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Numerical Mathematics III – Partial Differential Equations

Exercise Problems 02

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. Solve the following problems.

   (a) Show that
   $$v_{\mathring{x},i} = \frac{1}{2}\left(v_{x,i} + v_{\bar{x},i}\right), \qquad v_{\bar{x}x,i} = \left(v_{\bar{x},i}\right)_{x,i}.$$

   (b) Consider a function $v(x)$ at x_i and show the following consistency estimates
   $$v_{\mathring{x},i} = v'\left(x_i\right) + \mathcal{O}\left(h^2\right), \quad v_{\bar{x}x,i} = v''\left(x_i\right) + \mathcal{O}\left(h^2\right).$$

   (c) Compute the order of consistency of the following finite difference approximation
   $$u''\left(x\right) \sim \frac{1}{12\,h^2}\Big(-u\left(x+2\,h\right)+16\,u\left(x+h\right)-30\,u\left(x\right)+16\,u\left(x-h\right)-u\left(x-2\,h\right)\Big).$$

2. Consider the differential operator $Lu = \frac{\partial}{\partial x}\left(k\left(x\right)\frac{\partial u}{\partial x}\right)$ and its finite difference approximation
   $$\left(L_h u_h\right)_i = \frac{1}{h}\left(a_{i+1}\frac{u_{i+1}-u_i}{h} - a_i\frac{u_i - u_{i-1}}{h}\right).$$
   Show that $a_i = \frac{k_i + k_{i-1}}{2}$ and $a_i = k\left(x_i - \frac{h}{2}\right)$ satisfy the conditions for second order consistency
   $$\frac{a_{i+1}-a_i}{h} = k'\left(x_i\right) + \mathcal{O}\left(h^2\right), \qquad \frac{a_{i+1}+a_i}{2} = k\left(x_i\right) + \mathcal{O}\left(h^2\right),$$
   which were derived in the lecture.

3. *Finite difference approximation of the second order derivative at a non-equidistant grid.* Consider the interval $[x - h_x^-, x + h_x^+]$ with $h_x^-, h_x^+ > 0$, $h_x^- \neq h_x^+$.

   (a) Assume $u \in C^3\left([x - h_x^-, x + h_x^+]\right)$. Show the following consistency estimate
   $$\left|u''\left(x\right) - \frac{2}{h_x^+ + h_x^-}\left(\frac{u\left(x+h_x^+\right) - u\left(x\right)}{h_x^+} - \frac{u\left(x\right) - u\left(x - h_x^-\right)}{h_x^-}\right)\right| \le C\left(h_x^+ + h_x^-\right).$$

   (b) Prove that there is no other approximation which satisfies
   $$\left|u''\left(x\right) - \left(\alpha\,u\left(x - h_x^-\right) + \beta\,u\left(x\right) + \gamma\,u\left(x + h_x^+\right)\right)\right| \le C\left(h_x^+ + h_x^-\right)$$
   with $\alpha = \alpha\left(h_x^-, h_x^+\right)$, $\beta = \beta\left(h_x^-, h_x^+\right)$, $\gamma = \gamma\left(h_x^-, h_x^+\right) \in \mathbb{R}$.

The exercise problems should be solved in groups of two or three students. They have to be submitted until **Tuesday, Apr. 23, 2013** either before or after one of the lectures or directly at the office of Mrs. Hardering.