

# Chapter 1

## Some Partial Differential Equations From Physics

*Remark 1.1. Contents.* This chapter introduces some partial differential equations (pde's) from physics to show the importance of this kind of equations and to motivate the application of numerical methods for their solution.  $\square$

### 1.1 The Heat Equation

*Remark 1.2. Derivation.* The derivation follows (Wladimirow, 1972, p. 39). Let  $\mathbf{x} = (x_1, x_2, x_3)^T \in \Omega \subset \mathbb{R}^3$ , where  $\Omega$  is a domain,  $t \in \mathbb{R}$ , and consider the following physical quantities

- $u(t, \mathbf{x})$  – temperature at time  $t$  and at the point  $\mathbf{x}$  with unit [K],
- $\rho(t, \mathbf{x})$  – density of the considered species with unit  $[\text{kg}/\text{m}^3]$ ,
- $c(t, \mathbf{x})$  – specific heat capacity of the species with unit  $[\text{J}/\text{kg K}] = [\text{W s}/\text{kg K}]$ ,
- $k(t, \mathbf{x})$  – thermal conductivity of the species with unit  $[\text{W}/\text{m K}]$ ,
- $F(t, \mathbf{x})$  – intensity of heat sources or sinks with unit  $[\text{W}/\text{m}^3]$ .

Consider the heat equilibrium in an arbitrary volume  $V \subset \Omega$  and in an arbitrary time interval  $(t, t + \Delta t)$ . First, there are sources or sinks of heat: heat can enter or leave  $V$  through the boundary  $\partial V$ , or heat can be produced or absorbed in  $V$ . Let  $\mathbf{n}(\mathbf{x})$  be the unit outer normal at  $\mathbf{x} \in \partial V$ . Due to Fourier's<sup>1</sup> law, one finds that the heat

$$Q_1 = \int_t^{t+\Delta t} \int_{\partial V} k \frac{\partial u}{\partial \mathbf{n}}(t, \mathbf{s}) \, d\mathbf{s} \, dt = \int_t^{t+\Delta t} \int_{\partial V} (k \nabla u \cdot \mathbf{n})(t, \mathbf{s}) \, d\mathbf{s} \, dt, \quad [\text{J}],$$

enters through  $\partial V$  into  $V$ . One obtains with integration by parts (Gaussian theorem)<sup>2</sup>

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<sup>1</sup> Jean Baptiste Joseph Fourier (1768 – 1830)

<sup>2</sup> The physical unit is computed as follows. Differentiation with respect to space gives the factor  $1/\text{m}$ , integration on  $V$  gives the factor  $\text{m}^3$ , and integration with respect to time the factor  $\text{s}$ . Thus, one obtains

$$Q_1 = \int_t^{t+\Delta t} \int_V \nabla \cdot (k \nabla u)(t, \mathbf{x}) \, d\mathbf{x} \, dt, \quad [\text{J}].$$

In addition, the heat

$$Q_2 = \int_t^{t+\Delta t} \int_V F(t, \mathbf{x}) \, d\mathbf{x} \, dt, \quad [\text{W s}] = [\text{J}],$$

is produced in  $V$ .

Second, a law for the change of the temperature in  $V$  has to be derived. Using a Taylor series expansion, one gets that the temperature at  $\mathbf{x}$  changes in  $(t, t + \Delta t)$  by

$$u(t + \Delta t, \mathbf{x}) - u(t, \mathbf{x}) = \frac{\partial u}{\partial t}(t, \mathbf{x}) \Delta t + \mathcal{O}((\Delta t)^2).$$

Now, a linear ansatz is utilized, i.e.,

$$u(t + \Delta t, \mathbf{x}) - u(t, \mathbf{x}) = \frac{\partial u}{\partial t}(t, \mathbf{x}) \Delta t.$$

With this ansatz and using the relation between temperature and heat, one has that for the change of the temperature in  $V$  and for arbitrary sufficiently small  $\Delta t$ , the heat

$$\begin{aligned} Q_3 &= \int_t^{t+\Delta t} \int_V c\rho \frac{u(t + \Delta t, \mathbf{x}) - u(t, \mathbf{x})}{\Delta t} \, d\mathbf{x} \, dt \\ &= \int_t^{t+\Delta t} \int_V c\rho \frac{\partial u}{\partial t}(t, \mathbf{x}) \, d\mathbf{x} \, dt, \quad [\text{J}], \end{aligned}$$

is needed. This heat has to be equal to the heat sources, i.e., it holds  $Q_3 = Q_2 + Q_1$ , from what follows that

$$\int_t^{t+\Delta t} \int_V \left[ c\rho \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) - F \right] (t, \mathbf{x}) \, d\mathbf{x} \, dt = 0.$$

Since the volume  $V$  was chosen to be arbitrary and  $\Delta t$  was arbitrary as well, the term in the integral has to vanish. One obtains the so-called heat equation

$$c\rho \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) = F \quad \text{in } (0, T) \times \Omega.$$

At this point of modeling one should check if the equation is dimensionally correct. One finds that all terms have the unit  $[\text{W}/\text{m}^3]$ .

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$$[\text{s m}^3 \text{ } ^{1/\text{m}} \text{ W}/(\text{m K}) \text{ } ^{1/\text{m}} \text{ K}] = [\text{W s}] = [\text{J}].$$

For a homogeneous species,  $c$ ,  $\rho$ , and  $k$  are positive constants. Then, the heat equation simplifies to

$$\frac{\partial u}{\partial t} - \varepsilon^2 \Delta u = f \quad \text{in } (0, T) \times \Omega, \quad (1.1)$$

with  $\varepsilon^2 = k/(c\rho)$ ,  $[\text{m}^2/\text{s}]$  and  $f = F/(c\rho)$ ,  $[\text{K}/\text{s}]$ . To obtain a well-posed problem, (1.1) has to be equipped with an initial condition  $u(0, \mathbf{x})$  and appropriate boundary conditions on  $(0, T) \times \partial\Omega$ .  $\square$

*Remark 1.3. Boundary conditions.* For the theory and the numerical simulation of boundary value problems with partial differential equations, the choice of boundary conditions is of utmost importance. For the heat equation (1.1), one can prescribe the following types of boundary conditions:

- Dirichlet<sup>3</sup> condition: The temperature  $u(t, \mathbf{x})$  at a part of the boundary is prescribed

$$u = g_1 \quad \text{on } (0, T) \times \partial\Omega_D$$

with  $\partial\Omega_D \subset \partial\Omega$ . In the context of the heat equation, the Dirichlet condition is also called essential boundary conditions.

- Neumann<sup>4</sup> condition: The heat flux is prescribed at a part of the boundary

$$-k \frac{\partial u}{\partial \mathbf{n}} = g_2 \quad \text{on } (0, T) \times \partial\Omega_N$$

with  $\partial\Omega_N \subset \partial\Omega$ . This boundary condition is a so-called natural boundary condition for the heat equation.

- Mixed boundary condition, Robin<sup>5</sup> boundary condition: At the boundary, there is a heat exchange according to Newton's<sup>6</sup> law

$$k \frac{\partial u}{\partial \mathbf{n}} + h(u - u_{\text{env}}) = 0 \quad \text{on } (0, T) \times \partial\Omega_m,$$

with  $\partial\Omega_m \subset \partial\Omega$ , the heat exchange coefficient  $h$ ,  $[\text{W}/\text{m}^2\text{K}]$ , and the temperature of the environment  $u_{\text{env}}$ .  $\square$

*Remark 1.4. The stationary case.* An important special case is that the temperature is constant in time  $u(t, \mathbf{x}) = u(\mathbf{x})$ . Then, one obtains the stationary heat equation

$$-\varepsilon^2 \Delta u = f \quad \text{in } \Omega. \quad (1.2)$$

<sup>3</sup> Johann Peter Gustav Lejeune Dirichlet (1805 – 1859)

<sup>4</sup> Carl Gottfried Neumann (1832 – 1925)

<sup>5</sup> Gustave Robin (1855 – 1897)

<sup>6</sup> Isaac Newton (1642 – 1727)

This equation is called Poisson<sup>7</sup> equation. Its homogeneous form, i.e., with  $f(\mathbf{x}) = 0$ , is called Laplace<sup>8</sup> equation. Solution of the Laplace equation are called harmonic functions. In many aspects, the Poisson equation is the simplest partial differential equation. The most part of this lecture will consider numerical methods for solving this equation.  $\square$

*Remark 1.5. Another application of the Poisson equation.* The stationary distribution of an electric field with charge distribution  $f(\mathbf{x})$  satisfies also the Poisson equation (1.2).  $\square$

*Remark 1.6. Non-dimensional equations.* The mathematical analysis as well as the application of numerical methods relies on equations for functions without physical units, the so-called non-dimensional equations. Let

- $L$  – a characteristic length scale of the problem, [m],
- $U$  – a characteristic temperature scale of the problem, [K],
- $T^*$  – a characteristic time scale of the problem, [s].

If the new coordinates and functions are denoted with a prime, one gets with the transformations

$$\mathbf{x}' = \frac{\mathbf{x}}{L}, \quad u' = \frac{u}{U}, \quad t' = \frac{t}{T^*}$$

from (1.1) the non-dimensional equation

$$\begin{aligned} \frac{\partial}{\partial t'} (U u') \frac{\partial t'}{\partial t} - \varepsilon^2 \sum_{i=1}^3 \frac{\partial}{\partial x'_i} \left( \frac{\partial}{\partial x'_i} (U u') \frac{\partial x'_i}{\partial x_i} \right) \frac{\partial x'_i}{\partial x_i} &= f \quad \text{in } \left(0, \frac{T}{T^*}\right) \times \Omega' \\ \iff \\ \frac{U}{T^*} \frac{\partial u'}{\partial t'} - \frac{\varepsilon^2 U}{L^2} \sum_{i=1}^3 \frac{\partial^2 u'}{\partial (x'_i)^2} &= f \quad \text{in } \left(0, \frac{T}{T^*}\right) \times \Omega'. \end{aligned}$$

Usually, one denotes the non-dimensional functions like the dimensional functions, leading to

$$\frac{\partial u}{\partial t} - \frac{\varepsilon^2 T^*}{L^2} \Delta u = \frac{T^*}{U} f \quad \text{in } \left(0, \frac{T}{T^*}\right) \times \Omega.$$

For the analysis, one sets  $L = 1$  m,  $U = 1$  K, and  $T^* = 1$  s which yields

$$\frac{\partial u}{\partial t} - \varepsilon^2 \Delta u = f \quad \text{in } (0, T) \times \Omega, \quad (1.3)$$

with a non-dimensional temperature diffusion  $\varepsilon^2$  and a non-dimensional right-hand side  $f(t, \mathbf{x})$ .

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<sup>7</sup> Siméon Denis Poisson (1781 – 1840)

<sup>8</sup> Pierre Simon Laplace (1749 – 1829)

The same approach can be applied to the stationary equation (1.2) and one gets

$$-\varepsilon^2 \Delta u = f \text{ in } \Omega, \quad (1.4)$$

with the non-dimensional temperature diffusion  $\varepsilon^2$  and the non-dimensional right-hand side  $f(\mathbf{x})$ .  $\square$

*Remark 1.7. A standard approach for solving the instationary equation.* The heat equation (1.3) is an initial value problem with respect to time and a boundary value problem with respect to space. Numerical methods for solving initial value problems were topic of Numerical Mathematics 2.

A standard approach for solving the instationary problem consists in using a so-called one-step  $\theta$ -scheme for discretizing the temporal derivative. Consider two consecutive discrete times  $t_n$  and  $t_{n+1}$  with  $\tau = t_{n+1} - t_n$ . Then, the application of a one-step  $\theta$ -scheme yields for the solution at  $t_{n+1}$

$$\frac{u_{n+1} - u_n}{\tau} - \theta \varepsilon^2 \Delta u_{n+1} - (1 - \theta) \varepsilon^2 \Delta u_n = \theta f_{n+1} + (1 - \theta) f_n,$$

where the subscript at the functions denotes the time level. This equation is equivalent to

$$u_{n+1} - \tau \theta \varepsilon^2 \Delta u_{n+1} = u_n + \tau(1 - \theta) \varepsilon^2 \Delta u_n + \tau \theta f_{n+1} + \tau(1 - \theta) f_n. \quad (1.5)$$

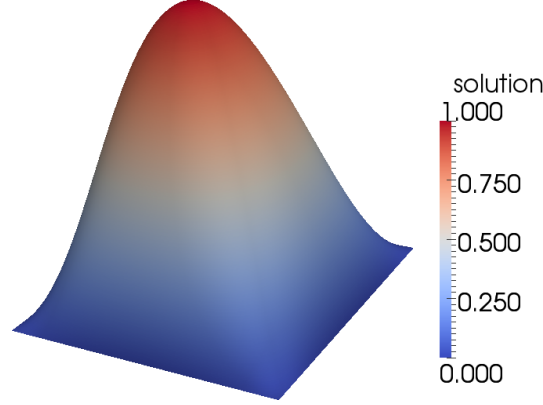
For  $\theta = 0$ , one obtains the forward Euler scheme, for  $\theta = 0.5$  the Crank–Nicolson scheme (trapezoidal rule), and for  $\theta = 1$  the backward Euler scheme.

Given  $u_n$ , (1.5) is a boundary value problem for  $u_{n+1}$ . That means, one has to solve in each discrete time a boundary value problem. For this reason, this course will concentrate on the numerical solution of boundary value problems.  $\square$

*Example 1.8. Poisson problems.* The simulations were performed with the code MOONMD JOHN & MATTHIES (2004).

- CONSIDER THE POISSON EQUATION (1.4) IN  $\Omega = (0, 1)^2$  WITH  $\varepsilon = 1$ . THE RIGHT-HAND SIDE AND THE DIRICHLET BOUNDARY CONDITIONS ARE CHOSEN SUCH THAT  $u(x, y) = \sin(\pi x) \sin(\pi y)$  IS THE PRESCRIBED SOLUTION, SEE FIGURE 1.1 HENCE, THIS SOLUTION SATISFIES HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS. DENOTE BY  $u_h(x, y)$  THE COMPUTED SOLUTION, WHERE  $h$  INDICATES THE REFINEMENT OF A MESH IN  $\Omega$ . THE ERRORS OBTAINED ON SUCCESSIVELY REFINED MESHES WITH THE SIMPLEST FINITE ELEMENT METHOD ARE PRESENTED IN TABLE 1.1.

ONE CAN OBSERVE IN TABLE 1.1 THAT  $\|u - u_h\|_{L^2(\Omega)}$  CONVERGES WITH SECOND ORDER AND  $\|\nabla(u - u_h)\|_{L^2(\Omega)}$  CONVERGES WITH FIRST ORDER. A MAIN TOPIC OF THE NUMERICAL ANALYSIS OF DISCRETIZATIONS FOR PARTIAL DIFFERENTIAL EQUATIONS CONSISTS IN SHOWING THAT THE



**Fig. 1.1** Solution of the two-dimensional example of Example 1.8.

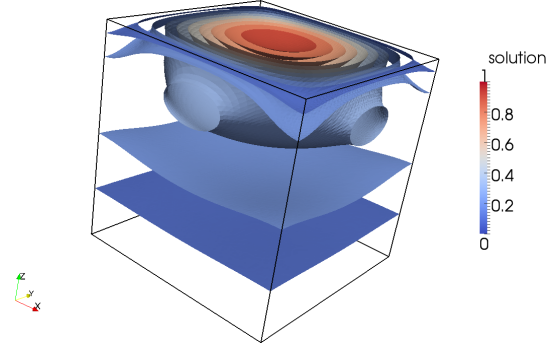
**Table 1.1** Example 1.8, two-dimensional example.

| $h$   | degrees of freedom | $\ u - u_h\ _{L^2(\Omega)}$ | $\ \nabla(u - u_h)\ _{L^2(\Omega)}$ |
|-------|--------------------|-----------------------------|-------------------------------------|
| 1/4   | 25                 | 8.522e-2                    | 8.391e-1                            |
| 1/8   | 81                 | 2.256e-2                    | 4.318e-1                            |
| 1/16  | 289                | 5.726e-3                    | 2.175e-1                            |
| 1/32  | 1089               | 1.437e-3                    | 1.089e-1                            |
| 1/64  | 4225               | 3.596e-4                    | 5.451e-2                            |
| 1/128 | 16641              | 8.993e-5                    | 2.726e-2                            |
| 1/256 | 66049              | 2.248e-5                    | 1.363e-2                            |
| 1/512 | 263169             | 5.621e-6                    | 6.815e-3                            |

COMPUTED SOLUTION CONVERGES TO THE SOLUTION OF AN APPROPRIATE CONTINUOUS PROBLEM IN APPROPRIATE NORMS. IN ADDITION, TO PROVE A CERTAIN ORDER OF CONVERGENCE (IN THE ASYMPTOTIC REGIME) IS OF INTEREST.

- CONSIDER THE POISSON EQUATION (1.4) IN  $\Omega = (0, 1)^3$  WITH  $\varepsilon = 1$  AND  $f = 0$ . AT  $z = 1$  THE TEMPERATURE PROFILE SHOULD BE  $u(x, y, 1) = 16x(1 - x)y(1 - y)$  AND AT THE OPPOSITE WALL SHOULD BE COOLED  $u(x, y, 0) = 0$ . AT ALL OTHER WALLS, THERE SHOULD BE AN UNDISTURBED TEMPERATURE FLUX  $\frac{\partial u}{\partial \mathbf{n}}(x, y, z) = 0$ . A APPROXIMATION OF THE SOLUTION COMPUTED WITH A FINITE ELEMENT METHOD IS PRESENTED IN FIGURE 1.2.

THE ANALYTIC SOLUTION IS NOT KNOWN IN THIS EXAMPLE (OR IT MAYBE HARD TO COMPUTE). IT IS IMPORTANT FOR APPLICATIONS THAT ONE OBTAINS, E.G., GOOD VISUALIZATIONS OF THE SOLUTION OR GOOD APPROXIMATE VALUES FOR QUANTITIES OF INTEREST. ONE KNOWS BY THE GENERAL THEORY THAT THE COMPUTED SOLUTION CONVERGES TO THE SOLUTION OF THE CONTINUOUS PROBLEM IN APPROPRIATE NORMS AND ONE HOPES THAT THE COMPUTED SOLUTION IS ALREADY SUFFICIENTLY CLOSE.



**Fig. 1.2** Contour lines of the solution of the three-dimensional example of Example 1.8.

□

## 1.2 The Diffusion Equation

*Remark 1.9. Derivation.* Diffusion is the transport of a species caused by the movement of particles. Instead of Fourier's law, Nernst's<sup>9</sup> law for the particle flux through  $\partial V$  per time unit is used

$$dQ = -D \nabla u \cdot \mathbf{n} \, ds$$

with

- $u(t, \mathbf{x})$  – particle density, concentration with unit  $[\text{mol}/\text{m}^3]$ ,
- $D(t, \mathbf{x})$  – diffusion coefficient with unit  $[\text{m}^2/\text{s}]$ .

The derivation of the diffusion equation proceeds in the same way as for the heat equation. It has the form

$$c \frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) + qu = F \quad \text{in } (0, T) \times \Omega, \quad (1.6)$$

where

- $c(t, \mathbf{x})$  – is the porosity of the species,  $[\cdot]$ ,
- $q(t, \mathbf{x})$  – is the absorption coefficient of the species with unit  $[1/\text{s}]$ ,
- $F(t, \mathbf{x})$  – describes sources and sinks,  $[\text{mol}/\text{s} \, \text{m}^3]$ .

The porosity and the absorption coefficient are positive functions. To obtain a well-posed problem, an initial condition and boundary conditions are necessary.

If the concentration is constant in time, one obtains

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<sup>9</sup> Walther Hermann Nernst (1864 – 1941)

$$-\nabla \cdot (D\nabla u) + qu = F \quad \text{in } \Omega. \quad (1.7)$$

Hence, the diffusion equation possesses a similar form as the heat equation (1.2).  $\square$

### 1.3 The Navier–Stokes Equations

*Remark 1.10. Generalities.* The Navier<sup>10</sup>–Stokes<sup>11</sup> equations are the fundamental equations of fluid dynamics. In this section, a viscous fluid (with internal friction) with constant density (incompressible) will be considered.  $\square$

*Remark 1.11. Conservation of mass.* The first basic principle of the flow of an incompressible fluid is the conservation of mass. Let  $V$  be an arbitrary volume. Then, the change of fluid in  $V$  satisfies

$$\underbrace{-\frac{\partial}{\partial t} \int_V \rho \, d\mathbf{x}}_{\text{change}} = \underbrace{\int_{\partial V} \rho \mathbf{v} \cdot \mathbf{n} \, ds}_{\text{flux through the boundary of } V} = \int_V \nabla \cdot (\rho \mathbf{v}) \, d\mathbf{x},$$

where

- $\mathbf{v}(t, \mathbf{x})$  – velocity  $(v_1, v_2, v_3)^T$  at time  $t$  and at point  $\mathbf{x}$  with unit  $[\text{m/s}]$ ,
- $\rho$  – density of the fluid,  $[\text{kg/m}^3]$ .

Since  $V$  is arbitrary, the terms in the volume integrals have to be the same. One gets the so-called continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{in } (0, T) \times \Omega.$$

Since  $\rho$  is constant, one obtains the first equation of the Navier–Stokes equations, the so-called incompressibility constraint,

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } (0, T) \times \Omega. \quad (1.8)$$

$\square$

*Remark 1.12. Conservation of linear momentum.* The second equation of the Navier–Stokes equations represents Newton’s second law of motion

$$\text{net force} = \text{mass} \times \text{acceleration}.$$

It states that the rate of change of the linear momentum must be equal to the net force acting on a collection of fluid particles.

<sup>10</sup> Claude Louis Marie Henri Navier (1785 – 1836)

<sup>11</sup> George Gabriel Stokes (1819 – 1903)



The forces acting on an arbitrary volume  $V$  are given by

$$F_V = \underbrace{\int_{\partial V} -P \mathbf{n} \, d\mathbf{s}}_{\text{outer pressure}} + \underbrace{\int_{\partial V} \mathbb{S}' \mathbf{n} \, d\mathbf{s}}_{\text{friction}} + \underbrace{\int_V \rho \mathbf{g} \, d\mathbf{x}}_{\text{gravitation}},$$

where

- $\mathbb{S}'(t, \mathbf{x})$  – stress tensor with unit  $[\text{N}/\text{m}^2]$ ,
- $P(t, \mathbf{x})$  – the pressure with unit  $[\text{N}/\text{m}^2]$ ,
- $\mathbf{g}(t, \mathbf{x})$  – standard gravity (directed),  $[\text{m}/\text{s}^2]$ .

The pressure possesses a negative sign since it is directed into  $V$ , whereas the stress acts outwardly.

The integral on  $\partial V$  can be transformed into an integral on  $V$  with integration by parts. One obtains the force per unit volume

$$-\nabla P + \nabla \cdot \mathbb{S}' + \rho \mathbf{g}.$$

On the basis of physical considerations (Landau & Lifschitz, 1966, p. 53) or (John, 2016, Chapter 2), one uses the following ansatz for the stress tensor

$$\mathbb{S}' = \eta \left( \nabla \mathbf{v} + \nabla \mathbf{v}^T - \frac{2}{3} (\nabla \cdot \mathbf{v}) \mathbb{I} \right) + \zeta (\nabla \cdot \mathbf{v}) \mathbb{I},$$

where

- $\eta$  – first order viscosity of the fluid,  $[\text{kg}/\text{m s}]$ ,
- $\zeta$  – second order viscosity of the fluid,  $[\text{kg}/\text{m s}]$ ,
- $\mathbb{I}$  – unit tensor.

For Newton's second law of motion one considers the movement of particles with velocity  $\mathbf{v}(t, \mathbf{x}(t))$ . One obtains the following equation

$$\underbrace{-\nabla P + \nabla \cdot \mathbb{S}' + \rho \mathbf{g}}_{\text{force per unit volume}} = \underbrace{\rho}_{\text{mass per unit volume}} \underbrace{\frac{d\mathbf{v}(t, \mathbf{x}(t))}{dt}}_{\text{acceleration}} = \rho (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}).$$

The second formula was obtained with the chain rule. The detailed form of the second term is

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = \begin{pmatrix} v_1 \partial_x v_1 + v_2 \partial_y v_1 + v_3 \partial_z v_1 \\ v_1 \partial_x v_2 + v_2 \partial_y v_2 + v_3 \partial_z v_2 \\ v_1 \partial_x v_3 + v_2 \partial_y v_3 + v_3 \partial_z v_3 \end{pmatrix}.$$

If both viscosities are constant, one gets

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla P}{\rho} = \mathbf{g} + \frac{1}{\rho} \left( \frac{\eta}{3} + \zeta \right) \nabla (\nabla \cdot \mathbf{v}),$$

where  $\nu = \eta/\rho, [m^2/s]$  is the kinematic viscosity. The second term on the right-hand side vanishes because of the incompressibility constraint (1.8).

One obtains the dimensional Navier–Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla P}{\rho} = \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in } (0, T) \times \Omega.$$

□

*Remark 1.13. Non-dimensional Navier–Stokes equations.* The final step in the modeling process is the derivation of non-dimensional equations. Let

- $L$  – a characteristic length scale of the problem, [m],
- $U$  – a characteristic velocity scale of the problem, [m/s],
- $T^*$  – a characteristic time scale of the problem, [s].

Denoting here the old coordinates with a prime, one obtains with the transformations

$$\mathbf{x} = \frac{\mathbf{x}'}{L}, \quad \mathbf{u} = \frac{\mathbf{v}}{U}, \quad t = \frac{t'}{T^*}$$

the non-dimensional equations

$$\frac{L}{UT^*} \partial_t \mathbf{u} - \frac{\nu}{UL} \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } (0, T) \times \Omega,$$

with the redefined pressure and the new right-hand side

$$p(t, \mathbf{x}) = \frac{P}{\rho U^2}(t, \mathbf{x}), \quad \mathbf{f}(t, \mathbf{x}) = \frac{L\mathbf{g}}{U^2}(t, \mathbf{x}).$$

This equation has two dimensionless characteristic parameters: the Strouhal<sup>12</sup> number  $St$  and the Reynolds<sup>13</sup> number  $Re$

$$St := \frac{L}{UT^*}, \quad Re := \frac{UL}{\nu}.$$

Setting  $T^* = L/U$ , one obtains the form of the incompressible Navier–Stokes equations which can be found in the literature

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - Re^{-1} \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0, T) \times \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } (0, T) \times \Omega. \end{aligned}$$

□

*Remark 1.14. About the incompressible Navier–Stokes equations.* The Navier–Stokes equations are not yet understood completely. For instance, the existence of an appropriately defined classical solution for  $\Omega \subset \mathbb{R}^3$  is not clear.

<sup>12</sup> Čeněk Strouhal (1850 – 1923)

<sup>13</sup> Osborne Reynolds (1842 – 1912)

This problem is among the so-called millennium problems of mathematics Fefferman (2000) and its answer is worth one million dollar. Also the numerical methods for solving the Navier–Stokes equations are by far not developed sufficiently well as it is required by many applications, e.g., for turbulent flows in weather prediction.  $\square$

*Remark 1.15. Slow flows.* An important special case is the case of slow flows which lead to a stationary (independent of time) flow field. In this case, the first term in the momentum balance equation vanishes. In addition, if the flow is very slow, the nonlinear term can be neglected as well. One gets the so-called Stokes equations

$$\begin{aligned} -Re^{-1}\Delta \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega. \end{aligned}$$

$\square$

## 1.4 Classification of Second Order Partial Differential Equations

**Definition 1.16. Semi-linear and linear second order partial differential equation.** Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ . A semi-linear second order partial differential equation defined on  $\Omega$  has the form

$$\sum_{j,k=1}^d a_{jk}(\mathbf{x}) \partial_j \partial_k u + F(\mathbf{x}, u, \partial_1 u, \dots, \partial_d u) = 0 \quad (1.9)$$

or in nabla notation

$$\nabla \cdot (A(\mathbf{x}) \nabla u) + \tilde{F}(\mathbf{x}, u, \partial_1 u, \dots, \partial_d u) = 0.$$

A linear second order partial differential equation has the form

$$\sum_{j,k=1}^d a_{jk}(\mathbf{x}) \partial_j \partial_k u + \mathbf{b}(\mathbf{x}) \cdot \nabla u + c(\mathbf{x})u = F(\mathbf{x}).$$

$\square$

*Remark 1.17. The matrix of the second order operator.* If  $u(\mathbf{x})$  is sufficiently regular, then the application of the Theorem of Schwarz<sup>14</sup> yields  $\partial_j \partial_k u(\mathbf{x}) = \partial_k \partial_j u(\mathbf{x})$ . It follows that equation (1.9) contains the coefficient  $\partial_j \partial_k u(\mathbf{x})$  twice, namely in  $a_{jk}(\mathbf{x})$  and  $a_{kj}(\mathbf{x})$ . For definiteness, one requires that

<sup>14</sup> Hermann Amandus Schwarz (1843 – 1921)

$$a_{jk}(\mathbf{x}) = a_{kj}(\mathbf{x}).$$

Now, one can write the coefficient of the second order derivative with the symmetric matrix

$$A(\mathbf{x}) = \begin{pmatrix} a_{11}(\mathbf{x}) & \cdots & a_{1d}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ a_{d1}(\mathbf{x}) & \cdots & a_{dd}(\mathbf{x}) \end{pmatrix}.$$

All eigenvalues of this matrix are real and the classification of semi-linear second order partial differential equations is based on these eigenvalues.  $\square$

**Definition 1.18. Classification of semi-linear second order partial differential equation.** On a subset  $\tilde{\Omega} \subset \Omega$  let  $\alpha$  be the number of positive eigenvalues of  $A(\mathbf{x})$ ,  $\beta$  be the number of negative eigenvalues, and  $\gamma$  be the multiplicity of the eigenvalue zero. The semi-linear second order partial differential equation (1.9) is said to be of type  $(\alpha, \beta, \gamma)$  on  $\tilde{\Omega}$ . It is called to be

- elliptic on  $\tilde{\Omega}$ , if it is of type  $(d, 0, 0) = (0, d, 0)$ ,
- hyperbolic on  $\tilde{\Omega}$ , if its type is  $(d-1, 1, 0) = (1, d-1, 0)$ ,
- parabolic on  $\tilde{\Omega}$ , if it is of type  $(d-1, 0, 1) = (0, d-1, 1)$ .

In the case of linear partial differential equations, one speaks of a parabolic equation if in addition to the requirement from above it holds that

$$\text{rank}(A(\mathbf{x}), \mathbf{b}(\mathbf{x})) = d$$

in  $\tilde{\Omega}$ .  $\square$

*Remark 1.19. Other cases.* Definition 1.18 does not cover all possible cases. However, the other cases are only of little interest in practice.  $\square$

*Example 1.20. Types of second order partial differential equations.*

- For the Poisson equation (1.4) one has  $a_{ii} = -\varepsilon^2 < 0$  and  $a_{ij} = 0$  for  $i \neq j$ . It follows that all eigenvalues of  $A$  are negative and the Poisson equation is an elliptic partial differential equation. The same reasoning can be applied to the stationary diffusion equation (1.7).
- In the heat equation (1.3) there is besides the spatial derivatives also the temporal derivative. The time as variable and its derivative has to be taken into account in the definition of the matrix  $A$ , i.e., the domain from Definition 1.18 is  $(0, T) \times \Omega$ . Since this derivative is only of first order, one obtains in  $A$  a zero row and a zero column. One has, e.g.,  $a_{ii} = -\varepsilon^2 < 0, i = 2, \dots, d+1, a_{11} = 0$ , and  $a_{ij} = 0$  for  $i \neq j$ . It follows that one eigenvalue is zero and the others have the same sign. The vector of the first order term has the form  $\mathbf{b} = (1, 0, \dots, 0)^T \in \mathbb{R}^{d+1}$ , where the 1 comes from  $\partial_t u(t, \mathbf{x})$ . Now, one can see immediately that  $(A, \mathbf{b})$  possesses full column rank. Hence, (1.3) is a parabolic partial differential equation.
- An example for a hyperbolic partial differential equation is the wave equation

$$\partial_{tt}u - \varepsilon^2 \Delta u = f \quad \text{in } (0, T) \times \Omega.$$

□

## 1.5 Literature

*Remark 1.21. Some books about the topic of this class.* Books about finite difference methods are

- Samarskij (1984), classic book, the English version is Samarskii (2001)
- LeVeque (2007)

Much more books can be found about finite element methods

- Ciarlet (2002), classic text,
- Strang & Fix (2008), classic text,
- Braess (2001), very popular book in Germany, English version available,
- Brenner & Scott (2008), rather abstract treatment, from the point of view of functional analysis,
- Ern & Guermond (2004), modern comprehensive book,
- Ern & Guermond (2021a,b,c), modern encyclopedic presentation
- Grossmann & Roos (2007)
- Šolín (2006), written by somebody who worked a lot in the implementation of the methods,
- Goering *et al.* (2010), introductory text, good for beginners,
- Ganesan & Tobiska (2017), introductory text, good for beginners,
- Deuffhard & Weiser (2012), strong emphasis on adaptive methods
- Dziuk (2010).

These lists are not complete.

These lecture notes are based in some parts on lecture notes from Sergej Rjasanow (Saarbrücken) and Manfred Dobrowolski (Würzburg). □