

## Numerical Mathematics III – Partial Differential Equations

### Exercise Problems 01

**Attention:** The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Basic properties of the nabla operator.* The following operators are defined for a scalar function  $u$  and a vector-valued function  $\mathbf{v} = (v_1, v_2, v_3)^T$  with the help of the nabla operator

$$\nabla = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right)^T = (\partial_{x_1}, \dots, \partial_{x_d}) :$$

- $\text{gradu} = \nabla u = (\partial_x u, \partial_y u, \partial_z u)^T$ ,
- $\text{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \partial_x v_1 + \partial_y v_2 + \partial_z v_3$ ,
- $\text{rot} \mathbf{v} = \nabla \times \mathbf{v} = \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix}$ .

Assuming that all considered functions are sufficiently smooth (sufficiently often differentiable), show the following identities:

- i)  $\nabla \cdot \nabla u = \Delta u = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u$ ,
- ii)  $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ ,
- iii)  $\nabla \times (\nabla u) = 0$ ,
- iv)  $\nabla \times (u \mathbf{v}) = u (\nabla \times \mathbf{v}) - (\mathbf{v} \times \nabla u)$ ,
- v)  $\nabla \cdot (u \mathbf{v}) = \nabla u \cdot \mathbf{v} + u \nabla \cdot \mathbf{v}$ .

2. *Analytic solution of a one-dimensional heat equation.* Show that

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0(z) e^{-\frac{(x-z)^2}{4t}} dz$$

is a solution of the one-dimensional heat equation

$$\begin{aligned} \partial_t u - \partial_{xx} u &= 0, & x \in \mathbb{R}, t > 0, \\ u(0, x) &= u_0(x), & x \in \mathbb{R}. \end{aligned}$$

It shall be assumed that  $u_0(x)$  is sufficiently smooth.

Hint. To check the initial condition, assume that  $u_0(x)$  can be expanded in a Fourier series

$$u_0(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos(n\omega x) + \beta_n \sin(n\omega x)),$$

with  $\omega \in \mathbb{R}$ , and use the following identities

$$\int_{-\infty}^{\infty} \sin(nz) e^{-\frac{(x-z)^2}{4t}} dz = \sqrt{4\pi t} e^{-n^2 t} \sin(nx),$$
$$\int_{-\infty}^{\infty} \cos(nz) e^{-\frac{(x-z)^2}{4t}} dz = \sqrt{4\pi t} e^{-n^2 t} \cos(nx), \quad n \in \mathbb{R}.$$

The exercise problems should be solved in groups of two or three students. The written parts have to be submitted until **Thursday, Apr. 18, 2019** to A. Jha. The executable codes have to be send by email to A. Jha.