

Berlin, 19.10.2026

Numerical Mathematics III – Partial Differential Equations

Exercise Problems 02

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present one correct solution (depending on the number of subproblems, a ‘solution’ might cover only a part of the subproblems) and obtain a total of 2 points from the presentation. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.

Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Harmonic function.* A function is called harmonic if its Laplacian vanishes. Find out whether the two-dimensional function

$$u(x, y) = \exp(x) \cos(y)$$

is harmonic.

2. *The Laplace equation in polar coordinates.*

- a) Transform the Laplace equation $\Delta u(x, y) = 0$ for $x, y > 0$ into polar coordinates.
- b) Confirm or falsify that $u(r, \varphi) = r^{\frac{2}{3}} \sin\left(\frac{2}{3}\right)$ is a solution of the Laplace equation.

3. *Basic properties of finite difference approximations.* Solve the following problems.

- i) Show that

$$v_{\bar{x},i} = \frac{1}{2} (v_{x,i} + v_{\bar{x},i}), \quad v_{\bar{x}x,i} = (v_{\bar{x},i})_{x,i}.$$

- ii) Consider a function $v(x)$ at x_i and show the following consistency estimates

$$v_{\bar{x},i} = v'(x_i) + \mathcal{O}(h^2), \quad v_{\bar{x}x,i} = v''(x_i) + \mathcal{O}(h^2).$$

- iii) Compute the order of consistency of the following finite difference approximation

$$u''(x) \sim \frac{1}{12h^2} \left(-u(x+2h) + 16u(x+h) - 30u(x) + 16u(x-h) - u(x-2h) \right).$$

4. *Finite difference approximation of the second order derivative for non-constant diffusion.* Consider the differential operator $Lu = \frac{\partial}{\partial x} \left(k(x) \frac{\partial u}{\partial x} \right)$ and its finite difference approximation

$$(L_h u_h)_i = \frac{1}{h} \left(a_{i+1} \frac{u_{i+1} - u_i}{h} - a_i \frac{u_i - u_{i-1}}{h} \right).$$

Show that $a_i = \frac{k_i + k_{i-1}}{2}$ and $a_i = k\left(x_i - \frac{h}{2}\right)$ satisfy the conditions for second order consistency

$$\frac{a_{i+1} - a_i}{h} = k'(x_i) + \mathcal{O}(h^2), \quad \frac{a_{i+1} + a_i}{2} = k(x_i) + \mathcal{O}(h^2),$$

which were derived in the lecture.

5. **This problem has to be solved until November 06, 2026!**

Code for five point stencil. Write a code for the numerical solution of

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0, 1)^2, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

The right-hand side and the boundary conditions should be chosen such that

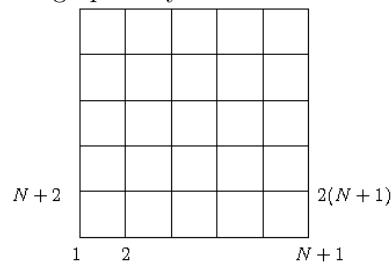
$$u(x, y) = x^4 y^5 - 17 \sin(xy)$$

is the solution of the boundary value problem.

Use the five point stencil for the discretizing the partial differential equation. The mesh widths should be chosen to be

$$h_x = h_y = h = 2^{-n} \quad n = 2, 3, 4, \dots, 8.$$

Order the nodes lexicographically and store the matrix in **sparse** format.



Compute the following errors

$$\|u - u_h\|_{l^\infty(\omega_h)}, \quad \|u - u_h\|_{l^2(\omega_h)},$$

and the orders of convergence, based on the errors on the two finest meshes, where the second norm is the standard Euclidean vector norm.

The exercise problems will be discussed at the tutorial on **Friday, October 30, 2026, 08-10.**