Appendix A
Functional Analysis

A.1 Metric Spaces, Banach Spaces, and Hilbert Spaces

Definition A.1. Metric space. Let $X \neq \emptyset$ be a set. A map $d : X \times X \to \mathbb{R}$ is called metric on X if for all $x, y, z \in X$ it is
i) $d(x, y) = 0 \iff x = y$,
ii) symmetry: $d(x, y) = d(y, x)$,
iii) triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$.
Then (X, d) is called a metric space. □

Definition A.2. Isometric metric space. Two metric spaces (X_1, d_1) and $X_2, d_2)$ are called isometric, if there is a surjective map $g : X_1 \to X_2$ such that for all $x, y \in X_1$ it is $d_1(x, y) = d_2(g(x), g(y))$. □

Definition A.3. Cauchy sequence, convergent sequence. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a metric space (X, d). It is called a Cauchy sequence if for each $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that

$$d(x_k, x_l) < \varepsilon \quad \forall\ k, l \geq N.$$

The sequence $\{x_n\}_{n=1}^{\infty}$ is said to convergence to $x \in X$, denoted by $x_n \to x$, if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

□

Definition A.4. Complete metric space. A metric space (X, d) is called complete, if each Cauchy sequence converges in X. That means, for each Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ there exists an element $x \in X$ such that $x_n \to x$. □

Definition A.5. Norm, triangle inequality, seminorm, normed space. Let X be a linear space over $\mathbb{R}$ (or $\mathbb{C}$). A mapping $\|\cdot\|_X : X \to \mathbb{R}$ is called a norm on X if

i) definiteness: $\|x\|_X = 0$ if and only if $x = 0$,
ii) homogeneity: $\|\alpha x\|_X = |\alpha| \, \|x\|_X$ for all $x \in X$, $\alpha \in \mathbb{R}$,
iii) the triangle inequality holds: $\|x + y\|_X \le \|x\|_X + \|y\|_X$ for all $x, y \in X$.

A mapping from X to $\mathbb{R}$ which satisfies only ii) and iii) is called a seminorm on X.

The space $(X, \|\cdot\|_X)$ is called normed space. □

Definition A.6. Equivalent norms. Two norms $\|\cdot\|_{X,1}$, $\|\cdot\|_{X,2}$ of a normed space X are called equivalent, if there are two positive constants C_1 and C_2 such that

$$C_1 \|x\|_{X,1} \le \|x\|_{X,2} \le C_2 \|x\|_{X,1} \quad \forall\, x \in X.$$

□

Remark A.7. Equivalent norms in finite dimensional spaces. It is well known that in finite dimensional spaces all norms are equivalent. □

Lemma A.8. A normed space is a metric space. *A normed space $(X, \|\cdot\|_X)$ becomes a metric space with the induced metric*

$$d(x_1, x_2) = \|x_1 - x_2\|_X \,, \quad x_1, x_2 \in X.$$

Definition A.9. Banach space. A normed space is called complete if it is a complete metric space with the induced metric. A complete normed space is called Banach space. □

Definition A.10. Inner product, scalar product. Let X be a linear space over $\mathbb{R}$. A map $(\cdot, \cdot)_X \; : \; X \times X \to \mathbb{R}$ is called symmetric sesquilinear form if for all $x, y, z \in X$ and all $\alpha \in \mathbb{R}$ it holds that
i) symmetry: $(x, y)_X = (y, x)_X$,
ii) $(\alpha x, y)_X = \alpha(x, y)$,
iii) $(x, y + z) = (x, y) + (x, z)$.

The symmetric sesquilinear form $(\cdot, \cdot)_X$ is called positive semidefinite if for all $x \in X$ it is $(x, x)_X \ge 0$. A positive semidefinite symmetric sesquilinear form with

$$(x, x)_X = 0 \quad \Longleftrightarrow \quad x = 0$$

is called inner product or scalar product on X. □

Definition A.11. Induced norm, inner product space, Hilbert space. Let $(\cdot, \cdot)_X$ be an inner product on X, then $(X, (\cdot, \cdot)_X$ is called pre Hilbert space. The inner product induces the norm

$$\|x\|_X = (x, x)_X^{1/2}$$

at the inner product space or pre-Hilbert space X. A complete inner product space is called Hilbert space.

For simplicity of notation, the subscript at the inner product symbol will be neglected if the inner product is clear from the context. □

Lemma A.12. Cauchy–Schwarz inequality. *Let $(X, (\cdot,\cdot))$ be an inner product space, then it holds the so-called Cauchy–Schwarz inequality*

$$|(x,y)| \le \|x\|_X \|y\|_X \quad \forall\, x,y \in X. \tag{A.1}$$

Example A.13. Cauchy–Schwarz inequality for sums. Consider $X = \mathbb{R}^n$ with the standard inner product for vectors, then one obtains with the triangle inequality and the Cauchy–Schwarz inequality (A.1)

$$\left|\sum_{i=1}^n x_i y_i\right| \le \sum_{i=1}^n |x_i|\,|y_i| \le \left(\sum_{i=1}^n x_i^2\right)^{1/2} \left(\sum_{i=1}^n y_i^2\right)^{1/2}, \tag{A.2}$$

for all $\boldsymbol{x} = (x_1,\ldots,x_n)^T, \boldsymbol{y} = (y_1,\ldots,y_n)^T \in \mathbb{R}^n$. □

Example A.14. Hölder inequality for sums. The Cauchy–Schwarz inequality (A.2) is a special case of the Hölder inequality

$$\sum_{i=1}^n |a_i b_i| \le \left(\sum_{i=1}^n |a_i|^p\right)^{1/p} \left(\sum_{i=1}^n |b_i|^q\right)^{1/q}, \quad 1 < p,q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1. \tag{A.3}$$

□

Definition A.15. Orthogonal elements, orthogonal complement of a subspace. Let X be a normed space endowed with an inner product $(\cdot,\cdot)$. Two elements $x, y \in X$ are said to be orthogonal if $(x,y) = 0$.

Let $Y \subset X$ be a subspace of X, then $Y^\perp = \{x \in X \,:\, (x,y) = 0 \text{ for all } y \in Y\}$ is the orthogonal complement of subspace to Y. □

Lemma A.16. Orthogonal complement is closed subspace. *Let $W \subset V$ be a subspace of a Hilbert space V. Then, $W^\perp$ is a closed subspace of V.*

Lemma A.17. Inequalities for real numbers. *Let $a, b \in \mathbb{R}$, then the following inequality is called Young's inequality:*

$$ab \le \frac{t}{p}a^p + \frac{t^{-q/p}}{q}b^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p,q < \infty, \quad t > 0. \tag{A.4}$$

The following inequalities for sums of non-negative real numbers hold:

$$\sum_{i=1}^n a_i \le \left(\sum_{i=1}^n a_i^{1/p}\right)^p \le n^{p/q} \sum_{i=1}^n a_i, \quad a_i \ge 0,\ p \in (1,\infty),\ \frac{1}{p} + \frac{1}{q} = 1. \tag{A.5}$$

The second inequality is just a consequence from (A.3).

Lemma A.18. Estimate for a Rayleigh quotient. *Let $A \in \mathbb{R}^{m\times n}$ be a matrix, then*

$$\inf_{\boldsymbol{x}\in\mathbb{R}^n, \boldsymbol{x}\neq\boldsymbol{0}} \frac{\boldsymbol{x}^T A^T A \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} = \lambda_{\min}\left(A^T A\right),$$

where $\lambda_{\min}\left(A^T A\right)$ *is the smallest eigenvalue of* $A^T A$. *The infimum is taken, i.e., it is even a minimum. The quotient on the left-hand side is called Rayleigh quotient.*

Proof. The matrix $A^T A$ is symmetric and positive semi-definite. Hence, all eigenvalues are non-negative and the (normalized eigenvectors) $\{\boldsymbol{\phi}_i\}_{i=1}^n$, form a basis of $\mathbb{R}^n$ and they are mutually orthonormal. Let the eigenvalues be ordered such that

$$0 \le \lambda_{\min}\left(A^T A\right) = \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n.$$

Each vector $\boldsymbol{x} \in \mathbb{R}^n$ can be written in the form $\boldsymbol{x} = \sum_{i=1}^n x_i \boldsymbol{\phi}_i$. Using that the eigenvectors are orthonormal, it follows that

$$\boldsymbol{x}^T \boldsymbol{x} = \sum_{i=1}^n x_i^2$$

and

$$\boldsymbol{x}^T A^T A \boldsymbol{x} = \boldsymbol{x}^T \sum_{i=1}^n x_i A^T A \boldsymbol{\phi}_i = \sum_{j=1}^n \sum_{i=1}^n \lambda_i x_j x_i \boldsymbol{\phi}_j \boldsymbol{\phi}_i = \sum_{i=1}^n \lambda_i x_i^2 \ge \lambda_{\min}\left(A^T A\right) \sum_{i=1}^n x_i^2.$$

Hence, one gets

$$\inf_{\boldsymbol{x}\in\mathbb{R}^n, \boldsymbol{x}\neq\boldsymbol{0}} \frac{\boldsymbol{x}^T A^T A \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} \ge \lambda_{\min}\left(A^T A\right).$$

Choosing $\boldsymbol{x} = x_1 \boldsymbol{\phi}_1$ leads to

$$\frac{\boldsymbol{x}^T A^T A \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} = \frac{\lambda_1 x_1^2}{x_1^2} = \lambda_1 = \lambda_{\min}\left(A^T A\right),$$

such that the equal sign holds. ■

A.2 Function Spaces

Remark A.19. Motivation. The study of the existence and uniqueness of solutions of the Navier–Stokes equations as well as the finite element error analysis requires tools from functional analysis, in particular the use of function spaces, certain inequalities, and imbedding theorems. There will be no difference in the notation for functions spaces for scalar, vector-valued, and tensor-valued functions.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a domain. That means, Ω is an open set. □

A.2.1 Continuous Functions and Functions with Classical Derivatives

Definition A.20. Derivatives and multi-index. A multi-index $\boldsymbol{\alpha}$ is a vector

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$$

with $\alpha_i \in \mathbb{N} \cup \{0\}$, $i = 1, \ldots, n$. Derivatives are denoted by

$$D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$

where

$$|\boldsymbol{\alpha}| = \sum_{i=1}^{n} \alpha_i.$$

Low order derivatives are also denoted by subscripts, e.g.,

$$\partial_x u = \frac{\partial u}{\partial x}.$$

□

Definition A.21. Spaces of continuously differentiable functions $C^m(\Omega)$, $C^m(\overline{\Omega})$, and $C_B^m(\Omega)$. Let $m \in \mathbb{N} \cup \{0\}$, then the space of m-times continuously differentiable functions in Ω is denoted by

$$C^m(\Omega) = \big\{ f \ : \ f \text{ and all its derivatives up to order } m \text{ are continuous in } \Omega \big\}.$$

It is

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega).$$

The space $C^m(\overline{\Omega})$ for $m < \infty$ is defined by

$$C^m(\overline{\Omega}) = \big\{ f \ : \ f \in C^m(\Omega) \text{ and all derivatives can be extended continuously to } \overline{\Omega} \big\}.$$

One defines

$$C^\infty(\overline{\Omega}) = \bigcap_{m=0}^{\infty} C^m(\overline{\Omega}).$$

Finally, the space

$$C_B^m(\Omega) = \big\{ f \ : \ f \in C^m(\Omega) \text{ and } f \text{ is bounded} \big\}.$$

is introduced. □

Remark A.22. Spaces of continuously differentiable functions $C^m(\Omega)$, $C^m(\overline{\Omega})$, and $C_B^m(\Omega)$.

- If Ω is bounded, then $C^m(\overline{\Omega})$, equipped with the norm

$$\|f\|_{C^m(\overline{\Omega})} = \sum_{0\le|\boldsymbol{\alpha}|\le m} \max_{\boldsymbol{x}\in\overline{\Omega}} |D^{\boldsymbol{\alpha}} f(\boldsymbol{x})| ,$$

  is a Banach space.
- The space $C_B^m(\Omega)$ becomes a Banach space with the norm

$$\|f\|_{C_B^m(\Omega)} = \max_{0\le|\boldsymbol{\alpha}|\le m} \sup_{\boldsymbol{x}\in\Omega} |D^{\boldsymbol{\alpha}} f(\boldsymbol{x})| .$$

- It is

$$C^m(\overline{\Omega}) \subset C_B^m(\Omega) \subset C^m(\Omega).$$

  Consider, e.g., $\Omega = (0,1)$ and $f(x) = \sin(1/x)$, then $f \in C_B(\Omega)$ but $f \notin C(\overline{\Omega})$.

□

Definition A.23. Support. Let $f \in C(\Omega)$, then

$$\operatorname{supp}(f) = \overline{\{\boldsymbol{x} \ : \ f(\boldsymbol{x}) \neq 0\}}$$

is the support of $f(\boldsymbol{x})$. The closure is taken with respect to $\mathbb{R}^d$. A function $f \in C(\Omega)$ is said to have a compact support, if the support of $f(\boldsymbol{x})$ is bounded in $\mathbb{R}^d$ and if $\operatorname{supp}(f) \subset \Omega$. □

Definition A.24. The space $C_0^m(\Omega)$. The space $C_0^m(\Omega)$ is given by

$$C_0^m(\Omega) = \{f \ : f \in C^m(\Omega) \text{ and } \operatorname{supp}(f) \text{ is compact in } \Omega\}.$$

In the literature, this space $C_0^\infty(\Omega)$ is often denoted by $\mathcal{D}(\Omega)$. □

Definition A.25. The spaces $C^{m,\alpha}(\overline{\Omega})$, spaces of Hölder continuous functions. Let $M \in \mathbb{R}^d$, $d \in \{2,3\}$, be a set and let $\alpha \in (0,1]$. Then, the constant

$$|f|_{C^{0,\alpha}(M)} = \sup_{\boldsymbol{x}\neq\boldsymbol{y}\in M} \left\{ \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\alpha} \right\}.$$

is called Hölder coefficient or Hölder constant. For $\alpha = 1$, it is usually called Lipschitz constant.

Let Ω be bounded. For $m \in \mathbb{N}\cup\{0\}$, then the following spaces are defined

$$C^{m,\alpha}(\overline{\Omega}) = \left\{ f \in C^m(\overline{\Omega}) \ : \ \left|D^{\boldsymbol{\beta}} f\right|_{C^{0,\alpha}(\overline{\Omega})} < \infty, |\boldsymbol{\beta}| = m \right\}.$$

For $m = 0$, these spaces are called spaces of Hölder continuous functions, and for $\alpha = 1$, space of Lipschitz continuous functions. □

Remark A.26. The spaces $C^{m,\alpha}(\overline{\Omega})$. The spaces $C^{m,\alpha}(\overline{\Omega})$ are Banach spaces if they are equipped with the norm

$$\|f\|_{C^{m,\alpha}(\overline{\Omega})} = \|f\|_{C^m(\overline{\Omega})} + \sum_{|\boldsymbol{\beta}|=m} [D^{\boldsymbol{\beta}} f]_{C^{0,\alpha}(\overline{\Omega})}.$$

□

A.2.2 Lebesgue and Sobolev Spaces

Definition A.27. Spaces of (Lebesgue) integrable functions $L^p(\Omega)$. The Lebesgue spaces are defined by

$$L^p(\Omega) = \left\{ f \ : \ \int_\Omega |f(\boldsymbol{x})|^p \, d\boldsymbol{x} < \infty \right\}, \quad p \in [1, \infty),$$

where the integral is to be understood in the sense of Lebesgue. The space $L^\infty(\Omega)$ is the space of all functions which are bounded for almost all $\boldsymbol{x} \in \Omega$

$$L^\infty(\Omega) = \{ f \ : \ |f(\boldsymbol{x})| < \infty \text{ for almost all } \boldsymbol{x} \in \Omega \}.$$

□

Remark A.28. Lebesgue spaces.

- The space $L^p(\Omega)$ is a normed vector space with norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(\boldsymbol{x})|^p \, d\boldsymbol{x} \right)^{1/p}, \quad p \in [1, \infty).$$

- An important special case is $L^2(\Omega)$ since this space is a Hilbert space. The inner product $(f, g)_{L^2(\Omega)}$ of $L^2(\Omega)$ and the induced norm are given by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(\boldsymbol{x}) g(\boldsymbol{x}) \, d\boldsymbol{x}, \quad \|f\|_{L^2(\Omega)} = (f, f)^{1/2}_{L^2(\Omega)}.$$

- For functions from $L^\infty(\Omega)$, there may be some points where the value of $|f(\boldsymbol{x})|$ is infinity but the measure of the set of these points has to be zero, which is given, e.g., if this set consists only a finite number of points. Th space $L^\infty(\Omega)$ becomes a Banach space if it is equipped with the norm

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\boldsymbol{x} \in \Omega} |f(\boldsymbol{x})|,$$

  where $\operatorname{ess\,sup}_{\boldsymbol{x} \in \Omega}$ is the essential supremum.

□

Example A.29. Cauchy–Schwarz inequality, Hölder inequality. Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $p, q \in [1, \infty]$ and $1/p + 1/q = 1$. Then it is $fg \in L^1(\Omega)$ and the Hölder inequality holds

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}. \tag{A.6}$$

For $p = q = 2$, this inequality is called Cauchy–Schwarz inequality

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \tag{A.7}$$

□

Definition A.30. Sobolev spaces $W^{k,p}(\Omega)$. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ consists of all integrable functions $f : \Omega \to \mathbb{R}$ such that for each multi-index $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \leq k$, the derivative $D^{\boldsymbol{\alpha}} f$ exists in the weak sense and it belongs to $L^p(\Omega)$. □

Remark A.31. Sobolev spaces.

- It is $L^p(\Omega) = W^{0,p}(\Omega)$.
- A norm in Sobolev spaces is defined by

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\boldsymbol{\alpha}| \leq k} \|D^{\boldsymbol{\alpha}} f\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } p \in [1, \infty), \\ \sum_{|\boldsymbol{\alpha}| \leq k} \operatorname{ess\,sup}_{\boldsymbol{x} \in \Omega} |D^{\boldsymbol{\alpha}} f| & \text{if } p = \infty. \end{cases}$$

  Sobolev spaces equipped with this norm are Banach spaces, e.g., see (Evans, 2010, p. 262).
- The Sobolev spaces for $p = 2$ are Hilbert spaces. They are often denoted by $W^{m,2}(\Omega) = H^m(\Omega)$ and they are equipped with the inner product

$$(f, g)_{H^k(\Omega)} = \sum_{|\boldsymbol{\alpha}| \leq k} (D^{\boldsymbol{\alpha}} f, D^{\boldsymbol{\alpha}} g)_{L^2(\Omega)}.$$

- In particular, the Sobolev spaces of first order are important for the study of the Navier–Stokes equations

$$W^{1,p}(\Omega) = \left\{ f \ : \ \int_\Omega |f(\boldsymbol{x})|^p + |\nabla f(\boldsymbol{x})|^p \ d\boldsymbol{x} < \infty \right\}, \quad p \in [1, \infty),$$

  which are equipped with the norm

$$\|f\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |f(\boldsymbol{x})|^p + |\nabla f(\boldsymbol{x})|^p \ d\boldsymbol{x} \right)^{1/p}, \quad p \in [1, \infty).$$

- The definition of Sobolev spaces can be extended to $k \in \mathbb{R}$, e.g., see Adams (1975).

□

Definition A.32. Sobolev spaces $W_0^{k,p}(\Omega)$. The Sobolev spaces $W_0^{k,p}(\Omega)$ are defined by the closure of $C_0^\infty(\Omega)$ in the norm of $W^{k,p}(\Omega)$. □

Remark A.33. On the smoothness of the boundary. The Sobolev imbedding theorem in Adams Adams (1975) requires that Ω has the so-called cone property or the strong local Lipschitz property. In the case that Ω is bounded, these assumptions reduce to the requirement that Ω has a locally Lipschitz boundary, (Adams, 1975, p. 67). That means, each point $\boldsymbol{x}$ in the boundary $\partial\Omega$ of Ω has a neighborhood $U_{\boldsymbol{x}}$ such the $\partial\Omega \cap U_{\boldsymbol{x}}$ is the graph of a Lipschitz continuous function. □

Theorem A.34. Trace theorem, *(Evans, 2010, p. 272). geht mit Lipschitz boundary, anderer Verweis. Let Ω be a bounded domain with C^1 boundary $\partial\Omega$. Then, there is a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that*

i) $Tf = f|_{\partial\Omega}$ if $f \in W^{1,p}(\Omega) \cap C(\bar(\Omega)$,
ii) $\|Tf\|_{L^p(\partial\Omega)} \le C \|f\|_{W^{1,p}(\Omega)}$ for each $f \in W^{1,p}(\Omega)$, with the constant C depending only on p and Ω.

Theorem A.35. Functions with vanishing trace, *(Evans, 2010, p. 273). Let the assumptions of Theorem A.34 be given. Then $f \in W_0^{1,p}(\Omega)$ if and only if $Tf = 0$ on $\partial\Omega$.*

Theorem A.36. Poincaré's inequality, Poincaré–Friedrichs' inequality, *(Gilbarg and Trudinger, 1983, p. 164). Let $f \in W_0^{1,p}(\Omega)$, then*

$$\|f\|_{L^p(\Omega)} \le \left(\frac{|\Omega|}{\omega_d}\right)^{1/d} \|\nabla f\|_{L^p(\Omega)} \quad p \in [1,\infty), \tag{A.8}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Remark A.37. Poincaré's inequality. Poincaré's inequality (A.8) holds also for functions $v \in H^1(\Omega)$ with $v = 0$ on $\Gamma_0 \subset \Gamma$ with $\mathrm{meas}(\Gamma_0) > 0$.

Poincaré's inequality stays valid for vector-valued functions $\boldsymbol{v}$ if Ω is bounded with a locally Lipschitz boundary, $\boldsymbol{v} \in W^{1,q}(\Omega)$, $1 \le q < \infty$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, see (Galdi, 1994, Section II.4).

there are some results on the constants, see Payne, Weinberger (1960), Laugesen, Siudeja (2010) □

Theorem A.38. Density of continuous functions in Sobolev spaces, *(Gilbarg and Trudinger, 1983, p. 154). The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Remark A.39. Density of continuous functions in Sobolev spaces. For $C^\infty(\overline{\Omega})$ to be dense in $W^{k,p}(\Omega)$, one needs some smoothness assumptions on the boundary $\partial\Omega$, e.g., $\partial\Omega$ is C^1 or the so-called segment property, Gilbarg and Trudinger (1983). This segment property follows from the strong local Lipschitz property, see (Adams, 1975, p. 67). □

Theorem A.40. Interpolation theorem for Sobolev spaces, *(Adams, 1975, Theorem 4.17). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a locally Lipschitz boundary and let $p \in [1,\infty)$. Then there exists a constant $C(m,p,\Omega)$ such that for $0 \le j \le m$ and any $u \in W^{m,p}(\Omega)$*

$$\|u\|_{W^{j,p}(\Omega)} \le C(m,p,\Omega)\, \|u\|_{W^{m,p}(\Omega)}^{j/m} \|u\|_{L^p(\Omega)}^{(m-j)/m}. \tag{A.9}$$

In addition, (A.9) *is valid for all $u \in W_0^{m,p}(\Omega)$ with a constant $C(m,p,d)$ independent of Ω.*

Remark A.41. Imbedding theorems. Imbedding theorems for Sobolev spaces are used frequently in the analysis of partial differential equations. They can be found, e.g., in the book Adams (1975). The imbedding theorems state that all functions belonging to a certain space do belong also to another space and that the norm of the functions in the larger space can be estimated by the norm in the smaller space. Let V be a Banach space such that an imbedding $W^{m,p}(\Omega) \to V$ holds. Then, there is a constant C depending on Ω such that

$$\|v\|_V \le C\, \|v\|_{W^{m,p}(\Omega)}$$

for all functions $v \in W^{m,p}(\Omega)$. The validity of imbeddings depends on the dimension d of the domain Ω. The larger the dimension, the less imbeddings are valid.

The presentation of the Sobolev imbedding theorem follows Adams (Adams, 1975, Theorem 5.4, Remark 5.5. (6)). Here, only the case of bounded domains will be presented. □

Theorem A.42. The Sobolev imbedding theorem. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a locally Lipschitz boundary. Let j and m be nonnegative integers and let p satisfy $1 \le p < \infty$.*

i) Let $mp < d$, then the imbedding

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega), \quad 1 \le q \le \frac{dp}{d-mp} \tag{A.10}$$

holds. In particular, it is

$$W^{m,p}(\Omega) \to L^q(\Omega), \quad 1 \le q \le \frac{dp}{d-mp}. \tag{A.11}$$

ii) Suppose $mp = d$. Then the imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega), \quad 1 \le q < \infty \tag{A.12}$$

is valid. If in addition $p = 1$, then this imbedding holds also for $q = \infty$

$$W^{d,1}(\Omega) \to L^\infty(\Omega)$$

and even

$$W^{d,1}(\Omega) \to C_B(\Omega).$$

iii) Suppose that $mp > d$, *then the imbedding*

$$W^{m,p}(\Omega) \to C_B(\Omega)$$

holds.

iv) Suppose $mp > d > (m-1)p$, *then*

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}\left(\overline{\Omega}\right) \quad \text{for} \quad 0 < \lambda \le m - \frac{d}{p}. \tag{A.13}$$

v) Suppose $d = (m-1)p$, *then*

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}\left(\overline{\Omega}\right) \quad \text{for} \quad 0 < \lambda < 1.$$

This imbedding holds for $\lambda = 1$ *if* $p = 1$ *and* $d = m - 1$.

vi) All imbeddings are true for arbitrary domains provided the W *spaces undergoing the imbedding are replaced with the corresponding* W_0 *spaces.*

Example A.43. Important Sobolev imbeddings. Let $d = 2$. Then it follows from (A.12) that

$$H^1(\Omega) = W^{1,2}(\Omega) \to L^q(\Omega), \quad q \in [1, \infty). \tag{A.14}$$

For $d = 3$, one gets with (A.11) that

$$H^1(\Omega) = W^{1,2}(\Omega) \to L^q(\Omega), \quad q \in [1, 6]. \tag{A.15}$$

□

A.2.3 Other Spaces

Remark A.44. Spaces of divergence-free functions. These spaces are denoted by the subscript div, e.g.

$$L^2_{\text{div}}(\Omega) = \left\{ \boldsymbol{f} \ : \ \boldsymbol{f} \in L^2(\Omega) \text{ and } \nabla \cdot \boldsymbol{f} = 0 \text{ in the sense of distributions} \right\}. \tag{A.16}$$

□

Remark A.45.

$$H(\text{div}, \Omega) = \left\{ \boldsymbol{v} \in L^2(\Omega) \ : \ \nabla \cdot \boldsymbol{v} \in L^2(\Omega) \right\} \tag{A.17}$$

$$H(\textbf{curl}, \Omega) = \left\{ v \in L^2(\Omega) \ : \ \textbf{curl}\, v \in L^2(\Omega) \right\} \tag{A.18}$$

$$H(\textbf{curl}, \Omega) = \left\{ \boldsymbol{v} \in L^2(\Omega) \ : \ \nabla \times \boldsymbol{v} \in L^2(\Omega) \right\} \tag{A.19}$$

□

Remark A.46. Spaces of functions defined in space-time domains. Let X be any normed space introduced above which is equipped with the norm $\|\cdot\|_X$ and let (t_0, t_1) be a time interval. Then

$$L^p(t_0, t_1; X) = \left\{ f(t, \boldsymbol{x}) \ : \ \int_{t_0}^{t_1} \|f\|_X^p(\tau) \ d\tau < \infty \right\}, \quad p \in [1, \infty).$$

The norm of $L^p(t_0, t_1; X)$ is

$$\|f\|_{L^p(t_0,t_1;X)} = \left(\int_{t_0}^{1t} \|f\|_X^p(\tau) \ d\tau \right)^{1/p}, \quad p \in [1, \infty).$$

The modifications for $p = \infty$ are the same as for the Lebesgue spaces. □

Important spaces for the study of the Navier–Stokes equations are

$$\begin{aligned} C_0^\infty(\Omega) &= \{f \ : \ f \in C^\infty(\Omega), f \text{ has compact support}\}, \\ C_{0,\mathrm{div}}^\infty(\Omega) &= \{\boldsymbol{f} \ : \ \boldsymbol{f} \in C_0^\infty(\Omega), \nabla \cdot \boldsymbol{f} = 0\}. \end{aligned} \tag{A.20}$$

A.3 Bounded Linear Operators in Banach Spaces

Definition A.47. *Linear operator, range, kernel.* Let X and Y be real Banach spaces. A mapping $A \ : \ X \to Y$ is a linear operator if

$$A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2 \quad \forall \ x_1, x_2 \in X, \alpha, \beta \in \mathbb{R}.$$

The range or image of A is given by

$$\mathrm{range}(A) = \{y \in Y \ : \ y = Ax \text{ for some } x \in X\}.$$

The kernel or the null space of A is defined by

$$\ker(A) = \{x \in X \ : \ Ax = 0\}.$$

Definition A.48. *Bounded operator, continuous operator.* An operator $A \ : X \to Y$, X, Y Banach spaces, is bounded if

$$\|A\| = \sup_{x \in X} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{x \in X, \|x\|_X \le 1} \|Ax\|_Y = \sup_{x \in X, \|x\|_X = 1} \|Ax\|_Y < \infty. \tag{A.21}$$

The operator A is continuous in $x_0 \in X$ if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$ with $\|x - x_0\|_X < \delta$ if follows that $\|Ax - Ax_0\|_Y < \varepsilon$. The operator A is called a continuous operator if A is continuous for all $x \in X$.

Remark A.49. Equivalent definition of a continuous operator. The operator $A : X \to Y$, X, Y Banach spaces, is continuous in $x_0 \in X$ if and only if for all sequences $\{x_n\}_{n=1}^{\infty}$, $x_n \in X$, with $x_n \to x_0$ it holds that $Ax_n \to Ax_0$ in Y. □

Lemma A.50. Properties of bounded linear operators. *Let X, Y be Banach spaces.*

i) A bounded linear operator $A : X \to Y$ is continuous.
ii) A continuous linear operator $A : X \to Y$ is bounded.
iii) The set

$$\mathcal{L}(X,Y) = \{A : A \text{ is a bounded linear operator from } X \text{ to } Y\}$$

is a Banach space endowed with the norm (A.21).

Proof. See (Kolmogorov and Fomin, 1975, §4.5.2, §4.5.3). □

Definition A.51. *Linear functional.* A (real) linear functional f defined on a Banach space X is a linear operator with range$(f) \subset \mathbb{R}$.

A.4 Functionals and Bilinear Forms

Definition A.52. Bounded bilinear form, coercive bilinear form, V-elliptic bilinear form. Let $b(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bilinear form on the Banach space V. Then it is bounded if

$$|b(u,v)| \le M \|u\|_V \|v\|_V \quad \forall\, u,v \in V, M > 0, \tag{A.22}$$

where the constant M is independent of u and v. The bilinear form is coercive or V-elliptic if

$$b(u,u) \ge m \|u\|_V^2 \quad \forall\, u \in V, m > 0, \tag{A.23}$$

where the constant m is independent of u. □

Remark A.53. Application to an inner product. Let V be a Hilbert space. Then the inner product $a(\cdot,\cdot)$ is a bounded and coercive bilinear form, since by the Cauchy–Schwarz inequality

$$|a(u,v)| \le \|u\|_V \|v\|_V \quad \forall\, u,v \in V,$$

and obviously $a(u,u) = \|u\|_V^2$. Hence, the constants can be chosen to be $M = 1$ and $m = 1$. □

Theorem A.54. Theorem of Banach on the inverse operator, *(Kolmogorov and Fomin, 1975, p. 225). Let* $A : X \to Y$ *be a bounded linear operator which defines a one-to-one mapping between the Banach spaces* X *and* Y*. Then, the inverse operator* A^{-1} *is bounded.*

Theorem A.55. Closed Range Theorem of Banach, **?**. *Let* X, Y *be Banach spaces and let* $A : X \to Y$ *be a bounded linear operator and let* $A' : Y' \to X'$ *be its dual. Then, the following statements are equivalent:*

i) $\mathrm{range}(A)$ *is closed,*
ii) $\mathrm{range}(A) = \{y \in Y : \langle y', y\rangle_{Y',Y} = 0 \text{ for all } y' \in \ker(A')\}$,
iii) $\mathrm{range}(A')$ *is closed,*
iv) $\mathrm{range}(A') = \{x' \in X' : \langle x', x\rangle_{X',X} = 0 \text{ for all } x \in \ker(A)\}$.

Theorem A.56. Hahn–Banach Theorem, *(Yosida, 1980, IV,1), (***?***, p. 67). Let* X *be a Banach space, let* Y *be a subspace of* X*, and let* f *be a bounded linear functional defined on* Y*. Then, there exists an extension* g *of* f *to* X*, where* g *is a linear functional with the same norm as* f*.*