Chapter 5
Finite Element Methods

5.1 Finite Element Spaces

Remark 5.1. Mesh cells, faces, edges, vertices. As initial step of the application of finite element methods, the considered domain is decomposed or triangulated with a mesh or a grid. A mesh cell K is a compact polyhedron in $\mathbb{R}^d$, $d \in \{2, 3\}$, whose interior is not empty. The boundary ∂K of K consists of m-dimensional linear manifolds (points, pieces of straight lines, pieces of planes), $0 \le m \le d-1$, which are called m-faces. The 0-faces are the vertices of the mesh cell, the 1-faces are the edges, and the $(d-1)$-faces are called facets. □

Remark 5.2. Finite-dimensional spaces defined on K. Let $s \in \mathbb{N}$. Finite element methods use finite-dimensional spaces $P(K) \subset C^s(K)$ that are defined on K. In general, $P(K)$ consists of polynomials. The dimension of $P(K)$ will be denoted by $N_K = \dim P(K)$. □

Example 5.3. The space $P(K) = P_1(K)$. The space consisting of linear polynomials on a mesh cell K is denoted by $P_1(K)$:

$$P_1(K) = \left\{ a_0 + \sum_{i=1}^{d} a_i x_i \ : \ \boldsymbol{x} = (x_1, \ldots, x_d)^T \in K \right\}.$$

There are $d+1$ unknown coefficients a_i, $i = 0, \ldots, d$, so that $\dim P_1(K) = N_K = d + 1$. □

Remark 5.4. Linear functionals defined on $P(K)$, nodal functionals. For the definition of finite elements, linear functionals that are defined on $P(K)$ are of importance. These functionals are called nodal functionals.

Consider linear and continuous functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K} \ : \ C^s(K) \to \mathbb{R}$ that are linearly independent. There are different types of functionals that can be utilized in finite element methods:

- point values: $\Phi(v) = v(\boldsymbol{x})$, $\boldsymbol{x} \in K$,
- point values of a first partial derivative: $\Phi(v) = \partial_i v(\boldsymbol{x})$, $\boldsymbol{x} \in K$,
- point values of the normal derivative on a facet E of K: $\Phi(v) = \nabla v(\boldsymbol{x}) \cdot \boldsymbol{n}_E$, $\boldsymbol{n}_E$ is the outward pointing unit normal vector on E,
- integral mean values on K: $\Phi(v) = \frac{1}{|K|} \int_K v(\boldsymbol{x})\ d\boldsymbol{x}$, where $|K|$ is the measure of K,
- integral mean values on faces E: $\Phi(v) = \frac{1}{|E|} \int_E v(\boldsymbol{s})\ d\boldsymbol{s}$ with $|E|$ being the measure of E.

The smoothness parameter s has to be chosen in such a way that the functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K}$ are continuous. If, e.g., a functional requires the evaluation of a partial derivative or a normal derivative, then one has to choose at least $s = 1$. For the other functionals given above, $s = 0$ is sufficient. □

Definition 5.5. Unisolvence of $P(K)$ with respect to the functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K}$. The space $P(K)$ is called unisolvent with respect to the functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K}$ if there is for each $\underline{a} \in \mathbb{R}^{N_K}$, $\underline{a} = (a_1, \ldots, a_{N_K})^T$, exactly one $p \in P(K)$ with

$$\Phi_{K,i}(p) = a_i, \quad 1 \le i \le N_K.$$

□

Remark 5.6. Local basis. Unisolvence means that if the value for each functional is known, then one can identify a unique element from $P(K)$ that takes these values.

Choosing in particular the Cartesian[1] unit vectors for $\underline{a}$, then it follows from the unisolvence that a set $\{\phi_{K,i}\}_{i=1}^{N_K}$ exists with $\phi_{K,i} \in P(K)$ and

$$\Phi_{K,i}(\phi_{K,j}) = \delta_{ij}, \quad i, j = 1, \ldots, N_K.$$

Consequently, the set $\{\phi_{K,i}\}_{i=1}^{N_K}$ forms a basis of $P(K)$ (*exercise*). This basis is called local basis. □

Remark 5.7. Transform of an arbitrary basis to the local basis. If an arbitrary basis $\{p_i\}_{i=1}^{N_K}$ of $P(K)$ is known, then the local basis can be computed by solving a linear system of equations. To this end, represent the local basis in terms of the known basis

$$\phi_{K,j} = \sum_{k=1}^{N_K} c_{jk} p_k, \quad c_{jk} \in \mathbb{R},\ j = 1, \ldots, N_K,$$

with unknown coefficients c_{jk}. Applying the definition of the local basis leads to the linear system of equations

[1] René Descartes (1596 – 1650)

$$\Phi_{K,i}(\phi_{K,j}) = \sum_{k=1}^{N_K} c_{jk} a_{ik} = \delta_{ij}, \quad i,j = 1,\ldots,N_K, \quad a_{ik} = \Phi_{K,i}(p_k).$$

Because of the unisolvence, the matrix $A = (a_{ij})$ is non-singular and the coefficients c_{jk} are determined uniquely. □

Example 5.8. Local basis for the space of linear functions on the reference triangle. Consider the reference triangle $\hat{K}$ with the vertices $(0,0)$, $(1,0)$, and $(0,1)$. A linear space on $\hat{K}$ is spanned by the functions $1, \hat{x}, \hat{y}$. Let the functionals be defined by the values of the functions in the vertices of the reference triangle. Then, the given basis is not a local basis because the function 1 does not vanish at the vertices.

Consider first the vertex $(0,0)$. A linear basis function $a\hat{x} + b\hat{y} + c$ that has the value 1 in $(0,0)$ and that vanishes in the other vertices has to satisfy the following set of equations

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The solution is $a = -1, b = -1, c = 1$. The two other basis functions of the local basis are $\hat{x}$ and $\hat{y}$, so that the local basis has the form $\{1 - \hat{x} - \hat{y}, \hat{x}, \hat{y}\}$. □

Remark 5.9. Triangulation, grid, mesh, grid cell. For the definition of global finite element spaces, a decomposition of the domain Ω into polyhedra K is needed, see Figure 5.1 for an example. This decomposition is called triangulation $\mathcal{T}^h$ and the polyhedra K are called mesh cells. The union of the polyhedra is called grid or mesh.

A triangulation is called admissible, see the definition in (Ciarlet, 1978, p. 38, p. 51), if:

- It holds $\overline{\Omega} = \cup_{K \in \mathcal{T}^h} K$.
- Each mesh cell $K \in \mathcal{T}^h$ is closed and the interior $\mathring{K}$ is non-empty.
- For distinct mesh cells K_1 and K_2 there holds $\mathring{K}_1 \cap \mathring{K}_2 = \emptyset$.
- For each $K \in \mathcal{T}^h$, the boundary ∂K is Lipschitz continuous.
- The intersection of two mesh cells is either empty or a common m-face, $m \in \{0, \ldots, d-1\}$.

□

Remark 5.10. Global and local functionals. Let

$$\Phi_1, \ldots, \Phi_N \; : \; \{v \in L^\infty(\Omega) \; : \; v|_K \in P(K)\} \to \mathbb{R}$$

be continuous linear functionals of the same types as given in Remark 5.4, where for each K, $v|_K \in P(K)$ has to be understood in the sense that the polynomial in K is extended continuously to the boundary of K. The

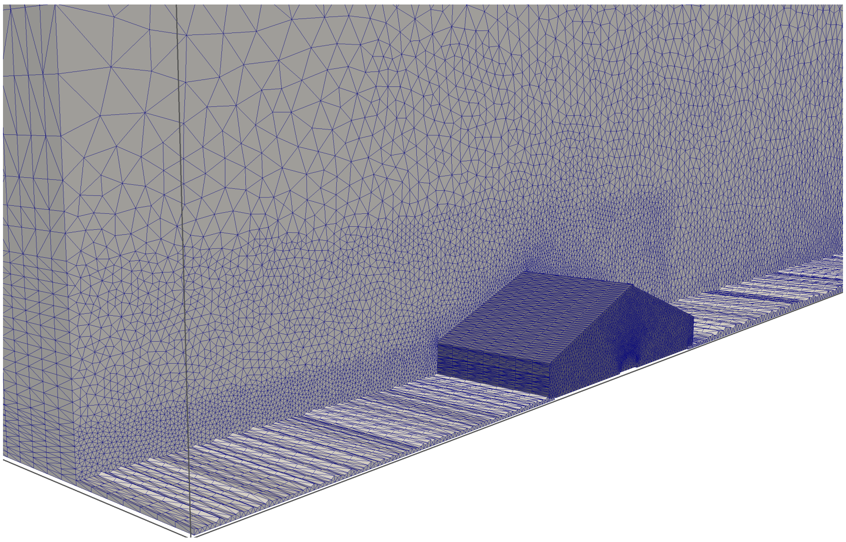

Fig. 5.1 Triangulation of a domain for the simulation of the flow through and around a dairy barn, from Janke *et al.* (2020).

restriction of the functionals to $C^s(K)$ defines a set of local functionals $\Phi_{K,1}, \ldots, \Phi_{K,N_K}$, where it is assumed that the local functionals are unisolvent on $P(K)$. The union of all mesh cells K_j, for which there is a $p \in P(K_j)$ with $\Phi_i(p) \neq 0$, will be denoted by ω_i. □

Example 5.11. On subdomains ω_i. Consider the two-dimensional case and let Φ_i be defined as nodal value of a function in $\boldsymbol{x} \in K$. If $\boldsymbol{x} \in \mathring{K}$, then $\omega_i = K$. In the case that $\boldsymbol{x}$ is on a face of K but not in a vertex, then ω_i is the union of K and the other mesh cell whose boundary contains this face. Last, if $\boldsymbol{x}$ is a vertex of K, then ω_i is the union of all mesh cells that possess this vertex, see Figure 5.2. □

Definition 5.12. Finite element space, global basis. A function $v(\boldsymbol{x})$ defined on Ω with $v|_K \in P(K)$ for all $K \in \mathcal{T}^h$ is called continuous with respect to a global functional Φ_i defined in Remark 5.10 if

$$\Phi_i(v|_{K_1}) = \Phi_i(v|_{K_2}), \quad \forall\, K_1, K_2 \in \omega_i.$$

The space

$$S = \Big\{ v \in L^\infty(\Omega) \ : \ v|_K \in P(K) \text{ and } v \text{ is continuous with respect to } \Phi_i, i = 1, \ldots, N \Big\}$$

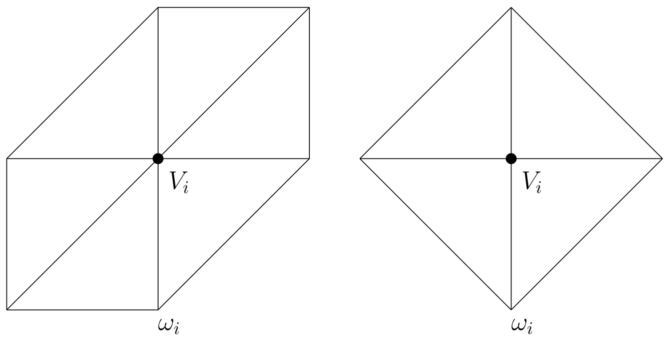

Fig. 5.2 Subdomains ω_i.

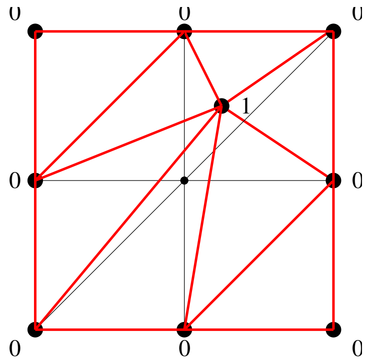

Fig. 5.3 Piecewise linear global basis function (red lines), hat function.

is called finite element space.

The global basis $\{\phi_j\}_{j=1}^N$ of S is defined by the condition

$$\phi_j \in S, \quad \Phi_i(\phi_j) = \delta_{ij}, \quad i, j = 1, \ldots, N.$$

□

Example 5.13. Piecewise linear global basis function. Figure 5.3 shows a piecewise linear global basis function in two dimensions. Because of its form, such a function is called hat function. □

Remark 5.14. On global basis functions, conforming and non-conforming finite element spaces. A global basis function coincides on each mesh cell with a local basis function. This property implies the uniqueness of the global basis functions.

Whether the continuity with respect to $\{\Phi_i\}_{i=1}^N$ implies the continuity of the finite element functions depends on the functionals that define the finite element space.

A finite element space whose functions belong to the space that appears in the definition of the weak problem is called conforming, otherwise it is called non-conforming. □

Definition 5.15. Parametric finite elements. Let $\hat{K}$ be a reference mesh cell with the local space $\hat{P}(\hat{K})$, the local functionals $\hat{\Phi}_1, \ldots, \hat{\Phi}_{\hat{N}}$, and a class of bijective mappings $\{F_K : \hat{K} \to K\}$. A finite element space is called a parametric finite element space if:

- The images $\{K\}$ of $\{F_K\}$ form the set of mesh cells.
- The local spaces are given by

$$P(K) = \left\{ p \ : \ p = \hat{p} \circ F_K^{-1}, \hat{p} \in \hat{P}(\hat{K}) \right\}. \tag{5.1}$$

- The local functionals are defined by

$$\Phi_{K,i}(v(\boldsymbol{x})) = \hat{\Phi}_i \left(\hat{v}(\hat{\boldsymbol{x}}) \right) = \hat{\Phi}_i \left(v(F_K(\hat{\boldsymbol{x}})) \right), \tag{5.2}$$

  where $\hat{\boldsymbol{x}} = (\hat{x}_1, \ldots, \hat{x}_d)^T$ are the coordinates of the reference mesh cell and it holds $\boldsymbol{x} = F_K(\hat{\boldsymbol{x}})$, $\hat{v} = v \circ F_K$, compare (5.1).

□

Remark 5.16. Motivations for using parametric finite elements. Definition 5.12 of finite elements spaces is very general. For instance, different types of mesh cells are allowed. However, as well the finite element theory as the implementation of finite element methods become much simpler if only parametric finite elements are considered. □

5.2 Finite Elements on Simplices

Definition 5.17. d-simplex. A d-simplex $K \subset \mathbb{R}^d$ is the convex hull of $(d+1)$ points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{d+1} \in \mathbb{R}^d$ that form the vertices of K. □

Remark 5.18. On d-simplices. It will be always assumed that the simplex is not degenerated, i.e., its d-dimensional measure is positive. This property is equivalent to the non-singularity of the matrix (*exercise*)

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1,d+1} \\ a_{21} & a_{22} & \ldots & a_{2,d+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \ldots & a_{d,d+1} \\ 1 & 1 & \ldots & 1 \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)},$$

where $\boldsymbol{a}_i = (a_{1i}, a_{2i}, \ldots, a_{di})^T$, $i = 1, \ldots, d+1$.

For $d = 2$, simplices are triangles and for $d = 3$ they are tetrahedra. □

Definition 5.19. Barycentric coordinates. Since K is the convex hull of the points $\{\boldsymbol{a}_i\}_{i=1}^{d+1}$, the parameterization of K with a convex combination of the vertices reads as follows

$$K = \left\{ \boldsymbol{x} \in \mathbb{R}^d \ : \ \boldsymbol{x} = \sum_{i=1}^{d+1} \lambda_i \boldsymbol{a}_i, \ 0 \le \lambda_i \le 1, \ \sum_{i=1}^{d+1} \lambda_i = 1 \right\}.$$

The coefficients $\lambda_1, \ldots, \lambda_{d+1}$ are called barycentric coordinates of $\boldsymbol{x} \in K$. □

Remark 5.20. On barycentric coordinates.

- From the definition, it follows that the barycentric coordinates are the solution of the linear system of equations

$$\sum_{i=1}^{d+1} a_{ji}\lambda_i = x_j, \quad 1 \le j \le d, \quad \sum_{i=1}^{d+1} \lambda_i = 1.$$

  Since the system matrix is non-singular, see Remark 5.18, the barycentric coordinates are determined uniquely.
- The barycentric coordinates of the vertex $\boldsymbol{a}_i$, $i = 1, \ldots, d+1$, of the simplex are $\lambda_i = 1$ and $\lambda_j = 0$ if $i \ne j$. Since $\lambda_i(\boldsymbol{a}_j) = \delta_{ij}$, the barycentric coordinate λ_i can be identified with the linear function that has the value 1 in the vertex $\boldsymbol{a}_i$ and that vanishes in all other vertices $\boldsymbol{a}_j$ with $j \ne i$.
- The barycenter of the simplex is given by

$$S_K = \frac{1}{d+1} \sum_{i=1}^{d+1} \boldsymbol{a}_i = \sum_{i=1}^{d+1} \frac{1}{d+1} \boldsymbol{a}_i.$$

  Hence, its barycentric coordinates are $\lambda_i = 1/(d+1)$, $i = 1, \ldots, d+1$.

□

Remark 5.21. Simplicial reference mesh cells. A commonly used reference mesh cell for triangles and tetrahedra is the unit simplex

$$\hat{K} = \left\{ \hat{\boldsymbol{x}} \in \mathbb{R}^d \ : \ \sum_{i=1}^{d} \hat{x}_i \le 1, \ \hat{x}_i \ge 0, \ i = 1, \ldots, d \right\},$$

see Figure 5.4. The class $\{F_K\}$ of admissible mappings consists of the bijective affine mappings

$$F_K \hat{\boldsymbol{x}} = B_K \hat{\boldsymbol{x}} + \boldsymbol{b}, \quad B_K \in \mathbb{R}^{d \times d}, \ \det(B_K) \ne 0, \ \boldsymbol{b} \in \mathbb{R}^d. \tag{5.3}$$

The images of these mappings generate the set of the non-degenerated simplices $\{K\} \subset \mathbb{R}^d$. □

Definition 5.22. Affine family of simplicial finite elements. Given a simplicial reference mesh cell $\hat{K}$, affine mappings $\{F_K\}$, and an unisolvent set

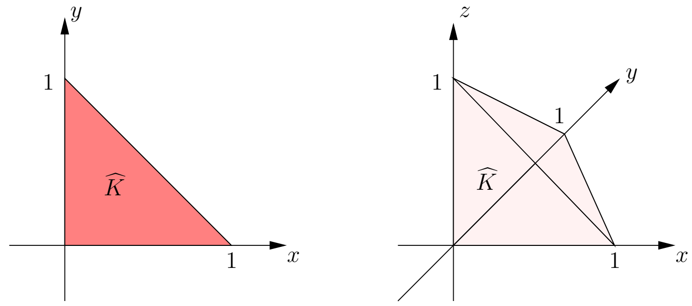

Fig. 5.4 The unit simplices in two and three dimensions.

of functionals on $\hat{K}$. Using (5.1) and (5.2), one obtains a local finite element space on each non-degenerated simplex. The set of these local spaces is called affine family of simplicial finite elements. □

Definition 5.23. Polynomial space P_k. Let $\boldsymbol{x} = (x_1, \ldots, x_d)^T$, $k \in \mathbb{N} \cup \{0\}$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)^T$. Then, the polynomial space P_k is given by

$$P_k = \operatorname{span}\left\{ \prod_{i=1}^{d} x_i^{\alpha_i} = \boldsymbol{x}^{\boldsymbol{\alpha}} \ : \ \alpha_i \in \mathbb{N} \cup \{0\} \ \text{ for } \ i = 1, \ldots, d, \ \sum_{i=1}^{d} \alpha_i \le k \right\}.$$

□

Remark 5.24. Lagrangian[2] *finite elements.* In many examples given below, the linear functionals on the reference mesh cell $\hat{K}$ are the values of the polynomials with the same barycentric coordinates as on the general mesh cell K. Finite elements whose linear functionals are values of the polynomials at certain points in K are called Lagrangian finite elements. □

Example 5.25. P_0 : piecewise constant finite element. The piecewise constant finite element space consists of discontinuous functions. The linear functional is the value of the polynomial in the barycenter of the mesh cell, see Figure 5.5. It is $\dim P_0(K) = 1$. □

Example 5.26. P_1 : conforming piecewise linear finite element. This finite element space is a subspace of $C(\overline{\Omega}) \subset H^1(\Omega)$. The linear functionals are the values of the function in the vertices of the mesh cells, see Figure 5.6. It follows that $\dim P_1(K) = d + 1$.

The local basis for the functionals $\{\Phi_i(v) = v(\boldsymbol{a}_i),\ i = 1, \ldots, d+1\}$ is $\{\lambda_i\}_{i=1}^{d+1}$ since $\Phi_i(\lambda_j) = \delta_{ij}$, compare Remark 5.20. Since a local basis exists, the functionals are unisolvent with respect to the polynomial space $P_1(K)$.

Now, it will be shown that the corresponding finite element space consists of continuous functions. Let K_1, K_2 be two mesh cells with the common facet

[2] Joseph-Louis de Lagrange (1736 – 1813)

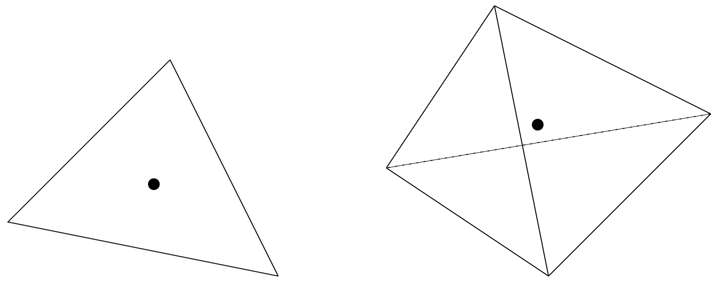

Fig. 5.5 The finite element $P_0(K)$.

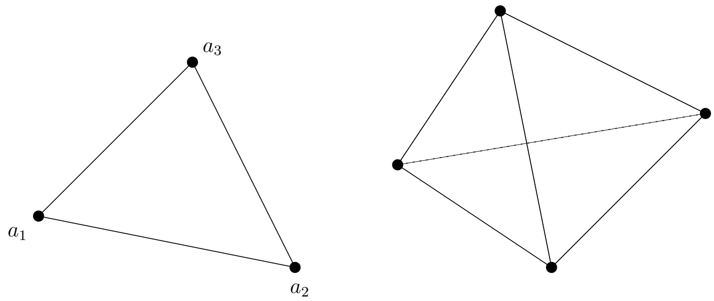

Fig. 5.6 The finite element $P_1(K)$.

E and let $v \in P_1(= S)$. The restriction of v_{K_1} on E is a linear function on E as well as the restriction of v_{K_2} on E. It has to be shown that both linear functions are identical. A linear function on the $(d-1)$-dimensional face E is uniquely determined with d linearly independent functionals that are defined on E. These functionals can be chosen to be the values of the function in the d vertices of E. The functionals in S are continuous by the definition of S. Thus, it must hold that both restrictions on E have the same values in the vertices of E. Hence, it is $v_{K_1}|_E = v_{K_2}|_E$ and the functions from P_1 are continuous. □

Example 5.27. P_2 : conforming piecewise quadratic finite element. This finite element space is also a subspace of $C(\overline{\Omega})$. It consists of piecewise quadratic functions. The functionals are the values of the functions in the $d+1$ vertices of the mesh cell and the values of the functions in the centers of the edges, see Figure 5.7. Since each vertex is connected to each other vertex, there are $\sum_{i=1}^{d} i = d(d+1)/2$ edges. Alternatively, the number of edges can be calculated by the observation that it is equal to the number of pairs that can be formed out of $(d+1)$ values, which is

$$\binom{d+1}{2} = \frac{(d+1)!}{(d-1)!2!} = \frac{d(d+1)}{2}.$$

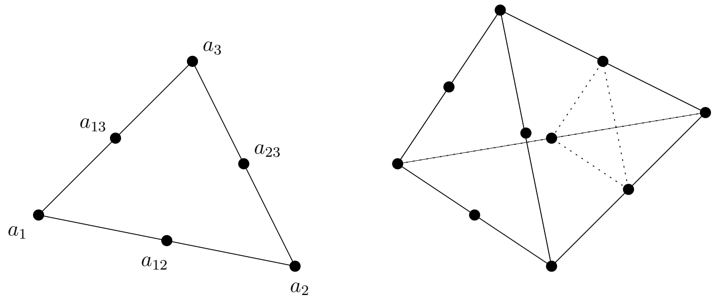

Fig. 5.7 The finite element $P_2(K)$.

Altogether, it follows that $\dim P_2(K) = (d+1)(d+2)/2$.

The part of the local basis that belongs to the functionals $\{\Phi_i(v) = v(\boldsymbol{a}_i),$ $i = 1, \ldots, d+1\}$, is given by

$$\{\phi_i(\lambda) = \lambda_i(2\lambda_i - 1), \quad i = 1, \ldots, d+1\}.$$

Denote the center of the edge between the vertices $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ by $\boldsymbol{a}_{ij}$. The corresponding part of the local basis is given by

$$\{\phi_{ij} = 4\lambda_i\lambda_j, \quad i,j = 1, \ldots, d+1, \ i < j\}.$$

The unisolvence follows from the fact that there exists a local basis. The continuity of the corresponding finite element space is shown in the same way as for the P_1 finite element. The restriction of a quadratic function defined in a mesh cell to a facet E is a quadratic function on that facet. Hence, the function on E is determined uniquely with $d(d+1)/2$ linearly independent functionals on E.

The functions ϕ_{ij} are called in two dimensions edge bubble functions. □

Example 5.28. P_3 : conforming piecewise cubic finite element. This finite element space consists of continuous piecewise cubic functions. It is a subspace of $C(\overline{\Omega})$. The functionals in a mesh cell K are defined to be the values in the vertices ($(d+1)$ values), two values on each edge (dividing the edge in three parts of equal length) ($2\sum_{i=1}^{d} i = d(d+1)$ values), and the values in the barycenter of the 2-faces of K, see Figure 5.8. Each 2-face of K is defined by three vertices. Then, the number of 2-faces corresponds to the number of triples that can be formed out of $(d+1)$ values, which is given by

$$\binom{d+1}{3} = \frac{(d+1)!}{(d-2)!3!} = \frac{(d-1)d(d+1)}{6}. \tag{5.4}$$

The dimension of $P_3(K)$ is given by

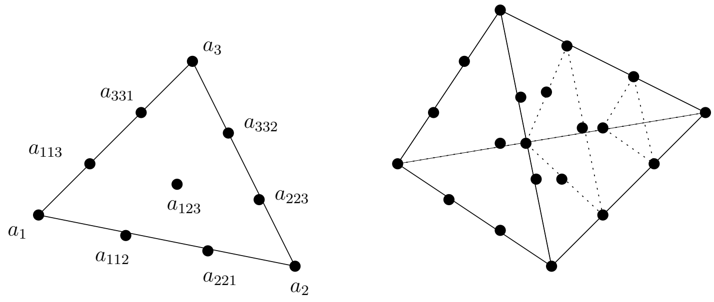

Fig. 5.8 The finite element $P_3(K)$.

$$\dim P_3(K) = (d+1) + d(d+1) + \frac{(d-1)d(d+1)}{6} = \frac{(d+1)(d+2)(d+3)}{6}. \tag{5.5}$$

For the functionals

$$\left\{ \begin{array}{ll} \Phi_i(v) = v(\boldsymbol{a}_i),\ i = 1, \ldots, d+1, & \text{(vertex)}, \\ \Phi_{iij}(v) = v(\boldsymbol{a}_{iij}),\ i, j = 1, \ldots, d+1, i \neq j, & \text{(point on edge)}, \\ \Phi_{ijk}(v) = v(\boldsymbol{a}_{ijk}),\ i = 1, \ldots, d+1, i < j < k, & \text{(point on 2-face)} \end{array} \right\},$$

the local basis is given by

$$\left\{ \phi_i(\lambda) = \frac{1}{2}\lambda_i(3\lambda_i - 1)(3\lambda_i - 2), \quad \phi_{iij}(\lambda) = \frac{9}{2}\lambda_i\lambda_j(3\lambda_i - 1), \right.$$
$$\left. \phi_{ijk}(\lambda) = 27\lambda_i\lambda_j\lambda_k \right\}.$$

In two dimensions, the function $\phi_{ijk}(\lambda)$ is called cell bubble function. □

Example 5.29. Cubic Hermite[3] *element.* This finite element space is a subspace of $C(\overline{\Omega})$, the dimension of the local space is $(d+1)(d+2)(d+3)/6$ and the functionals are the values of the function in the vertices of the mesh cell ($(d+1)$ values), the value of the barycenter at the 2-faces of K ($(d+1)(d-1)d/6$ values, compare (5.4)), and the partial derivatives at the vertices ($d(d+1)$ values), see Figure 5.9. The dimension is the same as for the $P_3(K)$ element, see (5.5). Hence, the local polynomials can be defined to be cubic.

This finite element does not define an affine family in the strict sense, because partial derivatives on the reference cell are mapped to directional derivatives on the physical cell. Concretely, the functionals for the partial derivatives $\hat{\Phi}_i(\hat{v}) = \partial_i \hat{v}(\mathbf{0})$ on the reference cell are mapped to the functionals

[3] Charles Hermite (1822 – 1901)

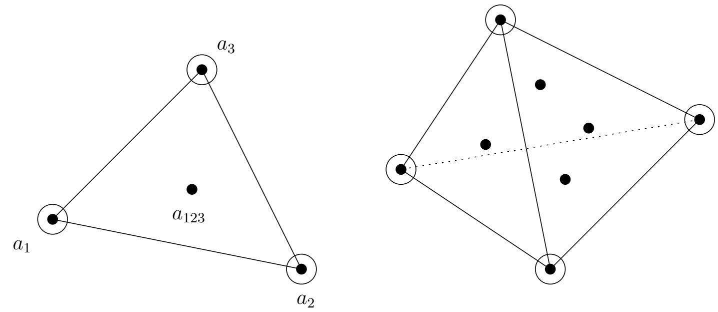

Fig. 5.9 The cubic Hermite element.

$\Phi_i(v) = \partial_{\boldsymbol{t}_i} v(\boldsymbol{a})$, where $\boldsymbol{a} = F_K(\boldsymbol{0})$ and $\boldsymbol{t}_i$ are the directions of edges which are adjacent to $\boldsymbol{a}$, i.e., $\boldsymbol{a}$ is an end point of this edge. This property suffices to control all first derivatives. One has to take care of this property in the implementation of this finite element.

Because of this property, one can use the derivatives in the direction of the edges as functionals

$$\begin{aligned} \Phi_i(v) &= v(\boldsymbol{a}_i), && \text{(vertices)} \\ \Phi_{ij}(v) &= \nabla v(\boldsymbol{a}_i) \cdot (\boldsymbol{a}_j - \boldsymbol{a}_i),\ i,j = 1,\ldots,d-1, i \neq j, && \text{(directional deriv.)} \\ \Phi_{ijk}(v) &= v(\boldsymbol{a}_{ijk}),\ i<j<k, && \text{(2-faces)} \end{aligned}$$

with the corresponding local basis

$$\begin{aligned} \phi_i(\lambda) &= -2\lambda_i^3 + 3\lambda_i^2 - 7\lambda_i \textstyle\sum_{j<k, j\neq i, k\neq i} \lambda_j\lambda_k, \\ \phi_{ij}(\lambda) &= \lambda_i\lambda_j(2\lambda_i - \lambda_j - 1), \\ \phi_{ijk}(\lambda) &= 27\lambda_i\lambda_j\lambda_k. \end{aligned}$$

The proof of the unisolvence can be found in the literature.

Here, the continuity of the functions will be shown only for $d = 2$. Let K_1, K_2 be two mesh cells with the common edge E and the unit tangential vector $\boldsymbol{t}$. Let V_1, V_2 be the end points of E. The restrictions v_{K_1}, v_{K_2} to E satisfy four conditions

$$v_{K_1}(V_i) = v_{K_2}(V_i), \quad \partial_{\boldsymbol{t}} v_{K_1}(V_i) = \partial_{\boldsymbol{t}} v_{K_2}(V_i),\ i = 1, 2.$$

Since both restrictions are cubic polynomials and four conditions have to be satisfied, their values coincide on E.

The cubic Hermite finite element possesses an advantage in comparison with the P_3 finite element. For $d = 2$, it holds for a regular triangulation $\mathcal{T}^h$ that

$$\#(K) \approx 2\#(V), \quad \#(E) \approx 2\#(V),$$

where $\#(\cdot)$ denotes the number of triangles, nodes, and edges, respectively. Hence, the dimension of P_3 is approximately $\#(V) + 2\#(E) + \#(K) \approx$

$7\#(V)$, whereas the dimension of the cubic Hermite element is approximately $3\#(V)+\#(K) \approx 5\#(V)$. This difference comes from the fact that both spaces are different proper subspaces of the space of all continuous piecewise cubic functions. The elements of both spaces are continuous functions, but for the functions of the cubic Hermite finite element, in addition, the first derivatives are continuous at the nodes. That means, these two spaces are different finite element spaces whose degree of the local polynomial space is the same (cubic). One can see at this example the importance of the functionals for the definition of the global finite element space. □

Example 5.30. P_1^{nc} : non-conforming linear finite element, Crouzeix–Raviart finite element, Crouzeix & Raviart (1973). This finite element space consists of piecewise linear but discontinuous functions. The functionals are given by the values of the functions in the barycenters of the $(d-1)$-faces so that $\dim P_1^{\mathrm{nc}}(K) = (d+1)$. It follows from the definition of the finite element space, Definition 5.12, that the functions from P_1^{nc} are continuous in the barycenter of the facets

$$P_1^{\mathrm{nc}} = \big\{v \in L^2(\Omega) \ : \ v|_K \in P_1(K),\ v(\boldsymbol{x}) \text{ is continuous at the barycenter of all facets}\big\}. \tag{5.6}$$

Equivalently, the functionals can be defined to be the integral mean values on the facets and then the global space is defined to be

$$P_1^{\mathrm{nc}} = \left\{ v \in L^2(\Omega) \ : \ v|_K \in P_1(K), \int_E v|_K \, d\boldsymbol{s} = \int_E v|_{K'} \, d\boldsymbol{s} \ \forall\, E \in \mathcal{E}(K) \cap \mathcal{E}(K') \right\}, \tag{5.7}$$

where $\mathcal{E}(K)$ is the set of all $(d-1)$-dimensional facets of K. Notice that the integral of a linear function on an edge E in $2d$ or on a triangular facet in $3d$ can be computed by

$$\int_E v \, d\boldsymbol{s} = |E|\, v(\boldsymbol{x}_E),$$

where $\boldsymbol{x}_E$ is the barycenter of E. Hence, the values of the integrals are equal if and only if the values in the barycenters are equal.

For the description of this finite element, one defines the functionals by

$$\Phi_i(v) = v(\boldsymbol{a}_{i-1,i+1}) \text{ for } d=2, \quad \Phi_i(v) = v(\boldsymbol{a}_{i-2,i-1,i+1}) \text{ for } d=3,$$

where the points are the barycenters of the facets with the vertices that correspond to the indices, see Figure 5.10. This set of functionals is unisolvent with the local basis

$$\phi_i(\lambda) = 1 - d\lambda_i, \quad i = 1, \ldots, d+1.$$

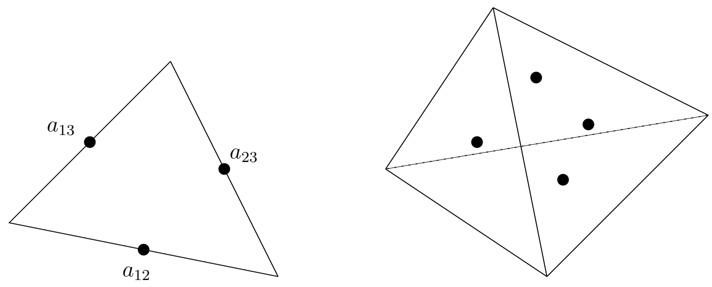

Fig. 5.10 The finite element $P_1^{\mathrm{nc}}(K)$.

□

5.3 Finite Elements on Parallelepipeds and Quadrilaterals

Remark 5.31. Reference mesh cells, reference map to parallelepipeds. One can find in the literature two reference cells: the unit cube $[0,1]^d$ and the large unit cube $[-1,1]^d$. It does not matter which reference cell is chosen. Here, the large unit cube will be used: $\hat{K} = [-1,1]^d$. The class of admissible reference maps $\{F_K\}$ to parallelepipeds consists of bijective affine mappings of the form

$$F_K \hat{\boldsymbol{x}} = B_K \hat{\boldsymbol{x}} + \boldsymbol{b}, \quad B_K \in \mathbb{R}^{d\times d},\ \boldsymbol{b} \in \mathbb{R}^d.$$

Affine mappings map parallel lines to parallel lines. If B_K is a diagonal matrix, then $\hat{K}$ is mapped to a d-rectangle K.

The class of mesh cells that is obtained in this way is not sufficient to triangulate general domains. If one wants to use more general mesh cells than parallelepipeds, then the class of admissible reference maps has to be enlarged, see Remark 5.40. □

Definition 5.32. Polynomial space Q_k. Let $\boldsymbol{x} = (x_1, \ldots, x_d)^T$ and denote by $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)^T$ a multi-index. Then, the polynomial space Q_k is given by

$$Q_k = \operatorname{span}\left\{ \prod_{i=1}^d x_i^{\alpha_i} = \boldsymbol{x}^{\boldsymbol{\alpha}} \ :\ 0 \le \alpha_i \le k \ \text{ for } \ i = 1, \ldots, d \right\}.$$

□

Example 5.33. Q_1 vs. P_1. The space Q_1 consists of all polynomials that are d-linear. Let $d = 2$, then it is

$$Q_1 = \operatorname{span}\{1, x, y, xy\},$$

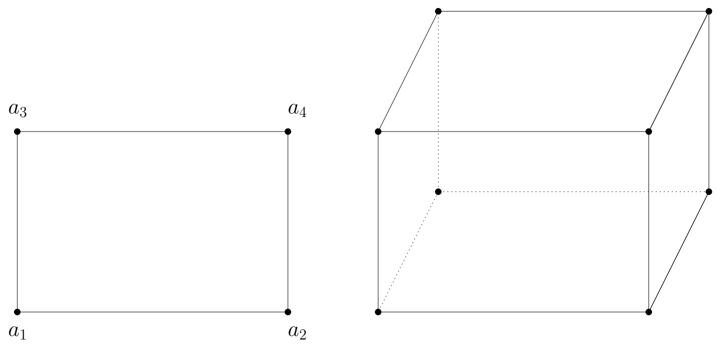

Fig. 5.11 The finite element $Q_1(K)$.

whereas

$$P_1 = \text{span}\{1, x, y\}.$$

□

Remark 5.34. Finite elements on d-rectangles. For simplicity of presentation, the examples below consider d-rectangles. In this case, the finite elements are just tensor products of one-dimensional finite elements. In particular, the basis functions can be written as products of one-dimensional basis functions. □

Example 5.35. Q_0 *: piecewise constant finite element.* Similarly to the P_0 space, the space Q_0 consists of piecewise constant, discontinuous functions. The functional is the value of the function in the barycenter of the mesh cell K and it holds $\dim Q_0(K) = 1$. □

Example 5.36. Q_1 *: conforming piecewise d-linear finite element.* This finite element space is a subspace of $C(\overline{\Omega})$. The functionals are the values of the function in the vertices of the mesh cell, see Figure 5.11. Hence, it is $\dim Q_1(K) = 2^d$.

The one-dimensional local basis functions, which will be used for the tensor product, are given by

$$\hat{\phi}_1(\hat{x}) = \frac{1}{2}(1 - \hat{x}), \quad \hat{\phi}_2(\hat{x}) = \frac{1}{2}(1 + \hat{x}).$$

With these functions, e.g., the basis functions in two dimensions are computed by

$$\hat{\phi}_1(\hat{x})\hat{\phi}_1(\hat{y}),\ \hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{y}),\ \hat{\phi}_2(\hat{x})\hat{\phi}_1(\hat{y}),\ \hat{\phi}_2(\hat{x})\hat{\phi}_2(\hat{y}).$$

The continuity of the functions of the finite element space Q_1 is proved in the same way as for simplicial finite elements. It is used that the restriction of a function from $Q_k(K)$ to a k-face E is a function from the space $Q_k(E)$, $k \geq 1$. □

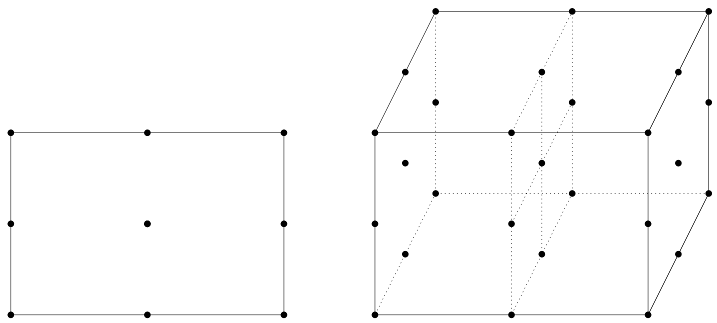

Fig. 5.12 The finite element $Q_2(K)$.

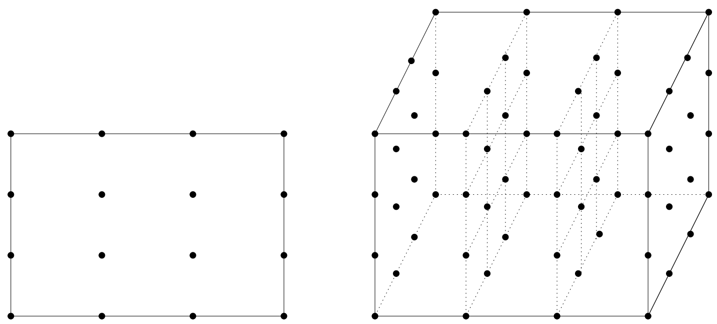

Fig. 5.13 The finite element $Q_3(K)$.

Example 5.37. Q_2 : conforming piecewise d-quadratic finite element. It holds that $Q_2 \subset C(\overline{\Omega})$. The functionals in one dimension are the values of the function at both ends of the interval and in the center of the interval. The one-dimensional basis function on the reference interval are defined by

$$\hat{\phi}_1(\hat{x}) = -\frac{1}{2}\hat{x}(1-\hat{x}), \quad \hat{\phi}_2(\hat{x}) = (1-\hat{x})(1+\hat{x}), \quad \hat{\phi}_3(\hat{x}) = \frac{1}{2}(1+\hat{x})\hat{x}.$$

In d dimensions, the functionals are the corresponding values of the tensor product of the intervals, see Figure 5.12. It follows that $\dim Q_2(K) = 3^d$.

The basis function $\prod_{i=1}^{d} \hat{\phi}_2(\hat{x}_i)$ is called cell bubble function. □

Example 5.38. Q_3 : conforming piecewise d-cubic finite element. This finite element space is a subspace of $C(\overline{\Omega})$. The functionals on the reference interval are given by the values at the end of the interval and the values at the points $\hat{x} = -1/3$, $\hat{x} = 1/3$. In multiple dimensions, it is the corresponding tensor product, see Figure 5.13. The dimension of the local space is $\dim Q_3(K) = 4^d$.

The one-dimensional basis functions in the reference interval are given by

$$\hat{\phi}_1(\hat{x}) = -\frac{1}{16}(3\hat{x}+1)(3\hat{x}-1)(\hat{x}-1), \quad \hat{\phi}_2(\hat{x}) = \frac{9}{16}(\hat{x}+1)(3\hat{x}-1)(\hat{x}-1),$$

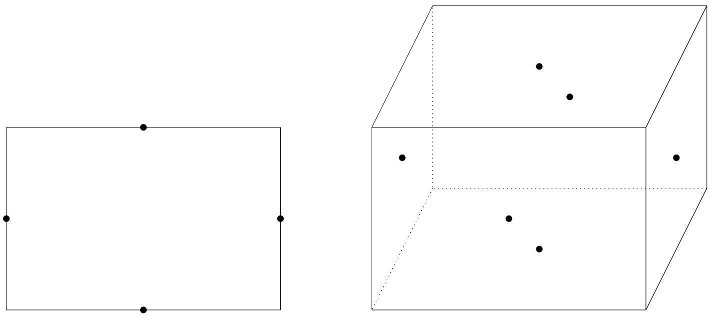

Fig. 5.14 The finite element $Q_1^{\text{rot}}(K)$.

$$\hat{\phi}_3(\hat{x}) = -\frac{9}{16}(\hat{x}+1)(3\hat{x}+1)(\hat{x}-1), \quad \hat{\phi}_4(\hat{x}) = \frac{1}{16}(3\hat{x}+1)(3\hat{x}-1)(\hat{x}+1).$$

□

Example 5.39. Q_1^{rot} : rotated non-conforming element of lowest order, Rannacher–Turek element, Rannacher & Turek (1992). For interested students only. This finite element space is a generalization of the P_1^{nc} finite element to quadrilateral and hexahedral mesh cells. It consists of discontinuous functions that are continuous at the barycenter of the faces. The dimension of the local finite element space is $\dim Q_1^{\text{rot}}(K) = 2d$. The space on the reference mesh cell is defined by

$$\begin{aligned} Q_1^{\text{rot}}\left(\hat{K}\right) &= \left\{\hat{p} \;:\; \hat{p} \in \operatorname{span}\{1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\}\right\} && \text{for } d = 2, \\ Q_1^{\text{rot}}\left(\hat{K}\right) &= \left\{\hat{p} \;:\; \hat{p} \in \operatorname{span}\{1, \hat{x}, \hat{y}, \hat{z}, \hat{x}^2 - \hat{y}^2, \hat{y}^2 - \hat{z}^2\}\right\} && \text{for } d = 3. \end{aligned}$$

Note that the transformed space

$$Q_1^{\text{rot}}(K) = \{p = \hat{p} \circ F_K^{-1}, \hat{p} \in Q_1^{\text{rot}}(\hat{K})\}$$

contains polynomials of the form $ax^2 - by^2$, where a, b depend on F_K.

For $d = 2$, the local basis on the reference cell is given by

$$\begin{aligned} \hat{\phi}_1(\hat{x},\hat{y}) &= -\frac{3}{8}(\hat{x}^2 - \hat{y}^2) - \frac{1}{2}\hat{y} + \frac{1}{4}, \quad \hat{\phi}_2(\hat{x},\hat{y}) = \frac{3}{8}(\hat{x}^2 - \hat{y}^2) + \frac{1}{2}\hat{x} + \frac{1}{4}, \\ \hat{\phi}_3(\hat{x},\hat{y}) &= -\frac{3}{8}(\hat{x}^2 - \hat{y}^2) + \frac{1}{2}\hat{y} + \frac{1}{4}, \quad \hat{\phi}_4(\hat{x},\hat{y}) = \frac{3}{8}(\hat{x}^2 - \hat{y}^2) - \frac{1}{2}\hat{x} + \frac{1}{4}. \end{aligned} \tag{5.8}$$

Analogously to the Crouzeix–Raviart finite element, the functionals can be defined as point values of the functions in the barycenters of the facets, see Figure 5.14, or as integral mean values of the functions at the facets. Consequently, the finite element spaces are defined in the same way as (5.6) or (5.7), with $P_1^{\text{nc}}(K)$ replaced by $Q_1^{\text{rot}}(K)$.

In the code ParMooN Wilbrandt *et al.* (2017), the mean value oriented Q_1^{rot} finite element space is implemented for two dimensions and the point value oriented Q_1^{rot} finite element space for three dimensions. For $d = 3$, the integrals on the facets of mesh cells, whose equality is required in the mean value oriented Q_1^{rot} finite element space, involve a weighting function which depends on the particular mesh cell K. The computation of these weighting functions for all mesh cells is an additional computational overhead. For this reason, it was suggested in (Schieweck, 1997, p. 21) to use for $d = 3$ the simpler point value oriented form of the Q_1^{rot} finite element. □

Remark 5.40. Parametric mappings. The image of an affine mapping of the reference mesh cell $\hat{K} = [-1, 1]^d$, $d \in \{2, 3\}$, is a parallelepiped. If one wants to consider finite elements on general d-quadrilaterals, then the class of admissible reference maps has to be enlarged.

The simplest non-affine parametric finite element on quadrilaterals in two dimensions uses bilinear mappings. Let $\hat{K} = [-1, 1]^2$ and let

$$F_K(\hat{\boldsymbol{x}}) = \begin{pmatrix} F_K^1(\hat{\boldsymbol{x}}) \\ F_K^2(\hat{\boldsymbol{x}}) \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12}\hat{x} + a_{13}\hat{y} + a_{14}\hat{x}\hat{y} \\ a_{21} + a_{22}\hat{x} + a_{23}\hat{y} + a_{24}\hat{x}\hat{y} \end{pmatrix}, \ F_K^i \in Q_1, \ i = 1, 2,$$

be a bilinear mapping from $\hat{K}$ on the class of admissible quadrilaterals. A quadrilateral K is called admissible if

- the length of every edge of K is larger than zero,
- the interior angles of K are smaller than π, i.e., K is convex.

This class contains, e.g., trapezoids and rhombi. □

Remark 5.41. Parametric finite element functions. The functions of the local space $P(K)$ on the mesh cell K are defined by $p = \hat{p} \circ F_K^{-1}$. These functions are in general rational functions if F_K is not affine. However, using d-linear mappings, then the restriction of F_K to an edge of $\hat{K}$ is an affine map, since $(d - 1)$ coordinates are fixed for an edge on $\hat{K}$. For instance, in the case of the Q_1 finite element, the functions on K are linear functions on each edge of K. It follows that the functions of the corresponding global finite element space Q_1 are continuous, compare Example 5.26. □

5.4 Transform of Integrals

Remark 5.42. Motivation. The transformation of integrals from the reference mesh cell to mesh cells of the grid and vice versa is used as well for the analysis as for the implementation of finite element methods. This section provides an overview of the most important formulae for transformations.

Let $\hat{K} \subset \mathbb{R}^d$ be the reference mesh cell, K be an arbitrary mesh cell, and $F_K : \hat{K} \to K$ with $\boldsymbol{x} = F_K(\hat{\boldsymbol{x}})$ be the reference map. It is assumed that the reference map is a continuous differentiable one-to-one map. The inverse map is denoted by $F_K^{-1} : K \to \hat{K}$. For the integral transforms, the derivatives (Jacobians) of F_K and F_K^{-1} are needed

$$DF_K(\hat{\boldsymbol{x}})_{ij} = \frac{\partial x_i}{\partial \hat{x}_j}, \quad DF_K^{-1}(\boldsymbol{x})_{ij} = \frac{\partial \hat{x}_i}{\partial x_j}, \quad i,j = 1,\ldots,d. \tag{5.9}$$

□

Remark 5.43. Integral with a function without derivatives. This integral transforms with the standard rule of integral transforms

$$\int_K v(\boldsymbol{x})\ d\boldsymbol{x} = \int_{\hat{K}} \hat{v}(\hat{\boldsymbol{x}})\,|\det DF_K(\hat{\boldsymbol{x}})|\ d\hat{\boldsymbol{x}}, \tag{5.10}$$

where $\hat{v}(\hat{\boldsymbol{x}}) = v(F_K(\hat{\boldsymbol{x}}))$. □

Remark 5.44. Transform of derivatives. Using the chain rule and (5.9), one obtains

$$\begin{aligned}
\frac{\partial v}{\partial x_i}(\boldsymbol{x}) &= \sum_{j=1}^{d} \frac{\partial \hat{v}}{\partial \hat{x}_j}(\hat{\boldsymbol{x}})\frac{\partial \hat{x}_j}{\partial x_i} = \nabla_{\hat{\boldsymbol{x}}}\hat{v}(\hat{\boldsymbol{x}}) \cdot \left(\left(DF_K^{-1}(\boldsymbol{x})\right)^T\right)_i \\
&= \nabla_{\hat{\boldsymbol{x}}}\hat{v}(\hat{\boldsymbol{x}}) \cdot \left(\left(DF_K^{-1}(F_K(\hat{\boldsymbol{x}}))\right)^T\right)_i \\
&= \left(\left(DF_K^{-1}(F_K(\hat{\boldsymbol{x}}))\right)^T\right)_i \cdot \nabla_{\hat{\boldsymbol{x}}}\hat{v}(\hat{\boldsymbol{x}}), \qquad (5.11)\\
\frac{\partial \hat{v}}{\partial \hat{x}_i}(\hat{\boldsymbol{x}}) &= \sum_{j=1}^{d} \frac{\partial v}{\partial x_j}(\boldsymbol{x})\frac{\partial x_j}{\partial \hat{x}_i} = \nabla v(\boldsymbol{x}) \cdot \left(\left(DF_K(\hat{\boldsymbol{x}})\right)^T\right)_i \\
&= \nabla v(\boldsymbol{x}) \cdot \left(\left(DF_K(F_K^{-1}(\boldsymbol{x}))\right)^T\right)_i. \qquad (5.12)
\end{aligned}$$

The index i denotes the i-th row of a matrix. Derivatives on the reference mesh cell are marked with a symbol on the operator. □

Remark 5.45. Integrals with gradients. Using the rule for transforming integrals and (5.11) gives

$$\begin{aligned}
&\int_K \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\ d\boldsymbol{x} \\
&= \int_{\hat{K}} \boldsymbol{b}\left(F_K(\hat{\boldsymbol{x}})\right) \cdot \left[\left(DF_K^{-1}\right)^T\left(F_K(\hat{\boldsymbol{x}})\right)\right] \nabla_{\hat{\boldsymbol{x}}}\hat{v}(\hat{\boldsymbol{x}})\,|\det DF_K(\hat{\boldsymbol{x}})|\ d\hat{\boldsymbol{x}}. \qquad (5.13)
\end{aligned}$$

Similarly, one obtains

$$\begin{aligned}
&\int_K \nabla v(\boldsymbol{x}) \cdot \nabla w(\boldsymbol{x})\ d\boldsymbol{x} \\
&= \int_{\hat{K}} \left[\left(DF_K^{-1}\right)^T\left(F_K(\hat{\boldsymbol{x}})\right)\right] \nabla_{\hat{\boldsymbol{x}}}\hat{v}(\hat{\boldsymbol{x}}) \cdot \left[\left(DF_K^{-1}\right)^T\left(F_K(\hat{\boldsymbol{x}})\right)\right] \nabla_{\hat{\boldsymbol{x}}}\hat{w}(\hat{\boldsymbol{x}})
\end{aligned}$$

$$\times \left|\det DF_K(\hat{\boldsymbol{x}})\right| \; d\hat{\boldsymbol{x}}. \tag{5.14}$$

□

Example 5.46. Affine transform. The most important class of reference maps are affine transforms (5.3), where the invertible matrix B_K and the vector $\boldsymbol{b}$ are constants. It follows that

$$\hat{\boldsymbol{x}} = B_K^{-1}(\boldsymbol{x} - \boldsymbol{b}) = B_K^{-1}\boldsymbol{x} - B_K^{-1}\boldsymbol{b}.$$

In this case, there are

$$DF_K = B_K, \quad DF_K^{-1} = B_K^{-1}, \quad \det DF_K = \det(B_K).$$

One obtains for the integral transforms from (5.10), (5.13), and (5.14)

$$\int_K v(\boldsymbol{x}) \; d\boldsymbol{x} = \left|\det(B_K)\right| \int_{\hat{K}} \hat{v}(\hat{\boldsymbol{x}}) \; d\hat{\boldsymbol{x}}, \tag{5.15}$$

$$\int_K \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \; d\boldsymbol{x} = \left|\det(B_K)\right| \int_{\hat{K}} \boldsymbol{b}\left(F_K(\hat{\boldsymbol{x}})\right) \cdot B_K^{-T} \nabla_{\hat{\boldsymbol{x}}} \hat{v}(\hat{\boldsymbol{x}}) \; d\hat{\boldsymbol{x}}, \tag{5.16}$$

$$\int_K \nabla v(\boldsymbol{x}) \cdot \nabla w(\boldsymbol{x}) \; d\boldsymbol{x} = \left|\det(B_K)\right| \int_{\hat{K}} B_K^{-T} \nabla_{\hat{\boldsymbol{x}}} \hat{v}(\hat{\boldsymbol{x}}) \cdot B_K^{-T} \nabla_{\hat{\boldsymbol{x}}} \hat{w}(\hat{\boldsymbol{x}}) \; d\hat{\boldsymbol{x}}. \tag{5.17}$$

Setting $v(\boldsymbol{x}) = 1$ in (5.15) yields

$$\left|\det(B_K)\right| = \frac{|K|}{\left|\hat{K}\right|}. \tag{5.18}$$

□