Department for Mathematics and Computer Science
Freie Universität Berlin
Prof. Dr. V. John, john@wias-berlin.de
André-Alexander Zepernick, a.zepernick@fu-berlin.de

Berlin, 26.10.2026

Numerical Mathematics III – Partial Differential Equations

Exercise Problems 03

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present one correct solution (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 2 points from the presentation. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.

Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Finite difference approximation of the second order derivative at a non-equidistant grid.* Consider the interval $[x - h_x^-, x + h_x^+]$ with $h_x^-, h_x^+ > 0$, $h_x^- \neq h_x^+$.

   i) Assume $u \in C^3([x - h_x^-, x + h_x^+])$. Show the following consistency estimate

   $$\left| u''(x) - \frac{2}{h_x^+ + h_x^-} \left(\frac{u(x + h_x^+) - u(x)}{h_x^+} - \frac{u(x) - u(x - h_x^-)}{h_x^-} \right) \right| \leq C \left(h_x^+ + h_x^- \right).$$

   ii) Prove that there is no other approximation which satisfies

   $$\left| u''(x) - \left(\alpha\, u\left(x - h_x^-\right) + \beta\, u(x) + \gamma\, u\left(x + h_x^+\right) \right) \right| \leq C \left(h_x^+ + h_x^- \right),$$

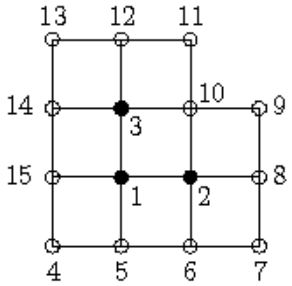
   where the weights depend only on the mesh width, i.e., $\alpha = \alpha(h_x^-, h_x^+)$, $\beta = \beta(h_x^-, h_x^+)$, $\gamma = \gamma(h_x^-, h_x^+) \in \mathbb{R}$, and the constant C does not explicitly depend on $u(\boldsymbol{x})$, $u'(\boldsymbol{x})$, and $u''(\boldsymbol{x})$.

2. *Second order FD approximation of first derivative on non-equidistant grid.* Let $v(\mathbf{x})$ be a sufficiently smooth function. Derive a second order finite difference approximation of the first derivative in the node x_i for a non-equidistant grid with the points $x_i - h^-, x_i, x_i + h^+$. Is a third order approximation possible? Hint: For an equidistant grid, the central difference is of second order, which is just the arithmetic average of the forward and the backward difference. To solve this problem, one has to find a different linear combination of the forward and backward difference.

3. *Five point stencil.* Consider the Dirichlet problem for the Poisson equation

   $$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

   and the corresponding finite difference discretization with the five point stencil on the following grid:



Compute the matrices $A \in \mathbb{R}^{3\times 3}$ and $B \in \mathbb{R}^{3\times 12}$ for the finite difference equation

$$A\underline{u} = \underline{f} + B\underline{g}$$

with $\underline{u} = (u_1, u_2, u_3)^T$ and $\underline{g} = (u_4, \ldots, u_{15})^T$.

4. *Comparison lemma.* Prove the comparison lemma (Corollary 2.24).

5. Do not forget the programming problem from exercise sheet 02.

The exercise problems will be discussed at the turorial on **Friday, November 06, 2026, 08-10**.