all t in some interval $[t_0, t_0 + a]$, $0 < a \le T$,

$$y_m(t) = y_{0m} + \int_{t_0}^{t} f_m(s, y_1(s), \ldots, y_n(s))\, ds, \quad m = 1, \ldots, n. \tag{A.33}$$

At each point where the term in the integral is continuous, the functions satisfy the ordinary differential equation

$$\frac{d}{dt} y_m(t) = f_m(t, y_1, \ldots, y_n), \quad m = 1, \ldots, n. \tag{A.34}$$

If in addition, for any two points $(t, \overline{y}_1, \ldots, \overline{y}_n), (t, \hat{y}_1, \ldots, \hat{y}_n) \in \Omega_T$ the Lipschitz condition

$$|f_m(t, \overline{y}_1, \ldots, \overline{y}_n) - f_m(t, \hat{y}_1(t), \ldots, \hat{y}_n(t))| \le G(t) \sum_{l=1}^{n} |\overline{y}_l - \hat{y}_l(t)|,$$

$m = 1, \ldots, n$, with a Lebesgue integrable function $G(t)$ is satisfied, then there exists exactly one solution of (A.33) *in $[t_0, t_0 + a]$.*

Remark A.67. On Carathéodory's theorem. The theorem of Carathéodory is an extension of the famous theorem of Peano to ordinary differential equations of type (A.34) with discontinuous right-hand side. □

Remark A.68. On Gronwall's lemma. Gronwall's lemma is an important tool for the analysis and finite element analysis of time-dependent problems. Two versions of this lemma in the continuous setting will be given below, see Emmrich (1999) for proofs and a discussion of the differences of these versions. discrete Gronwall □

Lemma A.69. Gronwall's lemma in integral form. *Let $T \in \mathbb{R}^+ \cup \infty$, $f, g, \in L^\infty(0, T)$, and $\lambda \in L^1(0, T)$, $\lambda(t) \ge 0$ for almost all $t \in [0, T]$. Then*

$$f(t) \le g(t) + \int_0^t \lambda(s) f(s)\, ds \quad a.e.\ in\ [0, T] \tag{A.35}$$

implies for almost all $t \in [0, T]$ that

$$f(t) \le g(t) + \int_0^t \exp\left(\int_s^t \lambda(\tau)\, d\tau\right) \lambda(s) g(s)\, ds. \tag{A.36}$$

If $g \in W^{1,1}(0, T)$, it follows that

$$f(t) \le \exp\left(\int_0^t \lambda(\tau)\, d\tau\right) \left(g(0) + \int_0^t \exp\left(-\int_0^s \lambda(\tau)\, d\tau\right)\right) g'(s)\, ds.$$

Moreover, if $g(t)$ is a monotonically increasing continuous function, it holds

$$f\,(t) \le \exp\left(\int_0^t \lambda\,(\tau)\;\;d\tau\right) g\,(t)\,. \tag{A.37}$$

Proof. Let

$$\tilde{f}(t) = \exp\left(-\int_0^t \lambda\,(\tau)\,d\tau\right) \int_0^t \lambda(s) f(s)\;ds,$$

then one obtains for almost all $t \in [0,T]$ with the product rule, the Leibniz integral rule, (A.35), and $\lambda(t) \ge 0$

$$\begin{aligned}\tilde{f}'(t) &= \exp\left(-\int_0^t \lambda\,(\tau)\,d\tau\right)\left(-\lambda(t)\int_0^t \lambda(s)f(s)\;ds + \lambda(t)f(t)\right)\\ &\le \exp\left(-\int_0^t \lambda\,(\tau)\,d\tau\right)\left(\lambda(t)\left(g(t)-f(t)\right)+\lambda(t)f(t)\right)\\ &= \exp\left(-\int_0^t \lambda\,(\tau)\,d\tau\right)\lambda(t)g(t).\end{aligned}$$

Integration yields, using $\tilde{f}(0) = 0$,

$$\tilde{f}(t) \le \int_0^t \exp\left(-\int_0^s \lambda\,(\tau)\,d\tau\right)\lambda(s)g(s)\;ds.$$

With (A.35), one obtains

$$\begin{aligned}\exp\left(-\int_0^t \lambda\,(\tau)\,d\tau\right)(f(t)-g(t)) &\le \exp\left(-\int_0^t \lambda\,(\tau)\,d\tau\right)\int_0^t \lambda\,(s)\,f\,(s)\;\;ds\\ &= \tilde{f}(t) \le \int_0^t \exp\left(-\int_0^s \lambda\,(\tau)\,d\tau\right)\lambda(s)g(s)\;ds.\end{aligned}$$

Multiplying this inequality with $\exp\left(\int_0^t \lambda\,(\tau)\,d\tau\right)$ and putting $g(t)$ to the right-hand side gives (A.36).

second estimate

If $g(t)$ is a monotonically increasing continuous function, one gets from (A.36), using that $g(t)$ takes its largest value at the final time and that $\lambda(t) \ge 0$, and applying the fundamental theorem of calculus

$$\begin{aligned}f(t) &\le g(t)\left(1 + \int_0^t \exp\left(\int_s^t \lambda\,(\tau)\;\;d\tau\right)\lambda\,(s)\;\;ds\right)\\ &= g(t)\left(1+\exp\left(\int_0^t \lambda\,(\tau)\;\;d\tau\right)\int_0^t \exp\left(-\int_0^s \lambda\,(\tau)\;\;d\tau\right)\lambda\,(s)\;\;ds\right)\\ &= g(t)\left(1+\exp\left(\int_0^t \lambda\,(\tau)\;\;d\tau\right)\int_0^t \frac{d}{ds}\left(-\exp\left(-\int_0^s \lambda\,(\tau)\;\;d\tau\right)\right)\;ds\right)\\ &= g(t)\left(1+\exp\left(\int_0^t \lambda\,(\tau)\;\;d\tau\right)\left(-\exp\left(-\int_0^t \lambda\,(\tau)\;\;d\tau\right)+1\right)\right)\\ &= \exp\left(\int_0^t \lambda\,(\tau)\;\;d\tau\right)g(t).\end{aligned}$$

■

Lemma A.70. Gronwall's lemma in differential form. *Let* $T \in \mathbb{R}^+ \cup \infty$*,* $f \in W^{1,1}\,(0,T)$ *and* $g, \lambda \in L^1\,(0,T)$*. Then*

$$f'(t) \leq g(t) + \lambda(t) f(t) \quad a.e.\ in\ [0,T] \tag{A.38}$$

implies for almost all $t \in [0,T]$

$$f(t) \leq \exp\left(\int_0^t \lambda(\tau)\, d\tau\right) f(0) + \int_0^t \exp\left(\int_s^t \lambda(\tau)\, d\tau\right) g(s)\, ds. \tag{A.39}$$

Proof. Defining

$$\tilde{f}(t) = \exp\left(-\int_0^t \lambda(\tau)\, d\tau\right) f(t) = \exp(-\Lambda(t)) f(t), \tag{A.40}$$

applying the chain rule, the Leibniz integral rule, and (A.38) gives

$$\begin{aligned}\tilde{f}'(t) &= -\Lambda'(t)\exp(-\Lambda(t))f(t) + \exp(-\Lambda(t))f'(t) = \exp(-\Lambda(t))\left(f'(t) - \lambda(t)f(t)\right)\\ &\leq \exp(-\Lambda(t))g(t).\end{aligned}$$

Integration in $(0,t)$ and using (A.40) yields

$$\tilde{f}(t) - \tilde{f}(0) = \exp\left(-\int_0^t \lambda(\tau)\, d\tau\right) f(t) - f(0) \leq \int_0^t \exp(-\Lambda(s))g(s)\ ds.$$

Multiplication with $\exp\left(\int_0^t \lambda(\tau)\, d\tau\right)$ gives (A.39). ■

Lemma A.71. Variation of Gronwall's lemma in differential form. *Let* $T \in \mathbb{R}^+ \cup \infty$, $f \in W^{1,1}(0,T)$ *and* $h, g, \lambda \in L^1(0,T)$ *and* $h(t), \lambda(t) \geq 0$ *a.e. in* $(0,T)$. *Then,*

$$f'(t) + h(t) \leq g(t) + \lambda(t) f(t) \quad a.e.\ in\ [0,T] \tag{A.41}$$

implies for almost all $t \in [0,T]$

$$\begin{aligned} &f(t) + \int_0^t h(s)\ ds \\ &\leq \exp\left(\int_0^t \lambda(\tau)\ d\tau\right) f(0) + \int_0^t \exp\left(\int_s^t \lambda(\tau)\ d\tau\right) g(s)\ ds. \end{aligned} \tag{A.42}$$

Moreover, if $g(t) \geq 0$ *a.e. in* $(0,T)$, *it holds*

$$f(t) + \int_0^t h(s)\ ds \leq \exp\left(\int_0^t \lambda(\tau)\ d\tau\right)\left(f(0) + \int_0^t g(s)\ ds\right). \tag{A.43}$$

Proof. From (A.41) it follows a.e. in $[0,T]$ that

$$f'(s) - \lambda(s) f(s) + h(s) \leq g(s).$$

The positivity of the exponential implies