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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 08

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof. If tools from AI are used to solve the problems, then this has to be indicated.

1. *Local basis.* Let $P(K)$ be unisolvent with respect to the functionals $\{\Phi_{K,i}\}_{i=1}^{N_K}$. Show that the set $\{\phi_{K,i}\}_{i=1}^{N_K} \subset P(K)$ with $\Phi_{K,i}(\varphi_{K,j}) = \delta_{ij}$ forms a basis of $P(K)$. **2 points**

2. *Local basis of $P_2(\hat{K})$.* Consider the reference triangle $\widehat{K}$ with the vertices $(0,0)$, $(1,0)$, and $(0,1)$. The space of polynomials of degree two is spanned by
$$1,\ \widehat{x},\ \widehat{y},\ \widehat{x}\widehat{y},\ \widehat{x}^2,\ \widehat{y}^2.$$
Use as functionals the values of the functions in the vertices and the barycenters of the edges. Compute the local basis with respect to these functionals. **3 points**

3. *Non-degenerated simplex.* Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{d+1} \in \mathbb{R}^d$ be the vertices of a simplex with $\boldsymbol{a}_i = (a_{1i}, a_{2i}, \ldots, a_{di})^T$, $i = 1, \ldots, d+1$. Show that the simplex is not degenerated, i.e., its d-dimensional measure is positive, if and only if the matrix
$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1,d+1} \\ a_{21} & a_{22} & \ldots & a_{2,d+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \ldots & a_{d,d+1} \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$
is non-singular. **3 points**

4. *Affine transform.* Let K be a triangle in $\mathbb{R}^2$ with the vertices $(x_i, y_i), i = 1,2,3$. Compute the affine transform of the reference triangle $\hat{K}$ to K, which maps $(0,0)$ to (x_1,y_1), $(1,0)$ to (x_2,y_2), and $(0,1)$ to (x_3,y_3). Which geometric property of K is connected to the absolute value of the determinant of the matrix of the transform? **2 points**

5. *Gradient of a linear function on a triangle.* Consider a linear function u^h on the triangle K with the vertices $P_i = (x_i, y_i), i = 1,2,3$, which has the values $u^h(P_i)$. Find a formula for ∇u^h. **3 points**

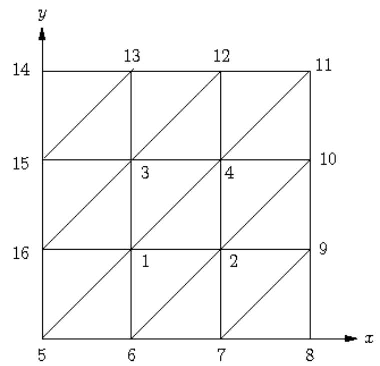
6. *Code with P_1 finite elements.* This problem has to be solved until June 30 !
Write a code for the numerical solution of
$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega = (0,1)^2, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

The right-hand side and the Dirichlet boundary conditions should be chosen such that

$$u(x,y) = x^4 y^5 - 17\sin(xy)$$

is the solution of the boundary value problem.

Use for discretizing the problem the P_1 finite element method on the following grid



Choose the mesh width to be

$$h_x = h_y = h = 2^{-n} \quad n = 2,3,4,\ldots,8.$$

Store the matrix in `sparse` format. The vertices should be enumerated analogously as in the sketch, i.e., the interior nodes are enumerated lexicographically and then follow the vertices on the boundary, counter clockwise and starting with $(0,0)$.

Evaluate

$$\|u - u_h\|_{L^2(\Omega)} \quad \text{and} \quad \|\nabla u - \nabla u_h\|_{L^2(\Omega)}.$$

10 points

Hint: the FEM problem is: Find $u^h \in P_1 +$ boundary condition with

$$(\nabla u^h, \nabla v^h) = (f, v^h) \quad \forall\, v^h \in P_1 + \text{ zero boundary condition.}$$

One can write the integrals as a sum over the mesh cells, for instance

$$(\nabla u^h, \nabla v^h) = \sum_{K \in \mathcal{T}^h} (\nabla u^h, \nabla v^h)_K.$$

For this reason, one should use in FEM an approach for assembling the matrices and the right-hand side which is based on a loop over the mesh cells (and not over the vertices as in finite difference methods):

- write a loop over the mesh cells,
- compute for each mesh cell K the numbers of the degrees of freedom (unknowns), which are for the P_1 finite element the numbers of the vertices,
- compute the local update of the matrix entries

$$(\nabla \phi_j, \nabla \phi_i)_K$$

  and add this update to the global matrix

$$a_{ij} := a_{ij} + (\nabla \phi_j, \nabla \phi_i)_K.$$

  Do the same for the right-hand side.

Concerning the matrix, one can compute alternatively the entries by hand and just sets them in the correct positions.

In the rows of the matrix, which correspond to the nodes on the boundary, replace the diagonal entry with one and set all other entries to be zero. The respective entry on the right-hand side gets the value of the boundary condition in this node.

In this way, one has obtained the linear system of equations whose solution gives the coefficient of the finite element solution u^h.

For the computation of the errors, use the same approach as for assembling the matrix:

- write a loop over the mesh cells,
- compute for each mesh cell K the numbers of the degrees of freedom (unknowns), which are for the P_1 finite element the numbers of the vertices,
- compute the squares of the local errors

$$l2 := (u - u^h, u - u^h), \quad h1 := (\nabla u - \nabla u^h, \nabla u - \nabla u^h),$$

  (formula for ∇u^h see previous problem)
- update the square of the global errors

$$L2 := L2 + l2, \quad H1 := H1 + h1.$$

Finally, take the roots of $L2$ and $H1$.

For the numerical quadrature in assembling the matrix and the right-hand side, and for computing the errors, use the edge midpoint rule

$$\int_K v(\boldsymbol{x})\, d\boldsymbol{x} \approx \frac{|K|}{3} \left(v(\boldsymbol{x}_1) + v(\boldsymbol{x}_2) + v(\boldsymbol{x}_3) \right),$$

where $|K|$ is the volume (area) of the triangle K and $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are the midpoints of the edges of K.

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, June 23, 2025, 10:00 a.m.** via the whiteboard.