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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 05

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof. If tools from AI are used to solve the problems, then this has to be indicated.

1. *Weak derivative in one dimension.* Solve the following problems.

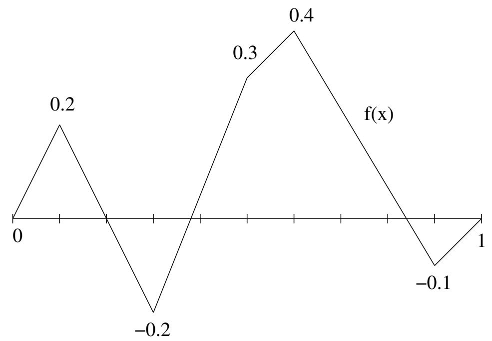
   i) Let $f(x) = 1$ in Ω. Investigate whether or not $f \in L^1(\Omega)$, $f \in L^1_{\text{loc}}(\Omega)$ for $\Omega = (0,1)$ and $\Omega = \mathbb{R}$. **2 points**

   ii) Show with the help of the definition that

   $$f'(x) = \begin{cases} -1 & x < 0, \\ 0 & x = 0, \\ 1 & x > 0, \end{cases}$$

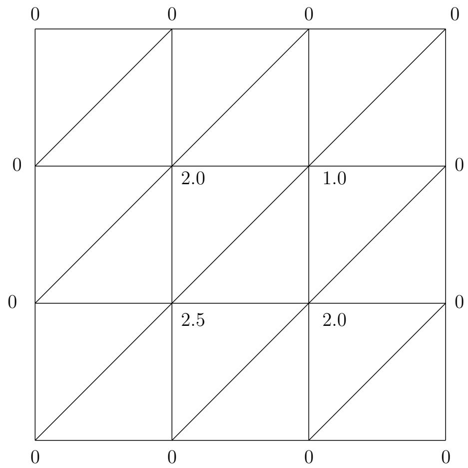
   is the weak derivative of $f(x) = |x|$. **2 points**

   iii) Compute the weak derivative of the following function $f : \Omega \to \mathbb{R}$, $\Omega = (0,1)$.



1 point

2. *Weak derivative in two dimensions.* Compute the first weak derivatives of the following function $f : \Omega \to \mathbb{R}$, $\Omega = (0,1)^2$, which is continuous and piecewise (with respect to the grid) linear and which is therefore completely determined by the values in the nodes.



2 points

3. *Hölder's inequality.*

   i) Let $r \in [1, \infty)$, $p, q \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, $u \in L^{rp}(\Omega)$, $v \in L^{rq}(\Omega)$. Show that

   $$\|uv\|_{L^r(\Omega)} \leq \|u\|_{L^{rp}(\Omega)} \|v\|_{L^{rq}(\Omega)}.$$

   1 point

   ii) Show that for $p \in (2, \infty)$

   $$\|uv\|_{L^2(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^{2p/(p-2)}(\Omega)}$$

   holds. **1 point**

4. *Rational functions in Lebesgue spaces in a ball.* Solve the following problems.

   i) For which values of $a \in \mathbb{R}$ is the function $f : (-1, 1) \to \mathbb{R}$

   $$f(x) = \begin{cases} |x|^a & x \neq 0, \\ 0 & x = 0 \end{cases}$$

   an element of $L^p((-1, 1))$ with $p \in [1, \infty]$? **2 points**

   ii) Let

   $$B_1(\mathbf{0}) = \{\boldsymbol{x} : \|\boldsymbol{x}\|_2 < 1\}$$

   be the d-dimensional unit ball, $d > 1$. Find the values $a \in \mathbb{R}$ for which the function $f : B_1(\mathbf{0}) \to \mathbb{R}$ with

   $$f(x) = \begin{cases} \|\boldsymbol{x}\|_2^a & \boldsymbol{x} \neq \mathbf{0}, \\ 0 & \boldsymbol{x} = \mathbf{0}, \end{cases}$$

   belongs to $L^p(B_1(\mathbf{0}))$ with $p \in [1, \infty]$! **2 points**

The exercise problems should be solved in groups of four students. The solutions have to be submitted until **Monday, May 26, 2025, 10:00 a.m.** via the whiteboard.