Department for Mathematics and Computer Science
Freie Universität Berlin
Prof. Dr. V. John, john@wias-berlin.de
André-Alexander Zepernick, a.zepernick@fu-berlin.de

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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 05

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present one correct solution (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 2 points from the presentation. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.

Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Weak derivative in one dimension.* Solve the following problems.

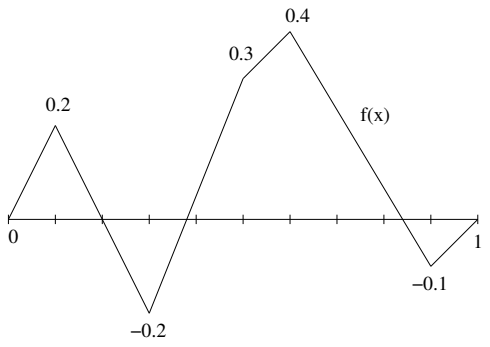
   i) Let $f(x) = 1$ in Ω. Investigate whether or not $f \in L^1(\Omega)$, $f \in L^1_{\text{loc}}(\Omega)$ for $\Omega = (0,1)$ and $\Omega = \mathbb{R}$.

   ii) Show with the help of the definition that

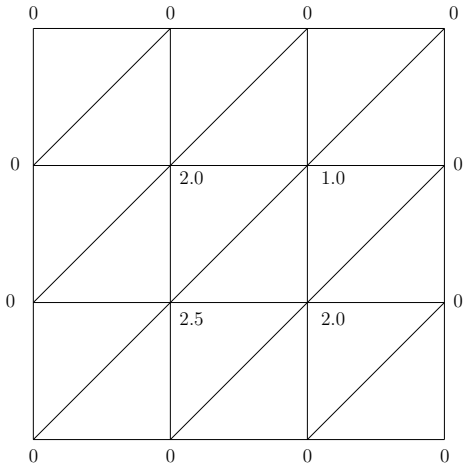
   $$f'(x) = \begin{cases} -1 & x < 0, \\ 0 & x = 0, \\ 1 & x > 0, \end{cases}$$

   is the weak derivative of $f(x) = |x|$.

   iii) Compute the weak derivative of the following function $f : \Omega \to \mathbb{R}$, $\Omega = (0,1)$.



2. *Weak derivative in two dimensions.* Compute the first weak derivatives of the following function $f : \Omega \to \mathbb{R}$, $\Omega = (0,1)^2$, which is continuous and piecewise (with respect to the grid) linear and which is therefore completely determined by the values in the nodes.



3. *Hölder's inequality.*

   i) Let $r \in [1, \infty)$, $p, q \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, $u \in L^{rp}(\Omega)$, $v \in L^{rq}(\Omega)$. Show that
   $$\|uv\|_{L^r(\Omega)} \leq \|u\|_{L^{rp}(\Omega)} \|v\|_{L^{rq}(\Omega)}.$$

   ii) Show that for $p \in (2, \infty)$
   $$\|uv\|_{L^2(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^{2p/(p-2)}(\Omega)}$$
   holds.

   iii) Prove the generalized Hölder inequality. Let $1 \leq p_1, ..., p_m \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_m} = 1$ and assume $u_k \in L^{p_k}(\Omega)$ for $k = 1, ..., m$, then
   $$\int_\Omega |u_1 \cdots u_m| \, \mathrm{d}x \leq \prod_{k=1}^{m} ||u_k||_{L^{p_k}(\Omega)}.$$

4. *Minkowski's inequality.* Let $1 \leq p, q \leq \infty$ and $u, v \in L^p(\Omega)$, then
   $$||u + v||_{L^p(\Omega)} \leq ||u||_{L^p(\Omega)} + ||v||_{L^p(\Omega)}.$$

The exercise problems will be discussed at the turorial on **Friday, November 20, 2026, 08-10**.