1/2} \left\| \left[\left|\nu \frac{\partial u_h}{\partial t_E}\right|\right]_E \right\|_{0,E} \leq C\nu |u - u_h|_{h,\omega_E}. \tag{4.8}$$

Proof: With the same steps like in the proof of Lemma 4.2 we obtain

$$\begin{aligned} \left\| \left[\left|\nu \frac{\partial u_h}{\partial t_E}\right|\right]_E \right\|_{0,E}^2 &\leq C \left\| \sum_{T \subset \omega_E} \int_T \nabla \left(\left[\left|\nu \frac{\partial u_h}{\partial t_E}\right|\right]_E^C \mathrm{B}_E \right) \cdot (\mathrm{curl}\,(\nu(u - u_h)))\, dx \right\| \\ &\leq C h_E^{-1/2} \nu |u - u_h|_{h,\omega_E} \left\| \left[\left|\nu \frac{\partial u_h}{\partial t_E}\right|\right]_E \right\|_{0,E}. \end{aligned}$$

□

Remark 4.4 Jumps of the nonconforming function. For the piecewise linear nonconforming element we have

$$\left\| \left[\left|\frac{\partial u_h}{\partial t_E}\right|\right]_E \right\|_{0,E}^2 = \frac{4}{h_E} \left(u_h(P_1) - u_h(P_2) + u_h(P_3) - u_h(P_4)\right)^2$$

and

$$\| [|u_h|]_E \|_{0,E}^2 = \frac{h_E}{3} \left(u_h(P_1) - u_h(P_2) + u_h(P_3) - u_h(P_4)\right)^2,$$

where the notations of Figure 4 have been used. That means, it is possible to replace the jumps of the tangential derivative across the edges with the jumps of the nonconforming function. Under the assumptions of Lemma 4.3 it follows from (4.8)

$$h_E^{-1/2} \|\nu\, [|u_h|]_E \|_{0,E} \leq C\nu |u - u_h|_{h,\omega_E} \tag{4.9}$$

for each edge E.

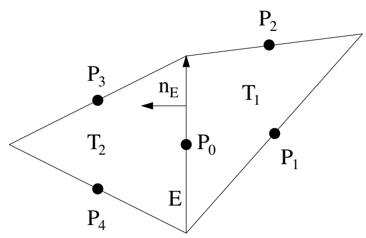

Figure 4: Figure for Remark 4.4.

Remark 4.5 Uniform boundedness of $\|u_h\|_{0,T}$. We assume that $\|u_h\|_{0,T}$ is bounded uniformly with respect to h_T and ν for all elements. If u_h is computed using a scheme which guarantees uniform L^2–stability, i.e. $\|u_h\|_0 \le C\|f\|_0$ with C independent of ν and the mesh then the assumption of uniform boundness of $\|u_h\|_T$ is fulfilled. The uniform L^2–stability of u_h can be proven if the discrete system is coercive with respect to a discrete energy norm, e.g. for nonconforming streamline diffusion finite element methods using Lemma 1 of [JMT96] and for upwind methods using Theorem 3 of Ohmori and Ushijima [OU84].

We are interested in considering the terms with $\|f-f_h\|_{0,\omega_T}$, $\|b-b_h\|_{\infty,\omega_T}$, and $\|c-c_h\|_{\infty,\omega_T}$ as higher order and assume the approximation properties

$$\begin{array}{l}\|b-b_h\|_{\infty,\omega_T}\|u_h\|_{0,\omega_T} \le Ch_T^2|u-u_h|_{h,\omega_T}, \quad \|f-f_h\|_{0,\omega_T} \le Ch_T|u-u_h|_{h,\omega_T}, \\ \|c-c_h\|_{\infty,\omega_T}\|u_h\|_{0,\omega_T} \le Ch_T|u-u_h|_{h,\omega_T}.\end{array} \tag{4.10}$$

THEOREM 4.6 *Under the regularity assumption (2.2), the approximation properties (4.10), a uniform boundness with respect to h_T and ν of $\|u_h\|_{0,T}$ for all elements, and the shape regularity of the mesh the estimate*

$$\begin{aligned}\eta_{T,i} \quad \le \quad & C\left[\nu|u-u_h|_{h,\omega_T} + h_T\left(\|b\|_{\infty,\omega_T}|u-u_h|_{h,\omega_T} + \|c\|_{\infty,\omega_T}\|u-u_h\|_{0,\omega_T}\right)\right] \\ & +\mathcal{O}(h_T^2|u-u_h|_{h,\omega_T}), \; i=1,2\end{aligned} \tag{4.11}$$

holds for $u_h \in V_h$ and each $T \in \mathcal{T}_h$.

If h_T is sufficiently small in the sense $h_T\|b\|_{\infty,\omega_T} < \nu$ and if $\|u-u_h\|_{0,\omega_T} \le \mathcal{O}(h_T|u-u_h|_{h,\omega_T})$ then

$$\eta_{T,i} \le C\nu|u-u_h|_{h,\omega_T} + \mathcal{O}(h_T^2|u-u_h|_{h,\omega_T}), \; i=1,2$$

holds for $u_h \in V_h$.

Proof: The statements are combinations of the results from Lemma 4.1, Lemma 4.2, Lemma 4.3, and (4.9). □

Remark 4.7 The rôle of the nonconformity. The estimate (4.11) holds for conforming as well as nonconforming P_1–finite elements. This can be seen from a detailed inspection of the proof. Of course, in the conforming case the jumps of the discrete function across the edges vanish.

Remark 4.8 The approximation properties. Theorem 4.6 is valid for any function $u_h \in V_h$, for which $\|u_h\|_{0,T}$ is uniformly bounded with respect to h_T and ν, independent if u_h solves a discrete problem approximating (2.1) or not. Therefore, the local error estimate can be applied on the solution u_h of a scheme including stabilisation or numerical integration. The approximation properties (4.10) are only necessary for the computation of the local error estimators (4.1) and (4.2).

Remark 4.9 Generalisations of the results. The extension of the results to higher order finite elements requires a more technical extension of the jumps $[|\nu\nabla u_h \cdot n_E|]_E$ and $\left[\left|\frac{\partial u_h}{\partial t_E}\right|\right]_E$ onto ω_E in such a way that inequalities of the form (4.5) and (4.7) are still valid, see e.g. Verfürth [Ver96b].

5 A Local L^2–A Posteriori Error Estimate

On each element T we define the local a posteriori error estimators

$$\tilde\eta_{T,1}^2 = h_T^4\|f_h+\nu\Delta u_h - b_h\cdot\nabla u_h - c_h u_h\|_{0,T}^2 + \sum_{E\subset\partial T} h_E^3\|\,[\![\nu\nabla u_h\cdot n_E]\!]_E\,\|_{0,E}^2, \qquad (5.1)$$

$$\begin{aligned}\tilde\eta_{T,2}^2 &= h_T^4\|f_h+\nu\Delta u_h - b_h\cdot\nabla u_h - c_h u_h\|_{0,T}^2 \\ &\quad + \sum_{E\subset\partial T}\left(h_E^3\|\,[\![\nu\nabla u_h\cdot n_E]\!]_E\,\|_{0,E}^2 + h_E\|\nu\,[\![u_h]\!]_E\,\|_{0,E}^2\right).\end{aligned} \qquad (5.2)$$

The functions f_h, b_h, c_h are again approximations of f, b, c such that that $f_h+\nu\Delta u_h - b_h\cdot\nabla u_h - c_h u_h \in P_k(T)$ for some k. The structure of $\tilde\eta_{T,i}$ is the same like that of $\eta_{T,i}$, $i=1,2$. The only difference consists in the weights of the terms.

In this section we are going to show that under appropriate approximation assumptions on f_h, b_h, c_h a local a posteriori error estimate of the form

$$\tilde\eta_{T,2} \le C(\nu+h_T)||u-u_h||_{0,\omega_T} + \mathcal{O}(h_T^3\|u-u_h\|_{0,\omega_T}) \quad \forall T\in\mathcal{T}_h$$

holds and for a constant convective vector b even

$$\tilde\eta_{T,1} \le C(\nu+h_T)\|u-u_h\|_{0,\omega_T} + \mathcal{O}(h_T^3\|u-u_h\|_{0,\omega_T}) \quad \forall T\in\mathcal{T}_h.$$

The basic technique for deriving such estimates is the same as in Section 4. However, due to additional integrating by parts new line integrals on the boundary of the local domains ω_T and ω_E as well as on the edge E in ω_E occur. To handle these new terms the special bubble functions B_T^* and B_E^* have to be used.

Lemma 5.1 *Let u be the solution of (2.1) and $u_h\in V_h$ an arbitrary function. Then the inequality*

$$\begin{aligned} & h_T^2\|f_h+\nu\Delta u_h - b_h\cdot\nabla u_h - c_h u_h\|_{0,T} \\ \le\ & C\Big[\nu\|u-u_h\|_{0,T} + h_T\|b\|_{\infty,T}\|u-u_h\|_{0,T} + h_T^2\left(\|\nabla\cdot b\|_{\infty,T}+\|c\|_{\infty,T}\right)\|u-u_h\|_{0,T} \\ & + h_T^2\|f-f_h\|_{0,T} + h_T(\|b-b_h\|_{\infty,T} + h_T\|c-c_h\|_{\infty,T})\|u_h\|_{0,T}\Big] \end{aligned} \qquad (5.3)$$

holds $\forall T\in\mathcal{T}_h$.

Proof: The proof starts like that of Lemma 4.1 with B_T replaced by B_T^*. We obtain with $w_h = -\nu\Delta u_h + b_h\cdot\nabla u_h + c_h u_h$

$$\begin{aligned} & \|f_h-w_h\|_{0,T}^2 \\ \le\ & C\Bigg[\int_T \nu\nabla(u-u_h)\cdot\nabla\left((f_h-w_h)\mathrm{B}_T^*\right)dx + \int_T (b\cdot\nabla(u-u_h)+c(u-u_h))(f_h-w_h)\mathrm{B}_T^* dx \\ & + \int_T ((b-b_h)\cdot\nabla u_h + (c-c_h)u_h)(f_h-w_h)\mathrm{B}_T^* dx + \int_T (f_h-f)(f_h-w_h)\mathrm{B}_T^* dx\Bigg]. \end{aligned}$$

Now, all terms which have derivatives of u and u_h are integrated by parts. The line integral on $\partial\omega_T$ vanishes as a consequence of (3.3) and (3.4)

$$\|f_h-w_h\|_{0,T}^2$$

$$
\begin{aligned}
\leq\; & C\Bigg[-\int_T \nu(u-u_h)\Delta\left((f_h-w_h)\mathrm{B}_T^*\right)dx-\int_T((u-u_h)\nabla\cdot((f_h-w_h)\mathrm{B}_T^* b)\,dx \\
& +\int_T c(u-u_h)(f_h-w_h)\mathrm{B}_T^* dx+\int_T((b-b_h)\cdot\nabla u_h+(c-c_h)u_h)(f_h-w_h)\mathrm{B}_T^* dx \\
& +\int_T (f_h-f)(f_h-w_h)\mathrm{B}_T^* dx\Bigg].
\end{aligned}
$$

Applying the Cauchy–Schwarz inequality, (3.1), and (3.9) we get

$$
\begin{aligned}
& \|f_h-w_h\|_{0,T}^2 \\
& \leq C\left[\nu\|u-u_h\|_{0,T}\|\Delta\left((f_h-w_h)\mathrm{B}_T^*\right)\|_{0,T}+\|u-u_h\|_{0,T}\left(\|\nabla\cdot b\|_{\infty,T}\|(f_h-w_h)\mathrm{B}_T^*\|_{0,T}\right.\right. \\
& \quad +\|b\|_{\infty,T}\|\nabla(f_h-w_h)\mathrm{B}_T^*\|_{0,T})+\|c\|_{\infty,T}\|u-u_h\|_{0,T}\|(f_h-w_h)\mathrm{B}_T^*\|_{0,T} \\
& \quad +\left(\|b-b_h\|_{\infty,T}|u_h|_{1,T}+\|c-c_h\|_{\infty,T}\|u_h\|_{0,T}\right)\|(f_h-w_h)\mathrm{B}_T^*\|_{0,T} \\
& \quad \left.+\|f-f_h\|_{0,T}\|(f_h-w_h)\mathrm{B}_T^*\|_{0,T}\right] \\
& \leq C\Big[\nu h_T^{-2}\|u-u_h\|_{0,T}\|+h_T^{-1}\|b\|_{\infty,T}\|u-u_h\|_{0,T}+\left(\|\nabla\cdot b\|_{\infty,T}+\|c\|_{\infty,T}\right)\|u-u_h\|_{0,T} \\
& \quad +\|f-f_h\|_{0,T}+\left(h_T^{-1}\|b-b_h\|_{\infty,T}+\|c-c_h\|_{\infty,T}\right)\|u_h\|_{0,T}\Big]\|f_h-w_h\|_{0,T}.
\end{aligned}
$$

This is the statement of the lemma. □

Lemma 5.2 *Assume the shape regularity of the triangulation, the regularity of the exact solution given by (2.2), and $u_h \in V_h$ an arbitrary function. Then, for each edge E we have*

$$
h_E^{1/2}\|\nu\,[|u_h|]_E\,\|_{0,E}\leq C\left(\nu\|u-u_h\|_{0,\omega_E}+\nu h_E^2|u|_{2,\omega_E}\right). \tag{5.4}
$$

Proof: Let $R_{X_h}u \in X_h$ be the interpolant of u and $\omega_E = T_1 \cup T_2$. Using (3.2) and (2.6) we get

$$
\begin{aligned}
\|\nu\,[|u_h|]_E\,\|_{0,E}^2 &= \int_E\left[\nu\left(u_h-R_{X_h}u\right)|_{T_1}-\nu\left(u_h-R_{X_h}u\right)|_{T_2}\right]^2 ds \\
&\leq 2\left\{\int_E\left[\nu\left(u_h-R_{X_h}u\right)|_{T_1}\right]^2+\int_E\left[\nu\left(u_h-R_{X_h}u\right)|_{T_2}\right]^2\right\} \\
&\leq Ch_E^{-1}\|\nu(u_h-R_{X_h}u)\|_{0,\omega_E}^2 \\
&\leq C\left(\nu^2 h_E^{-1}\|u-u_h\|_{0,\omega_E}^2+\nu^2 h_E^3|u|_{2,\omega_E}^2\right).
\end{aligned}
$$

□

Lemma 5.3 *Assume the shape regularity of the triangulation, the regularity of the exact solution given by (2.2), and $u_h \in V_h$ an arbitrary function. Then, for each edge E we have*

$$
\begin{aligned}
& h_E^{3/2}\|\,[|\nu\nabla u_h\cdot n_E|]_E\,\|_{0,E} \\
& \leq C\Big[\nu\|u-u_h\|_{0,\omega_E}+h_E\|b\|_{\infty,\omega_E}\|u-u_h\|_{0,\omega_E}+h_E^4|b|_{1,\infty,\omega_E}|u|_{2,\omega_E} \\
& \quad +h_E^2\left(\|\nabla\cdot b\|_{\infty,\omega_E}+|b|_{1,\infty,\omega_E}+\|c\|_{\infty,\omega_E}\right)\|u-u_h\|_{0,\omega_E}+h_E^2\|f-f_h\|_{0,\omega_E} \\
& \quad +h_E(\|b-b_h\|_{\infty,\omega_E}+h_E\|c-c_h\|_{\infty,\omega_E})\|u_h\|_{0,\omega_E}\Big].
\end{aligned} \tag{5.5}
$$

Proof: The proof starts like in Lemma 4.2 with B_E replaced by B_E^*,

$$\| [|\nu \nabla u_h \cdot n_E|]_E \|_{0,E}^2$$
$$\leq \; C \left\| \sum_{T \subset \omega_E} \left(\int_T \nu \nabla (u_h - u) \cdot \nabla ([|\nu \nabla u_h \cdot n_E|]_E^C \, \mathrm{B}_E^*) dx + \int_T \nu \Delta (u_h - u) \, [|\nu \nabla u_h \cdot n_E|]_E^C \, \mathrm{B}_E^* dx \right) \right\|.$$

Integrating the first term by parts we obtain

$$\sum_{T \subset \omega_E} \int_T \nu \nabla (u_h - u) \cdot \nabla ([|\nu \nabla u_h \cdot n_E|]_E^C \, \mathrm{B}_E^*) dx$$
$$= \; - \sum_{T \subset \omega_E} \int_T \nu (u_h - u) \, [|\nu \nabla u_h \cdot n_E|]_E^C \, \Delta \mathrm{B}_E^* dx + \int_{\partial \omega_E} \nu (u_h - u) \, [|\nu \nabla u_h \cdot n_E|]_E^C \, (\nabla \mathrm{B}_E^* \cdot n_{\omega_E}) ds$$
$$+ \int_E \nu \, [|u_h|]_E \, [|\nu \nabla u_h \cdot n_E|]_E \, (\nabla \mathrm{B}_E^* \cdot n_E) ds.$$

The line integral on $\partial \omega_E$ vanishes since (3.4) holds. The line integral on E is a result of the nonconformity of u_h. It vanishes as a consequence of (3.5). Using the Cauchy–Schwarz inequality, (3.1), and (3.13) the estimate of the first term is finished as follows

$$\begin{aligned} \sum_{T \subset \omega_E} \int_T \nu \nabla (u_h - u) \cdot \nabla ([|\nu \nabla u_h \cdot n_E|]_E^C \, \mathrm{B}_E^*) dx \;&\leq\; \nu \|u - u_h\|_{0,\omega_E} \| \, [|\nu \nabla u_h \cdot n_E|]_E^C \, \Delta \mathrm{B}_E^* \|_{0,\omega_E} \\ &\leq\; C h_E^{-3/2} \nu \|u - u_h\|_{0,\omega_E} \| \, [|\nu \nabla u_h \cdot n_E|]_E \, \|_{0,E}. \end{aligned}$$

In the second term we use (2.1) and integrate the convective term element–wise by parts. Since B_E^* vanishes on $\partial \omega_E$ we have

$$\begin{aligned} &\sum_{T \subset \omega_E} \int_T \nu \Delta (u_h - u) \, [|\nu \nabla u_h \cdot n_E|]_E^C \, \mathrm{B}_E^* dx \\ &= \sum_{T \subset \omega_E} \Bigg[\int_T ((f - f_h) - c(u - u_h)) \, [|\nu \nabla u_h \cdot n_E|]_E^C \, \mathrm{B}_E^* dx \\ &\quad + \int_T (u - u_h) \, [|\nu \nabla u_h \cdot n_E|]_E^C \, \nabla \cdot (\mathrm{B}_E^* b) dx \\ &\quad + \int_T (f_h + \nu \Delta u_h - b_h \cdot \nabla u_h - c_h u_h) \, [|\nu \nabla u_h \cdot n_E|]_E^C \, \mathrm{B}_E^* dx \\ &\quad - \int_T ((b - b_h) \cdot \nabla u_h + (c - c_h) u_h) \, [|\nu \nabla u_h \cdot n_E|]_E^C \, \mathrm{B}_E^* dx \Bigg] \\ &\quad + \int_E [|u_h|]_E \, [|\nu \nabla u_h \cdot n_E|]_E \, \mathrm{B}_E^* (b \cdot n_E) ds. \end{aligned}$$

We consider the integral on the inner edge E. Let B_E be the midpoint of E and b_0 the constant function with the value $b_0 = b(B_E)$. Then it follows

$$\int_E [|u_h|]_E \, [|\nu \nabla u_h \cdot n_E|]_E \, \mathrm{B}_E^* (b \cdot n_E) ds$$
$$= \int_E [|u_h|]_E \, [|\nu \nabla u_h \cdot n_E|]_E \, \mathrm{B}_E^* (b_0 \cdot n_E) ds + \int_E [|u_h|]_E \, [|\nu \nabla u_h \cdot n_E|]_E \, \mathrm{B}_E^* ((b - b_0) \cdot n_E) ds.$$

The first integral vanishes since $[|u_h|]_E$ is an odd function with respect to the midpoint of E and B_E^* is an even function

$$\int_E [|u_h|]_E\, [|\nu \nabla u_h \cdot n_E|]_E\, \mathrm{B}_E^*(b_0 \cdot n_E)ds = [|\nu \nabla u_h \cdot n_E|]_E\, (b_0 \cdot n_E) \int_E [|u_h|]_E\, \mathrm{B}_E^* ds = 0. \tag{5.6}$$

From the assumptions on b it follows $b \in (W^{1,\infty}(E))^2$ and with Taylor expansion

$$\|(b - b_0) \cdot n_E\|_{\infty,E} \le C h_E |b \cdot n_E|_{1,\infty,E}.$$

Using (5.4) with $\nu = 1$ we obtain the estimate

$$\begin{aligned}
&\int_E [|u_h|]_E\, [|\nu \nabla u_h \cdot n_E|]_E\, \mathrm{B}_E^*(b \cdot n_E)ds \\
&\quad\le\; C h_E |b \cdot n_E|_{1,\infty,E} \|\, [|u_h|]_E\, \|_{0,E} \|\, [|\nu \nabla u_h \cdot n_E|]_E\, \|_{0,E} \\
&\quad\le\; C h_E^{1/2} |b|_{1,\infty,\omega_E} \left(\|u - u_h\|_{0,\omega_E} + h_E^2 |u|_{2,\omega_E} \right) \|\, [|\nu \nabla u_h \cdot n_E|]_E\, \|_{0,E}.
\end{aligned}$$

The remaining integrals can be estimated using the Cauchy–Schwarz inequality, (4.4), and (5.3). We get for the individual terms

$$\begin{aligned}
&\sum_{T \subset \omega_E} \int_T ((f - f_h) - c(u - u_h))\, [|\nu \nabla u_h \cdot n_E|]_E^C\, \mathrm{B}_E^* dx \\
&\quad\le\; C h_E^{1/2} (\|f - f_h\|_{0,\omega_E} + \|c\|_{\infty,\omega_E} \|u - u_h\|_{0,\omega_E}) \|\, [|\nu \nabla u_h \cdot n_E|]_E\, \|_{0,E},
\end{aligned}$$

$$\begin{aligned}
&\sum_{T \subset \omega_E} \int_T (u - u_h)\, [|\nu \nabla u_h \cdot n_E|]_E^C\, \nabla \cdot (\mathrm{B}_E^* b) dx \\
&\quad\le\; C \|u - u_h\|_{0,\omega_E} (h_E^{-1/2} \|b\|_{\infty,\omega_E} + h_E^{1/2} \|\nabla \cdot b\|_{\infty,\omega_E}) \|\, [|\nu \nabla u_h \cdot n_E|]_E\, \|_{0,E},
\end{aligned}$$

$$\begin{aligned}
&\sum_{T \subset \omega_E} \int_T ((b - b_h) \cdot \nabla u_h + (c - c_h) u_h)\, [|\nu \nabla u_h \cdot n_E|]_E^C\, \mathrm{B}_E^* dx \\
&\quad\le\; C h_E^{1/2} \|u_h\|_{0,\omega_E} \left(h_E^{-1} \|b - b_h\|_{\infty,\omega_E} + \|c - c_h\|_{\infty,\omega_E} \right) \|\, [|\nu \nabla u_h \cdot n_E|]_E\, \|_{0,E},
\end{aligned}$$

$$\begin{aligned}
&\sum_{T \subset \omega_E} \int_T (f_h + \nu \Delta u_h - b_h \cdot \nabla u_h - c_h u_h)\, [|\nu \nabla u_h \cdot n_E|]_E^C\, \nabla \cdot (\mathrm{B}_E^* b) dx \\
&\quad\le\; C h_E^{1/2} \|f_h + \nu \Delta u_h - b_h \cdot \nabla u_h - c_h u_h\|_{0,\omega_E} \|\, [|\nu \nabla u_h \cdot n_E|]_E\, \|_{0,E}.
\end{aligned}$$

In collecting all estimates the lemma is proven. □

We again assume that $\|u_h\|_{0,T}$ is bounded uniformly with respect to h_T and ν for all elements, see Remark 4.5. In order to consider the terms with $\|f - f_h\|_{0,\omega_T}$, $\|b - b_h\|_{\infty,\omega_T}$, and $\|c - c_h\|_{\infty,\omega_T}$ as higher order we assume the approximation properties

$$\begin{aligned}
&\|b - b_h\|_{\infty,\omega_T} \|u_h\|_{0,\omega_T} \le C h_T^2 \|u - u_h\|_{0,\omega_T}, \quad \|f - f_h\|_{0,\omega_T} \le C h_T \|u - u_h\|_{0,\omega_T}, \\
&\|c - c_h\|_{\infty,\omega_T} \|u_h\|_{0,\omega_T} \le C h_T \|u - u_h\|_{0,\omega_T}.
\end{aligned} \tag{5.7}$$

THEOREM 5.4 *Under the regularity assumption (2.2), the approximation properties (5.7), a uniform boundedness with respect to h_T and ν of $\|u_h\|_{0,T}$ for all elements, and the shape regularity of the mesh the estimate*

$$\begin{aligned}\tilde{\eta}_{T,2} \quad \leq \quad & C\left[\left(\nu + h_T\|b\|_{\infty,\omega_T} + h_T^2|b|_{1,\infty,\omega_T} + h_T^2\|c\|_{\infty,\omega_T}\right)\|u-u_h\|_{0,\omega_T}\right. \\ & \left. +h_T^2\left(\nu + h_T^2|b|_{1,\infty,\omega_T}\right)|u|_{2,\omega_T}\right] + \mathcal{O}(h_T^3\|u-u_h\|_{0,\omega_T}) \end{aligned} \tag{5.8}$$

holds for $u_h \in V_h$ and each $T \in \mathcal{T}_h$. Exists a constant C independent of h_T and ν with

$$h_T^2|u|_{2,\omega_T} \leq C\|u-u_h\|_{0,\omega_T}$$

then

$$\tilde{\eta}_{T,2} \leq C\left(\nu + h_T\|b\|_{\infty,\omega_T} + h_T^2|b|_{1,\infty,\omega_T} + h_T^2\|c\|_{\infty,\omega_T}\right)\|u-u_h\|_{0,\omega_T} + \mathcal{O}(h_T^3\|u-u_h\|_{0,\omega_T})$$

for $u_h \in V_h$. If in addition h_T is small enough such that $h_T\|b\|_{\infty,\omega_T} < \nu$, $h_T^2|b|_{1,\infty,\omega_T} < \nu$, and $h_T^2\|c\|_{\infty,\omega_T} < \nu$ the estimate

$$\tilde{\eta}_{T,2} \leq C\nu\|u-u_h\|_{0,\omega_T} + \mathcal{O}(h_T^3\|u-u_h\|_{0,\omega_T})$$

holds for $u_h \in V_h$.

If b is a constant function with $b = b_h$ then we have

$$\tilde{\eta}_{T,1} \leq C\left[\left(\nu + h_T\|b\|_{\infty,\omega_T} + h_T^2\|c\|_{\infty,\omega_T}\right)\|u-u_h\|_{0,\omega_T}\right] + \mathcal{O}(h_T^3\|u-u_h\|_{0,\omega_T})$$

for $u_h \in V_h$ and each $T \in \mathcal{T}_h$. If h_T is small enough in the sense $h_T\|b\|_{\infty,\omega_T} < \nu$ and $h_T^2\|c\|_{\infty,\omega_T} < \nu$ then it follows for $u_h \in V_h$

$$\tilde{\eta}_{T,1} \leq C\nu\|u-u_h\|_{0,\omega_T} + \mathcal{O}(h_T^3\|u-u_h\|_{0,\omega_T}).$$

Proof: The local a posteriori error estimate (5.8) follows directly from the previous lemmata.

Let b be a constant function and $b = b_h$. Then Lemma 5.3 can be proven without using (5.4) since (5.6) holds with $b_0 = b$. Jumps of u_h which have to be estimated do not occur. Thus, even $\eta_{T,1}$ provides a local a posteriori error estimate. □

Remark 5.5 The rôle of the nonconformity. In the proof of Lemma 5.3 the nonconformity of u_h is a critical point. The special choice of B_E^* is the essential tool to overcome the difficulties.

Remark 5.6 The local $\boldsymbol{H^2}$–norm of $\boldsymbol{u}$. The estimate $|u|_{2,\omega_T} \leq Ch_T$ holds uniformly in ν where the solution u is smooth, i.e. where $|u|_{2,\omega_T}$ does not depend on ν. However, the situation in boundary layers is different. We consider an example which has been investigated by Roos [Roo96]. Assuming sufficiently smooth data in (2.1), $\Omega = (0,1)^2$, and a compatibility of the right hand side at the corners of $\bar{\Omega}$. Then u can be expressed as $u = v + w$ where $|v|_2$ is bounded uniformly in ν. Roos gives pointwise estimates of the absolute value of derivatives of w in boundary layers. Using these estimates and considering a right–angled element with length h_T of the two edges forming the right angle we find $|u|_{2,T} = \mathcal{O}(h_T^{1/2}\nu^{-3/2})$ for $h_T \geq \nu$. Thus, the term $\nu|u|_{2,\omega_T}$ in (5.8) will be bounded uniformly in ν not before $h_T = \mathcal{O}(\nu)$.

The statements of Remark 4.8 and 4.9 hold also for Theorem 5.4. The jumps of the nonconforming function across the edges may be replaced by the jumps of the tangential derivative across the edges, see Remark 4.4.

6 A Global L^2–A Posteriori Error Estimate

We apply a technique developed by Johnson et. all, see e.g. [EEHJ95]. It consists of four steps:

1. error representation using the solution of a (linearized) continuous dual problem,
2. Galerkin orthogonality of the discretization,
3. interpolation estimates for the solution of the dual problem,
4. strong stability of the dual problem.

The dual problem of (2.1) is
Given $w \in L^2(\Omega)$, find $z_w \in H_0^1(\Omega)$ with

$$(\nu \nabla z_w, \nabla v) - (b \cdot \nabla z_w, v) + ((c - \nabla \cdot b) z_w, v) \quad = \quad (w, v) \quad \forall v \in V. \tag{6.1}$$

For the solution z_w of this problem we assume H^2–regularity, i.e.

$$z_w \in H^2(\Omega) \cap H_0^1(\Omega) \tag{6.2}$$

and strong stability, i.e. there is a constant $C_S = C_S(\nu)$ with

$$\|z_w\|_2 \leq C_S \|w\|_0 \quad \forall w \in L^2(\Omega). \tag{6.3}$$

A technical tool is the function $\hat{f}_h$ which is defined as the right hand side of the standard Galerkin formulation of (2.3) in the space X_h for a given function $u_h \in V_h$

$$\sum_{T \in \mathcal{T}_h} (\nu \nabla u_h, \nabla v_h)_T + (b \cdot \nabla u_h, v_h)_T + (c u_h, v_h)_T = (\hat{f}_h, v_h) \quad \forall v_h \in X_h. \tag{6.4}$$

If it is possible to compute the integrals on the left hand side exactly then $\hat{f}_h$ is computable.

THEOREM 6.1 *Assume shape regularity of the triangulation, the regularity of the exact solution given by (2.2), the regularity and the stability of the solution of the dual problem given by (6.2) and (6.3). Then the global a posteriori error estimate*

$$\begin{aligned} \|u - u_h\|_0 &\leq CC_I C_S \hat{\eta} + CC_I C_S \left(\sum_{T \in \mathcal{T}_h} h_T^4 \|f - f_h\|_{0,T}^2 \right)^{1/2} + CC_S \|f - \hat{f}_h\|_0 \\ &= CC_I C_S \left(\sum_{T \in \mathcal{T}_h} \hat{\eta}_T^2 \right)^{1/2} + CC_I C_S \left(\sum_{T \in \mathcal{T}_h} h_T^4 \|f - f_h\|_{0,T}^2 \right)^{1/2} + CC_S \|f - \hat{f}_h\|_0 \end{aligned} \tag{6.5}$$

with

$$\hat{\eta}_T^2 = h_T^4 \|f_h + \nu \Delta u_h - b \cdot \nabla u_h - c u_h\|_{0,T}^2 + \sum_{E \subset \partial T} h_E^3 \| [\![\nu \nabla u_h \cdot n_E]\!]_E \|_{0,E}^2 + h_E \| [\![u_h]\!]_E \|_{0,E}^2 \tag{6.6}$$

holds for all $u_h \in V_h$.

Proof: We start with the continuous dual problem (6.1), set $w = u - u_h \in L^2(\Omega)$ and denote the solution by z. In general $u - u_h$ does not belong to V, that's why we cannot use $u - u_h$ as a test function in (6.1). For nonconforming test functions one obtains by going from the strong to the weak formulation of the dual problem an additional term. Testing the strong formulation of the dual problem with $u-u_h$ and integrating by parts we get

$$\begin{aligned}\|u-u_h\|_0^2 &= \sum_{T\in\mathcal{T}_h} (\nu\nabla z, \nabla(u-u_h))_T - (b\cdot\nabla z, u-u_h)_T + ((c-\nabla\cdot b)z, u-u_h)_T \\ &\quad - \sum_{E\subset\partial T}\int_E \nu\nabla z\cdot n_E(u-u_h)ds. \end{aligned}\tag{6.7}$$

Since $X_h = V\cap V_h$ it follows from (2.3)

$$\sum_{T\in\mathcal{T}_h} a(u,v_h)_T + (b\cdot\nabla u, v_h)_T + (cu, v_h)_T = (f, v_h) \quad \forall v_h\in X_h.$$

Using this and (6.4) we obtain the approximated Galerkin orthogonality

$$0 = \sum_{T\in\mathcal{T}_h} (\nu\nabla(u-u_h), \nabla v_h)_T + (b\cdot\nabla(u-u_h), v_h)_T + (c(u-u_h), v_h)_T - (f-\hat{f}_h, v_h)_T \quad \forall v_h\in X_h.$$

We choose $v_h = R_{X_h}z = z_h$ and subtract this from (6.7). Since z is continuous and ∇u is continuous a.e. it follows with integration by parts

$$\begin{aligned}&\|u-u_h\|_0^2 \\ &\quad= \sum_{T\in\mathcal{T}_h} (\nu\nabla(z-z_h), \nabla(u-u_h))_T - (b\cdot\nabla z, u-u_h)_T - (b\cdot\nabla(u-u_h), z_h)_T \\ &\qquad -((\nabla\cdot b)z, u-u_h)_T + (c(u-u_h), z-z_h)_T + (f-\hat{f}_h, z_h)_T + \sum_{E\subset\partial T}\int_E \nu\nabla z\cdot n_E u_h ds \\ &\quad= \sum_{T\in\mathcal{T}_h} (-\nu\Delta(u-u_h) + b\cdot\nabla(u-u_h) + c(u-u_h), z-z_h)_T \\ &\qquad + \sum_{E\subset\partial T}\int_E \nu\nabla(u-u_h)\cdot n_E(z-z_h)ds + \int_E \nu\nabla z\cdot n_E u_h ds - \int_E (b\cdot n_E)(u-u_h)z ds \\ &\qquad +(f-\hat{f}_h, z_h).\end{aligned}$$

The estimation of the first two terms using (2.6) or (2.7), respectively, and (6.3) is standard

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} (-\nu\Delta(u-u_h) + b\cdot\nabla(u-u_h) + c(u-u_h), z-z_h)_T \\ &\quad\le CC_IC_S\left(\sum_{T\in\mathcal{T}_h} h_T^4\|f_h+\nu\Delta u_h - b\cdot\nabla u_h - cu_h\|_{0,T}^2 + h_T^4\|f-f_h\|_{0,T}^2\right)^{1/2} \|u-u_h\|_0\,, \\ &\sum_{T\in\mathcal{T}_h}\sum_{E\subset\partial T}\int_E \nu\nabla(u-u_h)\cdot n_E(z-z_h)ds \\ &\quad\le CC_IC_S\left(\sum_E h_E^3\|\,[|\nu\nabla u_h\cdot n_E|]_E\,\|_{0,E}^2\right)^{1/2} \|u-u_h\|_0.\end{aligned}$$

Using (2.4), (2.8) and (6.3) we have for the third term

$$\sum_{T\in\mathcal{T}_h}\sum_{E\subset\partial T}\int_E \nu\nabla z\cdot n_E u_h ds = \sum_E\int_E \nu\nabla z\cdot n_E\,[|u_h|]_E\,ds$$

$$
\begin{aligned}
&= \sum_E \int_E \nu(\nabla z - \mathcal{M}_E^0(\nabla z)) \cdot n_E \, [|u_h|]_E \, ds \\
&\le C \sum_E h_E^{1/2} \|\nu \, [|u_h|]_E \, \|_{0,E} |\nabla z \cdot n_E|_{1,\omega_E} \\
&\le C\, C_S \left(\sum_E h_E \|\nu \, [|u_h|]_E \, \|_{0,E}^2 \right)^{1/2} \|u - u_h\|_0.
\end{aligned}
$$

The estimation of the fourth term starts with a splitting of this term

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} \sum_{E\subset\partial T} \int_E (b\cdot n_E) u_h z ds \\
&\quad = \sum_E \int_E (b(B_E)\cdot n_E) \, [|u_h|]_E \, z ds + \sum_E \int_E ((b - b(B_E))\cdot n_E) \, [|u_h|]_E \, z ds,
\end{aligned}
$$

where B_E is the midpoint of E. The first term of this sum is estimated using (2.4) and (2.8)

$$
\begin{aligned}
\sum_E \int_E (b(B_E)\cdot n_E) \, [|u_h|]_E \, z ds &= \sum_E (b(B_E)\cdot n_E) \int_E [|u_h|]_E \, (z - \mathcal{M}_E^0(z)) ds \\
&\le C \sum_E h_E^{1/2} \|b(B_E)\cdot n_E\|_{\infty,E} \| \, [|u_h|]_E \, \|_{0,E} |z|_{1,\omega_E}.
\end{aligned}
$$

The estimation of the second term of the sum starts with

$$
\sum_E \int_E ((b - b(B_E))\cdot n_E) \, [|u_h|]_E \, z ds \le \sum_E \|(b - b(B_E))\cdot n_E\|_{\infty,E} \|z\|_{0,E} \| \, [|u_h|]_E \, \|_{0,E}.
$$

Schieweck and Tobiska have proven in [ST96] $\forall z \in H^2(\omega_E)$ the inequality

$$
\|z\|_{0,E} \le C \left(h_E^{-1/2} \|z\|_{0,\omega_E} + h_E^{1/2} |z|_{1,\omega_E} \right).
$$

Because of the smoothness of b it holds $b \in \left(W^{1,\infty}(E)\right)^2$ and using Taylor expansion we obtain

$$
\|(b - b(B_E))\cdot n_E\|_{\infty,E} \le C h_E |b\cdot n_E|_{1,\infty,E}.
$$

Using this we have

$$
\sum_E \int_E ((b - b(B_E))\cdot n_E) \, [|u_h|]_E \, z ds \le C \sum_E |b\cdot n_E|_{1,\infty,E} \| \, [|u_h|]_E \, \|_{0,E} h_E^{1/2} \|z\|_{1,\omega_E},
$$

where the terms of higher order in h_E are absorbed in the constant C. Since $\|b(B_E)\cdot n_E\|_{\infty,E} \le \|b\cdot n_E\|_{\infty,E}$ we obtain with the help of the strong stability (6.3) the estimate

$$
\sum_{T\in\mathcal{T}_h} \sum_{E\subset\partial T} \int_E (b\cdot n_E) u_h z ds \le C\, C_S \left(\sum_E h_E \|b\cdot n_E\|_{1,\infty,E}^2 \| \, [|u_h|]_E \, \|_{0,E}^2 \right)^{1/2} \|u - u_h\|_0.
$$

The last term is estimated by

$$
(f - \hat{f}_h, z_h) \le \|f - \hat{f}_h\|_0 \left(\|z - z_h\|_0 + \|z\|_0\right) \le C C_S \|f - \hat{f}_h\|_0 \|u - u_h\|_0.
$$

The statement of the theorem is obtained by summarizing all estimates. $\square$

Remark 6.2 The term $\|f - \hat{f}_h\|_0$. The term $\|f - \hat{f}_h\|_0$ measures all effects which result from the fact that u_h is not the solution of the standard Galerkin discretization of (2.3) in X_h. These are the errors which come from the stabilization, from numerical integration in assembling the matrix and the right hand side of the discrete system, and from the termination of the iterative solver at an approximation of the exact solution of the discrete system. The term $\|f - \hat{f}_h\|_0$ in principle is computable but in the numerical practice it will be considered as higher order term. At this point it is necessary that u_h is a good approximation of the solution of the discrete problem.

Remark 6.3 The rôle of the conforming subspace $X_h \subset V_h$. We do not need a solution of a discrete problem belonging to the dual problem (6.1), but only an interpolant z_h of z. Here we make use of the fact that V_h contains the conform subspace X_h and we can choose $z_h \in X_h$. Thus, line integrals for the reason of discontinuities of z_h do not occur in the following integration by parts. So to speak, the dual problem is treated in a conforming way.

Remark 6.4 Jumps of the tangential derivative across edges. The jumps of the nonconforming function across the edges may be replaced by the jumps of the tangential derivative across the edges, compare Remark 4.4.

Remark 6.5 The estimation of the convective term. For problems without convection, $b \equiv (0,0)^T$, the local parts of the error estimator $\hat{\eta}$ change to

$$\hat{\eta}_T^2 = h_T^4 \|f_h + \nu\Delta u_h - b\cdot\nabla u_h - cu_h\|_{0,T}^2 + \sum_{E\subset\partial T} h_E^3 \| [\![\nu\nabla u_h \cdot n_E]\!]_E \|_{0,E}^2 + h_E \|\nu\, [\![u_h]\!]_E \|_{0,E}^2. \quad (6.8)$$

Now, the jumps of the nonconforming functions are weighted by the factor ν. In fact, in the proof of Theorem 6.1 the estimate of the term

$$\sum_{T\in\mathcal{T}_h} \sum_{E\subset\partial T} \int_E (b\cdot n_E) u_h z\, ds$$

seems not to be optimal because only H^1–regularity of z has been used instead of H^2–regularity. Thus, possibly there are lost powers of h_E. In this case the jumps of the nonconforming function would have the weight $\max\{\nu, h_E\|b\|_{\infty,\omega_T}\}$ which corresponds to the local a posteriori error estimator $\tilde{\eta}_{T,2}$, see (5.2).

Remark 6.6 On the robustness of $\hat{\eta}$. The error estimator $\hat{\eta}$ is not robust in the sense of Definition 1.1. For singularly perturbed problems the worst case a priori error estimate

$$\nu^{3/2}\|z\|_2 + \nu^{1/2}\|z\|_1 + \|z\|_0 \le C\|u - u_h\|_0 \quad (6.9)$$

holds with C independent of ν, see e.g. Roos, Stynes, Tobiska [RST96], Chapter III, Lemma 1.18. Thus, it is possible that $C_S(\nu) = \mathcal{O}(\nu^{-3/2})$. Then C_S is very large for small ν and the computation of $\hat{\eta}$ alone, without knowing the magnitude of C_S at least approximately, does not contain any information about the error $\|u - u_h\|_0$, see also Section 8.

7 Numerical Results With Local A Posteriori Error Estimators

In this section we present numerical results which have been obtained using the local a posteriori error estimators $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$ investigated in Section 4 and 5. In the numerical tests we have found only very small differences in the meshes generated by using $\eta_{T,2}$ instead of $\eta_{T,1}$ and $\tilde{\eta}_{T,2}$ instead of $\tilde{\eta}_{T,1}$, see [Joh97].

The sequence

$$\text{assemble} \to \text{solve} \to \text{estimate the error} \to \text{refine} \to \text{interpolate}$$

is called a *step*. Below we give errors in different norms which are calculated by numerical integration. The errors in $\|\cdot\|_0$ are computed using an integration rule which involves the values at the vertices of the triangle, the midpoints of the edges, and the barycentre, see Ciarlet [Cia91] formulae (25.19). The errors in $|\cdot|_h$ are computed by applying the edge–midpoint–rule, see Ciarlet [Cia91] formulae (25.17). We found in numerical tests that the edge–midpoint–rule for computing the L^2–error can lead to completely wrong results.

The results of numerical tests depend on parameters of the algorithms. Especially for the adaptive grid refinement some parameters have to be chosen, like criteria for choosing elements for refinement or coarsening, tolerances for these criteria, a minimal amount of refinement and a starting point for the adaptive refinement process. We refer to [Joh97] for details.

In all examples we have used Samarskij upwinding for stabilisation. The linear systems have been solved by a multigrid algorithm with L^2–projections as grid transfer operations and ILU_β–iterations as smoother.

Example 7.1 A solution with layers at the outflow boundary. The right hand side f and the boundary conditions g have been chosen such that

$$u(x,y,\nu) = xy^2 - y^2\exp\left(\frac{2(x-1)}{\nu}\right) - x\exp\left(\frac{3(y-1)}{\nu}\right) + \exp\left(\frac{2(x-1)+3(y-1)}{\nu}\right)$$

is the exact solution of

$$\begin{aligned} -\nu\Delta u + (2,3)\cdot\nabla u + u &= f \quad \text{in } (0,1)\times(0,1) \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

We have used $\nu = 10^{-6}$. The solution and the coarse grid (level 0) are presented in Figure 5.

The results using uniform refinement are given in Table 1 where the convergence rate is computed using the results of the level 6 and 7.

Lev.	unknowns	$\|u-u_h\|_0$
4	3008	3.627-2
5	12160	2.533-2
6	48896	1.780-2
7	196096	1.254-2
rate		0.510

Table 1: Example 7.1, error and convergence rate in $\|\cdot\|_0$ for uniform refinement.

An improvement of these results is only possible with a better resolution of the layers. That's why a local a posteriori error estimator should generate grids where above all the region of the

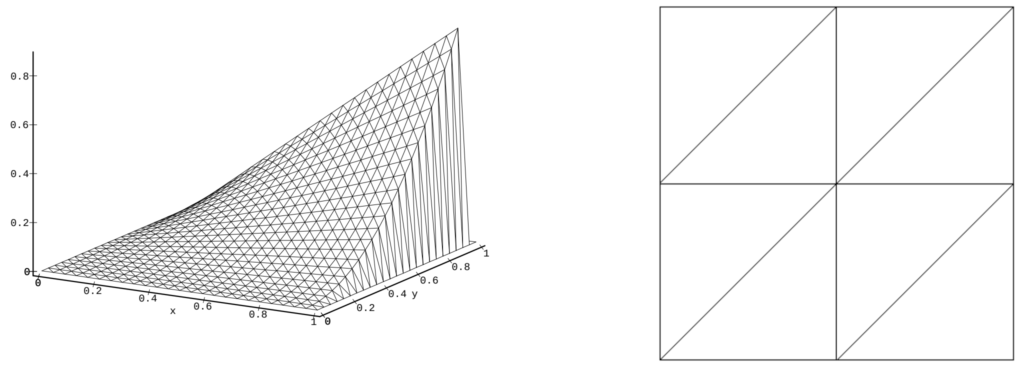

Figure 5: Example 7.1, exact solution and coarse grid (level 0).

	$\eta_{T,1}$			$\tilde{\eta}_{T,1}$		
step	unknowns	$\|u-u_h\|_0$	$\|u-u_h\|_h$	unknowns	$\|u-u_h\|_0$	$\|u-u_h\|_h$
6	2798	5.036-2	4.975+4	2650	5.029-2	4.996+4
9	11521	1.725-2	1.853+4	9842	2.465-2	2.553+4
12	51544	5.008-3	8.390+3	33101	1.099-2	1.460+4
13	88310	3.272-3	6.398+3	48198	1.223-2	1.281+4
15				101781	8.582-3	9.116+3

Table 2: Example 7.1, errors using locally refined grids generated by $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$.

layers is refined. In Table 2 the results using $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$ are presented which are clearly better than with uniform refinement. Using uniform refinement 50 000 000 unknowns are necessary in order to obtain the same accuracy in the L^2–norm as with $\eta_{T,1}$ and 88 000 unknowns (step 13) !

The mode of working of the local error estimators $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$ is different. The error estimator $\eta_{T,1}$ refines only inside the boundary layers, see Figure 6. In each refinement step the errors in $\|\cdot\|_0$ and $|\cdot|_h$ decrease. The error estimator $\tilde{\eta}_{T,1}$ refines also mainly inside the layers but with a lower depth of refinement. In those steps where $\tilde{\eta}_{T,1}$ does not increase the depth of refinement inside the layer the error in $\|\cdot\|_0$ does not decrease, see Table 2, from step 12 to 13.

Example 7.2 A solution with weak regularity. We consider the problem

$$\begin{aligned} -10^{-3}\Delta u + (1,0)\cdot\nabla u &= 0 \quad \text{in } (0,1)\times(0,1) \\ u &= 0 \quad \text{for } y=0, 0\le x\le 1 \\ u &= 0 \quad \text{for } y=1, 0\le x\le 1 \\ u &= 0 \quad \text{for } x=0, |y-0.5|>0.05 \\ u &= 1 \quad \text{for } x=0, |y-0.5|<0.05 \\ \nu\partial u/\partial n &= 0 \quad \text{for } x=1, 0\le y\le 1. \end{aligned} \tag{7.1}$$

The solution of (7.1), see Figure 7, can be represented as the Fourier series

$$u = \sum_{k=1}^{\infty} \varphi_k(x)\chi_k(y) = \sum_{k=1}^{\infty} (\theta_k\varphi_{1k}(x) + \varrho_k\varphi_{2k}(x))\chi_k(y),$$

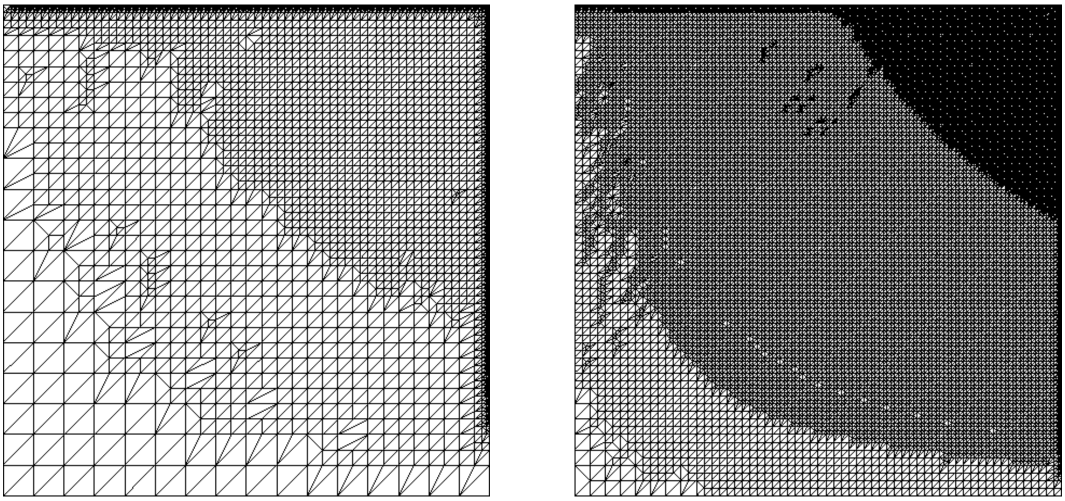

Figure 6: Example 7.1, locally refined grids; left using $\eta_{T,1}$ step 13; right using $\tilde{\eta}_{T,1}$ step 15.

with

$$\varphi_{1k}(x) = e^{x/(2\nu)}\frac{\sinh(\zeta_k(1-x))}{\sinh(\zeta_k)}, \quad \varphi_{2k}(x) = e^{(x-1)/(2\nu)}\frac{\sinh(\zeta_k x)}{\sinh(\zeta_k)}, \quad \zeta_k = \sqrt{\frac{1}{(2\nu)^2} + (k\pi)^2}.$$

The constants are

$$\theta_k = \frac{2}{k\pi}(\cos(0.45k\pi) - \cos(0.55k\pi)), \quad \varrho_k = \theta_k\left(\frac{\zeta_k e^{1/(2\nu)}}{\sinh(\zeta_k)}\right) \Big/ \left(\frac{1}{2\nu} + \zeta_k\frac{\cosh(\zeta_k)}{\sinh(\zeta_k)}\right),$$

and the functions which only depend on y are

$$\chi_k(y) = \sin(k\pi y).$$

We have used the same coarse grid as in Example 7.1.

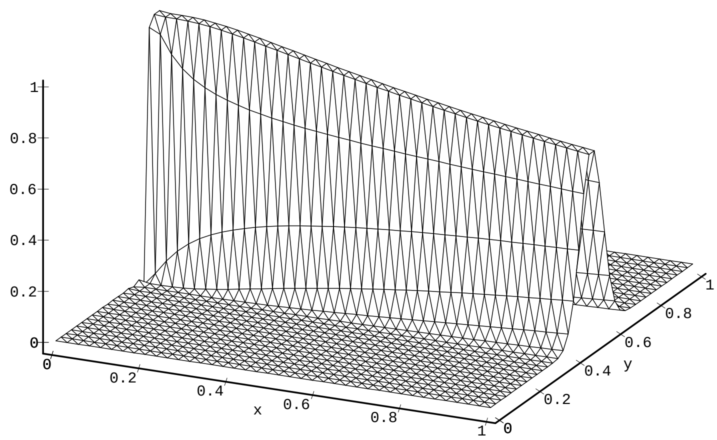

Figure 7: Solution of Example 7.2.

For the reason of the jumps in the boundary condition the solution u does not belong to $H^1(\Omega)$. Thus, the assumptions of Theorem 4.6 and 5.4 are not fulfilled. A Neumann boundary condition is given at the outflow boundary. The theoretical investigations in Section 4 and 5 have been restricted to Dirichlet problems. Similar to conforming finite element methods we include the error in the Neumann boundary conditions as an additional term into the local a posteriori error estimators, see e.g. Verfürth [Ver96b].

The results in Table 3 show once more that the solutions obtained with the adaptive h–method are superior to those obtained with uniform refinement. Both local a posteriori error estimators can be applied with success in spite of the weak regularity of the solution. Again it

	$\eta_{T,1}$		$\tilde{\eta}_{T,1}$	
step	unknowns	$\|u-u_h\|_0$	unknowns	$\|u-u_h\|_0$
6	2411	7.155-2	2413	5.255-2
9	7939	1.594-2	7194	3.327-2
12	27023	2.013-3	22339	9.908-3
15	93411	1.026-3	73586	5.563-3
16	137456	5.858-4	109296	5.733-3
17			156105	3.282-3
unif. 7	196352	2.009-2		

Table 3: Example 7.2, errors using locally refined grids generated by $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$.

can be observed that $\eta_{T,1}$ refines only inside the layers whereas the local refinement generated with $\tilde{\eta}_{T,1}$ covers a larger area, see Figure 8. As in Example 7.1 the results using $\eta_{T,1}$ are better. Refinement steps which do not increase the depth of refinement do not result in a decreasing error.

Using conforming finite element methods and a residual a posteriori error estimator for the local H^1–error appropriate parameters for the error estimation process have to be chosen in order to obtain good results. Otherwise the refinement is only at the jumps of the Dirichlet condition, see [Joh94]. Using $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$ this is not the case ! They produce good grids even with parameters where the conforming a posteriori error estimator for the local H^1–error fails.

Example 7.3 A solution with different singularities. We consider the problem in a L–shaped domain

$$\begin{aligned} -10^{-6}\Delta u + (3,1)\cdot\nabla u + u &= 100r(r-0.5)(r-\sqrt{2}/2) && \text{in } (0,1)\times(0,1)\setminus[0.5,1]\times[0,0.5] \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $r = \sqrt{(x-0.5)^2+(y-0.5)^2}$, see Adam [Ada95]. An analytical representation of the solution is not known. The solution possesses singularities at the reentrant corner $(0.5, 0.5)$, boundary layers at the outflow boundary, and an inner layer starting at the reentrant corner in the direction of the convection. An important criterion of the quality of the computed solution is the resolution of the inner layer.

Figure 9 presents the generated grids and the computed solutions. The local a posteriori error estimator $\eta_{T,1}$ refines inside the inner layer less than at the other singularities. This results in a bad resolution of this layer in comparison to the results using $\tilde{\eta}_{T,1}$. The local refinement generated by $\tilde{\eta}_{T,1}$ again covers a larger area than this produced by $\eta_{T,1}$. The presented solutions has been calculated using about 125 000 unknowns.

In [DJR$^+$96] this example is presented using a conforming finite element discretization. It has been shown that an a posteriori error estimator for the local H^1–error fails to resolve the inner layer without appropriate parameters in the refinement process. For the same parameters $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$ do not fail and generate similar grids like in Figure 9.

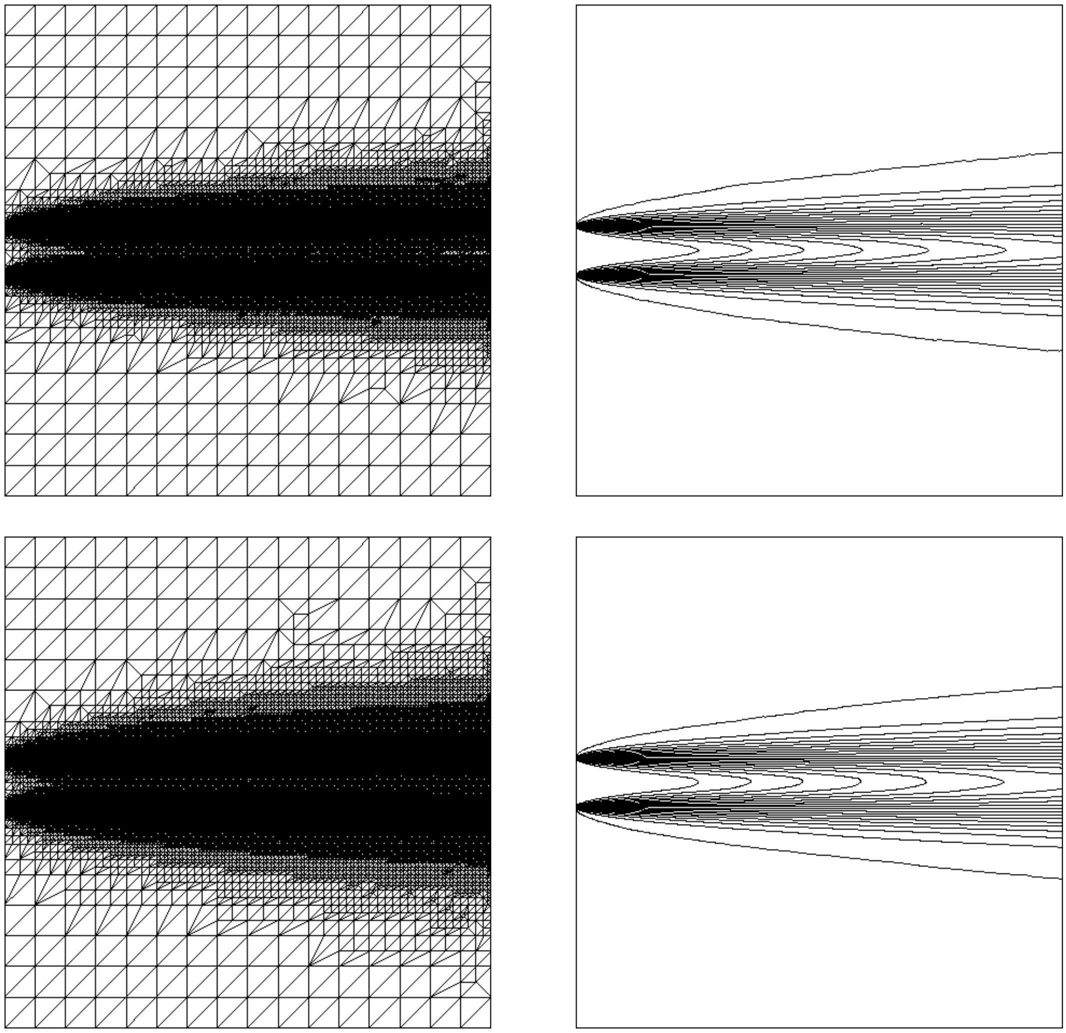

Figure 8: Example 7.2, locally refined grids and computed solutions, $\eta_{T,1}$ step 16 (above) and $\tilde{\eta}_{T,1}$ step 17 (below).

Remark 7.4 Summary of the numerical tests.

1. **Uniform vs. adaptive refinement.** The solutions of the presented examples have singularities. The use of $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$ has led in each example to clearly better results than using uniform refinement.
2. $\boldsymbol{\eta_{T,1}}$ **vs.** $\boldsymbol{\tilde{\eta}_{T,1}}$**.** $\eta_{T,1}$ has generated better grids for problems with one singularity, Example 7.1, and with several singularities of the same kind, Example 7.2, whereas $\tilde{\eta}_{T,1}$ has produced better grids for the problem with different singularities, Example 7.3. Considering the same number of steps the error estimator $\tilde{\eta}_{T,1}$ has generated in all examples grids with fewer unknowns than $\eta_{T,1}$. The locally refined grids of $\tilde{\eta}_{T,1}$ have covered a larger area and have had a lower depth of refinement.
3. **Assumptions of the theory are not fulfilled.** The local a posteriori error estimators $\eta_{T,1}$ and $\tilde{\eta}_{T,1}$ can be applied with success to problems where the solution does not possess the regularity which has been assumed in the theory, see Example 7.2.
4. **Computing times and numerical costs.** We denote with t_{uni} the time which is needed to solve a problem on the finest level of a uniform grid. Using adaptive methods only a small percentage of the number of unknowns has been necessary to obtain similar results

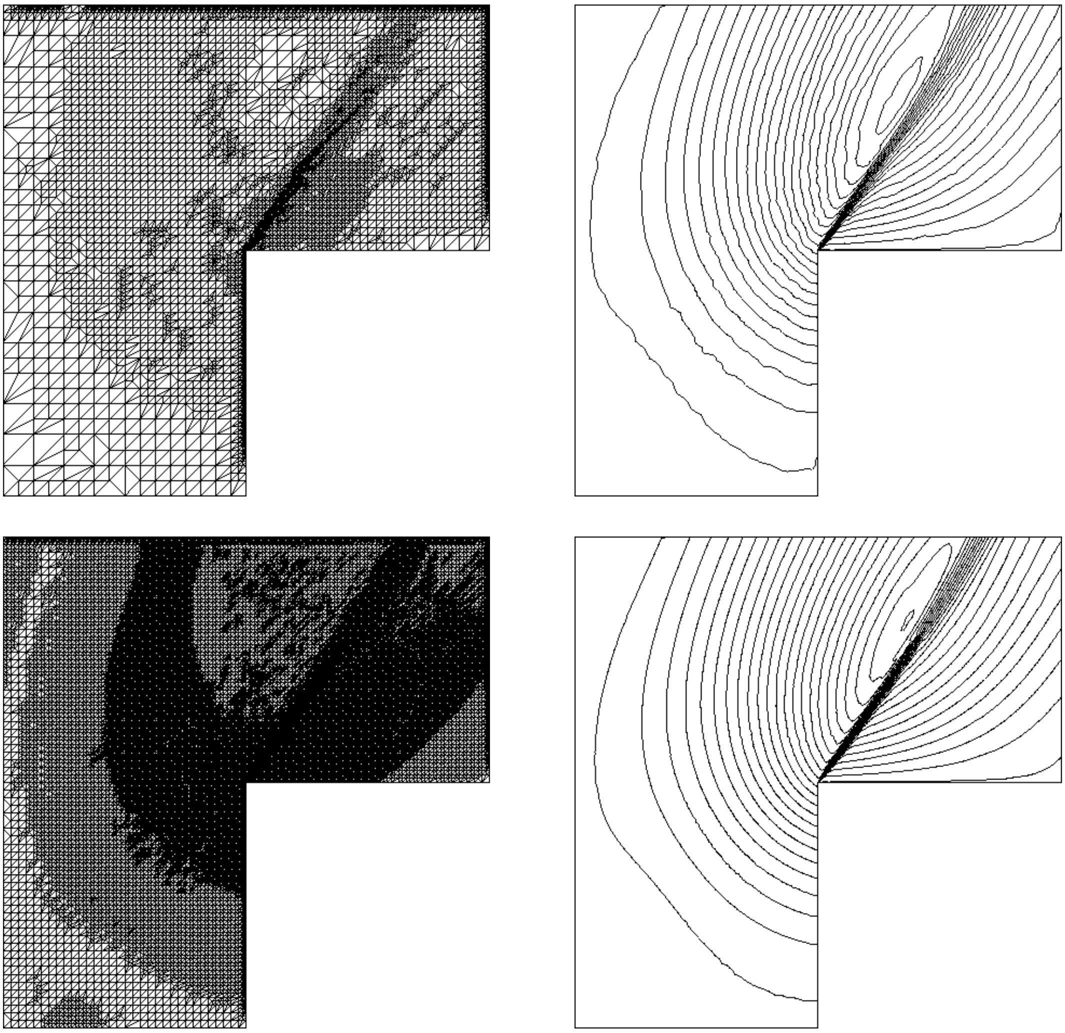

Figure 9: Example 7.3, locally refined grids and computed solutions, $\eta_{T,1}$ (above) and $\tilde{\eta}_{T,1}$ (below).

as with uniform refinement. Up to this point the computing times have been a small percentage of t_{uni}, too. However, a multiple of t_{uni} has been needed to compute the results on the finest adaptive refinements given in the tables above. This comes from the higher number of solution processes and from the fact that the iterative solver has performed in general more iterations to solve the discrete problems on locally refined grids than on uniform grids. The numerical costs of the computation of the error estimators is negligible in comparison to the total costs, see also [Joh94].

8 The Numerical Estimation of the Global L^2–Error

The numerical examples in this section compare the estimated error $\hat{\eta}$ from (6.5) and the (numerical approximated) real error $\|u - u_h\|_0$. This comparison also includes a way to estimate the stability constant C_S from (6.3).

As stated in Remark 6.6 the a posteriori error estimator $\hat{\eta}$ is not robust. If the stability constant C_S of the continuous dual problem (6.1) is large then the computation of $\hat{\eta}$ gives no information about the real error. It is necessary to have an idea of the size of C_S. In general C_S is not computable since the solution of the continuous dual problem (6.1) is not known and can not be computed. The heuristic way we have chosen with the aim to approximate C_S starts with a stabilized discrete dual problem. Let u_h be the computed solution on the grid with level l and u_{2h} the computed solution on the previous grid with level $l-1$. We have chosen the right hand side $u_h - u_{2h}$ for the discrete dual problem and denote the solution by $\hat{z}_h$. Let $\bar{z}_h$ be the conforming, piecewise linear function which is computed by a weighted average of $\hat{z}_h$ in the corners P_i of the triangles

$$\bar{z}_h(P_i) = \sum_{T\in\mathcal{T}_h, P_i\in T} \frac{\hat{z}_h(P_i)}{|T|}.$$

Let $\hat{z}_h^1$ and $\bar{z}_h^1$ be the piecewise constant gradients of $\hat{z}_h$ and $\bar{z}_h$, respectively. These function are transformed to conforming, piecewise linear functions with a second averaging process after which derivatives of the results are computed and denoted by $\hat{z}_h^2$ and $\bar{z}_h^2$. The stability constant C_S now can be approximated by

$$\hat{C}_S = \frac{(\|\hat{z}_h^2\|_0^2 + \|\hat{z}_h^1\|_0^2 + \|\hat{z}_h\|_0)^{1/2}}{\|u_h - u_{2h}\|_0} \qquad \text{or} \qquad \bar{C}_S = \frac{(\|\bar{z}_h^2\|_0^2 + \|\bar{z}_h^1\|_0^2 + \|\bar{z}_h\|_0)^{1/2}}{\|u_h - u_{2h}\|_0}.$$

In the numerical tests we have found that the difference between $\hat{C}_S$ and $\bar{C}_S$ often is small. In the examples below $\hat{C}_S$ has been somewhat larger than $\bar{C}_S$.

A similar way of obtaining information about the stability of the dual problem for the Navier–Stokes equations is proposed by Becker and Rannacher [BR96]. There the local terms of the global a posteriori error estimator are weighted with appropriate local norms of the solution of a linearized discrete dual problem.

Example 8.1 A reaction–diffusion equation. We start with an example where $b \equiv (0,0)^T$ because of the supposition that the convective term has not been estimated optimally in the proof of Theorem 6.1, see Remark 6.5. The right hand side f is chosen such that

$$u(x,y,\nu) = 16x(1-x)y(1-y)\left[\frac{1}{2} + \frac{\arctan\left((2/\sqrt{\nu})(r_0^2 - (x-x_0)^2 - (y-y_0)^2)\right)}{\pi}\right]$$

with $r_0 = 0.25$ and $x_0 = y_0 = 0.5$ is the exact solution of

$$\begin{aligned} -\nu\Delta u + 2u &= f \quad \text{in } (0,1)\times(0,1) \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The exact solution for $\nu = 10^{-4}$ and the coarse grid are shown in Figure 10. The solution u depends on ν, in particular $\|u\|_2$ increases for $\nu \to 0$. That's why the asymptotical order of convergence is not reached for small ν in the results presented in Table 4. We have used uniform refinement.

The results in Table 4 show that $\hat{\eta}$ computed with (6.8) is in fact only for $\nu = 10^{-1}$ an upper bound of the global error. In all other cases $\hat{\eta}$ is smaller than the error and in particular for

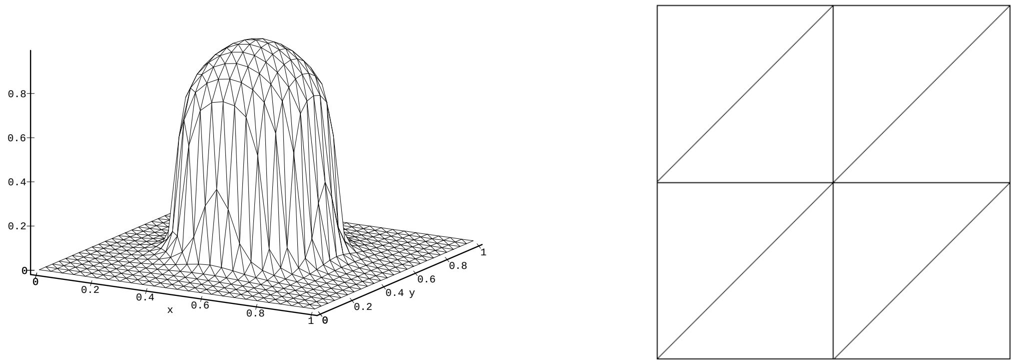

Figure 10: Example 8.1, exact solution for $\nu = 10^{-4}$ and coarse grid.

ν	level	$\|u-u_h\|_0$	rate	$\hat\eta$	rate	$\bar C_S$	$\bar C_S\hat\eta$	rate
10^{-1}	4	5.26-3	2.00	1.57-3	1.98	3.44	5.41-3	1.68
	5	1.32-4	2.00	3.94-4	1.99	3.67	1.45-3	1.90
	6	3.30-5	2.00	9.87-5	2.00	3.74	3.69-4	1.97
	7	8.35-6	1.98	2.47-5	2.00	3.75	9.26-5	2.00
10^{-3}	4	3.96-3	1.95	9.52-5	2.14	131.3	1.25-2	1.02
	5	1.02-3	1.95	2.79-5	1.77	174.8	4.89-3	1.36
	6	2.60-4	1.98	7.38-6	1.91	166.9	1.23-3	1.99
	7	6.53-5	1.99	1.87-6	1.98	194.4	3.64-4	1.76
10^{-5}	4	2.94-2	0.83	1.75-4	2.68	485.7	8.50-2	0.61
	5	1.38-2	1.09	2.21-5	2.98	1822.3	4.03-2	1.08
	6	5.39-3	1.36	2.51-6	3.14	5580.2	1.40-2	1.53
	7	1.71-3	1.65	4.15-7	2.60	12462.9	5.17-3	1.44
10^{-6}	4	4.14-2	0.63	2.27-4	2.55	494.8	1.12-1	0.43
	5	2.45-2	0.75	3.40-5	2.74	1977.0	6.72-2	0.74
	6	1.30-2	0.92	4.83-6	2.81	7351.2	3.55-2	0.92
	7	5.70-3	1.19	5.88-7	3.03	25924.0	1.52-2	1.22

Table 4: Example 8.1, exact and estimated global L^2–error.

$\nu \le 10^{-5}$ the estimated order of convergence is too large. The error estimates which have been computed using the heuristic way described above are always an upper bound of the error. The estimates are about 3 to 10 times larger than the real error. For small values of ν the reduced order of convergence is reflected very well.

Example 8.2 An example with convection. We consider

$$\begin{aligned} -\nu\Delta u + (2,3)\cdot\nabla u + 2u &= f \quad \text{in } (0,1)\times(0,1) \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where the right hand side is chosen such that

$$u(x,y) = 16x(1-x)y(1-y)\left[\frac{1}{2} + \frac{\arctan\left(200(r_0^2 - (x-x_0)^2 - (y-y_0)^2)\right)}{\pi}\right]$$

with $r_0 = 0.25$ and $x_0 = y_0 = 0.5$ is the exact solution, see Figure 10. The solution does not depend on ν. We have performed the calculations using Samarskij upwinding, uniform refinement, and the same coarse grid as in Example 8.1.

ν	level	$\|u-u_h\|_0$	rate	$\hat{\eta}$	rate	$\bar{C}_S$	$\bar{C}_S\hat{\eta}$	rate
10^{-1}	4	2.92-2	2.25	1.89-1	1.45	6.9	1.30+0	0.96
	5	1.16-2	1.34	6.32-2	1.58	7.1	4.48-1	1.53
	6	1.52-3	2.93	1.68-3	1.91	12.0	2.02-1	1.15
	7	3.85-4	1.98	4.29-3	1.97	4.9	2.12-2	3.25
10^{-3}	4	4.46-2	0.97	2.07-1	1.08	30.2	6.25	-0.30
	5	2.23-2	1.00	9.18-2	1.18	49.5	4.54	0.46
	6	1.08-2	1.04	3.74-2	1.30	82.4	3.08	0.56
	7	4.97-3	1.12	1.42-2	1.39	199.2	2.83	0.12
10^{-5}	4	4.72-2	0.92	2.22-1	1.01	31.6	7.03	-0.40
	5	2.48-2	0.93	1.07-1	1.05	53.8	5.78	0.28
	6	1.32-2	0.91	5.19-2	1.05	87.8	4.55	0.34
	7	7.04-3	0.91	2.54-2	1.03	225.8	5.73	-0.33
10^{-6}	4	4.73-2	0.92	2.23-1	1.01	31.6	7.03	-0.40
	5	2.48-2	0.93	1.08-1	1.05	53.8	5.79	0.28
	6	1.32-2	0.91	5.21-2	1.05	87.8	4.57	0.34
	7	7.06-3	0.90	2.65-2	1.03	225.8	5.78	-0.34

Table 5: Example 8.2, exact and estimated global L^2–error using $\hat{\eta}$.

In Table 5 we present results using the local a posteriori error estimator $\hat{\eta}$ from (6.5). The jumps of the nonconforming function are weighted by the factor 1. This term dominates in $\hat{\eta}$. Obviously, the order of convergence of this term is similar to that of $\|u-u_h\|_0$. This leads to the fact that $\hat{\eta}$ is always an upper bound of the global error. In contrast, the quantities $\bar{C}_s\hat{\eta}$ completely overestimate the error.

Following Remark 6.5 we have tested a modified a posteriori error estimator $\bar{\eta}$ whose local terms have the form

$$\bar{\eta}_T^2 = h_T^4\|f+\nu\Delta u_h - b\cdot\nabla u_h - cu_h\|_{0,T}^2 + \sum_{E\subset\partial T} h_E^3\|\,[\![\nu\nabla u_h\cdot n_E]\!]_E\,\|_{0,E}^2 + h_E\|(\nu+h_E\|b\|_{\infty,E})\,[\![u_h]\!]_E\,\|_{0,E}^2.$$

The behaviour of $\hat{\eta}$ and $\bar{\eta}$ differs considerably, compare Table 5 and 6. The jump terms of the nonconforming function are not longer dominant. $\bar{\eta}$ shows a similar behaviour like $\hat{\eta}$ in Example 8.1, i.e. $\bar{\eta}$ is an upper bound of the error for $\nu = 10^{-1}$ and for small values of ν it pretends a quadratic convergence instead of the actual linear one. The quantities $\bar{C}_S\bar{\eta}$ are always upper bounds of the error.

We have found in other numerical tests a similar behaviour like in this example, e.g. in Example 7.1 the quantities $\hat{\eta}$ and $\bar{C}_S\bar{\eta}$ have been nearly the same for small ν.

ν	level	$\|u-u_h\|_0$	rate	$\bar\eta$	rate	$\bar C_S$	$\bar C_S\bar\eta$	rate
10^{-1}	4	2.92-2	2.25	5.14-2	2.12	6.9	3.55-1	1.69
	5	1.16-2	1.34	1.35-2	1.93	7.1	9.59-2	1.89
	6	1.52-3	2.93	3.66-2	1.89	12.0	4.39-2	1.13
	7	3.85-4	1.98	9.65-4	1.92	4.9	4.76-3	3.21
10^{-3}	4	4.46-2	0.97	2.40-2	2.24	30.2	7.25-1	0.86
	5	2.23-2	1.00	5.05-3	2.25	49.5	2.50-1	1.54
	6	1.08-2	1.04	1.02-3	2.31	82.4	8.42-2	1.57
	7	4.97-3	1.12	2.03-4	2.33	199.2	4.04-2	1.06
10^{-5}	4	4.72-2	0.92	2.52-2	2.19	31.6	7.97-1	0.78
	5	2.48-2	0.93	5.70-3	2.15	53.8	3.06-1	1.38
	6	1.32-2	0.91	1.33-3	2.10	87.8	1.17-1	1.39
	7	7.04-3	0.91	3.23-4	2.04	225.8	7.29-2	0.68
10^{-6}	4	4.73-2	0.92	2.52-2	2.19	31.6	7.98-1	0.78
	5	2.48-2	0.93	5.70-3	2.15	53.8	3.07-1	1.38
	6	1.32-2	0.91	1.34-3	2.10	87.8	1.17-1	1.39
	7	7.06-3	0.90	3.25-4	2.04	225.8	7.34-2	0.68

Table 6: Example 8.2, exact and estimated global L^2–error using $\bar\eta$.

Remark 8.3 Summary of the numerical tests.

1. **Reaction–diffusion equations.** The error $\|u-u_h\|_0$ for small ν has been underestimated completely in using the local error estimator $\hat\eta$. A much more reliable error estimation could be achieved using the heuristic computed constant $\bar C_S$. The rate of convergence of $\|u-u_h\|_0$ has been reflected very well by $\bar C_S\hat\eta$.
2. **Convection–diffusion equations.** The local error estimator $\hat\eta$ has given good upper bounds of the error $\|u-u_h\|_0$ in the presented example. The error has been completely overestimated by $\bar C_S\hat\eta$. The modification $\bar\eta$ of the local error estimator $\hat\eta$ has shown a similar behaviour like $\hat\eta$ for reaction–diffusion equations.
3. **Open problems.** The heuristic approximation of C_S seems to give a robust a posteriori error estimation at least in some examples. This approximation requires the solution of an additional problem which is as hard to solve as the original problem or even more difficult. In our computations the multigrid solver had very bad convergence rates and using the nonconforming streamline–diffusion discretization it even diverged. We have presented only examples on uniform triangulations because locally refined meshes which are good for the original problem may not be adapted for the dual problem. In this case an additional "dual mesh" is necessary.

Acknowledgement: The author wishes to acknowledge Prof. L. Tobiska for inspiring discussions on the subject of this paper.

References

[Ada95] D. Adam. *Eine gleichmäßig konvergente Diskretisierungstechnik für konvektionsdominante Konvektions–Diffusions–Probleme.* PhD thesis, TU Dresden, 1995.

[Ang95] L. Angermann. Balanced a posteriori error estimates for finite–volume type discretizations of convection–dominated elliptic problems. *Computing*, 55:305 – 323, 1995.

[BR96] R. Becker and R. Rannacher. Weighted a posteriori error control in fe methods. Preprint 96 - 1, SFB 359, Universität Heidelberg, 1996.

[Cia91] P.G. Ciarlet. Basic error estimates for elliptic problems. In P.G. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis II*, pages 19–351. North–Holland Amsterdam, New York, Oxford, Tokyo, 1991.

[CR73] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations I. *R.A.I.R.O. Analyse Numérique*, 7:33–76, 1973.

[DDP95] E. Dari, R. Durán, and C. Padra. Error estimators for nonconforming finite element approximations of the Stokes problem. *Math. Comp.*, 64(211):1017 – 1033, 1995.

[DDPV96] E. Dari, R. Duran, C. Padra, and V. Vampa. A posteriori error estimators for nonconforming finite element methods. *M2AN*, 30(4):385 – 400, 1996.

[DJR+96] O. Dorok, V. John, U. Risch, F. Schieweck, and L. Tobiska. Parallel finite element methods for the incompressible Navier–Stokes equations. In E.H. Hirschel, editor, *Flow Simulation with High–Performance Computers II*, volume 52 of *Notes on Numerical Fluid Mechanics*, pages 20 – 33. Vieweg, 1996.

[EEHJ95] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson. Introduction to adaptive methods for differential equations. *Acta Numerica*, pages 105–158, 1995.

[JMT96] V. John, J.M. Maubach, and L. Tobiska. Nonconforming streamline–diffusion–finite–element–methods for convection–diffusion problems. Preprint 10/96, Otto-von-Guericke Universität Magdeburg, Fakultät für Mathematik, 1996. to appear in Numer. Math.

[Joh94] V. John. A comparison of some error estimators for convection–diffusion problems on a parallel computer. Preprint 12/94, Otto-von-Guericke Universität Magdeburg, Fakultät für Mathematik, 1994.

[Joh95] C. Johnson. On computability and error control in CFD. *Int. Jour. Num. Meth. Fluids*, 20:777–788, 1995.

[Joh97] V. John. *Parallele Lösung der inkompressiblen Navier–Stokes Gleichungen auf adaptiv verfeinerten Gittern.* PhD thesis, Otto-von-Guericke Universität Magdeburg, Fakultät für Mathematik, 1997. to appear.

[JR94] C. Johnson and R. Rannacher. On error control in CFD. In Wittum Hebeker, Rannacher, editor, *Numerical Methods for the Navier–Stokes Equations*, volume 47 of *Notes on Numerical Fluid Mechanics*, pages 133 – 144. Vieweg, 1994.

[JRB95] C. Johnson, R. Rannacher, and M. Boman. Numerics and hydrodynamics stability: Toward error control in computational fluid dynamics. *SIAM J. Numer. Anal.*, 32(4):1058 – 1079, 1995.

[OU84] K. Ohmori and T. Ushijima. A technique of upstream type applied to a linear nonconforming finite element approximation of convective diffusion equations. *R.A.I.R.O. Anal. Numer.*, 18(3):309–332, 1984.

[Roo96] H.-G. Roos. A priori estimates, asymptotic expansions and Shishkin decompositions. Report MATH–NM–21–96, Technical University of Dresden, 1996.

[RST96] H.-G. Roos, M. Stynes, and L. Tobiska. *Numerical Methods for Singularly Perturbed Differential Equations.* Springer, 1996.

[RT92] R. Rannacher and St. Turek. Simple nonconforming quadrilateral stokes element. *Numer. Meth. Part. Diff. Equ.*, 8:97 – 111, 1992.

[Sch96] F. Schieweck. *Parallele Lösung der stationären inkompressiblen Navier–Stokes Gleichungen.* Otto-von-Guericke Universität Magdeburg, Fakultät für Mathematik, 1996. Habilitation.

[ST96] F. Schieweck and L. Tobiska. An optimal order error estimate for an upwind discretization of the Navier–Stokes equations. *Num. Meth. Part. Diff. Equ.*, 12:407 – 421, 1996.

[Ver96a] R. Verfürth. A posteriori error estimates for nonlinear problems. L^r–estimates for finite element discretizations of elliptic equations. Bericht Nr. 199, Ruhr–Universität Bochum, Fakultät für Mathematik, 1996.

[Ver96b] R. Verfürth. *A Review of A Posteriori Error Estimation and Adaptive Mesh–Refinement Techniques.* Wiley and Teubner, 1996.

[Woh95] B. Wohlmuth. *Adaptive Multilevel–Finite–Elemente Methoden zur Lösung elliptischer Randwertprobleme.* PhD thesis, Technische Universität München, 1995.