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# Numerical Mathematics III – Partial Differential Equations

## Exercise Problems 04

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present one correct solution (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 2 points from the presentation. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.

**Prepare these presentations!** All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. Let the assumptions of the discrete maximum principle for the boundary value problem be satisfied. Prove the following stability estimate: For the solution of the problem

$$\begin{aligned} L_h u(\mathbf{x}) &= f(\mathbf{x}), \quad \mathbf{x} \in \omega_h, \\ u(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \gamma_h, \end{aligned}$$

with $d(\mathbf{x}) > 0$ for all $\mathbf{x} \in \omega_h$, it is

$$\|u\|_{l^\infty(\omega_h \cup \gamma_h)} \leq \left\| D^{-1} f \right\|_{l^\infty(\omega_h)}$$

with $D = \mathrm{diag}(d(\mathbf{x}))$ for $\mathbf{x} \in \omega_h$. The notations and the properties of the individual terms of the finite difference operator are the same as in the lecture, Section 2.3.

2. *An eigenvalue problem connected to the five point stencil.* Show that the vector $v_k = (v_{k,0}, \cdots, v_{k,n})$ with

$$v_{k,0} = v_{k,n} = 0, \quad v_{k,i} = \sqrt{2} \sin\left(\pi\, k\, x_i\right),$$

solves the eigenvalue problem

$$v_{k,i-1} + \left(\lambda_k\, h^2 - 2\right) v_{k,i} + v_{k,i+1} = 0$$

with

$$\lambda_k = \frac{2}{h^2}\left(1 - \cos\left(\pi\, k\, h\right)\right) = \frac{4}{h^2} \sin^2\left(\frac{\pi\, k\, h}{2}\right).$$

3. *Matrix property, detail of the proof of Theorem 2.45.* In the proof of Theorem 2.45, the following argument is used: Let $A, B \in \mathbb{R}^{n \times n}$ be two symmetric and positive definite matrices with $AB = BA$ and

$$(A\underline{x}, \underline{x}) \geq (B\underline{x}, \underline{x}) \quad \forall\ \underline{x} \in \mathbb{R}^n,$$

then $\|A\underline{x}\|_2 \geq \|B\underline{x}\|_2$ for all $\underline{x} \in \mathbb{R}^n$. The symbols denote the Euclidean inner product and the Euclidean vector norm. Prove this statement.

4. This problem has to be solved until November 20, 2026!
   *Code for nine point stencil.* The problem is formulated analogously as Problem 5 from exercise sheet 02. The only difference is that the finite difference scheme with the nine point stencil should be used (instead of using the five point stencil).

The exercise problems will be discussed at the turorial on **Friday, November 13, 2026, 08-10**.