Chapter 7

Finite Element Methods for Second Order Elliptic Equations

7.1 General Convergence Theorems

Remark 7.1 *Motivation.* In Section 5.1, non-conforming finite element methods have been introduced, i.e., methods where the finite element space V^h is not a subspace of V, which is the space in the definition of the continuous variational problem. The property $V^h \not\subset V$ is given for the Crouzeix–Raviart and the Rannacher–Turek element. Another case of non-conformity is given if the domain does not possess a polyhedral boundary and one has to apply some approximation of the boundary.

For non-conforming methods, the finite element approach is not longer a Ritz method. Hence, the convergence proof from Theorem 4.14 cannot be applied in this case. The abstract convergence theorem, which will be proved in this section, allows the numerical analysis of complex finite element methods. □

Remark 7.2 *Notations, Assumptions.* Let $\{h > 0\}$ be a set of mesh widths and let S^h, V^h normed spaces of functions which are defined on domains $\{\Omega^h \subset \mathbb{R}^d\}$. It will be assumed that the space S^h has a finite dimension and that S^h and V^h possess a common norm $\|\cdot\|_h$. In the application of the abstract theory, S^h will be a finite element space and V^h is defined such that the restriction and/or extension of the solution of the continuous problem to Ω^h is contained in V^h. Strictly speaking, the modified solution of the continuous problem does not solve the given problem any longer. Hence, it is consequent that the continuous problem does not appear explicitly in the abstract theory.

Given the bilinear forms

$$\begin{aligned} a^h &: S^h \times S^h \to \mathbb{R}, \\ \tilde{a}^h &: (S^h + V^h) \times (S^h + V^h) \to \mathbb{R}. \end{aligned}$$

Let the bilinear form a^h be regular in the sense that there is a constant $m > 0$, which is independent of h, such that for each $v^h \in S^h$ there is a $w^h \in S^h$ with $\|w^h\|_h = 1$ such that

$$m \|v^h\|_h \le a^h(v^h, w^h). \tag{7.1}$$

This condition is equivalent to the requirement that the stiffness matrix A with the entries $a_{ij} = a^h(\phi_j, \phi_i)$, where $\{\phi_i\}$ is a basis of S^h, is uniformly non-singular, i.e., its regularity is independent of h. For the second bilinear form, only its boundedness will be assumed

$$\tilde{a}^h(u, v) \le M \|u\|_h \|v\|_h \quad \forall\ u, v \in S^h + V^h. \tag{7.2}$$

Let the linear functionals $\{f^h(\cdot)\}\ :\ S^h \to \mathbb{R}$ be given. Then, the following discrete problems will be considered: Find $u^h \in S^h$ with

$$a^h(u^h, v^h) = f^h(v^h) \quad \forall\ v^h \in S^h. \tag{7.3}$$

Because the stiffness matrix is assumed to be non-singular, there is a unique solution of (7.3). □

Theorem 7.3 Abstract finite element error estimate. *Let the conditions (7.1) and (7.2) be satisfied and let u^h be the solution of (7.3). Then the following error estimate holds for each $\tilde{u} \in V^h$*

$$\begin{aligned} \left\|\tilde{u} - u^h\right\|_h &\le c \inf_{v^h \in S^h} \left\{ \left\|\tilde{u} - v^h\right\|_h + \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(v^h, w^h) - a^h(v^h, w^h)\right|}{\|w^h\|_h} \right\} \\ &\quad + c \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(\tilde{u}, w^h) - f^h(w^h)\right|}{\|w^h\|_h} \end{aligned} \tag{7.4}$$

with $c = c(m, M)$.

Proof: Because of (7.1) there is for each $v^h \in S^h$ a $w^h \in S^h$ with $\left\|w^h\right\|_h = 1$ and

$$m \left\|u^h - v^h\right\|_h \le a^h(u^h - v^h, w^h).$$

Using the definition of u^h from (7.3), one obtains

$$m \left\|u^h - v^h\right\|_h \le f^h(w^h) - a^h(v^h, w^h) + \tilde{a}^h(v^h, w^h) + \tilde{a}^h(\tilde{u} - v^h, w^h) - \tilde{a}^h(\tilde{u}, w^h).$$

From (7.2) and $\left\|w^h\right\|_h = 1$ it follows that

$$\tilde{a}^h(\tilde{u} - v^h, w^h) \le M \left\|\tilde{u} - v^h\right\|_h.$$

Rearranging the terms appropriately and using $\left\|w^h / \left\|w^h\right\|_h\right\|_h = 1$ gives

$$\begin{aligned} m \left\|u^h - v^h\right\|_h &\le M \left\|\tilde{u} - v^h\right\|_h + \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(v^h, w^h) - a^h(v^h, w^h)\right|}{\|w^h\|_h} \\ &\quad + \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(\tilde{u}, w^h) - f^h(w^h)\right|}{\|w^h\|_h}. \end{aligned}$$

Applying the triangle inequality

$$\left\|\tilde{u} - u^h\right\|_h \le \left\|\tilde{u} - v^h\right\|_h + \left\|u^h - v^h\right\|_h$$

and inserting the estimate from above gives (7.4). ■

Remark 7.4 *To Theorem 7.3.* An important special case of this theorem is the case that the stiffness matrix is uniformly positive definite, i.e., the condition

$$m \left\|v^h\right\|_h^2 \le a^h(v^h, v^h) \quad \forall\ v^h \in S^h \tag{7.5}$$

is satisfied. Dividing (7.5) by $\left\|v^h\right\|_h$ reveals that condition (7.1) is implied by (7.5).

If the continuous problem is also defined with the bilinear form $\tilde{a}^h(\cdot,\cdot)$, then

$$\sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(v^h, w^h) - a^h(v^h, w^h)\right|}{\|w^h\|_h}$$

can be considered as consistency error of the bilinear forms and the term

$$\sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(\tilde{u}, w^h) - f^h(w^h)\right|}{\|w^h\|_h}$$

as consistency error of the right-hand sides. □

Theorem 7.5 First Strang lemma *Let S^h be a conform finite element space, i.e., $S^h \subset V$, with $\|\cdot\|_h = \|\cdot\|_V$ and let the space V^h be independent of h. Consider a continuous problem of the form*

$$\tilde{a}^h(u, v) = f(v) \quad \forall\, v \in V,$$

then the following error estimate holds.

$$\begin{aligned}\left\|u - u^h\right\|_V \quad &\le \quad c \inf_{v^h \in S^h} \left\{ \left\|u - v^h\right\|_V + \sup_{w^h \in S^h} \frac{\left|\tilde{a}^h(v^h, w^h) - a^h(v^h, w^h)\right|}{\|w^h\|_V} \right\} \\ &\quad + c \sup_{w^h \in S^h} \frac{\left|f(w^h) - f^h(w^h)\right|}{\|w^h\|_V}.\end{aligned}$$

Proof: The statement of this theorem follows directly from Theorem 7.3. ■

7.2 Linear Finite Element Method on Non-Polyhedral Domains

Remark 7.6 *The continuous problem.* The abstract theory will be applied to the linear finite element method for the solution of second order elliptic partial differential equations.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with Lipschitz boundary, which does not need to be polyhedral. Let

$$Lu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{7.6}$$

where the operator is given by

$$Lu = -\nabla \cdot (A \nabla u)$$

with

$$A(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{i,j=1}^d, \quad a_{ij} \in W^{1,p}(\Omega), p > d, \tag{7.7}$$

It will be assumed that there are two positive real numbers m, M such that

$$m \|\boldsymbol{\xi}\|_2^2 \le \boldsymbol{\xi}^T A(\mathbf{x}) \boldsymbol{\xi} \le M \|\boldsymbol{\xi}\|_2^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d, \mathbf{x} \in \overline{\Omega}. \tag{7.8}$$

From the Sobolev inequality it follows that $a_{ij} \in L^\infty(\Omega)$. With

$$a(u, v) = \int_\Omega (A(\mathbf{x}) \nabla u(\mathbf{x})) \cdot \nabla v(\mathbf{x})\, d\mathbf{x}$$

and the Cauchy–Schwarz inequality, one obtains

$$|a(u, v)| \le \|A\|_{L^\infty(\Omega)} \int_\Omega |\nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})|\, d\mathbf{x} \le c \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

for all $u, v \in H_0^1(\Omega)$. In addition, it follows that

$$m \|\nabla u\|_{L^2(\Omega)}^2 \le a(u,u) \quad \forall\, u \in H_0^1(\Omega).$$

Hence, the bilinear form is bounded and elliptic. Using the Theorem of Lax–Milgram, Theorem 4.5, it follows that there es a unique weak solution $u \in H_0^1(\Omega)$ of (7.6) with

$$a(u,v) = f(v) \quad \forall\, v \in H_0^1(\Omega).$$

$\square$

Remark 7.7 *The finite element problem.* Let $\mathcal{T}^h$ be a regular triangulation consisting of simplices $\{K\}$ such that the vertices of the simplices belong to $\overline{\Omega}$, see Figure 7.1, and define $\overline{\Omega^h} = \cup_{K \in \mathcal{T}^h} K$.

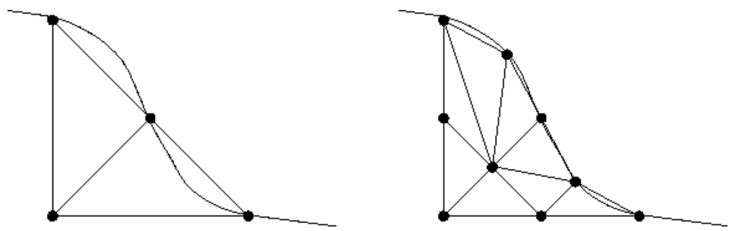

Figure 7.1: Approximation of the boundary by the finite element mesh.

The space of continuous and piecewise linear functions that vanish at the boundary of Ω^h will be denoted by P_1. It will be assumed that for the data of the problem $a_{ij}(\mathbf{x}), f(\mathbf{x})$ there exist extensions $\tilde{a}_{ij}(\mathbf{x}), \tilde{f}(\mathbf{x})$ to a larger domain $\tilde{\Omega} \supset \overline{\Omega^h}$ with

$$\|\tilde{a}_{ij}\|_{W^{1,p}(\tilde{\Omega})} \le c\,\|a_{ij}\|_{W^{1,p}(\Omega)}\,, \quad \left\|\tilde{f}\right\|_{L^2(\tilde{\Omega})} \le c\,\|f\|_{L^2(\Omega)}\,. \tag{7.9}$$

In addition, it will be assumed that the coefficients $\tilde{a}_{ij}(\mathbf{x})$ satisfy the ellipticity condition (7.8) on $\tilde{\Omega}$.

Obviously, $f(\mathbf{x})$ can be continued simply by zero. The extensions of $a_{ij}(\mathbf{x})$ have to be weakly differentiable. It is possible to show that such extensions exist, see the literature.

The finite element method is defined as follows: Find $u^h \in P_1$ with

$$a^h(u^h, v^h) = f^h(v^h) \quad \forall\, v^h \in P_1,$$

where

$$a^h(u^h, v^h) = \int_{\Omega^h} \left(\tilde{A}(\mathbf{x})\nabla u^h(\mathbf{x})\right) \cdot \nabla v^h(\mathbf{x})\; d\mathbf{x}, \quad f^h(v^h) = \int_{\Omega^h} \tilde{f}(\mathbf{x}) v^h(\mathbf{x})\; d\mathbf{x}.$$

In practice, it might be hard to apply the method in this form. From the existence of the extension operators for $a_{ij}(\mathbf{x})$ it is not yet clear how to compute them. On the other hand, in practice often the coefficients $a_{ij}(\mathbf{x})$ are constant or at least piecewise constant. In these case, the extension is trivial. As remedy in the general case, one can use quadrature rules whose nodes are situated within $\overline{\Omega}$, see the literature. $\square$

Remark 7.8 *Goal of the analysis, further assumptions.* The goal consists in proving the linear convergence of the finite element method in $\|\cdot\|_h = \|\cdot\|_{H^1(\Omega^h)}$. In the analysis, one has to pay attention to the fact that in general neither holds $\Omega^h \subset \Omega$ nor $\Omega \subset \Omega^h$. It will be assumed that there is an extension $\tilde{u} \in H^2(\tilde{\Omega})$ of $u(\mathbf{x})$ with

$$\|\tilde{u}\|_{H^2(\tilde{\Omega})} \le c\,\|u\|_{H^2(\Omega)}\,. \tag{7.10}$$

In addition, it will be assumed that Ω^h is a sufficiently good approximation of Ω in the following sense

$$\max_{\mathbf{x}\in\partial\Omega^h} \operatorname{dist}(\mathbf{x},\partial\Omega) \le ch^2. \tag{7.11}$$

One can show that (7.11) is satisfied for $d = 2$ if the boundary of Ω is piecewise C^2 and the corners of Ω are vertices of the triangulation. In this case, one can rotate the coordinate system locally such that the distance between $\partial\Omega$ and $\partial\Omega^h$ can be represented as the error of a one-dimensional interpolation problem with continuous, piecewise linear finite elements. Using error estimates for this kind of problem, e.g., see Goering et al. (2010), one can estimate the error by ch^2. For three-dimensional domains, with piecewise C^∞ boundary, one needs in addition a smoothness assumption for the edges. □

Lemma 7.9 Estimate of a function on the difference of the domains. *Let the condition* (7.11) *be satisfied. Then, for all* $v \in W^{1,1}(\Omega)$ *it holds the estimate*

$$\int_{\Omega_s} |v(\mathbf{x})|\ d\mathbf{x} \le ch^2 \int_\Omega \left(|v(\mathbf{x})| + \|\nabla v(\mathbf{x})\|_2\right)\ d\mathbf{x}, \tag{7.12}$$

where Ω_s *is the set* $\Omega\setminus\Omega^h$ *or* $\Omega^h\setminus\Omega$.

Proof: At the beginning, a one-dimensional estimate will be shown. Let $f \in C^1([0,1])$, then one obtains with the fundamental theorem of calculus

$$f(x) = \int_y^x f'(\xi)\ d\xi + f(y) \quad \forall\, x,y \in [0,1].$$

It follows that

$$|f(x)| \le \int_0^1 \left|f'(\xi)\right|\ d\xi + |f(y)|\,.$$

Integrating this inequality with respect to y in $[0,1]$ and with respect to x in $[0,a]$ with $a \in (0,1]$ yields

$$\int_0^a |f(x)|\ dx \le a\int_0^1 \left|f'(\xi)\right|\ d\xi + a\int_0^1 |f(y)|\ dy = a\int_0^1 \left(|f(x)| + \left|f'(x)\right|\right)\ dx. \tag{7.13}$$

Consider the case $\Omega_s = \Omega\setminus\Omega^h$. Since Ω has a Lipschitz boundary, it can be shown that $\partial\Omega$ can be covered with a finite number of open sets $U_1,\ldots,U_N$. After a rotation of the coordinate system, one can represent $\partial\Omega\cap U_i$ as a Lipschitz continuous function $g_i(\mathbf{y}')$ of $(d-1)$ arguments $\mathbf{y}' = (y_1,\ldots,y_{d-1}) \in U_i' \subset \mathbb{R}^{d-1}$.

In the next step of the proof, sets will be constructed whose union covers the difference $\Omega\setminus\Omega^h$. Let

$$S_{i,\sigma} = \left\{(\mathbf{y}',y_d)\ :\ g_i(\mathbf{y}') - \sigma < y_d < g_i(\mathbf{y}'),\ \mathbf{y}' \in U_i'\right\}, \quad i = 1,\ldots,N,$$

see Figure 7.2. Then, using (7.11) it is $(\Omega\setminus\Omega^h)\cap U_i \subset S_{i,c_1h^2}$, where c_1 depends on $g_i(\mathbf{y}')$ but not on h. In addition, there is a σ_0 such that $S_{i,\sigma_0} \subset \Omega$ for all i.

The transform of (7.13) to the interval $[0,\sigma_0]$ gives for sufficiently small h, such that $c_1h^2 \le 1$,

$$\int_0^{c_1h^2} |f(x)|\ dx \le ch^2 \int_0^{\sigma_0} \left(|f(x)| + \left|f'(x)\right|\right)\ dx.$$

For $v \in C^1(\overline{\Omega})$, one applies this estimate to the rotated function $v(\mathbf{y}',x)$

$$\begin{aligned}
\int_{S_{i,c_1h^2}} |v(\mathbf{y})|\ d\mathbf{y} &= \int_{U_i'}\int_0^{c_1h^2} \left|v(\mathbf{y}',x)\right|\ dx\ d\mathbf{y}' \\
&\le ch^2 \int_{U_i'}\int_0^{\sigma_0} \left(\left|\partial_x v(\mathbf{y}',x)\right| + \left|v(\mathbf{y}',x)\right|\right)\ dx\ d\mathbf{y}' \\
&\le ch^2 \int_\Omega \left(|\partial_{y_d} v(\mathbf{y})| + |v(\mathbf{y})|\right)\ d\mathbf{y},
\end{aligned}$$

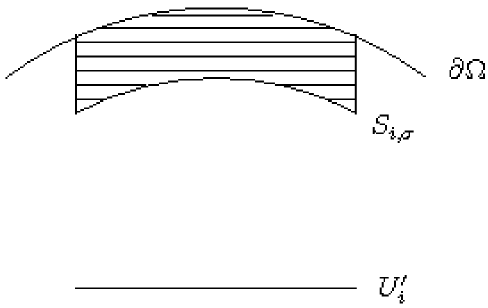

Figure 7.2: $S_{i,\sigma}$.

where in the first step the theorem of Fubini was used. Taking the sum over i proves the lemma for functions from $C^1(\overline{\Omega})$. Since $C^1(\overline{\Omega})$ is dense in $W^{1,1}(\Omega)$, the statement of the lemma holds also for $v \in W^{1,1}(\Omega)$.

The case $\Omega_s = \Omega^h \setminus \Omega$ is proved analogously. ■

Theorem 7.10 Error estimate, linear convergence. *Let the assumptions* (7.7)*,* (7.8)*,* (7.9)*,* (7.10)*, and* (7.11) *be satisfied. Then, it holds the error estimate*

$$\left\| \tilde{u} - u^h \right\|_{H^1(\Omega^h)} \le ch \left\| u \right\|_{H^2(\Omega)},$$

where c does not depend on u, f, and h.

Proof: For proving the error estimate, the abstract error estimate, Theorem 7.3, is used with $S^h = P_1$, $V^h = H^1(\Omega^h)$, $\|\cdot\|_h = \|\cdot\|_{H^1(\Omega^h)}$, and

$$a^h(u,v) = \tilde{a}^h(u,v) = \int_{\Omega^h} \left(\tilde{A}(\mathbf{x}) \nabla u(\mathbf{x}) \right) \cdot \nabla v(\mathbf{x}) \, d\mathbf{x}.$$

With this choice of $a^h(\cdot,\cdot)$ and $\tilde{a}^h(\cdot,\cdot)$, the middle term in the abstract error estimate (7.4) vanishes. Setting in the abstract error estimate $v^h = I_h \tilde{u}$, one obtains with the interpolation error estimate (6.5) and (7.10)

$$\left\| \tilde{u} - I_h \tilde{u} \right\|_{H^1(\Omega^h)} \le ch \left\| D^2 \tilde{u} \right\|_{L^2(\Omega^h)} \le ch \left\| u \right\|_{H^2(\Omega)}. \tag{7.14}$$

It remains to estimate the last term of (7.4).

The regularity and the boundedness of $a^h(\cdot,\cdot)$ can be proved easily using the ellipticity and the boundedness of the coefficients $\tilde a_{ij}(\mathbf{x})$.

The estimate of the last term of (7.4) starts with integration by parts

$$a^h(\tilde u, w^h) = \int_{\Omega^h} \left(\tilde A(\mathbf{x})\nabla \tilde u(\mathbf{x})\right)\cdot \nabla w^h(\mathbf{x})\ d\mathbf{x} = \int_{\Omega^h} g(\mathbf{x}) w^h(\mathbf{x})\ d\mathbf{x}$$

with $g(\mathbf{x}) = -\nabla\cdot(\tilde A \nabla \tilde u)(\mathbf{x})$. Because of $g(\mathbf{x}) = \tilde f(\mathbf{x}) = f(\mathbf{x})$ in Ω it is

$$a^h(\tilde u, w^h) - f^h(w^h) = \int_{\Omega^h\setminus\Omega} \left(g(\mathbf{x}) - \tilde f(\mathbf{x})\right) w^h(\mathbf{x})\ d\mathbf{x}.$$

Using the extension of $w^h(\mathbf{x})$ by zero on $\Omega\setminus\Omega^h$, one obtains with (7.12), and noting that in general $\Omega^h \not\subset \Omega$,

$$\begin{aligned}\int_{\Omega^h\setminus\Omega} \left|w^h(\mathbf{x})\right|^2\ d\mathbf{x} &\le ch^2 \int_{\Omega} \left(\left\|\nabla w^h(\mathbf{x})\right\|_2^2 + \left|w^h(\mathbf{x})\right|^2\right)\ d\mathbf{x}\\ &\le ch^2 \int_{\Omega^h} \left(\left\|\nabla w^h(\mathbf{x})\right\|_2^2 + \left|w^h(\mathbf{x})\right|^2\right)\ d\mathbf{x} = ch^2 \left\|w^h\right\|_{H^1(\Omega^h)}^2 .\end{aligned}$$

Applying the Cauchy–Schwarz inequality and the triangle inequality yields

$$\begin{aligned}\left|a^h(\tilde u, w^h) - f^h(w^h)\right| &\le \left\|g - \tilde f\right\|_{L^2(\Omega^h\setminus\Omega)} \left\|w^h\right\|_{L^2(\Omega^h\setminus\Omega)}\\ &\le ch\left(\|g\|_{L^2(\tilde\Omega)} + \left\|\tilde f\right\|_{L^2(\tilde\Omega)}\right)\left\|w^h\right\|_{H^1(\Omega^h)},\end{aligned}$$

where $\tilde\Omega$ was introduced in Remark 7.7. Now, a bound for $\|g\|_{L^2(\tilde\Omega)}$ is needed. Using the product rule and the triangle inequality, one gets

$$\left\|\nabla\cdot(\tilde A\nabla\tilde u)\right\|_{L^2(\tilde\Omega)} \le \left\|\sum_{i,j=1}^d \tilde a_{ij}\frac{\partial^2 \tilde u}{\partial x_i \partial x_j}\right\|_{L^2(\tilde\Omega)} + \left\|\left(\nabla\cdot\tilde A\right)\cdot\nabla\tilde u\right\|_{L^2(\tilde\Omega)}.$$

Because of the Sobolev imbedding $W^{1,p}(\tilde\Omega) \to L^\infty(\tilde\Omega)$ for $p > d$, Theorem 3.53, it follows that $\left\|\tilde A\right\|_{L^\infty(\tilde\Omega)} \le c$. One obtains for the first term

$$\left\|\sum_{i,j=1}^d \tilde a_{ij}\frac{\partial^2 \tilde u}{\partial x_i \partial x_j}\right\|_{L^2(\tilde\Omega)} \le c\left\|D^2\tilde u\right\|_{L^2(\tilde\Omega)}.$$

The estimate of the second term uses Hölders inequality (*exercise*)

$$\left\|\left(\nabla\cdot\tilde A\right)\cdot\nabla\tilde u\right\|_{L^2(\tilde\Omega)} \le \left\|\nabla\cdot\tilde A\right\|_{L^p(\tilde\Omega)}^2 \|\nabla\tilde u\|_{L^{2p/(p-2)}(\tilde\Omega)}^2 \le c\|\nabla\tilde u\|_{L^{2p/(p-2)}(\tilde\Omega)}^2.$$

Using a Sobolev inequality, e.g., see Adams (1975), one obtains the estimate

$$\|\nabla\tilde u\|_{L^{2p/(p-2)}(\tilde\Omega)} \le c\|\tilde u\|_{H^2(\tilde\Omega)}.$$

Inserting all estimates, one obtains with (7.9) and (7.10)

$$\begin{aligned}\left|a^h(\tilde u, w^h) - f^h(w^h)\right| &\le ch\left(\|\tilde u\|_{H^2(\tilde\Omega)} + \left\|\tilde f\right\|_{L^2(\tilde\Omega)}\right)\left\|w^h\right\|_{H^1(\Omega^h)}\\ &\le ch\left(\|u\|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)}\right)\left\|w^h\right\|_{H^1(\Omega^h)}\\ &\le ch\|u\|_{H^2(\Omega)}\left\|w^h\right\|_{H^1(\Omega^h)}.\end{aligned}$$

In the final step of this estimate, one uses the representation of $f(\mathbf{x})$ from (7.6), for which one can perform estimates that are analog to the estimates of $g(\mathbf{x})$.

The proof of the linear convergence is finished by using (7.4), (7.14), and the last estimate. ■

7.3 Finite Element Method with the Nonconforming Crouzeix–Raviart Element

Remark 7.11 *Assumptions and discrete problem.* The nonconforming Crouzeix–Raviart finite element $P_1^{\rm nc}$ was introduced in Example 5.30. To simplify the presentation, it will be restricted here on the two-dimensional case. In addition, to avoid the estimate of the error coming from approximating the domain, it will be assumed that Ω is a convex domain with polygonal boundary.

Let $\mathcal{T}^h$ be a regular triangulation of Ω with triangles. Let $P_1^{\rm nc}$ (nc – non conforming) be denote the finite element space of piecewise linear functions which are continuous at the midpoints of the edges. This space is nonconforming if it is applied for the discretization of a second order elliptic equation since the continuous problem is given in $H_0^1(\Omega)$ and the functions of $H_0^1(\Omega)$ do not possess jumps. The functions of $P_1^{\rm nc}$ have generally jumps, see Figure 7.3, and they are not weakly differentiable. In addition, the space is also nonconforming with respect to the boundary condition, which is not satisfied exactly. The functions from $P_1^{\rm nc}$ vanish in the midpoint of the edges at the boundary. However, in the other points at the boundary, their value is generally not equal to zero.

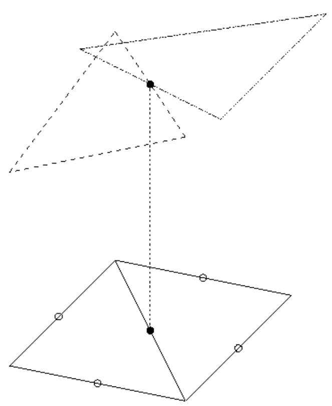

Figure 7.3: Function from $P_1^{\rm nc}$.

The bilinear form

$$a(u,v) = \int_\Omega (A(\mathbf{x})\nabla u(\mathbf{x}))\cdot\nabla v(\mathbf{x})\ d\mathbf{x}$$

will be extended to $H_0^1(\Omega) + P_1^{\rm nc}$ by

$$a^h(u,v) = \sum_{K\in\mathcal{T}^h}\int_K (A(\mathbf{x})\nabla u(\mathbf{x}))\cdot\nabla v(\mathbf{x})\ d\mathbf{x} \quad \forall\ u,v\in H_0^1(\Omega)+P_1^{\rm nc}.$$

Then the nonconforming finite element method is given by: Find $u^h \in P_1^{\rm nc}$ with

$$a^h(u^h,v^h) = (f,v^h) \quad \forall\ v^h\in P_1^{\rm nc}.$$

The goal of this section consists in proving the linear convergence with respect to h in the energy norm $\|\cdot\|_h = \left(a^h(\cdot,\cdot)\right)^{1/2}$. It will be assumed that the solution of the continuous problem (7.6) is smooth, i.e., that $u\in H^2(\Omega)$, that $f\in L^2(\Omega)$, and that the coefficients $a_{ij}(\mathbf{x})$ are weakly differentiable with bounded derivatives. $\square$

Remark 7.12 *The error equation.* The first step of proving an error estimate consists in deriving an equation for the error. To this end, multiply the continuous problem (7.6) with a test function from $v^h \in P_1^{\mathrm{nc}}$, integrate the product on Ω, and apply integration by parts on each triangle. This approach gives

$$\begin{aligned}
(f, v^h) &= -\sum_{K\in\mathcal{T}^h} \int_K \nabla\cdot(A(\mathbf{x})\nabla u(\mathbf{x}))\, v^h(\mathbf{x})\ d\mathbf{x} \\
&= \sum_{K\in\mathcal{T}^h} \int_K (A(\mathbf{x})\nabla u(\mathbf{x}))\cdot\nabla v^h(\mathbf{x})\ d\mathbf{x} \\
&\quad -\sum_{K\in\mathcal{T}^h} \int_{\partial K} (A(s)\nabla u(s))\cdot\mathbf{n}_K(s) v^h(s)\ ds \\
&= a^h(u, v^h) - \sum_{K\in\mathcal{T}^h} \int_{\partial K} (A(s)\nabla u(s))\cdot\mathbf{n}_K(s) v^h(s)\ ds,
\end{aligned}$$

where $\mathbf{n}_K$ is the unit outer normal at the edges of the triangles. Subtracting the finite element equation, one obtains

$$a^h(u-u^h, v^h) = \sum_{K\in\mathcal{T}^h} \int_{\partial K} (A(s)\nabla u(s))\cdot\mathbf{n}_K(s) v^h(s)\ ds \quad \forall\, v^h \in P_1^{\mathrm{nc}}. \tag{7.15}$$

□

Lemma 7.13 Estimate of the right-hand side of the error equation (7.15)**.** *Assume that $u \in H^2(\Omega)$ and $a_{ij} \in W^{1,\infty}(\Omega)$, then it is*

$$\left| \sum_{K\in\mathcal{T}^h} \int_{\partial K} A(s)\nabla u(s)\cdot\mathbf{n}_K(s) v^h(s)\ ds \right| \le ch \left\| u \right\|_{H^2(\Omega)} \left\| v^h \right\|_h .$$

Proof: Every edge of the triangulation which is in Ω appears exactly twice in the boundary integrals on ∂K. The corresponding unit normals possess opposite signs. One can choose for each edge one unit normal and then one can write the integrals in the form

$$\sum_E \int_E \left[\left| (A(s)\nabla u(s))\cdot\mathbf{n}_E(s) v^h(s) \right|\right]_E\ ds = \sum_E \int_E (A(s)\nabla u(s))\cdot\mathbf{n}_E(s) \left[\left| v^h \right|\right]_E (s)\ ds,$$

where the sum is taken over all edges $\{E\}$. Here, $\left[\left| v^h \right|\right]_E$ denotes the jump of v^h

$$\left[\left| v^h \right|\right]_E (s) = \begin{cases} v^h|_{K_1}(s) - v^h|_{K_2}(s) & s \in E \subset \Omega, \\ v^h(s) & s \in E \subset \partial\Omega, \end{cases}$$

where $\mathbf{n}_E$ is directed from K_1 to K_2 or it is the outer normal on $\partial\Omega$. For writing the integrals in this form, it was used that $\nabla u(s)$, $A(s)$, and $\mathbf{n}_E(s)$ are almost everywhere continuous, such that these functions can be written as factor in front of the jumps. Because of the continuity condition for the functions from P_1^{nc} and the homogeneous Dirichlet boundary condition, it is for all $v^h \in P_1^{\mathrm{nc}}$ that $\left[\left| v^h \right|\right]_E (P) = 0$ for the midpoints P of all edges. From the linearity of the functions on the edges, it follows that

$$\int_E \left[\left| v^h \right|\right]_E (s)\ ds = 0 \quad \forall\, E. \tag{7.16}$$

Let E be an arbitrary edge in Ω which belongs to the triangles K_1 and K_2. The next goal consists in proving the estimate

$$\begin{aligned}
&\left| \int_E (A(s)\nabla u(s))\cdot\mathbf{n}_E(s) \left[\left| v^h \right|\right]_E (s)\ ds \right| \\
&\quad \le \ ch \left\| u \right\|_{H^2(K_1)} \left(\left\| \nabla v^h \right\|_{L^2(K_1)} + \left\| \nabla v^h \right\|_{L^2(K_2)} \right).
\end{aligned} \tag{7.17}$$

To this end, one uses a reference configuration $\left(\hat{K}_1, \hat{K}_2, \hat{E}\right)$, where $\hat{K}_1$ is the unit triangle and $\hat{K}_2$ is the triangle which one obtains by reflecting the unit triangle at the y-axis. The common edge $\hat{E}$ is the interval $(0, 1)$ on the y-axis. The unit normal on $\hat{E}$ will be chosen to be the Cartesian unit vector $\mathbf{e}_x$, see Figure 7.4. This reference configuration can be transformed to (K_1, K_2, E) by a map which is continuous and on both triangles $\hat{K}_i$ affine. For this map one, can prove the same properties for the transform as proved in Chapter 6.

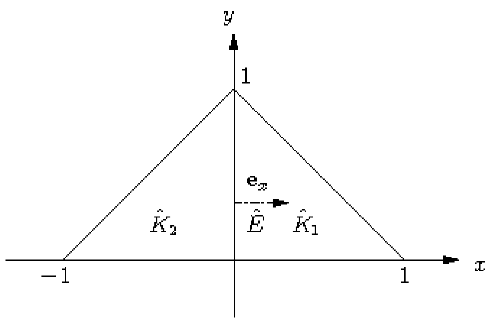

Figure 7.4: Reference configuration.

Using (7.16), the Cauchy–Schwarz inequality, and the trace theorem, one obtains for an arbitrary constant $\alpha \in \mathbb{R}$

$$\begin{aligned} \int_{\hat{E}} \left(\hat{A}(\hat{s})\nabla\hat{u}(\hat{s})\right) \cdot \mathbf{e}_x \left[\left|\hat{v}_1^h\right|\right]_{\hat{E}} \, d\hat{s} &= \int_{\hat{E}} \left(\left(\hat{A}(\hat{s})\nabla\hat{u}(\hat{s})\right) \cdot \mathbf{e}_x - \alpha\right) \left[\left|\hat{v}_1^h\right|\right]_{\hat{E}} \, d\hat{s} \\ &\le c \left\|\left(\hat{A}\nabla\hat{u}\right) \cdot \mathbf{e}_x - \alpha\right\|_{H^1(\hat{K}_1)} \left\|\left[\left|\hat{v}_1^h\right|\right]_{\hat{E}}\right\|_{L^2(\hat{E})} \end{aligned} \tag{7.18}$$

In particular, one can choose α such that

$$\int_{\hat{E}} \left(\left(\hat{A}(\hat{s})\nabla\hat{u}(\hat{s})\right) \cdot \mathbf{e}_x - \alpha\right) \, d\hat{s} = 0.$$

The $L^2(\Omega)$ term in the first factor of the right-hand side of (7.18) can be bounded using the estimate from Lemma 6.4 for $k = 0$ and $l = 1$

$$\begin{aligned} &\left\|\left(\hat{A}\nabla\hat{u}\right) \cdot \mathbf{e}_x - \alpha\right\|_{H^1(\hat{K}_1)} \\ &\quad\le c\left(\left\|\left(\hat{A}\nabla\hat{u}\right) \cdot \mathbf{e}_x - \alpha\right)\right\|_{L^2(\hat{K}_1)} + \left\|\nabla\left(\left(\hat{A}\nabla\hat{u}\right) \cdot \mathbf{e}_x - \alpha\right\|_{L^2(\hat{K}_1)}\right) \\ &\quad\le c\left\|\nabla\left(\left(\hat{A}\nabla\hat{u}\right) \cdot \mathbf{e}_x - \alpha\right)\right\|_{L^2(\hat{K}_1)} \\ &\quad= c\left\|\nabla\left(\left(\hat{A}\nabla\hat{u}\right) \cdot \mathbf{e}_x\right)\right\|_{L^2(\hat{K}_1)}. \end{aligned}$$

To estimate the second factor, in the first step, the trace theorem is applied

$$\begin{aligned} \left\|\left[\left|\hat{v}_1^h\right|\right]_{\hat{E}}\right\|_{L^2(\hat{E})} &\le c\left(\left\|\hat{v}^h\right\|_{H^1(\hat{K}_1)} + \left\|\hat{v}^h\right\|_{H^1(\hat{K}_2)}\right) \\ &\le c\left(\left\|\nabla\hat{v}^h\right\|_{L^2(\hat{K}_1)} + \left\|\nabla\hat{v}^h\right\|_{L^2(\hat{K}_2)}\right). \end{aligned}$$

The second estimate follows from the equivalence of all norms in finite dimensional spaces. To apply this argument, one has to prove that the terms in the last line are in fact norms. Let the terms in the last line be zero, then it follows that $\hat{v}^h = c_1$ in $\hat{K}_1$ and $\hat{v}^h = c_2$ in $\hat{K}_2$. Because $\hat{v}^h$ is continuous in the midpoint of $\hat{E}$, one finds that $c_1 = c_2$ and consequently that $\left[\left|\hat{v}^h\right|\right]_{\hat{E}} = 0$. Hence, also the left hand side of the estimate is zero. It follows that the right-hand side of this estimate defines a norm in the quotient space of P_1^{nc} with respect to $\left[\left|\hat{v}^h\right|\right]_{\hat{E}} = 0$.

Altogether, one obtains for the reference configuration

$$
\begin{aligned}
&\left| \int_{\hat{E}} \left(\hat{A}(\hat{s})\nabla u(\hat{s})\right)\cdot \mathbf{e}_x \left[\left|\hat{v}_1^h\right|\right]_{\hat{E}} \ d\hat{s}\right| \\
&\quad \le \ c\left\|\nabla\left(\left(\hat{A}\nabla u\right)\cdot \mathbf{e}_x\right)\right\|_{L^2(\hat{K}_1)} \left(\left\|\nabla \hat{v}^h\right\|_{L^2(\hat{K}_1)} + \left\|\nabla \hat{v}^h\right\|_{L^2(\hat{K}_2)}\right).
\end{aligned}
$$

This estimate has to be transformed to the triple (K_1, K_2, E). In this step, one gets for the integral on the edge the factor c (ch for ∇ and ch^{-1} for $d\hat{s}$). For the product of the norms on the right-hand side, one obtains the factor ch (ch for the first factor and c for the second factor). In addition, one uses that $A(s)$ and all first order derivatives of $A(s)$ are bounded to estimated the first term on the right-hand side (*exercise*). In summary, (7.17) is proved.

The statement of the lemma follows by summing over all edges and by applying on the right-hand side the Cauchy–Schwarz inequality. ■

Theorem 7.14 Finite element error estimate. *Let the assumptions of Lemma 7.13 be satisfied, then it holds the following error estimate*

$$
\left\|u - u^h\right\|_h^2 \le ch \left\|u\right\|_{H^2(\Omega)} \left\|u - u^h\right\|_h + ch^2 \left\|u\right\|_{H^2(\Omega)}^2 .
$$

Proof: Applying Lemma 7.13, it follows from the error equation (7.15) that

$$
\left|a^h(u - u^h, v^h)\right| \le ch \left\|u\right\|_{H^2(\Omega)} \left\|v^h\right\|_h \quad \forall\, v^h \in P_1^{\text{nc}}.
$$

Let $I_h : H_0^1(\Omega) \to P_1^{\text{nc}}$ be an interpolation operator with optimal interpolation order in $\|\cdot\|_h$. Then, one obtains with the Cauchy–Schwarz inequality, the triangle inequality, and the interpolation estimate

$$
\begin{aligned}
\left\|u - u^h\right\|_h^2 &= a^h(u - u^h, u - u^h) = a^h(u - u^h, u - I_h u) + a^h(u - u^h, I_h u - u^h) \\
&\le \left|a^h(u - u^h, u - I_h u)\right| + ch \left\|u\right\|_{H^2(\Omega)} \left\|I_h u - u^h\right\|_h \\
&\le \left\|u - u^h\right\|_h \left\|u - I_h u\right\|_h + ch \left\|u\right\|_{H^2(\Omega)} \left(\left\|I_h u - u\right\|_h + \left\|u - u^h\right\|_h\right) \\
&\le ch \left\|u - u^h\right\|_h \left\|u\right\|_{H^2(\Omega)} + ch \left\|u\right\|_{H^2(\Omega)} \left(h \left\|u\right\|_{H^2(\Omega)} + \left\|u - u^h\right\|_h\right).
\end{aligned}
$$

■

Remark 7.15 *To the error estimate.* If h is sufficiently small, than the second term of the error estimate is of higher order and this term can be absorbed into the constant of the first term. One obtains the asymptotic error estimate

$$
\left\|u - u^h\right\|_h \le ch \left\|u\right\|_{H^2(\Omega)} .
$$

□

7.4 $L^2(\Omega)$ Error Estimates

Remark 7.16 *Motivation.* A method is called quasi-optimal in a given norm, if the order of the method is the same as the optimal approximation order. Already for one dimension, one can show that at most linear convergence in $H^1(\Omega)$ can be achieved for the best approximation in P_1. This statement can be already verified with the function $v(x) = x^2$. Hence, all considered methods so far are quasi-optimal in the energy norm.

However, the best approximation error in $L^2(\Omega)$ is of one order higher than the best approximation error in $H^1(\Omega)$. A natural question is if finite element methods converge also of higher order with respect to the error in $L^2(\Omega)$ than with respect to the error in the energy norm.

In this section it will be shown that one can obtain for finite element methods a higher order of convergence in $L^2(\Omega)$ than in $H^1(\Omega)$. However, there are more restrictive assumptions to prove this property in comparison with the convergence prove for the energy norm. □

Remark 7.17 *Model problem.* Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a convex polyhedral domain with Lipschitz boundary. The model problem has the form

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{7.19}$$

For proving an error estimate in $L^2(\Omega)$, the regularity of the solution of (7.19) plays an essential role. □

Definition 7.18 m-regular differential operator. Let L be a second order differential operator. This operator is called m-regular, $m \geq 2$, if for all $f \in H^{m-2}(\Omega)$ the solutions of $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$, are in the space $H^m(\Omega)$ and the following estimate holds

$$\|u\|_{H^m(\Omega)} \leq c\,\|f\|_{H^{m-2}(\Omega)} + c\,\|u\|_{H^1(\Omega)}\,. \tag{7.20}$$

□

Remark 7.19 *On the m-regularity.*

- The definition is formulated in a way that it can be applied also if the solution of the problem is not unique.
- For the Laplacian, the term $\|u\|_{H^1(\Omega)}$ can be estimated by $\|f\|_{L^2(\Omega)}$ such that with (7.20) one obtains (*exercise*)
$$\|u\|_{H^2(\Omega)} \leq c\,\|f\|_{L^2(\Omega)}\,.$$
- Many regularity results can be found in the literature. Loosely speaking, they say that regularity is given if the data of the problem (coefficients of the operator, boundary of the domain) are sufficiently regular. For instance, an elliptic operator in divergence form ($\Delta = \nabla \cdot \nabla$) is 2-regular if the coefficients are from $W^{1,p}(\Omega)$, $p \geq 1$, and if $\partial\Omega$ is a C^2 boundary. Another important result is the 2-regularity of the Laplacian on a convex domain. A comprehensive overview on regularity results can be found in Grisvard (1985).

□

Remark 7.20 *Variational form and finite element formulation of the model problem.* The variational form of (7.19) is: Find $u \in H_0^1(\Omega)$ with

$$(\nabla u, \nabla v) = (f, v) \quad \forall\, v \in H_0^1(\Omega).$$

The P_1 finite element space, with zero boundary conditions, will be used for the discretization. Then, the finite element problem reads as follows: Find $u^h \in P_1$ such that

$$(\nabla u^h, \nabla v^h) = (f, v^h) \quad \forall\, v^h \in P_1. \tag{7.21}$$

□

Theorem 7.21 Finite element error estimates. *Let $u(\mathbf{x})$ be the solution of (7.19), let (7.19) be 2-regular, and let $u^h(\mathbf{x})$ be the solution of (7.21). Then, the following error estimates hold*

$$\begin{aligned} \left\|\nabla(u - u^h)\right\|_{L^2(\Omega)} &\leq ch\,\|f\|_{L^2(\Omega)}\,, \\ \left\|u - u^h\right\|_{L^2(\Omega)} &\leq ch^2\,\|f\|_{L^2(\Omega)}\,. \end{aligned}$$

Proof: With the error estimate in $H^1(\Omega)$, Corollary 6.16, and the 2-regularity, one obtains

$$\left\|\nabla(u-u^h)\right\|_{L^2(\Omega)} \le ch\left\|u\right\|_{H^2(\Omega)} \le ch\left\|f\right\|_{L^2(\Omega)}.$$

For proving the $L^2(\Omega)$ error estimate, let $w \in H_0^1(\Omega)$ be the unique solution of the so-called dual problem

$$(\nabla v, \nabla w) = (u-u^h, v) \quad \forall\, v \in H_0^1(\Omega).$$

For a symmetric differential operator, the dual problem has the same form like the original (primal) problem. Hence, the dual problem is also 2-regular and it holds the estimate

$$\left\|w\right\|_{H^2(\Omega)} \le c\left\|u-u^h\right\|_{L^2(\Omega)}.$$

For performing the error estimate, the Galerkin orthogonality of the error is utilized

$$(\nabla(u-u^h), \nabla v^h) = (\nabla u, \nabla v^h) - (\nabla u^h, \nabla v^h) = (f, v^h) - (f, v^h) = 0$$

for all $v^h \in P_1$. Now, the error $u-u^h$ is used as test function v in the dual problem. Let $I_h w$ be the interpolant of w in P_1. Using the Galerkin orthogonality, the interpolation estimate, and the regularity of w, one obtains

$$\begin{aligned}
\left\|u-u^h\right\|_{L^2(\Omega)}^2 &= (\nabla(u-u^h), \nabla w) = (\nabla(u-u^h), \nabla(w-I_h w)) \\
&\le \left\|\nabla(u-u^h)\right\|_{L^2(\Omega)} \left\|\nabla(w-I_h w)\right\|_{L^2(\Omega)} \\
&\le ch\left\|w\right\|_{H^2(\Omega)} \left\|\nabla(u-u^h)\right\|_{L^2(\Omega)} \\
&\le ch\left\|u-u^h\right\|_{L^2(\Omega)} \left\|\nabla(u-u^h)\right\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, division by $\left\|u-u^h\right\|_{L^2(\Omega)}$ and the application of the already known error estimate for $\left\|\nabla(u-u^h)\right\|_{L^2(\Omega)}$ are used for completing the proof of the theorem. ■

Chapter 8

Outlook

Remark 8.1 *Outlook to forthcoming classes.* This class provided an introduction to numerical methods for solving partial differential equations and the numerical analysis of these methods. There are many further aspects that will be covered in forthcoming classes.

Further aspects for elliptic problems.

- Adaptive methods and a posteriori error estimators. It will be shown how it is possible to estimate the error of the computed solution only using known quantities and in which ways one can decide where it makes sense to refine the mesh and where not. (Numerical Mathematics IV)
- Multigrid methods. Multigrid methods are for certain classes of problems optimal solvers. (probably Numerical Mathematics IV)
- Numerical analysis of problems with other boundary conditions or taking into account quadrature rules.

Time-dependent problems. As mentioned in Remark 1.7, standard approaches for the numerical solution of time-dependent problems are based on solving stationary problems in each discrete time.

- The numerical analysis of discretizations of time-dependent problems has some new aspects, but also many tools from the analysis of steady-state problems are used. (Numerical Mathematics IV)

Convection-diffusion equations. Convection-diffusion equations are of importance in many applications. Generally, the convection (first order differential operator) dominates the diffusion (second order differential operator).

- In the convection-dominated regime, the Galerkin method as presented in this class does not work. One needs new ideas for discretizations and these new discretizations create new challenges for the numerical analysis. (Numerical Mathematics IV)

Problems with more than one unknown function. The fundamental equation of fluid dynamics, the Navier–Stokes equations, Section 1.3, belong to this class.

- It will turn out that the discretization of the Navier–Stokes equations requires special care in the choice of the finite element spaces. The numerical analysis becomes rather involved. (special class)

$\square$