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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 08

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Alternative inner product in* $L^2(0,\infty)$. Show that

$$a(u,v) = \int_0^\infty e^{-x} u(x) v(x)\ dx$$

defines a (real) inner product in $L^2(0,\infty)$.
Hint: A map $a(\cdot,\cdot)\ :\ V \times V \to \mathbb{R}$ is a (real) inner product if it is bilinear, symmetric, and coercive, i.e.:

   i) $a(\alpha u+\beta v, w) = \alpha a(u,v)+\beta a(w,v),\ a(u,\alpha v+\beta w) = \alpha a(u,v)+\beta a(u,w),\ \forall\ u,v,w \in V, \alpha,\beta \in \mathbb{R}$,
   ii) $a(u,v) = a(v,u)\ \forall\ u,v \in V$,
   iii) $a(u,u) \ge 0\ \forall\ u \in V$ and $a(u,u) = 0 \iff u = 0$.

2. *Boundedness of bilinear form.* Let $a\ :\ H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the bilinear form

$$a(u,v) = \int_\Omega \nabla u(\mathbf{x})^T A(\mathbf{x}) \nabla v(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x})\ d\mathbf{x}.$$

with $a_{ij} \in L^\infty(\Omega)$, $i,j = 1,\ldots,d$, $c \in L^\infty(\Omega)$, $c \ge 0$. Show that this bilinear form is bounded, i.e., there is a constant C such that

$$|a(u,v)| \le C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall\ u,v \in H^1(\Omega).$$

3. *Connection of properties of matrices and bilinear forms.* Let

$$A = (a_{ij}) = a(\phi_j, \phi_i),$$

where $\{\phi_i\}_{i=1}^k$ is the basis of a finite-dimensional space V_k. Show that

   i)
$$A = A^T \quad \iff \quad a(v,w) = a(w,v) \quad \forall\ v,w \in V_k,$$
   ii)
$$\underline{x}^T A \underline{x} > 0 \quad \forall\ \underline{x} \ne \underline{0} \quad \iff \quad a(v,v) > 0 \quad \forall\ v \ne 0.$$

4. *Stability estimate for the Poisson problem.* Let $\Omega \in \mathbb{R}^d$, $d \in \{2,3\}$, a bounded domain with Lipschitz boundary. Consider the Poisson problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma.$$

Show the stability estimate

$$\|u\|_{H^1(\Omega)} \le C\, \|f\|_{L^2(\Omega)}.$$

Hint: Test the Poisson equation with the function u.

The exercise problems should be solved in groups of two or three students. The written parts have to be submitted until **Thursday, June 13, 2019**. The executable codes have to be send by email to A. Jha.