Department for Mathematics and Computer Science
Free University of Berlin
Prof. Dr. V. John, john@wias-berlin.de
Abhinav Jha, jha@wias-berlin.de

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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 06

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

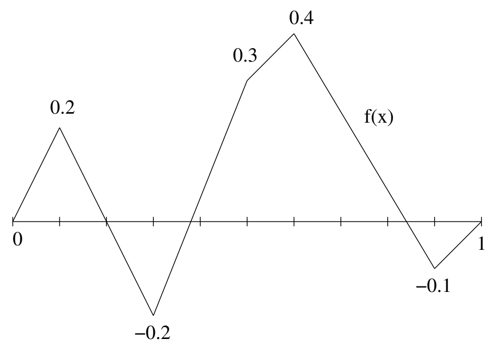
1. *Spaces $L^1(\Omega)$ and $L^1_{\mathrm{loc}}(\Omega)$.* Let $f(x) = 1$ in Ω. Investigate if $f \in L^1(\Omega)$, $f \in L^1_{\mathrm{loc}}(\Omega)$ for $\Omega = (0,1)$ and $\Omega = \mathbb{R}$.

2. *Weak derivative in one dimension.* Solve the following problems.

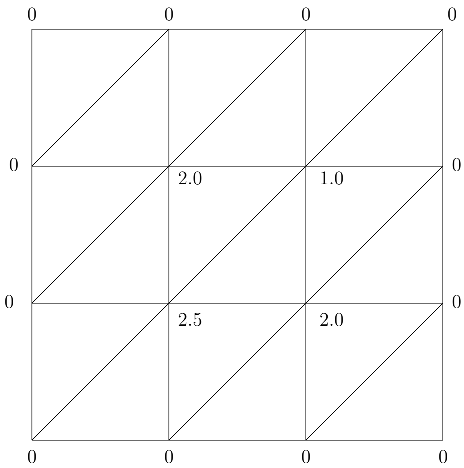
   i) Let $f(x) = 1$ in Ω. Investigate whether or not $f \in L^1(\Omega)$, $f \in L^1_{\mathrm{loc}}(\Omega)$ for $\Omega = (0,1)$ and $\Omega = \mathbb{R}$.

   ii) Show with the help of the definition that
   $$f'(x) = \begin{cases} -1 & x < 0, \\ 0 & x = 0, \\ 1 & x > 0, \end{cases}$$
   is the weak derivative of $f(x) = |x|$.

   iii) Compute the weak derivative of the following function $f : \Omega \to \mathbb{R}$, $\Omega = (0,1)$.



3. *Weak derivative in two dimensions.* Compute the weak derivative of the following function $f : \Omega \to \mathbb{R}$, $\Omega = (0,1)^2$, which is continuous and piecewise (with respect to the grid) linear and which is therefore completely determined by the values in the nodes.



4. *Hölder's inequality.* Let $r \in [1, \infty)$, $p, q \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, $u \in L^{rp}(\Omega)$, $v \in L^{rq}(\Omega)$. Show that

$$\|uv\|_{L^r(\Omega)} \leq \|u\|_{L^{rp}(\Omega)} \|v\|_{L^{rq}(\Omega)}.$$

The exercise problems should be solved in groups of two or three students. The written parts have to be submitted until **Monday, June 03, 2019** after the class. The executable codes have to be send by email to A. Jha.