Chapter 4
The Oseen Equations

Remark 4.1. Motivation. Oseen equations, which are linear equations, show up as an auxiliary problem in many numerical approaches for solving the Navier–Stokes equations. Applying an implicit method for the temporal discretization of the Navier–Stokes equations requires the solution of of a nonlinear problem in each discrete time. Likewise, the steady-state Navier–Stokes equations are nonlinear. Applying in either situation a so-called Picard method (a fixed point iteration) for solving the nonlinear problem, leads to an Oseen problem in each iteration. The application of semi-implicit time discretizations to the Navier–Stokes equations leads directly to an Oseen problem in each discrete time. Altogether, Oseen problems have to be solved in many methods for solving the Navier–Stokes equations numerically. In addition, some parts of the theory of the Oseen equations are used in the analysis of the Navier–Stokes equations, e.g., for the uniqueness of a weak solution of the steady-state Navier–Stokes equations in Theorem 5.15. For these reasons, the analysis and numerical analysis of Oseen problems is of fundamental interest.

In addition to the Stokes equations, the Oseen equations possess a convective term and a reactive term. Both of them might be dominant. Thus, besides the inf-sup condition, the second difficulty mentioned in Remark 1.19 has to be addressed. □

4.1 The Continuous Equations

Remark 4.2. The Oseen equations. The Oseen equations are linear and stationary equations. In comparison with the Stokes equations (3.1), they have a convective term (first order derivative of the velocity) and they might possess also a so-called reactive term (zeroth order derivative of the velocity) in the momentum equation. The Oseen equations have the form

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \text{ in } \Omega, \end{aligned} \tag{4.1}$$

where $\nu > 0$, $\boldsymbol{b}$ is a given convection field which has to be in some sense divergence-free, see Remark 4.6 for details, $c(\boldsymbol{x}) \geq c_0 \geq 0$ for almost all $\boldsymbol{x} \in \Omega$, and $\boldsymbol{f} \in H^{-1}(\Omega)$.

For the application of the abstract theory from Chapter 2 and for the finite element error analysis, the Oseen equations will be equipped for simplicity with homogeneous Dirichlet boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ on Γ. □

Remark 4.3. On the coefficients in (4.1).

- If the functions in (4.1) are considered with dimensions, then all terms in the momentum equation have the unit m/s^2. Hence, the function c must have the unit $1/\mathrm{s}$, i.e., if $c(t, \boldsymbol{x}) > 0$, then $c^{-1}(t, \boldsymbol{x})$ represents a time, see also the discussion below in this remark.
- Let $\boldsymbol{b} \in L^\infty(\Omega)$, then it will be always assumed in this chapter that the momentum equation is scaled such that one of the following situations is given:
  - $\|\boldsymbol{b}\|_{L^\infty(\Omega)} = \mathcal{O}(1)$ if $\nu \leq \|\boldsymbol{b}\|_{L^\infty(\Omega)}$,
  - $\nu = \mathcal{O}(1)$ if $\|\boldsymbol{b}\|_{L^\infty(\Omega)} \leq \nu$.

  The first situation is the interesting one which appears in applications.
- The following cases for the other coefficients of (4.1) are of interest:
  - *ν is of moderate size, $c = 0$.* The case $c = 0$ occurs in numerical methods for solving the steady-state Navier–Stokes equations, e.g., if a fixed point iteration is applied. From the point of view of applications, the limit $\nu \to 0$ is not of interest for the steady-state Navier–Stokes equations since for very small ν (or very large Reynolds numbers) one expects generally a time-dependent solution.
  - *ν is of arbitrary size, $c = \mathcal{O}\left((\Delta t)^{-1}\right)$.* For small ν, one expects a time-dependent solution, for very small ν even a turbulent solution. In time-dependent problems, a term of type $c\boldsymbol{u}$ comes from an implicit or semi-implicit temporal discretization. Then, $c = \mathcal{O}\left((\Delta t)^{-1}\right)$, where Δt is the length of the time step. Thus, for small time steps, $c\boldsymbol{u}$ might become a dominant term in (4.1). Altogether, this case is of interest for a wide range of ν and a wide range of c.
- In numerical methods where Oseen problems appear, $\boldsymbol{b}$ is a computed velocity field, often the currently available finite element approximation of the solution. Since generally $V_{\mathrm{div}}^h \not\subset V_{\mathrm{div}}$, this finite element approximation of the velocity is in general not weakly divergence-free. It is discretely divergence-free, if the system from which this approximation was computed was solved exactly. But even an exact solving is often not performed such that one can expect only that the computed velocity field for $\boldsymbol{b}$ is only approximately discretely divergence-free.

□

Remark 4.4. The weak form of the Oseen equations. For the weak formulation of the Oseen problem (4.1) with homogeneous Dirichlet boundary conditions, the same functions spaces are used as for the weak form of the Stokes problem. Let $V = H_0^1(\Omega)$ and $Q = L_0^2(\Omega)$. Then, the weak form of (4.1) reads as follows: Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V',V} \ \forall\, \boldsymbol{v} \in V, \\ -(\nabla \cdot \boldsymbol{u}, q) &= 0 \qquad\qquad \forall\, q \in Q. \end{aligned} \tag{4.2}$$

To cast (4.2) into the abstract framework of Section 2.1, the following bilinear forms are defined:

$$\begin{aligned} a \ &: \ V \times V \to \mathbb{R}, \quad a(\boldsymbol{u}, \boldsymbol{v}) = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}), \\ b \ &: \ V \times Q \to \mathbb{R}, \quad\ \ b(\boldsymbol{v}, q) = -(\nabla \cdot \boldsymbol{v}, q). \end{aligned} \tag{4.3}$$

The right-hand side of the second equation of the abstract problem (2.4) is $r = 0$. □

Remark 4.5. Minimal regularity of $\boldsymbol{b}$ *and* c *for* (4.2) *to be well posed.* Consider first the two-dimensional case. From the Sobolev imbedding (A.16) follows that $\boldsymbol{u}, \boldsymbol{v} \in L^q(\Omega)$ with $q \in [1, \infty)$. Hence, it is sufficient that $\boldsymbol{b} \in L^{2+\varepsilon_0}(\Omega)$ and $c \in L^{1+\varepsilon_1}(\Omega)$, $\varepsilon_0, \varepsilon_1 > 0$, such that all terms in (4.2) are well defined.

In three dimensions, the Sobolev imbedding (A.17) gives $\boldsymbol{u}, \boldsymbol{v} \in L^6(\Omega)$ such that $\boldsymbol{b} \in L^3(\Omega)$ and $c \in L^{3/2}(\Omega)$ have to be satisfied.

In addition to the regularity assumptions, the properties on $\boldsymbol{b}$ and c given in Remark 4.2 have to be satisfied. □

Remark 4.6. Property of the convective term. The key property of the convective term which will be used in the analysis is

$$((\boldsymbol{b} \cdot \nabla)\boldsymbol{v}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in V. \tag{4.4}$$

This property is given, e.g., if $\boldsymbol{b}$ satisfies the regularity assumptions from Remark 4.5, $\nabla \cdot \boldsymbol{b} \in L^2(\Omega)$, and $\nabla \cdot \boldsymbol{b} = 0$ almost everywhere in Ω, since integration by parts and the application of the product rule yields

$$((\boldsymbol{b} \cdot \nabla)\boldsymbol{v}, \boldsymbol{v}) = -(\nabla \cdot \boldsymbol{b}, \boldsymbol{v}^T \boldsymbol{v}) - ((\boldsymbol{b} \cdot \nabla)\boldsymbol{v}, \boldsymbol{v}). \tag{4.5}$$

The integral on Γ vanishes because $\boldsymbol{v} = \boldsymbol{0}$ on Γ. Since $\nabla \cdot \boldsymbol{b} = 0$, one obtains (4.4).

Property (4.4) also holds if $\boldsymbol{b}$ and $\nabla \cdot \boldsymbol{b}$ satisfy the same regularity assumptions as in the previous case, $\boldsymbol{b}$ is weakly divergence-free, and

$$\int_\Gamma (\boldsymbol{b} \cdot \boldsymbol{n})\,(\boldsymbol{s})\ d\boldsymbol{s} = 0.$$

For all $\boldsymbol{v} \in V$, it follows from the Sobolev embedding (A.12) that $\boldsymbol{v} \in L^4(\Omega)$. That means

$$\int_{\Omega} \|\boldsymbol{v}\|_2^4 \, d\boldsymbol{x} = \int_{\Omega} \left(\boldsymbol{v}^T\boldsymbol{v}\right)^2 \, d\boldsymbol{x} < \infty$$

and therefore that $\boldsymbol{v}^T\boldsymbol{v} \in L^2(\Omega)$. Then, there is a constant C such that $\boldsymbol{v}^T\boldsymbol{v} + C \in Q$. If $\boldsymbol{b}$ is weakly divergence-free, it holds

$$\begin{aligned} 0 &= \left(\nabla \cdot \boldsymbol{b}, \boldsymbol{v}^T\boldsymbol{v} + C\right) = \left(\nabla \cdot \boldsymbol{b}, \boldsymbol{v}^T\boldsymbol{v}\right) + (\nabla \cdot \boldsymbol{b}, C) \\ &= \left(\nabla \cdot \boldsymbol{b}, \boldsymbol{v}^T\boldsymbol{v}\right) - C \int_{\Gamma} \boldsymbol{b} \cdot \boldsymbol{n} \, d\boldsymbol{s}, \end{aligned}$$

where in the last step integration by parts was applied. If the boundary integral vanishes, then one gets that (4.4) follows from (4.5). A special case of this situation is that $\boldsymbol{b} \in V_{\text{div}}$.

Under the same conditions on $\boldsymbol{b}$ as in the previous cases and with the same arguments, one finds that

$$((\boldsymbol{b} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w}) = -((\boldsymbol{b} \cdot \nabla)\boldsymbol{w}, \boldsymbol{v}) \quad \forall \, \boldsymbol{v}, \boldsymbol{w} \in V. \tag{4.6}$$

□

Theorem 4.7. Existence and uniqueness of a weak solution of the Oseen equations. *Let Ω be a bounded domain in $\mathbb{R}^d$ with a Lipschitz continuous boundary Γ and let the conditions on the data of the Oseen problem from Remarks 4.5 and 4.6 be fulfilled. Then, there exists a unique solution $(\boldsymbol{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ of* (4.2).

Proof. One has to check the conditions on $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ which were stated in Lemma 2.19

With respect to the bilinear form $b(\cdot,\cdot)$, one has the same situation as for the Stokes problem, see Theorem 3.5. Hence, Theorem 2.40 states the fulfillment of the inf-sup condition.

For the bilinear form $a(\cdot,\cdot)$, one has to prove that $a(\cdot,\cdot)$ is coercive in V_{div}. Since (4.4), it follows that

$$a(\boldsymbol{v}, \boldsymbol{v}) = \nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{v}) + (c\boldsymbol{v}, \boldsymbol{v}) \geq \nu \|\boldsymbol{v}\|_V^2,$$

because of $c \geq 0$, thus $a(\cdot,\cdot)$ is even coercive in V. Hence, the coercivity of $a(\cdot,\cdot)$ is proved and the application of Lemma 2.19 gives the statement of the theorem. ■

Lemma 4.8. Stability of the solution. *Under the conditions of Theorem 4.7, the solution of the Oseen problem* (4.2) *depends continuously on the data of the problem*

$$\frac{\nu}{2} \|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2 \leq \frac{1}{2\nu} \|\boldsymbol{f}\|_{H^{-1}(\Omega)}^2 \tag{4.7}$$

and if additionally $\boldsymbol{b}, c \in L^\infty(\Omega)$ then

$$\|p\|_{L^2(\Omega)} \le \frac{1}{\beta_{\text{is}}}\left[\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \left(\nu^{1/2} + C_{\text{PF}}\frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{\nu^{1/2}} + C_{\text{PF}}\|c\|_{L^\infty(\Omega)}^{1/2}\right)\right.$$
$$\left.\times\left(\nu^{1/2}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)} + \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}\right)\right]. \tag{4.8}$$

If $\boldsymbol{f} \in L^2(\Omega)$ *and* $c_0 > 0$*, then also the stability bounds*

$$\nu\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2 \le \frac{1}{2c_0}\|\boldsymbol{f}\|_{L_2(\Omega)}^2 \tag{4.9}$$

and if in addtion $\boldsymbol{b}, c \in L^\infty(\Omega)$ *then*

$$\|p\|_{L^2(\Omega)} \le \frac{1}{\beta_{\text{is}}}\left[C_{\text{PF}}\|\boldsymbol{f}\|_{L^2(\Omega)} + \left(\nu^{1/2} + \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{c_0^{1/2}} + C_{\text{PF}}\|c\|_{L^\infty(\Omega)}^{1/2}\right)\right.$$
$$\left.\times\left(\nu^{1/2}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)} + \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}\right)\right] \tag{4.10}$$

hold. Here, C_{PF} *is the constant of the Poincaré–Friedrichs inequality* (A.9).

Proof. The proof follows in principle the proof of Lemma 3.6. To highlight the dependency of the stability estimates on the coefficients of the problem, it will be presented in detail.

Inserting $\boldsymbol{u} \in V$ as test function in (4.2), using (4.4), applying the estimate for the dual pairing, and using Young's inequality (A.4) leads to

$$\begin{aligned}\nu\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2 &\le \|\boldsymbol{f}\|_{H^{-1}(\Omega)}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)} \\ &\le \frac{1}{2\nu}\|\boldsymbol{f}\|_{H^{-1}(\Omega)}^2 + \frac{\nu}{2}\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2.\end{aligned}$$

This inequality gives (4.7).

If $\boldsymbol{f} \in L^2(\Omega)$ and $c(\boldsymbol{x}) \ge c_0 > 0$, then the velocity estimate can be performed by applying the Cauchy–Schwarz inequality (A.8)

$$\begin{aligned}\nu\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2 &\le \|\boldsymbol{f}\|_{L_2(\Omega)}\|\boldsymbol{u}\|_{L^2(\Omega)} \le \|\boldsymbol{f}\|_{L_2(\Omega)}\left\|\left(\frac{c}{c_0}\right)^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)} \\ &= \frac{1}{c_0^{1/2}}\|\boldsymbol{f}\|_{L_2(\Omega)}\left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)} \\ &\le \frac{1}{2c_0}\|\boldsymbol{f}\|_{L_2(\Omega)}^2 + \frac{1}{2}\left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}^2.\end{aligned}$$

Now (4.9) follows.

The stability estimate for the pressure starts with the inf-sup condition. Then, equation (4.2) is inserted, (4.6) is applied, the estimate for the dual pairing, the Cauchy–Schwarz inequality, and then the Poincaré–Friedrichs inequality are applied to obtain

$$
\begin{aligned}
\|p\|_{L^2(\Omega)} &\le \frac{1}{\beta_{\mathrm{is}}} \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{-(\nabla\cdot\boldsymbol{v},p)}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}} \\
&= \frac{1}{\beta_{\mathrm{is}}} \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{\langle\boldsymbol{f},\boldsymbol{v}\rangle_{V',V} - \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + ((\boldsymbol{b}\cdot\nabla)\boldsymbol{v},\boldsymbol{u}) - (c\boldsymbol{u},\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}} \\
&\le \frac{1}{\beta_{\mathrm{is}}}\Bigg(\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \nu\,\|\nabla\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{b}\|_{L^\infty(\Omega)}\,\|\boldsymbol{u}\|_{L^2(\Omega)} \\
&\quad + C_{\mathrm{PF}}\,\|c\|_{L^\infty(\Omega)}^{1/2} \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)} \Bigg).
\end{aligned}
$$

In order to be able to apply now the stability estimate for the velocity, $\|\boldsymbol{u}\|_{L^2(\Omega)}$ is estimated with the Poincaré–Friedrichs inequality, because without the uniform positivity of $c(\boldsymbol{x})$ the term $\|\boldsymbol{u}\|_{L^2(\Omega)}$ cannot be estimated with $\left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)}$. After having applied the Poincaré–Friedrichs inequality, the bound (4.7) is inserted which gives (4.8).

In the case $\boldsymbol{f}\in L^2(\Omega)$ and $c(\boldsymbol{x})\ge c_0>0$, one gets in the same way the estimate

$$
\begin{aligned}
\|p\|_{L^2(\Omega)} &\le \frac{1}{\beta_{\mathrm{is}}}\Bigg(C_{\mathrm{PF}}\,\|\boldsymbol{f}\|_{L^2(\Omega)} + \nu\,\|\nabla\boldsymbol{u}\|_{L^2(\Omega)} + \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{c_0^{1/2}} \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)} \\
&\quad + C_{\mathrm{PF}}\,\|c\|_{L^\infty(\Omega)}^{1/2} \left\|c^{1/2}\boldsymbol{u}\right\|_{L^2(\Omega)} \Bigg),
\end{aligned}
$$

from which (4.10) follows by inserting (4.9). ■

Remark 4.9. Discussion of the stability estimates. For the linearization of the steady-state Navier–Stokes equations with a fixed point iteration, where $c = 0$, estimates (4.7) and (4.8) are of interest. In particular, one has to study the dependency of the stability bounds on the viscosity. It can be seen that the bounds for $\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}$ and $\|p\|_{L^2(\Omega)}$ scale like $\mathcal{O}\left(\nu^{-1}\right)$. Even replacing $\|\boldsymbol{f}\|_{H^{-1}(\Omega)}$ by $\|\boldsymbol{f}\|_{L^2(\Omega)}$, if this regularity of the right-hand side is given, does not change the dependency of the stability bounds on ν. Thus, for small ν, the problem loses stability. But note that the limit $\nu\to 0$ is not of that much interest in applications, see Remark 4.3.

For numerical schemes applied to the time-dependent Navier–Stokes equations, where the regularity $\boldsymbol{f}\in L^2(\Omega)$ is often given in applications, estimates (4.9) and (4.10) are relevant for $c = c_0 = \mathcal{O}\left((\Delta t)^{-1}\right)$. In this case, the stability bound for $\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}$ scales like $\mathcal{O}\left(\left(\frac{\Delta t}{\nu}\right)^{1/2}\right)$, for $\|\boldsymbol{u}\|_{L^2(\Omega)}$ like $\mathcal{O}\left(\Delta t\right)$, and for $\|p\|_{L^2(\Omega)}$ like

$$
\begin{aligned}
&\mathcal{O}\left(\left(1 + \left(\nu^{1/2} + (\Delta t)^{1/2} + (\Delta t)^{-1/2}\right)\left(\nu^{1/2}\left(\frac{\Delta t}{\nu}\right)^{1/2} + (\Delta t)^{-1/2}(\Delta t)\right)\right)\right) \\
&\qquad = \mathcal{O}\left(\left(\nu\Delta t\right)^{1/2} + \Delta t + 1\right).
\end{aligned}
$$

If the length of the time step is sufficiently small, i.e., $\Delta t \le \nu \le 1$, then the stability bound for $\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}$ behaves like $\mathcal{O}\left(1\right)$. For small time steps, also the two other stability bounds are small. □

4.2 The Galerkin Finite Element Method

Remark 4.10. Goals. Besides existence, uniqueness, and stability of a solution of the finite element problem, finite element error estimates are the main topic of this section. The goals and principal approach for deriving such estimates are described in Remark 3.14. For the Oseen problem, the constants in the finite element error estimates will depend in particular on the coefficients of the equations. This dependency has to be tracked during performing the estimates. □

Remark 4.11. The Galerkin finite element method for conforming inf-sup stable pairs of finite element spaces. Let $V^h \subset V$, $Q^h \subset Q$ be inf-sup stable finite element spaces. Then, the Galerkin finite element method of (4.2) reads as follows: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned} a\left(\boldsymbol{u}^h, \boldsymbol{v}^h\right) + b\left(\boldsymbol{v}^h, p^h\right) &= \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle_{V',V} && \forall\, \boldsymbol{v}^h \in V^h, \\ b\left(\boldsymbol{u}^h, q^h\right) &= 0 && \forall\, q^h \in Q^h, \end{aligned} \tag{4.11}$$

with the bilinear forms defined in (4.3). □

Corollary 4.12. Unique solvability of the finite element problem. *Let the assumptions be as in Theorem 4.7 and let V^h and Q^h be conforming and inf-sup stable finite element spaces. Then, the finite element problem* (4.11) *has a unique solution.*

Proof. The statement of the corollary follows from Lemma 2.19 by combining the coercivity of $a(\cdot,\cdot)$ in V^h, see the proof of Theorem 4.7, and the discrete inf-sup condition. ■

Lemma 4.13. Stability of the finite element solution. *Let $V^h \times Q^h$ be inf-sup stable finite element spaces. Then, the solution of* (4.11) *fulfills*

$$\frac{\nu}{2} \left\| \nabla \boldsymbol{u}^h \right\|_{L^2(\Omega)}^2 + \left\| c^{1/2} \boldsymbol{u}^h \right\|_{L^2(\Omega)}^2 \le \frac{1}{2\nu} \left\| \boldsymbol{f} \right\|_{H^{-1}(\Omega)}^2 \tag{4.12}$$

and if in addtion $\boldsymbol{b}, c \in L^\infty(\Omega)$ then

$$\begin{aligned} \left\| p^h \right\|_{L^2(\Omega)} \le \frac{1}{\beta_{\mathrm{is}}^h} \Bigg[& \left\| \boldsymbol{f} \right\|_{H^{-1}(\Omega)} + \left(\nu^{1/2} + C_{\mathrm{PF}} \frac{\left\| \boldsymbol{b} \right\|_{L^\infty(\Omega)}}{\nu^{1/2}} + C_{\mathrm{PF}} \left\| c \right\|_{L^\infty(\Omega)}^{1/2} \right) \\ & \times \left(\nu^{1/2} \left\| \nabla \boldsymbol{u}^h \right\|_{L^2(\Omega)} + \left\| c^{1/2} \boldsymbol{u}^h \right\|_{L^2(\Omega)} \right) \Bigg]. \end{aligned}$$

If $\boldsymbol{f} \in L^2(\Omega)$ and $c(\boldsymbol{x}) \ge c_0 > 0$, then also the following stability estimates hold

$$\nu \left\| \nabla \boldsymbol{u}^h \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| c^{1/2} \boldsymbol{u}^h \right\|_{L^2(\Omega)}^2 \le \frac{1}{2c_0} \left\| \boldsymbol{f} \right\|_{L_2(\Omega)}^2$$

and if additionally $\boldsymbol{b}, c \in L^\infty(\Omega)$ *then*

$$\left\|p^h\right\|_{L^2(\Omega)} \le \frac{1}{\beta_{\mathrm{is}}^h}\left[C_{\mathrm{PF}}\left\|\boldsymbol{f}\right\|_{L^2(\Omega)} + \left(\nu^{1/2} + \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{c_0^{1/2}} + C_{\mathrm{PF}}\left\|c\right\|_{L^\infty(\Omega)}^{1/2}\right)\right.$$
$$\left.\times\left(\nu^{1/2}\left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} + \left\|c^{1/2}\boldsymbol{u}^h\right\|_{L^2(\Omega)}\right)\right].$$

The constant C_{PF} *comes from the Poincaré–Friedrichs inequality.*

Proof. The proof is performed exactly like the proof of Lemma 4.8. ■

Theorem 4.14. Finite element error estimate for the velocity. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with polyhedral and Lipschitz-continuous boundary and let* $(\boldsymbol{u}, p) \in V \times Q$ *be the unique solution of the Oseen problem* (4.2) *with the conditions on the data from Remarks 4.5 and 4.6 be fulfilled and in addition with* $\boldsymbol{b}, c \in L^\infty(\Omega)$. *Assume that inf-sup stable conforming finite element spaces* $V^h \times Q^h$ *are used for discretizing* (4.2) *and denote by* $\boldsymbol{u}^h \in V_{\mathrm{div}}^h$ *the velocity solution. Then, the following error estimate holds*

$$\nu^{1/2}\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} + \left\|c^{1/2}\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} \tag{4.13}$$
$$\le C\left[\left(1+\frac{1}{\beta_{\mathrm{is}}^h}\right)C_{\mathrm{os}}\inf_{\boldsymbol{v}^h\in V^h}\left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)} + \frac{1}{\nu^{1/2}}\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\right],$$

where

$$C_{\mathrm{os}} = \nu^{1/2} + \|c\|_{L^\infty(\Omega)}^{1/2} + \|\boldsymbol{b}\|_{L^\infty(\Omega)}\min\left\{\frac{1}{\nu^{1/2}}, \frac{1}{c_0^{1/2}}\right\} \tag{4.14}$$

and C *does not depend on the coefficients and the triangulation, but it depends on the domain.*

Proof. The principal ideas for proving the error estimate are the same as in the proof of the corresponding estimate for the Stokes problem, see Theorem 3.20. For the Oseen problem, there are some additional terms to be estimated and it is important to study the dependency of the error estimate on the coefficients of the equations.

From the abstract theory of linear saddle point problems, Remark 2.11, it is known that the discrete velocity can be computed by solving the problem: Find $\boldsymbol{u}^h \in V_{\mathrm{div}}^h$ such that

$$\nu\left(\nabla\boldsymbol{u}^h, \nabla\boldsymbol{v}^h\right) + \left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{u}^h + c\boldsymbol{u}^h, \boldsymbol{v}^h\right) = \langle\boldsymbol{f}, \boldsymbol{v}^h\rangle_{V',V}\ \forall\, \boldsymbol{v}^h \in V_{\mathrm{div}}^h. \tag{4.15}$$

Using the functions from $V_{\mathrm{div}}^h \subset V$ as test functions in the continuous equation (4.2) and subtracting (4.15) gives the error equation

$$\nu\left(\nabla(\boldsymbol{u}-\boldsymbol{u}^h), \nabla\boldsymbol{v}^h\right) + \left((\boldsymbol{b}\cdot\nabla)\left(\boldsymbol{u}-\boldsymbol{u}^h\right) + c\left(\boldsymbol{u}-\boldsymbol{u}^h\right), \boldsymbol{v}^h\right)$$
$$-\left(\nabla\cdot\boldsymbol{v}^h, p-q^h\right) = 0 \quad \forall\, \boldsymbol{v}^h \in V_{\mathrm{div}}^h,\ \forall\, q^h \in Q^h. \tag{4.16}$$

Again, the error is split into a best approximation error in V_{div}^h and the difference of the best approximation to the solution of (4.15)

$$\boldsymbol{u} - \boldsymbol{u}^h = (\boldsymbol{u} - I_h \boldsymbol{u}) - \left(\boldsymbol{u}^h - I_h \boldsymbol{u}\right) = \boldsymbol{\eta} - \boldsymbol{\phi}^h, \quad I_h \boldsymbol{u} \in V_{\mathrm{div}}^h.$$

Now, $\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h$ is used as test function in (4.16). Rearranging terms and using (4.4) gives

$$\begin{aligned} &\nu \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \\ &= \nu \left(\nabla \boldsymbol{\eta}, \nabla \boldsymbol{\phi}^h\right) + \left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\eta} + c\boldsymbol{\eta}, \boldsymbol{\phi}^h\right) - \left(\nabla \cdot \boldsymbol{\phi}^h, p - q^h\right). \end{aligned} \tag{4.17}$$

The first term on the right-hand side of (4.17) is estimated by the Cauchy–Schwarz inequality (A.8) and Young's inequality (A.4)

$$\nu \left|\left(\nabla \boldsymbol{\eta}, \nabla \boldsymbol{\phi}^h\right)\right| \le \nu \left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \le \nu \left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + \frac{\nu}{4}\left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.$$

Similarly, one obtains for the last term with (2.39)

$$\left|-\left(\nabla \cdot \boldsymbol{\phi}^h, p - q^h\right)\right| \le \frac{2}{\nu}\left\|p - q^h\right\|_{L^2(\Omega)}^2 + \frac{\nu}{8}\left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2,$$

where in addition (2.38) was used. Also the reactive term is estimated in this way

$$\begin{aligned} \left|\left(c\boldsymbol{\eta}, \boldsymbol{\phi}^h\right)\right| &\le \left\|c\right\|_{L^\infty(\Omega)}^{1/2} \left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)} \left\|c^{1/2}\boldsymbol{\phi}\right\|_{L^2(\Omega)} \\ &\le \left\|c\right\|_{L^\infty(\Omega)} \left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + \frac{1}{4}\left\|c^{1/2}\boldsymbol{\phi}\right\|_{L^2(\Omega)}^2. \end{aligned}$$

For the convective term, one obtains with (4.6)

$$\begin{aligned} \left|\left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\eta}, \boldsymbol{\phi}^h\right)\right| &= \left|-\left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\phi}^h, \boldsymbol{\eta}\right)\right| \le \left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)} \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)} \\ &\le \frac{2}{\nu}\left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)}^2 \left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + \frac{\nu}{8}\left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2. \end{aligned}$$

In the case $c_0 > 0$, there is the alternative estimate

$$\begin{aligned} \left|\left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\eta}, \boldsymbol{\phi}^h\right)\right| &\le \left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)} \left\|c^{-1/2}\right\|_{L^\infty(\Omega)} \left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)} \left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\ &\le \frac{\left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)}^2 \left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)}^2}{c_0} + \frac{\left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2}{4}. \end{aligned}$$

Inserting all estimates into (4.17) gives

$$\begin{aligned} &\frac{1}{2}\left(\nu \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2\right) \\ &\le \nu \left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + \left\|c\right\|_{L^\infty(\Omega)} \left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 \\ &\quad + \min\left\{\frac{2\left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)}^2 \left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2}{\nu}, \frac{\left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)}^2 \left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)}^2}{c_0}\right\} + \frac{2}{\nu}\left\|p - q^h\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.18}$$

From the triangle inequality, inequality (A.5), and the Poincaré–Friedrichs inequality (A.9), one obtains

$$\begin{aligned}
&\nu^{1/2}\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}+\left\|c^{1/2}\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\\
&\le \nu^{1/2}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}+\left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}+\nu^{1/2}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}+\left\|c^{1/2}\boldsymbol{\eta}\right\|_{L^2(\Omega)}\\
&\le \sqrt{2}\left(\nu\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2\right)^{1/2}+\nu^{1/2}\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}+\|c\|_{L^\infty(\Omega)}^{1/2}\left\|\boldsymbol{\eta}\right\|_{L^2(\Omega)}\\
&\le \sqrt{2}\left(\nu\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\left\|c^{1/2}\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2\right)^{1/2}+\left(\nu^{1/2}+C_{\mathrm{PF}}\|c\|_{L^\infty(\Omega)}^{1/2}\right)\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}.
\end{aligned}$$

Inserting (4.18), applying once more (A.5) and the Poincaré–Friedrichs inequality, and using (2.54) gives (4.13). ■

Theorem 4.15. Finite element error estimate for the $L^2(\Omega)$ norm of the pressure. *Let the assumption of Theorem 4.14 hold. Then the following error estimate holds for the pressure*

$$\begin{aligned}
\left\|p-p^h\right\|_{L^2(\Omega)} \le C\Bigg[&\frac{1}{\beta_{\mathrm{is}}^h}\left(1+\frac{1}{\beta_{\mathrm{is}}^h}\right)C_{\mathrm{os}}^2\inf_{\boldsymbol{v}^h\in V^h}\left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)}\\
&+\left(1+\frac{1}{\beta_{\mathrm{is}}^h}+\frac{1}{\beta_{\mathrm{is}}^h}\frac{C_{\mathrm{os}}}{\nu^{1/2}}\right)\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}\Bigg],
\end{aligned}\tag{4.19}$$

where C_{os} is defined in (4.14) and C does not depend on the coefficients and the triangulation, but it depends on the domain.

Proof. The way of proving (4.19) follows the proof of Theorem 3.24.

Using the discrete inf-sup condition (2.45), the discrete Oseen equations (4.11) as well as the continuous Oseen equations (4.2), one obtains in the same way as in the proof of Theorem 3.24

$$\left\|p^h-q^h\right\|_{L^2(\Omega)} \le \frac{1}{\beta_{\mathrm{is}}^h}\sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}}\frac{a\left(\boldsymbol{u}-\boldsymbol{u}^h,\boldsymbol{v}^h\right)+b\left(\boldsymbol{v}^h,p-q^h\right)}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}}\tag{4.20}$$

for all $q^h\in Q^h$. The bilinear forms are replaced by (4.3) and then the individual terms are estimated. Using the Cauchy–Schwarz inequality (A.8), (2.39), and the Poincaré–Friedrichs inequality (A.9), one obtains

$$\begin{aligned}
\left|\nu\left(\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\nabla\boldsymbol{v}^h\right)\right| &\le \nu\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)},\\
\left|\left(\nabla\cdot\boldsymbol{v}^h,p-q^h\right)\right| &\le \left\|p-q^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)},\\
\left|\left(c\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\boldsymbol{v}^h\right)\right| &\le C_{\mathrm{PF}}\|c\|_{L^\infty(\Omega)}^{1/2}\left\|c^{1/2}\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}.
\end{aligned}$$

The convective term can be estimated with the same tools

$$\left|\left((\boldsymbol{b}\cdot\nabla)\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\boldsymbol{v}^h\right)\right| \le C_{\mathrm{PF}}\|\boldsymbol{b}\|_{L^\infty(\Omega)}\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}$$

or with applying integration by parts in the first step of the estimate

$$\left|\left((\boldsymbol{b}\cdot\nabla)\left(\boldsymbol{u}-\boldsymbol{u}^h\right),\boldsymbol{v}^h\right)\right| \le \frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{c_0^{1/2}}\left\|c^{1/2}\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}.$$

Inserting these estimates into (4.20) leads for all $q^h \in Q^h$ to

$$\begin{aligned}
&\left\|p^h - q^h\right\|_{L^2(\Omega)} \\
&\le \frac{C}{\beta_{\mathrm{is}}^h}\left(\nu^{1/2} + \|c\|_{L^\infty(\Omega)}^{1/2} + \|\boldsymbol{b}\|_{L^\infty(\Omega)} \min\left\{\frac{1}{\nu^{1/2}}, \frac{1}{c_0^{1/2}}\right\}\right) \\
&\quad \times \left(\nu^{1/2}\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} + \left\|c^{1/2}\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}\right) + \frac{C}{\beta_{\mathrm{is}}^h}\left\|p - q^h\right\|_{L^2(\Omega)}.
\end{aligned}$$

The triangle inequality and the insertion of (4.13) conclude the proof. ∎

Corollary 4.16. Finite element error estimates for conforming pairs of finite element spaces. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with polyhedral and Lipschitz-continuous boundary which is decomposed by a regular and quasi-uniform family of triangulations $\{\mathcal{T}^h\}$. Let $(\boldsymbol{u}, p)$ be the solution of the Oseen equations* (4.2) *with $\boldsymbol{u} \in H^{k+1}(\Omega) \cap V$ and $p \in H^k(\Omega) \cap Q$. Then for the inf-sup stable pairs of finite element spaces*

- $P_k^{\mathrm{bubble}}/P_k$, $k = 1$ *(MINI element),*
- P_k/P_{k-1}, Q_k/Q_{k-1}, $k \ge 2$ *(Taylor–Hood element),*
- $P_k^{\mathrm{bubble}}/P_{k-1}^{\mathrm{disc}}$, $Q_k/P_{k-1}^{\mathrm{disc}}$, $k \ge 2$,

the following error estimates hold

$$\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)} \le \frac{C}{\nu^{1/2}} h^k \left(C_{\mathrm{os}}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \frac{1}{\nu^{1/2}}\|p\|_{H^k(\Omega)}\right) \tag{4.21}$$

$$\left\|p - p^h\right\|_{L^2(\Omega)} \le Ch^k\Bigg(C_{\mathrm{os}}^2\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} + \left(1 + \frac{C_{\mathrm{os}}}{\nu^{1/2}}\right)\|p\|_{H^k(\Omega)}\Bigg). \tag{4.22}$$

The constants C depend on the inverse of the discrete inf-sup constant β_{is}^h.

Proof. The estimates follow directly from estimating the best approximation errors in (4.13) and (4.19) with interpolation errors which are known for the finite element spaces, see Theorem C.15. ∎

Remark 4.17. Discussion of the dependency of the error estimates on the coefficients of the problem. This discussion considers in particular the two cases of interest which were described in Remark 4.3. Possible dependencies of $\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}$ and $\|p\|_{H^k(\Omega)}$ on the coefficients of the problem are not taken into account. The dependency of the error estimates on the coefficients of the problem will be discussed only for the interesting case $\nu \le 1$.

Consider first the case $c = c_0 = 0$, which appears in the iterative solution of the steady-state Navier–Stokes equations with a fixed point iteration. In this case there is $C_{\mathrm{os}} = \mathcal{O}\left(\nu^{-1/2}\right)$, see Figure 4.1. It follows that the constants in the estimates (4.21) and (4.22) behave both like $\mathcal{O}\left(\nu^{-1}\right)$ such that the estimates blow up for $\nu \to 0$. As already mentioned in Remark 4.3, the case

$\nu \to 0$ is not of interest in applications. However, ν can be nevertheless small and in this case the constants in the estimates (4.21) and (4.22) might become large.

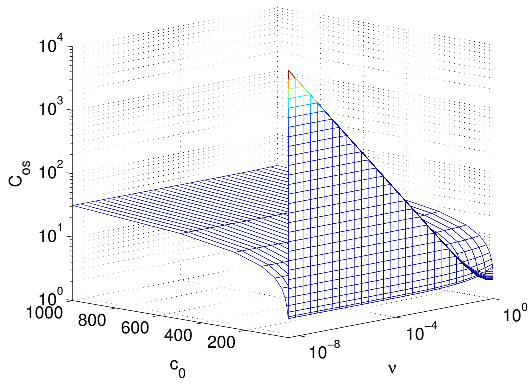

Fig. 4.1 C_{os} for $\|\boldsymbol{b}\|_{L^\infty(\Omega)} = 1$.

The other interesting situation is $c = c_0 = \mathcal{O}\left((\Delta t)^{-1}\right)$. This case arises in numerical methods for the time-dependent Navier–Stokes equations and thus also the situation that ν is very small is of interest.

- If the time step is sufficiently small, i.e., $\Delta t \le \nu^{-1}$ and $\Delta t \le (\Delta t)^{-1}$, then $C_{\mathrm{os}} = \mathcal{O}\left((\Delta t)^{-1/2}\right)$. It follows that the constant in (4.21) scales like $\mathcal{O}\left((\nu\Delta t)^{-1/2}\right) + \mathcal{O}\left((\nu^{-1})\right)$ and the constant in (4.22) like $\mathcal{O}\left((\Delta t)^{-1}\right) + \mathcal{O}\left((\nu\Delta t)^{-1/2}\right)$.
- For large time steps compared with the viscosity, i.e., $\Delta t > \nu^{-1}$, one has $C_{\mathrm{os}} = \mathcal{O}\left(\nu^{-1/2}\right)$, which leads to constants of size $\mathcal{O}\left(\nu^{-1}\right)$.

In all cases, the error bounds become large for a small viscosity ν (or a large Reynolds number). They become also large, if $\|c\|_{L^\infty(\Omega)}$ is large, in particular for small time steps. In summary, the error bounds are not uniform with respect to the coefficients of the problem. Thus, it can happen that the Galerkin finite element discretization for the Oseen problem becomes unstable. □

Example 4.18. Analytical example which supports the error estimates (4.21) *and* (4.22)*.* The numerical results which will be presented were obtained for Example D.2. That means, the solution and Sobolev norms of the solution do not depend on the viscosity. Of course, any divergence-free vector field can be chosen as convection field in the Oseen equations (4.1). Form the point of view of applications, the most important case is that the convection field is the solution of the Oseen equations itself. This case will be considered here, i.e., the convection field is given by (D.1). Thus, $\|\boldsymbol{b}\|_{L^\infty(\Omega)} = \mathcal{O}\,(1)$.

The simulations were performed for the Taylor–Hood element Q_2/Q_1 on the uniform quadrilateral grid shown in Figure 3.2.

Results will be presented for the two situations of interest described in Remark 4.3. In the first situation, it is $c = 0$ and different values of ν were chosen. The errors in $\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ are presented in Figure 4.2. The second order convergence for large viscosities and the increase of the errors for small ν on coarse grids can be clearly seen. On finer grids, this dependency becomes smaller and it tends to vanish. A similar behavior can be observed if instead of the regular grid the irregular grid from Figure 3.7 is used.

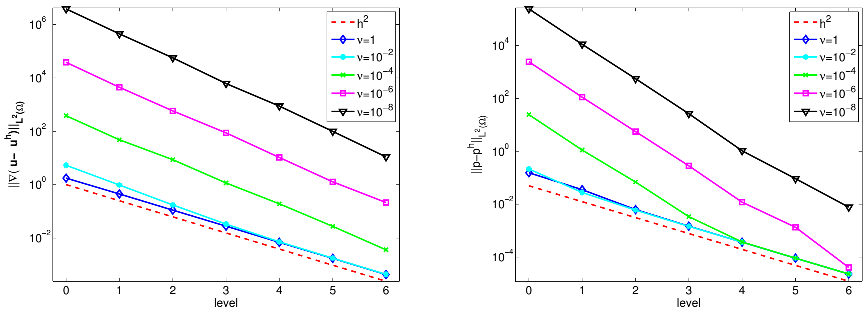

Fig. 4.2 Example 4.18. Convergence of the errors $\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ for $c = 0$ and different values of ν.two levels more

In the second numerical study, the coefficient of the zeroth order term was chosen to be $c = 100$ and the viscosity was varied, see Figure 4.3. With respect to the pressure, one obtains more or less the same accuracy for all values of the viscosity. This is also true for the error in the gradient of the velocity, but only on coarse grids. For finer grids, larger errors can be observed for small values of the viscosity. The asymptotic region seems not to be reaches for these values.

Finally, the case of constant $\nu = 10^{-4}$ and different values of c was studied, see Figure 4.4. With respect to the error in the velocity, it can be clearly seen that an increase of c decreases the error. For several values of c, the asymptotic region is not yet reached and the order of error redcution is higher than the predicted order of convergence. Concerning the error in the pressure, one obtains almost the same results for all values of c on finer grids. On coarser grids, in the pre-asymptotic region, the largest errors can be observed for $c = 0$ and $c = 10000$.

For obtaining the solutions of the linear systems of equations, the direct solver UMFPACK, see Davis (2004a), was used. In many cases, e.g., $c = 0$ and ν small, it was not possible to solve the linear systems of equations with the iterative methods described in Chapter **??**. This fact is a hint that the matrices have a high condition number. □

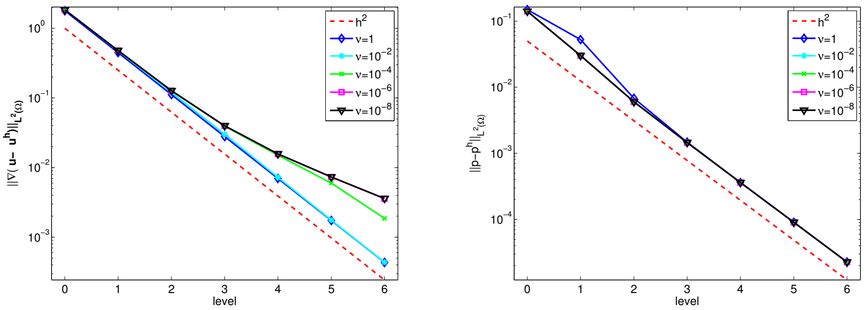

Fig. 4.3 Example 4.18. Convergence of the errors $\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ for $c=100$ and different values of ν.

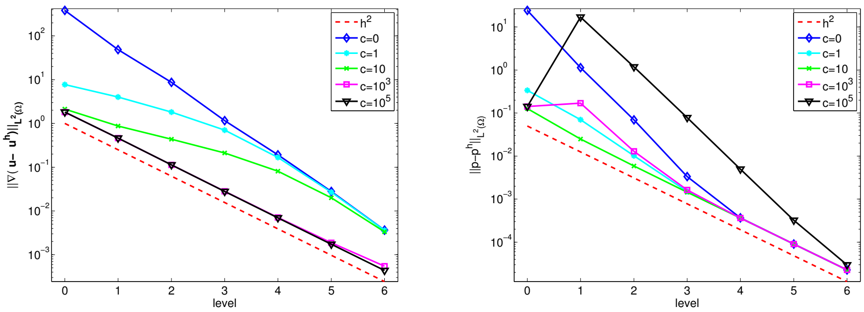

Fig. 4.4 Example 4.18. Convergence of the errors $\left\|\nabla(\boldsymbol{u}-\boldsymbol{u}^h)\right\|_{L^2(\Omega)}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ for $\nu=10^{-4}$ and different values of c.

4.3 Residual-Based Stabilizations

4.3.1 The Basic Idea

Remark 4.19. Residual-based stabilizations. Residual-based methods are one of the most popular approaches for obtaining stable discretizations of the Oseen equations, or more general of convection-dominated problems. Several types of residual-based stabilizations for the Oseen equations can be found in the literature, e.g., see Braack et al. (2007). In this section, residual-based stabilizations involving a Streamline-Upwind Petrov–Galerkin (SUPG) term, a pressure stabilization Petrov–Galerkin (PSPG) term, and a grad-div stabilization term will be presented. □

Remark 4.20. The basic idea of residual-based stabilizations. The basic idea consists in a penalization of large values of the so-called strong residual.

Given a linear partial differential equation in strong form

$$A_{\mathrm{str}} u_{\mathrm{str}} = f, \quad f \in L^2(\Omega),$$

and its Galerkin finite element discretization: Find $u^h \in V^h$ such that

$$a^h\left(u^h, v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h. \tag{4.23}$$

For residual-based stabilizations, a modification of A_{str} is needed which is well defined for finite element functions. This modification should be also a linear operator and it is denoted by $A_{\mathrm{str}}^h \,:\, V^h \to L^2(\Omega)$. The (strong) residual is now defined by

$$r^h\left(u^h\right) = A_{\mathrm{str}}^h u^h - f \ \in L^2(\Omega).$$

In general, it holds $r^h\left(u^h\right) \neq 0$, but a good numerical approximation of the solution of the continuous problem should have in some sense a small residual. Now, instead of finding the solution of (4.23), the minimizer of the residual is searched, i.e, the following optimization problem is considered

$$\operatorname*{arg\,min}_{u^h \in V^h} \left\| r^h\left(u^h\right)\right\|_{L^2(\Omega)}^2 = \operatorname*{arg\,min}_{u^h \in V^h} \left(r^h\left(u^h\right), r^h\left(u^h\right)\right). \tag{4.24}$$

The necessary condition for taking the minimum is the vanishing of the Gâteaux derivative. This derivative is computed by using the linearity of A_{str}^h and the bilinearity of the inner product in $L^2(\Omega)$

$$\begin{aligned}
0 &= \lim_{\varepsilon \to 0} \frac{\left(r^h\left(u^h + \varepsilon v^h\right), r^h\left(u^h + \varepsilon v^h\right)\right) - \left(r^h\left(u^h\right), r^h\left(u^h\right)\right)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\left(r^h\left(u^h\right) + \varepsilon A_{\mathrm{str}}^h v^h, r^h\left(u^h\right) + \varepsilon A_{\mathrm{str}}^h v^h\right) - \left(r^h\left(u^h\right), r^h\left(u^h\right)\right)}{\varepsilon} \\
&= 2\left(r^h\left(u^h\right), A_{\mathrm{str}}^h v^h\right) \quad \forall\, v^h \in V^h.
\end{aligned}$$

It follows that the necessary condition for the solution of (4.24) is

$$\left(r^h\left(u^h\right), A_{\mathrm{str}}^h v^h\right) = 0 \quad \forall\, v^h \in V^h.$$

A generalization consists in considering the minimization problem

$$\operatorname*{arg\,min}_{u^h \in V^h} \left\| \delta^{1/2} r^h\left(u^h\right)\right\|_{L^2(\Omega)}^2 = \operatorname*{arg\,min}_{u^h \in V^h} \left(\delta r^h\left(u^h\right), r^h\left(u^h\right)\right). \tag{4.25}$$

with the positive weighting function $\delta(\boldsymbol{x})$. Analogously to the derivation for the special case, one obtains as necessary condition for the minimum

$$\left(\delta r^h\left(u^h\right), A_{\mathrm{str}}^h v^h\right) = 0 \quad \forall\, v^h \in V^h. \tag{4.26}$$

The solutions of (4.24) or (4.25) will not be identical to the solution of the Galerkin discretization (4.23). It turns out that the reason for the Galerkin discretization to fail is that the solution possesses structures (scales) that are important but which are not resolved by the used finite element space (grid), which is in particular the main difficulty in the simulation of turbulent flows. see Section 7.1. For flow problems, such structures are layers, particularly at boundaries. The numerical methods should compute sharp layers. However the sharpness of layers in numerical solutions is restricted by the resolution, which is generally much coarser than the layer width. Hence, even for a numerical solution with sharp layers, the strong residual in the layer regions is very large. In particular, a numerical solution with sharp layers (with respect to the resolution of the finite element space) will not be the minimizer of (4.24) or (4.25), see Figure 4.5. The minimizers of (4.24) or (4.25) tend to possess strongly smeared layers and these solutions are useless in applications. For this reason, one considers in residual-based stabilizations a combination of the Galerkin discretization (4.23) and the minimization of the strong residual

$$a^h\left(u^h, v^h\right) + \left(\delta r^h\left(u^h\right), A^h_{\text{str}} v^h\right) = \left(f, v^h\right) \quad \forall\, v^h \in V^h. \tag{4.27}$$

The goal of numerical analysis consists in determining the weighting function δ optimally. This goal is generally not achieved completely, only asymptotically optimal choices for weighting functions are known, e.g. see Remark 4.41.

□

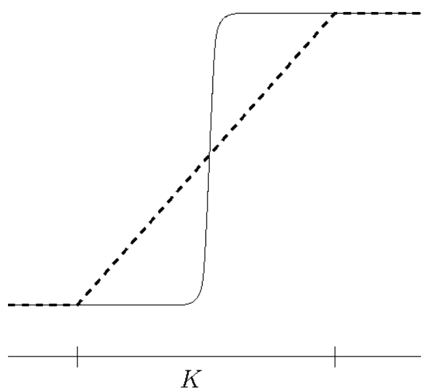

Fig. 4.5 Function with sharp layer (solid line) and optimal piecewise linear approximation in a mesh cell K (dashed line). The equation which is satisfied by the function in K is far from being satisfied by the piecewise linear approximation. Hence, despite the approximation is optimal, the strong residual will be large.

Example 4.21. Oseen equations. There are two equations in the Oseen equations and these equations should be treated separately. Given a triangulation $\mathcal{T}^h$ with mesh cells $\{K\}$. Then, one has for the momentum equation

$$
\begin{aligned}
A_{\mathrm{str,m}}(\boldsymbol{u}, p) &= -\nu \Delta \boldsymbol{u} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u} + \nabla p, \\
A^h_{\mathrm{str,m}}\left(\boldsymbol{u}^h, p^h\right) &= -\nu \Delta \boldsymbol{u}^h + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u}^h + c\boldsymbol{u}^h + \nabla p^h \quad \text{if} \quad \boldsymbol{x} \in \mathring{K}, \forall\, K \in \mathcal{T}^h.
\end{aligned}
$$

Thus, $A^h_{\mathrm{str,m}}$ is not defined on the faces of the mesh cells. However, the union of the faces is a set of Lebesgue measure zero and since $A^h_{\mathrm{str,m}}\left(\boldsymbol{u}^h, p^h\right) \in L^2(\Omega)$, it is not necessary to define this function on this set. For the continuity equation, one has

$$
\begin{aligned}
A_{\mathrm{str,c}}(\boldsymbol{u}) &= -\nabla \cdot \boldsymbol{u}, \\
A^h_{\mathrm{str,c}}\left(\boldsymbol{u}^h\right) &= -\nabla \cdot \boldsymbol{u}^h \quad \text{if} \quad \boldsymbol{x} \in \mathring{K}, \forall\, K \in \mathcal{T}^h.
\end{aligned}
$$

The corresponding residuals are denoted by $r^h_m\left(\boldsymbol{u}^h, p^h\right)$ and $r^h_c\left(\boldsymbol{u}^h\right)$, respectively.

The general framework for minimizing the residual leads to the following terms in (4.27), besides the terms coming from the Galerkin discretization,

$$
\begin{aligned}
&\sum_{K \in \mathcal{T}^h} \left[\left(\delta_c r^h_c\left(\boldsymbol{u}^h\right), A^h_{\mathrm{str,c}}(\boldsymbol{v}^h)\right)_K + \left(\delta_m r^h_m\left(\boldsymbol{u}^h, p^h\right), A^h_{\mathrm{str,m}}(\boldsymbol{v}^h, q^h)\right)_K \right] \\
&\qquad = \sum_{K \in \mathcal{T}^h} \Big[\left(\delta_c \nabla \cdot \boldsymbol{u}^h, \nabla \cdot \boldsymbol{v}^h\right)_K \qquad\qquad (4.28) \\
&\qquad + \Big(\delta_m \left(-\nu \Delta \boldsymbol{u}^h + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u}^h + c\boldsymbol{u}^h + \nabla p^h - \boldsymbol{f}\right), \\
&\qquad -\nu \Delta \boldsymbol{v}^h + (\boldsymbol{b} \cdot \nabla)\boldsymbol{v}^h + c\boldsymbol{v}^h + \nabla q^h\Big)_K \Big].
\end{aligned}
$$

The decomposition of the integrals is necessary since the terms are generally not well-defined on the boundaries of the mesh cells.

The terms in (4.28) are the prototype of residual-based stabilization terms for the Oseen equations. The actually used residual-based stabilizations contain some modifications. □

4.3.2 The SUPG/PSPG/grad-div Stabilization

Remark 4.22. Plan of this section. The numerical analysis of stabilized methods becomes technically much more involved than the numerical analysis of the Galerkin discretization. The contents of this section and the main steps of the numerical analysis are as follows:

- introduction of the stabilized method, Remark 4.24 and some general discussions of this method,
- consistency and Galerkin orthogonality of the method, Lemma 4.30,
- introduction of a norm for the numerical analysis, which is almost without pressure contribution, in (4.41),

- existence and uniqueness of a solution for this method, Theorem 4.33, and stability of the solution, Lemma 4.34,
- introduction of a norm with some pressure contribution in (4.49),
- proof of a discrete inf-sup condition for the bilinear form of the method where the constant does not depend on the data of the problem, Lemma 4.37,
- finite element error estimate, Theorem 4.40, which is based on the discrete inf-sup condition, and its discussion.

In summary, it will turn out that the numerical analysis can be performed for the velocity in a norm that is stronger than the norm used in the Galerkin method. However, for the stabilized method, it is difficult to incorporate the pressure appropriately in the analysis. A new idea in the finite element error analysis, compared with the Stokes equations, is to use a discrete inf-sup condition for the complete bilinear form as a starting point. □

Remark 4.23. Technical tools. In the analysis, the following technical tools are used, see Tobiska and Verfürth (1996). It is assumed that there are two interpolation operators $I^h : V \to V^h$ and $J^h : Q \to Q^h$ such that the following estimates hold for all $K \in \mathcal{T}^h$, $E \in \mathcal{E}^h$, with $E = K_1 \cap K_2$ for $K_1, K_2 \in \mathcal{T}^h$, $k \geq 1$, $0 \leq m \leq 2$, $\max\{m,1\} \leq l \leq k+1$, and $0 \leq i \leq 1$, $1 \leq j \leq k$:

$$\left\|\boldsymbol{u} - I^h\boldsymbol{u}\right\|_{H^m(K)} \leq Ch_K^{l-m} \left\|\boldsymbol{u}\right\|_{H^l(K)} \quad \forall\, \boldsymbol{u} \in H^l(K), \tag{4.29}$$

$$\left\|\boldsymbol{u} - I^h\boldsymbol{u}\right\|_{L^2(E)} \leq Ch_E^{l-1/2} \left\|\boldsymbol{u}\right\|_{H^l(K_1\cup K_2)} \quad \forall\, \boldsymbol{u} \in H^l(K_1 \cup K_2), \tag{4.30}$$

$$\left\|p - J^h p\right\|_{H^i(K)} \leq Ch_K^{j-i} \left\|p\right\|_{H^j(K)} \quad \forall\, p \in H^j(K), \tag{4.31}$$

$$\left\|p - J^h p\right\|_{L^2(E)} \leq Ch_E^{j-1/2} \left\|p\right\|_{H^j(K_1\cup K_2)} \quad \forall\, \boldsymbol{u} \in H^j(K_1 \cup K_2), \tag{4.32}$$

$$\left\|\left[\left|q^h\right|\right]_E\right\|_{L^2(E)} \leq Ch_E^{-1/2} \left\|q^h\right\|_{L^2(K_1\cup K_2)} \quad \forall\, q^h \in Q^h. \tag{4.33}$$

These conditions are satisfied if V^h and Q^h contain functions that are piecewise polynomials of degree k and $k-1$, respectively. The interpolation operator for the pressure is the Clément operator, see Remark C.18. Note that if the Clément operator is applied for a finite element space with functions whose integral mean value is zero, then one can equip this space with a basis of functions with this property and finally, it follows from (C.13) that the Clément interpolant has also integral mean value zero. description necessary? For the velocity, the modification of the Clément operator is used which preserves homogeneous Dirichlet boundary conditions, see Remark C.22. later, shortly before they are needed, is this remark used? □

Remark 4.24. SUPG/PSPG/grad-div (spg) method. The SUPG/PSPG/grad-div method for solving the Oseen problem (4.2) is obtained by adding both a control of the strong residual of the momentum equation and the strong residual of the continuity equation on each mesh cell to the Galerkin finite element formulation. This classical residual-based stabilization reads as follows: Given $\boldsymbol{f} \in L^2(\Omega)$ and a triangulation $\mathcal{T}^h$ of Ω, find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\text{spg}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \quad (4.34)$$

where the bilinear form $A_{\text{spg}} : \left(V \times \tilde{Q}\right) \times \left(V \times \tilde{Q}\right) \to \mathbb{R}$ is given by

$$\begin{aligned}
&A_{\text{spg}}\left((\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) \\
&= \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) + \left((\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}\right) - (\nabla \cdot \boldsymbol{v}, p) + (\nabla \cdot \boldsymbol{u}, q) \\
&\quad + \sum_{K \in \mathcal{T}^h} \mu_K \left(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}\right)_K + \sum_{E \in \mathcal{E}^h} \gamma_E \left([|p|]_E, [|q|]_E\right)_E \qquad (4.35) \\
&\quad + \sum_{K \in \mathcal{T}^h} \left(-\nu \Delta \boldsymbol{u} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u} + \nabla p, \delta_K^v (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} + \delta_K^p \nabla q\right)_K
\end{aligned}$$

and the linear form $L_{\text{spg}} : \left(V \times \tilde{Q}\right) \to \mathbb{R}$ by

$$L_{\text{spg}}\left((\boldsymbol{v}, q)\right) = (\boldsymbol{f}, \boldsymbol{v}) + \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K^v (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} + \delta_K^p \nabla q\right)_K . \qquad (4.36)$$

The correct asymptotic choice of the stabilization parameters $\mu_K \geq 0$ and γ_E, $\delta_K^v, \delta_K^p > 0$ in (4.35) and (4.36) will be determined by the finite element error analysis. The space $\tilde{Q}$ is defined in (3.73).

In contrast to the prototype stabilization (4.28) the following modifications are contained in the SUPG/PSPG/grad-div (spg) method:

- The term $\nu \Delta \boldsymbol{v}^h$ is missing. A motivation for this modification is that the term is of minor importance if ν is small, which is the interesting case for stabilized methods.
- Likewise, the term $c\boldsymbol{v}^h$ does not appear. This term does not improve important properties of the discretization, like stability or the order of convergence.
- The term with the jumps of the finite element pressure appears. This term will be necessary for defining an appropriate norm in the case of discontinuous pressure approximations, see Lemma 4.32. If $Q^h \subset C(\Omega)$, then this term vanishes anyway.
- The velocity and pressure test functions in the stabilization term for the momentum balance may possess different weights. However, in practice these weights are chosen to be the same, see Remark 4.25.

The different stabilization terms have different functions:

- The stabilization term with the test function $(\boldsymbol{b} \cdot \nabla)\boldsymbol{v}$ is called Streamline-Upwind Petrov–Galerkin (SUPG) term. Sometimes, it is referred also as streamline diffusion (SD) term. This term stabilizes dominating convection.

- The pressure stabilization Petrov–Galerkin (PSPG) term is the stabilization term with the test function ∇q and for discontinuous discrete pressures the term with the pressure jumps across the faces. With these terms, a violation of the discrete inf-sup conditions (2.45) is stabilized.
- Finally, the grad-div term is the stabilization term which involves the residual of the continuity equation. With this term, one gets an additional control on the violation of the conservation of mass. Applying integration by parts, one obtains for $\mu_K = \mu = \text{const}$

$$\mu\left(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}\right) = -\mu\left(\nabla\left(\nabla \cdot \boldsymbol{u}\right), \boldsymbol{v}\right) \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V.$$

The name of this stabilization term is derived from the term on the right-hand side of this equation.

Individual equations for the velocity and the pressure test functions are obtained for the test functions $\left(\boldsymbol{v}^h, 0\right)$ and $\left(\boldsymbol{0}, -q^h\right)$: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned}
&\nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) + \left(\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h + c\boldsymbol{u}^h, \boldsymbol{v}^h\right) - \left(\nabla\cdot\boldsymbol{v}^h, p^h\right)\\
&\quad + \sum_{K\in\mathcal{T}^h} \mu_K\left(\nabla\cdot\boldsymbol{u}^h, \nabla\cdot\boldsymbol{v}^h\right)_K\\
&\quad + \sum_{K\in\mathcal{T}^h}\left(-\nu\Delta\boldsymbol{u}^h + \left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h + c\boldsymbol{u}^h + \nabla p^h, \delta_K^v\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{v}^h\right)_K\\
&= \left(\boldsymbol{f}, \boldsymbol{v}^h\right) + \sum_{K\in\mathcal{T}^h}\left(\boldsymbol{f}, \delta_K^v\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{v}^h\right)_K \quad \forall\, \boldsymbol{v}^h \in V^h,
\end{aligned}$$

and

$$\begin{aligned}
&-\left(\nabla\cdot\boldsymbol{u}^h, q^h\right) - \sum_{E\in\mathcal{E}^h}\gamma_E\left(\left[|p^h|\right]_E, \left[|q^h|\right]_E\right)_E\\
&\quad - \sum_{K\in\mathcal{T}^h}\left(-\nu\Delta\boldsymbol{u}^h + \left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h + c\boldsymbol{u}^h + \nabla p^h, \delta_K^p\nabla q^h\right)_K\\
&= -\sum_{K\in\mathcal{T}^h}\left(\boldsymbol{f}, \delta_K^p\nabla q^h\right)_K \quad \forall\, q^h \in Q^h.
\end{aligned}$$

The second equation shows that for $\delta_K^p > 0$ the finite element solution is generally not discretely divergence-free. But on the other hand, the grad-div term gives a control on the $L^2(\Omega)$ norm of the divergence.

Considering Dirichlet boundary conditions, as it is done in this section, an additive constant of the finite element pressure has to be fixed in (4.34). For this reason $Q^h \subset L_0^2(\Omega)$ is still a correct condition for the finite element pressure space.

A finite element error analysis of the SUPG/PSPG/grad-div method was given in Tobiska and Verfürth (1996) and another presentation can be found in Roos et al. (2008). split in several remarks? □

Remark 4.25. On the stabilization parameters. In the following, it will be assumed that $\delta_K = \delta_K^v = \delta_K^p$ for all $K \in \mathcal{T}^h$ and it is set

$$\delta = \max_{K\in\mathcal{T}^h} \delta_K, \quad \mu = \max_{K\in\mathcal{T}^h} \mu_K, \quad \gamma = \max_{E\in\mathcal{E}^h} \gamma_E. \tag{4.37}$$

□

Remark 4.26. Stabilization parameters with dimensions. Considering the bilinear form (4.35) with dimensions, i.e., $\boldsymbol{u}, \boldsymbol{b}$ are velocities with $[\mathrm{m/s}]$, p a pressure divided by density with $[\mathrm{m^2/s^2}]$ and so on, one gets the following units for the stabilization parameters:

- μ_K : $[\mathrm{m^2/s}]$ – μ_K is a viscosity scale,
- γ_E : $[\mathrm{s/m}]$ – γ_E^{-1} is a velocity scale,
- δ_K : $[\mathrm{s}]$ – δ_K is a time scale.

□

Remark 4.27. Historical remarks to the SUPG method. The SUPG method was introduced in Hughes and Brooks (1979); Brooks and Hughes (1982) for stabilizing convection-dominated convection-diffusion equations. Stabilizations of the Oseen equations and the stationary Navier–Stokes equations which contain the SUPG term were analyzed independently in Hansbo and Szepessy (1990); Lube and Tobiska (1990); Tobiska and Lube (1991); Franca and Frey (1992). □

Remark 4.28. On the grad-div term. A grad-div term appeared already in the derivation of the Navier–Stokes equations, see (1.18), which reads for constant viscosities

$$\nabla \cdot \left(\left(\zeta - \frac{2\mu}{3} \right) \nabla \cdot \boldsymbol{v} \mathbb{I} \right) = \left(\zeta - \frac{2\mu}{3} \right) \nabla \left(\nabla \cdot \boldsymbol{v} \right).$$

In the process of non-dimensionalization, this term becomes

$$\frac{L}{U^2} \left(\frac{\zeta}{\rho} - \frac{2\nu}{3} \right) \nabla \left(\nabla \cdot \boldsymbol{u} \right).$$

The parameter in front of the grad-div term is just a constant. move to other place □

Remark 4.29. The SUPG/PSPG/grad-div method does not lead to a saddle point problem. The SUPG/PSPG/grad-div method contains a pressure-pressure coupling

$$- \sum_{E\in\mathcal{E}^h} \gamma_E \left([|p^h|]_E, [|q^h|]_E \right)_E - \sum_{K\in\mathcal{T}^h} \delta_K \left(\nabla p^h, \nabla q^h \right)_K. \tag{4.38}$$

This coupling leads to a non-zero pressure-pressure contribution in the second equations of (2.4) or (2.5). From this circumstance, it follows that (4.34) is not a saddle point problem. Hence, one can apply the finite element theory for elliptic partial differential equations.

As consequence, the inf-sup stability of the finite element spaces for velocity and pressure does not play a critical role if the SUPG/PSPG/grad-div method is used. However, the finite element error analysis will reveal that the asymptotically optimal choice of the stabilization parameter is affected by the concrete choice of the pair of finite element spaces. For instance, the optimal choice of the stabilization parameters will be fundamentally different, e.g., for inf-sup stable pairs of finite element spaces and for equal order finite element spaces for velocity and pressure.

Consider for simplicity the case $Q^h \subset H^1(\Omega)$ and $\delta_K = \delta$. The pressure-pressure coupling (4.38) is just a finite element discretization of the operator $\delta\Delta p$ which has to be equipped with boundary conditions. As discussed in Remark 3.58, these are Neumann boundary conditions on Γ. □

Lemma 4.30. Consistency and Galerkin orthogonality. *Let* $(\boldsymbol{u}, p) \in (V \cap H^2(\Omega)) \times (Q \cap H^1(\Omega))$ *be the solution of* (4.2) *and let* $\boldsymbol{b}, c \in L^\infty(\Omega)$, $\boldsymbol{f} \in L^2(\Omega)$. *Consider conforming finite element spaces* V^h *and* Q^h*, then, the SUPG/PSPG/grad-div method* (4.34) *is consistent, i.e.,*

$$A_{\text{spg}}\left((\boldsymbol{u}, p), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right)\right), \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \quad (4.39)$$

and the Galerkin orthogonality (projection property)

$$A_{\text{spg}}\left(\left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = 0, \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h \qquad (4.40)$$

holds.

Proof. With the assumed regularity for the solution of (4.2), the local residuals in A_{spg} and L_{spg} belong to $L^2(K)$ for all $K \in \mathcal{T}^h$. The pressure does not possess jumps almost everywhere across the faces of the mesh cells. From the first property, it follows that the local residuals vanish for the solution of (4.2). Hence, all stabilization terms vanish if a sufficiently smooth solution of (4.2) is inserted into (4.34). The remaining terms are identical to equation (4.2).

The Galerkin orthogonality follows by subtracting (4.34) from (4.39). ■

Remark 4.31. Norm for the first part of the analysis. Stabilized methods are analyzed generally in a norm which is connected to the stabilization. This norm is in general stronger than the norm which is used in the analysis of the Galerkin finite element method. The norm used in the finite element error analysis for stabilized methods reveals which norms of the error are controlled by the stabilization.

For the SUPG/PSPG/grad-div method (4.34), the following norm will be used in the first part of the analysis

$$\|(\boldsymbol{v},q)\|_{\text{spg}} = \Bigg\{\nu \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{v}\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}^h} \mu_K \|\nabla\cdot\boldsymbol{v}\|_{L^2(K)}^2 \qquad (4.41)$$

$$+ \sum_{E\in\mathcal{E}^h} \gamma_E \|[|q|]_E\|_{L^2(E)}^2 + \sum_{K\in\mathcal{T}^h} \delta_K \|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v} + \nabla q\|_{L^2(K)}^2 \Bigg\}^{1/2}.$$

This norm is mesh-dependent. For the Galerkin finite element discretization, only the first two terms in $\|\cdot\|_{\text{spg}}$ are involved in the error bounds, see Theorem 4.14. The error estimation in the norm $\|(\cdot,\cdot)\|_{\text{spg}}$ will give additional control on the error of the divergence, the term with the streamline derivative of the solution and the gradient of the pressure, and for discontinuous pressure approximations on the jumps of the discrete pressure. The amount of control depends on the parameters μ_K, δ_K, and γ_E, which have to be chosen such that

- the unique solvability of the finite element problem can be ensured and
- the asymptotic order of convergence of the error bounds becomes as high as possible.

□

Lemma 4.32. $\|(\cdot,\cdot)\|_{\text{spg}}$ defines a norm in $V^h \times Q^h$. *Let $\delta_K > 0$ for all $K \in \mathcal{T}^h$, in the case $Q^h \not\subset H^1(\Omega)$ let $\gamma_E > 0$ for all $E \in \mathcal{E}^h$, and let V^h, Q^h be conforming finite element spaces. Then, $\|(\cdot,\cdot)\|_{\text{spg}}$ defines a norm in $V^h \times Q^h$.*

Proof. Expression (4.41) is the square root of a sum of squares of semi-norms. Thus, it is clearly a semi-norm itself. It remains to prove that from $\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}} = 0$ it follows that $\boldsymbol{v}^h = \boldsymbol{0}$ and $q^h = 0$, see Definition A.5.

Let $\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}} = 0$, then all terms in (4.41) vanish. In particular it is $\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} = 0$. Since this expression is a norm in V^h, it follows that $\boldsymbol{v}^h = \boldsymbol{0}$. With this result, one gets

$$0 = \sum_{E\in\mathcal{E}^h} \gamma_E \left\|[|q|]_E\right\|_{L^2(E)}^2 + \sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla q^h\right\|_{L^2(K)}^2.$$

Because δ_K is assumed to be positive for all mesh cells, it follows that $\left\|\nabla q^h\right\|_{L^2(K)} = 0$ for all $K \in \mathcal{T}^h$. If $Q^h \in H^1(\Omega)$, then $\left\|[|q|]_E\right\|_{L^2(E)} = 0$ for all faces. Otherwise, one gets this property from the assumption $\gamma_E > 0$ for all faces. Altogether, it follows that q^h is a constant in Ω. The only globally constant function in Q^h is $q^h = 0$. Hence $\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}}$ defines a norm on $V^h \times Q^h$. ■

Theorem 4.33. Existence and uniqueness of a solution of (4.34). *Let $\boldsymbol{b}, c \in L^\infty(\Omega)$, let V^h and Q^h be conforming finite element spaces, and let the stabilization parameters be chosen such that*

$$0 < \delta_K \le \min\left\{\frac{h_K^2}{3\nu C_{\text{inv}}^2}, \frac{1}{3\|c\|_{L^\infty(K)}}\right\}, \quad 0 \le \mu_K \le \mu < \infty, \qquad (4.42)$$

and

$$0 < \gamma_E \leq \gamma < \infty$$

if $Q^h \not\subset H^1(\Omega)$. Then, the finite element problem (4.34) *possesses a unique solution.*

Proof. It will be shown that A_{spg} is coercive on $V^h \times Q^h$, it is bounded on $V^h \times Q^h$, and that L_{spg} is bounded as well. Then, the statement of the theorem follows from the Theorem of Lax–Milgram, Theorem B.3.

First, coercivity is shown. It is

$$\begin{aligned} & A_{\mathrm{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \\ & = \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{spg}}^2 + \sum_{K\in\mathcal{T}^h} \delta_K \left(-\nu\Delta\boldsymbol{v}^h + c\boldsymbol{v}^h, (\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + \nabla q^h\right)_K, \end{aligned} \tag{4.43}$$

where (4.4) was used. Now, the last terms on the right-hand side are estimated from above. One obtains with the Cauchy–Schwarz inequality (A.8), Young's inequality (A.4), and (4.42)

$$\begin{aligned} & \left|\sum_{K\in\mathcal{T}^h} \delta_K \left(c\boldsymbol{v}^h, (\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + \nabla q^h\right)_K\right| \\ & \leq \frac{3}{2}\sum_{K\in\mathcal{T}^h} \delta_K \left\|c\right\|_{L^\infty(K)} \left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(K)}^2 + \frac{1}{6}\sum_{K\in\mathcal{T}^h} \delta_K \left\|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)}^2 \\ & \leq \frac{1}{2}\left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + \frac{1}{6}\sum_{K\in\mathcal{T}^h} \delta_K \left\|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)}^2. \end{aligned} \tag{4.44}$$

Using in addition the inverse inequality (C.28) yields

$$\begin{aligned} & \left|\sum_{K\in\mathcal{T}^h} \delta_K \left(-\nu\Delta\boldsymbol{v}^h, (\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + \nabla q^h\right)_K\right| \\ & \leq \frac{3}{2}\sum_{K\in\mathcal{T}^h} \delta_K \frac{C_{\mathrm{inv}}^2}{h_K^2}\nu^2 \left\|\nabla\boldsymbol{v}^h\right\|_{L^2(K)}^2 + \frac{1}{6}\sum_{K\in\mathcal{T}^h} \delta_K \left\|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)}^2 \\ & \leq \frac{\nu}{2}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + \frac{1}{6}\sum_{K\in\mathcal{T}^h} \delta_K \left\|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)}^2. \end{aligned} \tag{4.45}$$

Subtracting these upper bounds in (4.43) gives

$$A_{\mathrm{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \geq \frac{1}{2}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{spg}}^2, \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h, \tag{4.46}$$

which is the coercivity of A_{spg} on $V^h \times Q^h$ since $\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{spg}}$ defines a norm on $V^h \times Q^h$.

Next, the boundedness of A_{spg} will be studied. Straightforward estimates, using the Cauchy–Schwarz inequality, Hölder's inequality, and the inverse estimate, give

$$
\begin{aligned}
&A_{\text{spg}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \\
&\leq \nu \left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} + \|\boldsymbol{b}\|_{L^\infty(\Omega)} \left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} \left\|\boldsymbol{v}^h\right\|_{L^2(\Omega)} \\
&\quad + \left\|c^{1/2}\boldsymbol{u}^h\right\|_{L^2(\Omega)} \left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(\Omega)} + \left\|p^h\right\|_{L^2(\Omega)} \left\|\nabla\cdot \boldsymbol{v}^h\right\|_{L^2(\Omega)} \\
&\quad + \left\|q^h\right\|_{L^2(\Omega)} \left\|\nabla\cdot \boldsymbol{u}^h\right\|_{L^2(\Omega)} + \sum_{K\in\mathcal{T}^h} \mu_K \left\|\nabla\cdot \boldsymbol{u}^h\right\|_{L^2(K)} \left\|\nabla\cdot \boldsymbol{v}^h\right\|_{L^2(K)} \\
&\quad + \sum_{E\in\mathcal{E}^h} \gamma_E \left\|\left[\left|p^h\right|\right]_E\right\|_{L^2(E)} \left\|\left[\left|q^h\right|\right]_E\right\|_{L^2(E)} \\
&\quad + \sum_{K\in\mathcal{T}^h} \delta_K \left(\frac{C_{\text{inv}}}{h_K}\nu \left\|\nabla \boldsymbol{u}^h\right\|_{L^2(K)} + \left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}^h + \nabla p^h\right\|_{L^2(K)}\right. \\
&\quad \left. + \|c\|_{L^\infty(K)}^{1/2} \left\|c^{1/2}\boldsymbol{u}^h\right\|_{L^2(K)}\right) \left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)}.
\end{aligned}
$$

Applying the definition of the stabilization parameters (4.42) and the Cauchy–Schwarz inequality for sums (A.2) leads to

$$
\begin{aligned}
&A_{\text{spg}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \\
&\leq C\left[\nu^{1/2} \left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} + \|\boldsymbol{b}\|_{L^\infty(\Omega)} \left\|\nabla \boldsymbol{u}^h\right\|_{L^2(\Omega)} + \left\|c^{1/2}\boldsymbol{u}^h\right\|_{L^2(\Omega)}\right. \\
&\quad + \left\|p^h\right\|_{L^2(\Omega)} + \left\|\nabla\cdot \boldsymbol{u}^h\right\|_{L^2(\Omega)} + \left(\sum_{K\in\mathcal{T}^h} \mu_K \left\|\nabla\cdot \boldsymbol{u}^h\right\|_{L^2(K)}^2\right)^{1/2} \\
&\quad \left. + \left(\sum_{E\in\mathcal{E}^h} \gamma_E \left\|\left[\left|p^h\right|\right]_E\right\|_{L^2(E)}^2\right)^{1/2} + \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}^h + \nabla p^h\right\|_{L^2(K)}^2\right)^{1/2}\right] \\
&\quad \times\left[\nu^{1/2} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)} + \left\|\boldsymbol{v}^h\right\|_{L^2(\Omega)} + \left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(\Omega)} + \left\|q^h\right\|_{L^2(\Omega)} + \left\|\nabla\cdot \boldsymbol{v}^h\right\|_{L^2(\Omega)}\right. \\
&\quad + \left(\sum_{K\in\mathcal{T}^h} \mu_K \left\|\nabla\cdot \boldsymbol{v}^h\right\|_{L^2(K)}^2\right)^{1/2} + \left(\sum_{E\in\mathcal{E}^h} \gamma_E \left\|\left[\left|q^h\right|\right]_E\right\|_{L^2(E)}^2\right)^{1/2} \\
&\quad \left. + \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)}^2\right)^{1/2}\right].
\end{aligned}
$$

Both expressions in the brackets are clearly norms in $V^h \times Q^h$ since they have all terms of $\|\cdot\|_{\text{spg}}$ and some additional non-negative terms. Because in finite-dimensional spaces all norms are equivalent, Remark A.7, it follows that

$$
A_{\text{spg}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \leq C \left\|\left(\boldsymbol{u}^h, p^h\right)\right\|_{\text{spg}} \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}},
$$

for all $\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$. Maybe easier argument: continuity of bilinear form, from continuity it follows boundedness.

Finally, the boundedness of L_{spg} follows in the same way

$$L_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right)\right) \le C\left(\frac{1}{\nu}\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \left(\sum_{K\in\mathcal{T}^h}\delta_K\|\boldsymbol{f}\|^2_{L^2(K)}\right)^{1/2}\right)\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}}.$$

This estimate does not require the use of the norm equivalence. ■

Lemma 4.34. Stability of the finite element solution. *Let the assumptions of Theorem 4.33 be satisfied, then, the solution of* (4.34) *fulfills*

$$\left\|\left(\boldsymbol{u}^h, p^h\right)\right\|^2_{\text{spg}} \le \frac{12}{5}\min\left\{\frac{\|\boldsymbol{f}\|^2_{H^{-1}(\Omega)}}{\nu}, \frac{\|\boldsymbol{f}\|^2_{L_2(\Omega)}}{c_0}\right\} + 4\sum_{K\in\mathcal{T}^h}\delta_K\|\boldsymbol{f}\|^2_{L^2(K)}. \tag{4.47}$$

If the finite element spaces fulfill the discrete inf-sup condition (2.45)*, then there holds the stability estimate*

$$\begin{aligned}
&\left\|p^h\right\|_{L^2(\Omega)} \\
&\le \frac{C}{\beta^h_{\text{is}}}\Bigg[\|\boldsymbol{f}\|_{H^{-1}(\Omega)} + \delta\|\boldsymbol{b}\|_{L^\infty(\Omega)}\|\boldsymbol{f}\|_{L^2(\Omega)} + \Bigg(\nu^{1/2} + \|c\|^{1/2}_{L^\infty(\Omega)} + \mu^{1/2} \\
&\quad + \|\boldsymbol{b}\|_{L^\infty(\Omega)}\min\left\{\frac{1}{\nu^{1/2}}, \frac{1}{c_0^{1/2}}\right\} + \delta^{1/2}\|\boldsymbol{b}\|_{L^\infty(\Omega)}\Bigg)\left\|\left(\boldsymbol{u}^h, p^h\right)\right\|_{\text{spg}}\Bigg],
\end{aligned} \tag{4.48}$$

where the constant C depends on the domain.

Proof. As usual, the solution of the equations will be used as test function. Inserting $\left(\boldsymbol{u}^h, p^h\right)$ in (4.34) and using (4.4) gives

$$\begin{aligned}
\left\|\left(\boldsymbol{u}^h, p^h\right)\right\|^2_{\text{spg}} &= \left(\boldsymbol{f}, \boldsymbol{u}^h\right) + \sum_{K\in\mathcal{T}^h}\delta_K\left(\boldsymbol{f}, \left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h + \nabla p^h\right)_K \\
&\quad - \sum_{K\in\mathcal{T}^h}\delta_K\left(-\nu\Delta\boldsymbol{u}^h + c\boldsymbol{u}^h, \left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h + \nabla p^h\right)_K.
\end{aligned}$$

The terms on the right-hand side are estimated by using the Cauchy–Schwarz inequality (A.8) and Young's inequality (A.4), see the proof of Lemma 4.8 for the first term and (4.44) and (4.45) for the other terms

$$\left|\left(\boldsymbol{f},\boldsymbol{u}^h\right)\right| \le \frac{3}{5\nu}\left\|\boldsymbol{f}\right\|_{H^{-1}(\Omega)}^2 + \frac{5\nu}{12}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}^2,$$

$$\left|\left(\boldsymbol{f},\boldsymbol{u}^h\right)\right| \le \frac{3}{5c_0}\left\|\boldsymbol{f}\right\|_{L^2(\Omega)}^2 + \frac{5}{12}\left\|c^{1/2}\boldsymbol{u}^h\right\|_{L^2(\Omega)}^2,$$

$$\left|\sum_{K\in\mathcal{T}^h}\delta_K\left(c\boldsymbol{u}^h,\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h+\nabla p^h\right)_K\right|$$
$$\le \frac{1}{3}\left\|c^{1/2}\boldsymbol{u}^h\right\|_{L^2(\Omega)}^2 + \frac{1}{4}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h+\nabla p^h\right\|_{L^2(K)}^2,$$

$$\left|\sum_{K\in\mathcal{T}^h}\delta_K\left(-\nu\Delta\boldsymbol{u}^h,\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h+\nabla p^h\right)_K\right|$$
$$\le \frac{\nu}{3}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}^2 + \frac{1}{4}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h+\nabla p^h\right\|_{L^2(K)}^2.$$

For the remaining term, one obtains

$$\left|\sum_{K\in\mathcal{T}^h}\delta_K\left(\boldsymbol{f},\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h+\nabla p^h\right)_K\right|$$
$$\le \sum_{K\in\mathcal{T}^h}\delta_K\left\|\boldsymbol{f}\right\|_{L^2(K)}^2 + \frac{1}{4}\sum_{K\in\mathcal{T}^h}\delta_K\left\|\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h+\nabla p^h\right\|_{L^2(K)}^2.$$

Absorbing all terms into the left-hand side gives the first statement of the lemma.

For inf-sup stable discretizations, the stability estimate for the finite element pressure starts with the discrete inf-sup condition (2.45), e.g., see the proof of Lemma 4.8. Then, $-(\nabla\cdot\boldsymbol{v}^h,p^h)$ is substituted by (4.34). In the next step, it is noted that the terms with q^h cancel, which follows from choosing in (4.34) as test functions $(\boldsymbol{0},q^h)$. For the terms coming from the stabilization, the Cauchy–Schwarz inequality for sums (A.2) is applied and the bounds for the stabilization parameters (4.42) are used to obtain, e.g.,

$$\sum_{K\in\mathcal{T}^h}\delta_K\frac{C_{\text{inv}}}{h_K}\nu\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(K)}\left\|\boldsymbol{b}\right\|_{L^\infty(K)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(K)}$$
$$\le C\delta^{1/2}\left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)}\nu^{1/2}\left\|\nabla\boldsymbol{u}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}.$$

In this way, and by using the Poincaré–Friedrichs inequality (A.9), one gets a factor $\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}$ in all terms of the numerator such that these terms cancel with the denominator. ■

Remark 4.35. On the stability estimate.

- The estimate (4.47) is in a stronger norm than for the Galerkin discretization, see (4.12).
- An estimate of $\left\|p^h\right\|_{L^2(\Omega)}$ for inf-sup stable pairs of finite element spaces in terms of the data of the problem is obtained by inserting (4.47) into the right-hand side of (4.48).

□

Remark 4.36. On the way of proving an error estimate in a norm which contains the $L^2(\Omega)$ norm of the pressure. The approach that was used in Tobiska and Verfürth (1996) to prove an error estimate for the SUPG/PSPG/grad-div method differs from the approach for the Galerkin discretization. This way utilizes a discrete inf-sup condition for the bilinear form A_{spg}.

So far, the norm $\|\cdot\|_{\text{spg}}$ was used in the analysis. This norm does not control the $L^2(\Omega)$ norm of the pressure. If $Q^h \subset H^1(\Omega)$, only the gradient of the pressure is contained in a mixed term with the streamline derivative of the velocity. However, from the analysis of the continuous problem, it is known that the natural norm for the pressure is the $L^2(\Omega)$ norm. The finite element error analysis which will be performed now uses a norm which, in addition to $\|\cdot\|_{\text{spg}}$, possesses a contribution of the pressure

$$\|(\boldsymbol{v}, q)\|_{\text{spg},p} = \left(\|(\boldsymbol{v}, q)\|_{\text{spg}}^2 + \omega_{\text{pres}}^{-2} \|q\|_{L^2(\Omega)}^2\right)^{1/2} \tag{4.49}$$

with

$$\omega_{\text{pres}} = \max\left\{1, \nu^{-1/2}, \|c\|_{L^\infty(\Omega)}^{1/2}\right\}. \tag{4.50}$$

However, for the interesting cases of small ν and large c (small time steps), the contribution of the pressure becomes small:

$$\omega_{\text{pres}}^{-2} = \min\left\{1, \nu, \|c\|_{L^\infty(\Omega)}^{-1}\right\}.$$

□

Lemma 4.37. Inf-sup condition for A_{spg}. *Let the assumptions of Theorem 4.33 be satisfied, let in addition exist a positive constants δ_0 such that for all triangulations in the family $\{\mathcal{T}^h\}_{h>0}$ it holds*

$$0 < \delta_0 h_K^2 \leq \delta_K \quad \forall\, K \in \mathcal{T}^h. \tag{4.51}$$

If $Q^h \not\subset H^1(\Omega)$, then let there be a positive constant γ_0 such that for all triangulations in the family $\{\mathcal{T}^h\}_{h>0}$ one has

$$0 < \gamma_0 h_E \leq \gamma_E \quad \forall\, E \in \mathcal{E}^h. \tag{4.52}$$

Then, there is a constant $\beta_{\text{spg}}^h > 0$, such that

$$\inf_{\substack{(\boldsymbol{v}^h, q^h) \in V^h \times Q^h \\ \|(\boldsymbol{v}^h, q^h)\|_{\text{spg},p} = 1}} \sup_{\substack{(\boldsymbol{w}^h, r^h) \in V^h \times Q^h \\ \|(\boldsymbol{w}^h, r^h)\|_{\text{spg},p} = 1}} A_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, r^h\right)\right) \geq \beta_{\text{spg}}^h. \tag{4.53}$$

The inf-sup constant is independent of h and ν, see also Remark 4.38.

Proof. The plan of the proof is as follows:

i) For all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$ a function $\boldsymbol{w}^h \in V^h$ will be constructed such that one obtains an estimate of the form

$$A_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, 0\right)\right) \geq \text{negative terms} + C \left\|q^h\right\|_{L^2(\Omega)}^2$$

with $C > 0$.

ii) A linear combination of the result of the first step and the coercivity condition (4.46) is constructed such that the negative terms from the first step are absorbed. The result is of the form

$$A_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left((1-\rho)\boldsymbol{v}^h + \rho \boldsymbol{w}^h, (1-\rho)q^h\right)\right) \geq C \left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg},p}^2$$

with $C > 0$ and $\rho \in (0,1)$.

iii) It is proved that

$$\left\|\left((1-\rho)\boldsymbol{v}^h + \rho \boldsymbol{w}^h, (1-\rho)q^h\right)\right\|_{\text{spg},p} \leq 2 \left\|\boldsymbol{v}^h, q^h\right\|_{\text{spg},p},$$

which is inserted into the right-hand side of the result from the second step. Then, the discrete inf-sup condition (4.53) follows with straighforward arguments.

Step i). Consider an arbitrary pair $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$ and set for abbreviation

$$X := \left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)}^2\right)^{1/2},$$

$$Y := \left(\sum_{K\in\mathcal{T}^h} \mu_K \left\|\nabla\cdot\boldsymbol{v}^h\right\|_{L^2(K)}^2\right)^{1/2},$$

$$Z := \left(\sum_{E\in\mathcal{E}^h} \gamma_E \left\|\left[\left|q^h\right|\right]_E\right\|_{L^2(E)}^2\right)^{1/2}.$$

From (4.46) it is known that

$$A_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \geq \frac{1}{2}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}}^2, \quad \forall \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h. \tag{4.54}$$

The spaces V and Q satisfy the inf-sup condition, see Theorem 2.40. From Corollary 2.38 it follows that for $q^h \subset Q$ there is a $\boldsymbol{w} \in V_{\text{div}}^{\perp} \subset V$ with $\nabla\cdot\boldsymbol{w} = -q^h$ and $\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \leq C\left\|q^h\right\|_{L^2(\Omega)}$. Let $\boldsymbol{w}^h = I^h\boldsymbol{w}$ be an interpolation of $\boldsymbol{w}$ which fulfills the standard interpolation properties, in particular the following interpolation properties are assumed

$$\left\|\boldsymbol{w} - I^h\boldsymbol{w}\right\|_{H^m(K)} \leq C h_K^{l-m}\|\boldsymbol{w}\|_{H^l(K)} \quad \forall\, \boldsymbol{w} \in H^l(K), \tag{4.55}$$

$$\left\|\boldsymbol{w} - I^h\boldsymbol{w}\right\|_{L^2(E)} \leq C h_E^{l-1/2}\|\boldsymbol{w}\|_{H^l(K_1\cup K_2)} \quad \forall\, \boldsymbol{w} \in H^l(K_1\cup K_2), \tag{4.56}$$

where E is the joint face of K_1 and K_2, $H^0(K) = L^2(K)$, $0 \leq m \leq 2$, $\max\{1, m\} \leq l$. in the literature, the Scott-Zhang operator is used, check It follows with $l = m = 1$ in (4.55) that

$$\begin{aligned}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(\Omega)} &\leq \left\|\nabla\boldsymbol{w} - \nabla\boldsymbol{w}^h\right\|_{L^2(\Omega)} + \|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \leq C\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} + \|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \\ &\leq C_1\left\|q^h\right\|_{L^2(\Omega)}.\end{aligned} \tag{4.57}$$

Adding $-\left(\nabla\cdot\boldsymbol{w}, q^h\right) + \left(\nabla\cdot\boldsymbol{w}, q^h\right)$ and using integration by parts yields

$$
\begin{aligned}
&A_{\mathrm{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, 0\right)\right) \\
&= \nu\left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{w}^h\right) + \left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h, \boldsymbol{w}^h\right) + \left(c\boldsymbol{v}^h, \boldsymbol{w}^h\right) - \left(\nabla\cdot\boldsymbol{w}, q^h\right) \\
&\quad - \sum_{K\in\mathcal{T}^h}\left(\boldsymbol{w}-\boldsymbol{w}^h, \nabla q^h\right)_K + \sum_{E\in\mathcal{E}^h}\left(\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\cdot\boldsymbol{n}_E, \left[|q^h|\right]_E\right)_E \\
&\quad + \sum_{K\in\mathcal{T}^h}\mu_K\left(\nabla\cdot\boldsymbol{v}^h, \nabla\cdot\boldsymbol{w}^h\right)_K \\
&\quad + \sum_{K\in\mathcal{T}^h}\delta_K\left(-\nu\Delta\boldsymbol{v}^h + (\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + c\boldsymbol{v}^h + \nabla q^h, (\boldsymbol{b}\cdot\nabla)\,\boldsymbol{w}^h\right)_K .
\end{aligned}
$$

Now, applying the Cauchy–Schwarz inequality (A.8), adding $(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h$, using the triangle inequality, the Poincaré–Friedrichs inequality (A.9), (2.39), (4.57), and $\nabla\cdot\boldsymbol{w} = -q^h$ leads to

$$
\begin{aligned}
&A_{\mathrm{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, 0\right)\right) \\
&\geq -C_1\Big(\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)} + \|\boldsymbol{b}\|_{L^\infty(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)}^{1/2}\left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(\Omega)}\Big)\left\|q^h\right\|_{L^2(\Omega)} \\
&\quad - \sum_{K\in\mathcal{T}^h}\left\|\boldsymbol{w}-\boldsymbol{w}^h\right\|_{L^2(K)}\Big(\left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)} + \left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h\right\|_{L^2(K)}\Big) \\
&\quad - \sum_{E\in\mathcal{E}^h}\left\|\boldsymbol{w}-\boldsymbol{w}^h\right\|_{L^2(E)}\left\|\left[|q^h|\right]_E\right\|_{L^2(E)} \\
&\quad - \sum_{K\in\mathcal{T}^h}\mu_K\left\|\nabla\cdot\boldsymbol{v}^h\right\|_{L^2(K)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(K)} \\
&\quad - \sum_{K\in\mathcal{T}^h}\delta_K\Bigg[\nu\left\|\Delta\boldsymbol{v}^h\right\|_{L^2(K)} + \left\|c^{1/2}\right\|_{L^\infty(K)}\left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(K)} \\
&\quad + \left\|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)}\Bigg]\|\boldsymbol{b}\|_{L^\infty(K)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(K)} + \left\|q^h\right\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

The individual terms are estimated seperately, always using the Cauchy–Schwarz inequality (A.8), with the goal to get the factor $\left\|q^h\right\|_{L^2(\Omega)}$. One obtains with (4.55), (4.51), the Poincaré–Friedrichs inequality, and (4.57)

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}^h}\left\|\boldsymbol{w}-\boldsymbol{w}^h\right\|_{L^2(K)}\Big(\left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)} + \left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h\right\|_{L^2(K)}\Big) \\
&\leq C\sum_{K\in\mathcal{T}^h}h_K\|\boldsymbol{w}\|_{H^1(K)}\Big(\left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h + \nabla q^h\right\|_{L^2(K)} + \left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}^h\right\|_{L^2(K)}\Big) \\
&\leq CX\left(\sum_{K\in\mathcal{T}^h}h_K^2\delta_K^{-1}\|\boldsymbol{w}\|_{H^1(K)}^2\right)^{1/2} \\
&\quad + C\|\boldsymbol{b}\|_{L^\infty(\Omega)}\left(\sum_{K\in\mathcal{T}^h}h_K^2\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(K)}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}^h}\|\boldsymbol{w}\|_{H^1(K)}^2\right)^{1/2} \\
&\leq C\left(\delta_0^{-1/2}X + h\|\boldsymbol{b}\|_{L^\infty(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\right)\|\nabla\boldsymbol{w}\|_{L^2(\Omega)} \\
&\leq CC_1\left(\delta_0^{-1/2}X + h\|\boldsymbol{b}\|_{L^\infty(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\right)\left\|q^h\right\|_{L^2(\Omega)} .
\end{aligned}
$$

For the next term, one gets with (4.55), (4.52), the Poincaré–Friedrichs inequality, and (4.57)

$$\begin{aligned}
&\sum_{E\in\mathcal{E}^h} \left\|\boldsymbol{w}-\boldsymbol{w}^h\right\|_{L^2(E)} \left\|\left[\left|q^h\right|\right]_E\right\|_{L^2(E)} \\
&\le C\sum_{E\in\mathcal{E}^h} \|\boldsymbol{w}\|_{H^1(K_1\cup K_2)} \left\|\left[\left|q^h\right|\right]_E\right\|_{L^2(E)} \le CZ\left(\sum_{E\in\mathcal{E}^h}\gamma_E^{-1}h_E\|\boldsymbol{w}\|^2_{H^1(K_1\cup K_2)}\right)^{1/2} \\
&\le CC_1\gamma_0^{-1/2}Z\left\|q^h\right\|_{L^2(\Omega)}.
\end{aligned}$$

The estimate of the next term uses again (4.57) leading to

$$\sum_{K\in\mathcal{T}^h}\mu_K\left\|\nabla\cdot\boldsymbol{v}^h\right\|_{L^2(K)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(K)} \le C_1\mu^{1/2}Y\left\|q^h\right\|_{L^2(\Omega)}.$$

For the SUPG/PSPG term, one obtains with the inverse estimate (C.28), (4.42), and (4.57)

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^h}\delta_K\nu\left\|\Delta\boldsymbol{v}^h\right\|_{L^2(K)}\|\boldsymbol{b}\|_{L^\infty(K)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(K)} \\
&\le \sum_{K\in\mathcal{T}^h}C_{\mathrm{inv}}\delta_K h_K^{-1}\nu\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(K)}\|\boldsymbol{b}\|_{L^\infty(K)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(K)} \\
&\le CC_1h\|\boldsymbol{b}\|_{L^\infty(\Omega)}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|q^h\right\|_{L^2(\Omega)}.
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^h}\delta_K\|c\|^{1/2}_{L^\infty(K)}\left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(K)}\|\boldsymbol{b}\|_{L^\infty(K)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(K)} \\
&\le C_1\delta\|c\|^{1/2}_{L^\infty(\Omega)}\|\boldsymbol{b}\|_{L^\infty(\Omega)}\left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|q^h\right\|_{L^2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^h}\delta_K\left\|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{v}^h+\nabla q^h\right\|_{L^2(K)}\|\boldsymbol{b}\|_{L^\infty(K)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(K)} \\
&\qquad\le C_1\delta^{1/2}\|\boldsymbol{b}\|_{L^\infty(\Omega)}X\left\|q^h\right\|_{L^2(\Omega)}.
\end{aligned}$$

Collecting terms and applying Young's inequality (A.4) term by term leads to

$$
\begin{aligned}
&A_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left(\boldsymbol{w}^h, 0\right)\right) \\
&\geq -\left[\left(C_1\frac{\nu}{\nu^{1/2}} + C_1\frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{\nu^{1/2}} + CC_1 h\frac{\|\boldsymbol{b}\|_{L^\infty(\Omega)}}{\nu^{1/2}}\right)\nu^{1/2}\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\right. \\
&\quad +\left(C_1\|c\|_{L^\infty(\Omega)}^{1/2} + C_1\delta\|c\|_{L^\infty(\Omega)}^{1/2}\|\boldsymbol{b}\|_{L^\infty(\Omega)}\right)\left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(\Omega)} \\
&\quad \left.+C_1\mu^{1/2}Y + CC_1\gamma_0^{-1/2}Z + \left(CC_1\delta_0^{-1/2} + C_1\delta^{1/2}\|\boldsymbol{b}\|_{L^\infty(\Omega)}\right)X\right]\left\|q^h\right\|_{L^2(\Omega)} \\
&\quad +\left\|q^h\right\|_{L^2(\Omega)}^2 \\
&\geq -\omega_{\text{pres}}\widehat{C}\left[\nu^{1/2}\|\nabla\boldsymbol{v}\|_{L^2(\Omega)} + \left\|c^{1/2}\boldsymbol{v}^h\right\|_{L^2(\Omega)} + Y + Z + X\right]\left\|q^h\right\|_{L^2(\Omega)} + \left\|q^h\right\|_{L^2(\Omega)}^2 \\
&\geq -2\omega_{\text{pres}}^2\widehat{C}^2\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}}^2 + \frac{3}{8}\left\|q^h\right\|_{L^2(\Omega)}^2, \qquad (4.58)
\end{aligned}
$$

where ω_{pres} is defined in (4.50) and

$$
\begin{aligned}
\widehat{C} = \max\Big\{&C_1\nu + C_1\|\boldsymbol{b}\|_{L^\infty(\Omega)} + CC_1 h\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \qquad (4.59)\\
&C_1 + C_1\delta\|\boldsymbol{b}\|_{L^\infty(\Omega)}, C_1\mu^{1/2}, CC_1\gamma_0^{-1/2}, \left(CC_1\delta_0^{-1/2} + C_1\delta^{1/2}\|\boldsymbol{b}\|_{L^\infty(\Omega)}\right)\Big\}.
\end{aligned}
$$

Step ii). Let $\rho > 0$, then adding $(1-\rho)$ times (4.54) and ρ times (4.58) gives

$$
\begin{aligned}
&A_{\text{spg}}\left(\left(\boldsymbol{v}^h, q^h\right), \left((1-\rho)\boldsymbol{v}^h + \rho\boldsymbol{w}^h, (1-\rho)q^h\right)\right) \\
&\geq \left(\frac{1}{2} - \frac{\rho}{2} - 2\rho\left(\omega_{\text{pres}}\widehat{C}\right)^2\right)\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\text{spg}}^2 + \frac{3\rho}{8}\left\|q^h\right\|_{L^2(\Omega)}^2.
\end{aligned}
$$

The term in the parentheses should be positive. Requiring that

$$
\frac{1}{2} - \frac{\rho}{2} - 2\rho\left(\omega_{\text{pres}}\widehat{C}\right)^2 \leq \frac{3\rho}{8}\omega_{\text{pres}}^2
$$

leads to

$$
\frac{1}{2} \leq \left(\frac{3}{8}\omega_{\text{pres}}^2 + \frac{1}{2} + 2\left(\omega_{\text{pres}}\widehat{C}\right)^2\right)\rho \leq \left(\frac{3}{8} + \frac{1}{2} + 2\widehat{C}^2\right)\omega_{\text{pres}}^2\rho
$$

because $\omega_{\text{pres}} \geq 1$. The following choice satisfies the last requirement

$$
0 < \rho = \frac{4}{7 + 16\widehat{C}^2}\omega_{\text{pres}}^{-2} < 1. \qquad (4.60)
$$

For the term in parentheses, one gets with this choice and $\omega_{\text{pres}}^{-2} \leq 1$

$$
\begin{aligned}
\frac{1}{2} - \frac{\rho}{2} - 2\rho\left(\omega_{\text{pres}}\widehat{C}\right)^2 &= \frac{1}{2} - \frac{2}{7 + 16\widehat{C}^2}\left(\omega_{\text{pres}}^{-2} + 4\widehat{C}^2\right) \\
&\geq \frac{1}{2} - \frac{2 + 8\widehat{C}^2}{7 + 16\widehat{C}^2} = \frac{1}{2}\left(1 - \frac{4 + 16\widehat{C}^2}{7 + 16\widehat{C}^2}\right) > 0.
\end{aligned}
$$

In this way, one obtains the estimate

$$
\begin{aligned}
& A_{\mathrm{spg}}\left(\left(\boldsymbol{v}^h, q^h\right),\left((1-\rho)\boldsymbol{v}^h+\rho\boldsymbol{w}^h,(1-\rho)q^h\right)\right) \\
& \geq \frac{3}{14+32\widehat{C}^2}\left(\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{spg}}^2+\omega_{\mathrm{pres}}^{-2}\left\|q^h\right\|_{L^2(\Omega)}^2\right) \\
& =\frac{3}{14+32\widehat{C}^2}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{spg},p}^2 .
\end{aligned} \tag{4.61}
$$

Step iii). On the other hand, one has by the triangle inequality, the definition of $\|(\cdot,\cdot)\|_{\mathrm{spg},p}$, (2.39), (4.57), (4.50), (4.59), (4.60), and once more the definition of $\|(\cdot,\cdot)\|_{\mathrm{spg},p}$

$$
\begin{aligned}
& \left\|\left((1-\rho)\boldsymbol{v}^h+\rho\boldsymbol{w}^h,(1-\rho)q^h\right)\right\|_{\mathrm{spg},p} \\
& \leq(1-\rho)\left\|\boldsymbol{v}^h, q^h\right\|_{\mathrm{spg},p}+\rho\Bigg(\nu\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(\Omega)}^2+\nu\left\|c^{1/2}\boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 \\
& \quad+\sum_{K\in\mathcal{T}^h}\mu_K\left\|\nabla\cdot\boldsymbol{w}^h\right\|_{L^2(K)}^2+\sum_{K\in\mathcal{T}^h}\delta_K\left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}^h\right\|_{L^2(K)}^2\Bigg)^{1/2} \\
& \leq\left\|\boldsymbol{v}^h, q^h\right\|_{\mathrm{spg},p}+\rho\left(C_1^2\nu+C_1^2\|c\|_{L^\infty(\Omega)}+C_1^2\mu d+C_1^2\delta\|\boldsymbol{b}\|_{L^\infty(\Omega)}^2\right)^{1/2}\left\|q^h\right\|_{L^2(\Omega)} \\
& \leq\left\|\boldsymbol{v}^h, q^h\right\|_{\mathrm{spg},p}+2\rho\max\left\{1,\nu^{-1/2},\|c\|_{L^\infty(\Omega)}^{1/2}\right\} \\
& \quad\times\max\left\{C_1\nu, C_1, C_1\mu^{1/2}, C_1\delta^{1/2}\|\boldsymbol{b}\|_{L^\infty(\Omega)}\right\}\left\|q^h\right\|_{L^2(\Omega)} \\
& \leq\left\|\boldsymbol{v}^h, q^h\right\|_{\mathrm{spg},p}+2\rho\omega_{\mathrm{pres}}\widehat{C}\left\|q^h\right\|_{L^2(\Omega)} \\
& \leq\left\|\boldsymbol{v}^h, q^h\right\|_{\mathrm{spg},p}+\frac{8\widehat{C}}{7+16\widehat{C}^2}\omega_{\mathrm{pres}}^{-1}\left\|q^h\right\|_{L^2(\Omega)} \\
& \leq(1+1)\left\|\boldsymbol{v}^h, q^h\right\|_{\mathrm{spg},p}=2\left\|\boldsymbol{v}^h, q^h\right\|_{\mathrm{spg},p},
\end{aligned}
$$

where $8\widehat{C}/(7+16\widehat{C}^2)\leq 1/\sqrt{7}<1$ has been used in the last step.

Inserting this estimate into (4.61) gives for all $\left(\boldsymbol{v}^h, q^h\right)\in V^h\times Q^h$

$$
\begin{aligned}
& A_{\mathrm{spg}}\left(\left(\boldsymbol{v}^h, q^h\right),\left((1-\rho)\boldsymbol{v}^h+\rho\boldsymbol{w}^h,(1-\rho)q^h\right)\right) \\
& \geq\frac{3}{28+64\widehat{C}^2}\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{spg},p}\left\|\left((1-\rho)\boldsymbol{v}^h+\rho\boldsymbol{w}^h,(1-\rho)q^h\right)\right\|_{\mathrm{spg},p}
\end{aligned}
$$

or

$$
\begin{aligned}
& A_{\mathrm{spg}}\left(\frac{\left(\boldsymbol{v}^h, q^h\right)}{\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{spg},p}},\frac{\left((1-\rho)\boldsymbol{v}^h+\rho\boldsymbol{w}^h,(1-\rho)q^h\right)}{\left\|\left((1-\rho)\boldsymbol{v}^h+\rho\boldsymbol{w}^h,(1-\rho)q^h\right)\right\|_{\mathrm{spg},p}}\right) \\
& \qquad\qquad\geq\frac{3}{28+64\widehat{C}^2}=\beta_{\mathrm{spg}}^h .
\end{aligned} \tag{4.62}
$$

The arguments of the bilinear form are normalized with respect to $\|\cdot\|_{\mathrm{spg},p}$. The inequality stays valid if, for each $\left(\boldsymbol{v}^h, q^h\right)\in V^h\times Q^h$, the supremum of normalized functions with respect to the second argument of the bilinear form is considered

$$
\sup_{\substack{\left(\boldsymbol{w}^h, r^h\right)\in V^h\times Q^h\\ \left\|\left(\boldsymbol{w}^h, r^h\right)\right\|_{\mathrm{spg},p}=1}} A_{\mathrm{spg}}\left(\frac{\left(\boldsymbol{v}^h, q^h\right)}{\left\|\left(\boldsymbol{v}^h, q^h\right)\right\|_{\mathrm{spg},p}},\left(\boldsymbol{w}^h, r^h\right)\right)\geq\beta_{\mathrm{spg}}^h .
$$

Since this inequality holds still for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$, it is valid also for the infimum of normalized functions with respect to the first argument, such that (4.53) is proved. ■

Remark 4.38. On β^h_{spg}. From the assumptions on the scaling of the Oseen equations, see Remark 4.3, it follows that the constant $\widehat{C}$ in (4.59) behaves like $\mathcal{O}(1)$ with respect to the coefficients of the Oseen problem. Therefore, one obtains from (4.62) also that $\beta^h_{\mathrm{spg}} = \mathcal{O}(1)$ with respect to the coefficients of the Oseen equations.

An inf-sup condition of form (4.53) does not hold for the Galerkin discretization. That means, one has to expect that $\beta^h_{\mathrm{spg}} \to 0$ if the stabilization terms vanish. The vanishing of the SUPG/PSPG term is described by the size of δ_0 from (4.51). It can be seen in (4.59) that $\widehat{C} = \mathcal{O}\left(\delta_0^{-1/2}\right)$ from what follows that $\beta^h_{\mathrm{spg}} = \mathcal{O}(\delta_0)$ for $\delta_0 \to 0$.

If $Z \neq 0$, then one obtains with the same arguments that $\beta^h_{\mathrm{spg}} = \mathcal{O}(\gamma_0)$ for $\gamma_0 \to 0$. □

Corollary 4.39. Existence and uniqueness of a solution of (4.34)**.** *Let the assumptions of Lemma 4.37 be satisfied, then problem* (4.34) *possesses a unique solution.*

Proof. The statement of the corollary is a direct consequence of the inf-sup condition (4.53) and Lemma B.15. ■

Theorem 4.40. Error estimate. *Let the assumptions of Lemma 4.37 be satisfied, let* $(\boldsymbol{u}.p)$ *be the solution of* (4.2) *and* $\left(\boldsymbol{u}^h, p^h\right)$ *be the solution of* (4.34). *With the assumptions* $\boldsymbol{u} \in H^{k+1}(\Omega)$, $k \geq 1$, *and* $p \in H^{l+1}(\Omega)$, $l \geq 0$, *the following error estimate holds*

$$\begin{aligned}
&\left\| \left(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h\right) \right\|_{\mathrm{spg},p} \\
&\leq C \Bigg[h^k \Bigg(\nu^{1/2} + \left(h + \delta^{1/2} h\right) \|c\|^{1/2}_{L^\infty(\Omega)} + \delta^{1/2} \|\boldsymbol{b}\|^{1/2}_{L^\infty(\Omega)} + \delta^{1/2} \\
&\quad + \delta_0^{-1/2} + \gamma_0^{-1/2} + \mu^{1/2} \Bigg) \|\boldsymbol{u}\|_{H^{k+1}(\Omega)} \\
&\quad + h^l \Bigg(\delta^{1/2} + h \min \left\{ \nu^{-1/2}, \max_{K \in \mathcal{T}^h} \left\{ \mu_K^{-1/2} \right\} \right\} + h \omega^{-1}_{\mathrm{pres}} \\
&\quad + \gamma^{1/2} \left(h + h^{1/2}\right) \Bigg) \|p\|_{H^{l+1}(\Omega)} \Bigg]
\end{aligned} \qquad (4.63)$$

with C *independent of the coefficients of the problem. The terms* $\gamma_0^{-1/2}$ *and* $\gamma^{1/2}\left(h + h^{1/2}\right)$ *do not appear if* $Q^h \subset H^1(\Omega)$.

Proof. Let $\boldsymbol{v}^h = I^h \boldsymbol{u}$ be the Lagrangian interpolation of $\boldsymbol{u}$ in V^h and $q^h = P^h_{L^2} q$ be the $L^2(\Omega)$ projection of q onto Q^h.

The triangle inequality gives

$$\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\mathrm{spg},p} \le \left\|\left(\boldsymbol{u}-\boldsymbol{v}^h, p-q^h\right)\right\|_{\mathrm{spg},p} + \left\|\left(\boldsymbol{u}^h-\boldsymbol{v}^h, p^h-q^h\right)\right\|_{\mathrm{spg},p}. \quad (4.64)$$

First, the interpolation error will be considered. Using the definition of the norm, the interpolation estimates (C.11), estimate (2.39), the estimates for the $L^2(\Omega)$ projection (C.24) and (C.25), estimate (**??**), for faces and (4.37) one obtains

$$\begin{aligned}
&\left\|\left(\boldsymbol{u}-\boldsymbol{v}^h, p-q^h\right)\right\|_{\mathrm{spg},p} \\
&= \Bigg(\nu\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}^h}\mu_K\left\|\nabla\cdot\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(K)}^2 \\
&\quad + \sum_{E\in\mathcal{E}^h}\gamma_E\left\|\left[\left|p-q^h\right|\right]_E\right\|_{L^2(E)}^2 + \sum_{K\in\mathcal{T}^h}\delta_K\left\|\left(\boldsymbol{b}\cdot\nabla\right)\left(\boldsymbol{u}-\boldsymbol{v}^h\right)+\nabla\left(p-q^h\right)\right\|_{L^2(K)}^2 \\
&\quad + \omega_{\mathrm{pres}}^{-2}\left\|p-q^h\right\|_{L^2(\Omega)}^2\Bigg)^{1/2} \\
&\le C\Bigg(\nu h^{2k}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \|c\|_{L^\infty(\Omega)}h^{2k+2}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \mu h^{2k}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 \\
&\quad + \gamma h^{2l+1}\|p\|_{H^{l+1}(\Omega)}^2 + \delta\|\boldsymbol{b}\|_{L^\infty(\Omega)}h^{2k}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}^2 + \delta h^{2l}\|p\|_{H^{l+1}(\Omega)}^2 \\
&\quad + \omega_{\mathrm{pres}}^{-2}h^{2l+2}\|p\|_{H^{l+1}(\Omega)}^2\Bigg)^{1/2} \\
&\le C\left(\nu + h^2\|c\|_{L^\infty(\Omega)} + \mu + \delta\|\boldsymbol{b}\|_{L^\infty(\Omega)}\right)^{1/2} h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} \\
&\quad + C\left(\gamma h + \delta + \omega_{\mathrm{pres}}^{-2}h^2\right)^{1/2} h^l\|p\|_{H^{l+1}(\Omega)} \\
&\le C\left(\nu^{1/2} + h\|c\|_{L^\infty(\Omega)}^{1/2} + \mu^{1/2} + \delta^{1/2}\|\boldsymbol{b}\|_{L^\infty(\Omega)}^{1/2}\right) h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)} \\
&\quad + C\left(\gamma^{1/2}h^{1/2} + \delta^{1/2} + \omega_{\mathrm{pres}}^{-1}h\right) h^l\|p\|_{H^{l+1}(\Omega)}. \qquad (4.65)
\end{aligned}$$

Next, the second term of (4.64) will be estimated. Scaling the inf-sup condition (4.53) and using the Galerkin orthogonality (4.40) leads to

$$\begin{aligned}
&\left\|\left(\boldsymbol{u}^h-\boldsymbol{v}^h, p^h-q^h\right)\right\|_{\mathrm{spg},p} \\
&\le \frac{1}{\beta_{\mathrm{spg}}^h}\sup_{\substack{\left(\boldsymbol{w}^h,r^h\right)\in V^h\times Q^h\\ \left\|\left(\boldsymbol{w}^h,r^h\right)\right\|_{\mathrm{spg},p}=1}} A_{\mathrm{spg}}\left(\left(\boldsymbol{u}^h-\boldsymbol{v}^h, p^h-q^h\right),\left(\boldsymbol{w}^h,r^h\right)\right) \\
&= \frac{1}{\beta_{\mathrm{spg}}^h}\sup_{\substack{\left(\boldsymbol{w}^h,r^h\right)\in V^h\times Q^h\\ \left\|\left(\boldsymbol{w}^h,r^h\right)\right\|_{\mathrm{spg},p}=1}} A_{\mathrm{spg}}\left(\left(\boldsymbol{u}-\boldsymbol{v}^h, p-q^h\right),\left(\boldsymbol{w}^h,r^h\right)\right). \qquad (4.66)
\end{aligned}$$

With this step, one got rid of $\left(\boldsymbol{u}^h, p^h\right)$ in the estimate. Now, all terms on the right-hand side of (4.66) are estimate individually. The goal is to estimate the terms with $\left(\boldsymbol{w}^h, r^h\right)$ with terms that are included in $\left\|\left(\boldsymbol{w}^h, r^h\right)\right\|_{\mathrm{spg},p}$ and then to estimate this norm with 1.

With the Cauchy–Schwarz inequality (A.8), $\left\|\left(\boldsymbol{w}^h, r^h\right)\right\|_{\mathrm{spg},p} = 1$, and the interpolation estimate (C.11), one obtains

$$\begin{aligned}\nu\left(\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right),\nabla\boldsymbol{w}^h\right) &\le \nu\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(\Omega)}\\ &\le C\nu^{1/2}h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\left\|\left(\boldsymbol{w}^h,r^h\right)\right\|_{\mathrm{spg},p}\\ &= C\nu^{1/2}h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\end{aligned}\tag{4.67}$$

and

$$\begin{aligned}\left(c\left(\boldsymbol{u}-\boldsymbol{v}^h\right),\boldsymbol{w}^h\right) &\le \left\|c^{1/2}\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}\left\|c^{1/2}\boldsymbol{w}^h\right\|_{L^2(\Omega)}\\ &\le C\|c\|_{L^\infty(\Omega)}^{1/2}h^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}.\end{aligned}\tag{4.68}$$

The estimate of the next term starts with integration by parts. Then, $\nabla\cdot\boldsymbol{b}=0$ is used, the Cauchy–Schwarz inequalities for integrals (A.8) and for sums (A.2) are applied, $\left\|\left(\boldsymbol{w}^h,r^h\right)\right\|_{\mathrm{spg},p}=1$ is utilized, the conditions (4.51) and (4.52) on the stabilization parameters are used, the interpolation estimate (C.11), the estimate (**??**), hier wird eine Absch"atzung von Kante auf Gitterzelle gemacht and $\left\|\left(\boldsymbol{w}^h,r^h\right)\right\|_{\mathrm{spg},p}=1$ are applied to get

$$\begin{aligned}&\left(\boldsymbol{b}\cdot\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right),\boldsymbol{w}^h\right)+\left(\nabla\cdot\left(\boldsymbol{u}-\boldsymbol{v}^h\right),r^h\right)\\ &=-\left(\boldsymbol{u}-\boldsymbol{v}^h,(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}^h\right)-\left(\boldsymbol{u}-\boldsymbol{v}^h,\nabla r^h\right)+\sum_{K\in\mathcal{T}^h}\sum_{E\subset\partial K}\left(\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\cdot\boldsymbol{n}_E,r^h\right)_E\\ &=-\left(\boldsymbol{u}-\boldsymbol{v}^h,(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}^h+\nabla r^h\right)+\sum_{E\in\mathcal{E}^h}\left(\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\cdot\boldsymbol{n}_E,\left[|r^h|\right]_E\right)_E\\ &\le\left(\sum_{K\in\mathcal{T}^h}\delta_K^{-1}\left\|\boldsymbol{u}-\boldsymbol{v}^h\right\|_{L^2(K)}^2\right)^{1/2}\left(\sum_{K\in\mathcal{T}^h}\delta_K\left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}^h+\nabla r^h\right\|_{L^2(K)}^2\right)^{1/2}\\ &\quad+\left(\sum_{E\in\mathcal{E}^h}\gamma_E^{-1}\left\|\boldsymbol{u}-\boldsymbol{v}^h\right\|_{L^2(E)}^2\right)^{1/2}\left(\sum_{E\in\mathcal{E}^h}\gamma_E\left\|\left[|r^h|\right]_E\right\|_{L^2(E)}^2\right)^{1/2}\\ &\le\left(C\delta_0^{-1}\sum_{K\in\mathcal{T}^h}h_K^{-2}h_K^{2k+2}\|\boldsymbol{u}\|_{H^{k+1}(K)}^2\right)^{1/2}\\ &\quad+\left(C\gamma_0^{-1}\sum_{E\in\mathcal{E}^h}h_E^{-1}h_E^{2k+1}\|\boldsymbol{u}\|_{H^{k+1}(K_1\cup K_2)}^2\right)^{1/2}\\ &\le C\left(\delta_0^{-1/2}+\gamma_0^{-1/2}\right)h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)},\end{aligned}\tag{4.69}$$

where $E=K_1\cap K_2$. The next part of the right-hand side of (4.66) can be estimated with the Cauchy–Schwarz inequality, (2.39), the norm property of $\left(\boldsymbol{w}^h,r^h\right)$, and the projection estimate (C.24)

$$\begin{aligned}\left(\nabla\cdot\boldsymbol{w}^h,p-q^h\right) &\le\left\|p-q^h\right\|_{L^2(\Omega)}\left\|\nabla\cdot\boldsymbol{w}^h\right\|_{L^2(\Omega)}\\ &\le\left\|p-q^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(\Omega)}\\ &=\nu^{-1/2}\left\|p-q^h\right\|_{L^2(\Omega)}\nu^{1/2}\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(\Omega)}\\ &\le\nu^{-1/2}\left\|p-q^h\right\|_{L^2(\Omega)}\\ &\le C\nu^{-1/2}h^{l+1}\|p\|_{H^{l+1}(\Omega)}.\end{aligned}$$

For the estimate of this term, also a different part of $\left\|\left(\boldsymbol{w}^h, r^h\right)\right\|_{\mathrm{spg},p}$ can be utilized, which gives

$$\begin{aligned}\left(\nabla\cdot\boldsymbol{w}^h, p-q^h\right) &\le \left\|p-q^h\right\|_{L^2(\Omega)}\left\|\nabla\cdot\boldsymbol{w}^h\right\|_{L^2(\Omega)}\\ &\le \max_{K\in\mathcal{T}^h}\left\{\mu_K^{-1/2}\right\}\left\|p-q^h\right\|_{L^2(\Omega)}\sum_{K\in\mathcal{T}^h}\mu_K^{1/2}\left\|\nabla\cdot\boldsymbol{w}^h\right\|_{L^2(K)}\\ &\le \max_{K\in\mathcal{T}^h}\left\{\mu_K^{-1/2}\right\}\left\|p-q^h\right\|_{L^2(\Omega)}\left(\sum_{K\in\mathcal{T}^h}\mu_K\left\|\nabla\cdot\boldsymbol{w}^h\right\|_{L^2(K)}^2\right)^{1/2}\\ &\le \max_{K\in\mathcal{T}^h}\left\{\mu_K^{-1/2}\right\}\left\|p-q^h\right\|_{L^2(\Omega)}\\ &\le C\max_{K\in\mathcal{T}^h}\left\{\mu_K^{-1/2}\right\}h^{l+1}\left\|p\right\|_{H^{l+1}(\Omega)}.\end{aligned}$$

Altogether, one obtains

$$\left(\nabla\cdot\boldsymbol{w}^h, p-q^h\right) \le C\min\left\{\nu^{-1/2}, \max_{K\in\mathcal{T}^h}\left\{\mu_K^{-1/2}\right\}\right\}h^{l+1}\left\|p\right\|_{H^{l+1}(\Omega)}. \qquad (4.70)$$

The term with the pressure jumps is estimated as follows details have to be worked out

$$\sum_{E\in\mathcal{E}^h}\gamma_E\left(\left[\left|p-q^h\right|\right]_E, \left[\left|r^h\right|\right]_E\right)_E \le C\gamma^{1/2}h^{l+1}\left\|p\right\|_{H^{l+1}(\Omega)}. \qquad (4.71)$$

Finally, the SUPG/PSPG terms have to be considered. One obtains with the Cauchy–Schwarz inequality for integrals and for sums, $\left\|\left(\boldsymbol{w}^h, r^h\right)\right\|_{\mathrm{spg},p}=1$, the condition (4.42) on the stabilization parameter, and the interpolation estimate (C.11)

$$\begin{aligned}&\sum_{K\in\mathcal{T}^h}\delta_K\left(-\nu\Delta\left(\boldsymbol{u}-\boldsymbol{v}^h\right)+c\left(\boldsymbol{u}-\boldsymbol{v}^h\right), (\boldsymbol{b}\cdot\nabla)\boldsymbol{w}^h+\nabla r^h\right)_K\\ &\le \sum_{K\in\mathcal{T}^h}\left(\nu\delta_K^{1/2}\left\|\Delta\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(K)}+\|c\|_{L^\infty(K)}\delta_K^{1/2}\left\|\boldsymbol{u}-\boldsymbol{v}^h\right\|_{L^2(K)}\right)\\ &\quad\times\delta_K^{1/2}\left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}^h+\nabla r^h\right\|_{L^2(K)}\\ &\le\left[\left(\sum_{K\in\mathcal{T}^h}\nu^2\delta_K\left\|\Delta\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(K)}^2\right)^{1/2}\right.\\ &\quad\left.+\left(\sum_{K\in\mathcal{T}^h}\|c\|_{L^\infty(K)}^2\delta_K\left\|\boldsymbol{u}-\boldsymbol{v}^h\right\|_{L^2(K)}^2\right)^{1/2}\right]\\ &\quad\times\left(\sum_{K\in\mathcal{T}^h}\delta_K\left\|(\boldsymbol{b}\cdot\nabla)\boldsymbol{w}^h+\nabla r^h\right\|_{L^2(K)}^2\right)^{1/2}\\ &\le C\left(\sum_{K\in\mathcal{T}^h}\nu h_K^2\left\|\Delta\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(K)}^2\right)^{1/2}+\delta^{1/2}\|c\|_{L^\infty(\Omega)}\left\|\boldsymbol{u}-\boldsymbol{v}^h\right\|_{L^2(\Omega)}\\ &\le C\nu^{1/2}hh^{k-1}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}+C\delta^{1/2}\|c\|_{L^\infty(\Omega)}h^{k+1}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\\ &\le C\left(\nu^{1/2}+\delta^{1/2}\|c\|_{L^\infty(\Omega)}h\right)h^k\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}. \qquad (4.72)\end{aligned}$$

In a similar way, one gets

$$\begin{aligned}
&\sum_{K\in\mathcal{T}^h} \delta_K \left(\left(\boldsymbol{b}\cdot\nabla\right)\left(\boldsymbol{u}-\boldsymbol{v}^h\right)+\nabla\left(p-q^h\right), \left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{w}^h+\nabla r^h\right)_K \\
&\le \left[\left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\boldsymbol{b}\right\|_{L^\infty(K)}^2 \left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(K)}^2\right)^{1/2}\right. \\
&\quad \left.+\left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\nabla\left(p-q^h\right)\right\|_{L^2(K)}^2\right)^{1/2}\right]\left(\sum_{K\in\mathcal{T}^h} \delta_K \left\|\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{w}^h+\nabla r^h\right\|_{L^2(K)}^2\right)^{1/2} \\
&\le C\delta^{1/2}\left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)} h^k \left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)} + C\delta^{1/2}h^l \left\|p\right\|_{H^{l+1}(\Omega)}. \qquad (4.73)
\end{aligned}$$

Collecting the estimates (4.65) and (4.67) – (4.73) gives the error bound (4.63). ■

Remark 4.41. Optimal asymptotics for the stabilization parameters. Based on the error estimate (4.63), one tries to determine the asymptotic form of the stabilization parameters such that the error bound becomes asymptotically optimal. However, one should be aware that changes of the stabilization parameters also change the norm in the left-hand side of (4.63). Thus, small stabilization parameters provide only a weak control on certain individual norms of $\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\mathrm{spg},p}$.

Let $\left\{\mathcal{T}^h\right\}$ be a family of uniform triangulations.

- Consider first the case $\nu < h$. In this case, it is more precise to speak of order of error reduction than of order of convergence. It will be assumed that $\mu \ge \nu$ such that

$$\min\left\{\nu^{-1/2}, \mu^{-1/2}\right\} = \mu^{-1/2}.$$

  - *Inf-sup stable pairs with $k = l+1$.* The optimal order of error reduction which can be achieved is k. Hence, the term in the first parentheses of the right-hand side of (4.63) should be $\mathcal{O}(1)$ and the term in the second parentheses $\mathcal{O}(h)$. Concentrating on the most important terms, one has to calibrate

$$\delta^{1/2}, \mu^{1/2}, \frac{\delta^{1/2}}{h}, \frac{1}{\mu^{1/2}}, \frac{\gamma^{1/2}}{h^{1/2}},$$

    where the last three terms come from the second parentheses, which has to be scaled by h^{-1} such that the order of error reduction becomes k. It can be seen that one has to choose $\delta = \mathcal{O}\left(h^2\right)$, $\mu = \mathcal{O}(1)$, and, if $Q^h \not\subset H^1(\Omega)$ then $\gamma = \mathcal{O}(h)$.
  - *Equal-order pairs with $k = l \ge 1$.* In this case, the terms in both parentheses on the right-hand side of (4.63) should scale the same way. The most important terms apart of $\delta_0^{-1/2}$ and $\gamma_0^{-1/2}$ are

$$\delta^{1/2}, \mu^{1/2}, \frac{h}{\mu^{1/2}}, \gamma^{1/2} h^{1/2}.$$

Thus, one gets $\mu = \mathcal{O}(h)$ such that $\mu^{1/2} = \mathcal{O}\left(h^{1/2}\right)$ and $h/\mu^{1/2} = \mathcal{O}\left(h^{1/2}\right)$. Choosing in addition $\delta = \mathcal{O}(h)$ and $\gamma = \mathcal{O}(1)$, all terms in the parentheses, apart of $\delta_0^{-1/2}$ and $\gamma_0^{-1/2}$, are of order $k + 1/2$. In view of (4.51), one can think of $\delta_0 = \mathcal{O}\left(h^{-1}\right)$ if $\delta = \mathcal{O}(h)$ such that $\delta_0^{-1/2} = \mathcal{O}\left(h^{1/2}\right)$. Similarly, one has from (4.52) that $\gamma_0 = \mathcal{O}\left(h^{-1}\right)$ if $\gamma = \mathcal{O}(1)$ such that $\gamma_0^{-1/2} = \mathcal{O}\left(h^{1/2}\right)$. Altogether, one can expect an error reduction of order $k + 1/2$.

The obtained parameter choices do not contradict the assumption $\mu \geq \nu$.

- Consider now the case $\nu \geq h$. The first parentheses can be at best of order $\mathcal{O}(1)$, because of the term $\nu^{1/2}$, such that the optimal order of convergence is k.
  - *Inf-sup stable pairs with $k = l + 1$.* The situation is similar as in the case $\nu < h$, only for μ there is more freedom. Since choosing μ large leads to a strong norm on the left-hand side of (4.63), it makes sense to choose the same asymptotic scalings of the stabilization parameters like for $\nu < h$.
  - *Equal-order pairs with $k = l \geq 1$.* Now it is sufficient that both parentheses scale with $\mathcal{O}(1)$. There is now some freedom for choosing the parameters. Small parameters lead to a lower bound on the right-hand side of (4.63) and large parameters to stronger norms on the left-hand side of (4.63). To get a small error bound, within the assumptions of the numerical analysis, like (4.51) and (4.52), one has to choose $\delta = \mathcal{O}\left(h^2\right)$ and $\gamma = \mathcal{O}(h)$. For the grad-div parameter $\mu = \mathcal{O}(h)$ is still a correct choice, in particular if ν is small, but for large ν also other choices are possible.

□

Example 4.42. Analytical example which supports the error estimate (4.63). The same problem as in Example 4.18 is considered. Results for the inf-sup stable Q_2/Q_1 Taylor–Hood finite element and for the equal-order pair P_1/P_1 will be presented. For both pairs it is $Q^h \subset H^1(\Omega)$ such that $\gamma_E = 0$ and the terms with γ_0 and γ do not appear on the right-hand side of (4.63). Note that the norm $\|\cdot\|_{\text{spg}}$ changes if the coefficients ν and c of the problem are changed.

The Q_2/Q_1 finite element. For the Q_2/Q_1 finite element, the optimal asymptotic choice of stabilization parameters is $\delta_K = \mathcal{O}\left(h_K^2\right)$ and $\mu_K = \mathcal{O}(1)$. Numerical studies in Matthies et al. (2009) investigated the sensitivity of the errors for wide ranges of parameters for several examples. From these studies, it can be concluded that $\mu_K = 0.2$ is a good choice. The SUPG/PSPG parameter is not that important. It should not be too large only. The results presented in Figures 4.6 – 4.8 were obtained with $\delta_K = 0.1h_K^2$.

The norm on the left-hand side of the error estimate (4.63) can be split into two parts, namely into $\left\|\left(\boldsymbol{u}-\boldsymbol{u}^h, p-p^h\right)\right\|_{\mathrm{spg}}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$. Both parts will be studied separately. Second order reduction of the error in the norm $\|\cdot\|_{\mathrm{spg}}$ was predicted by the numerical analysis. This order can be clearly seen in Figures 4.6 – 4.8 for different combinations of the coefficients of the Oseen equations. The errors do not depend on the inverse of ν. In addition, one can see also a second order error reduction for the pressure error in $L^2(\Omega)$. The bound for this error is the right-hand side of (4.63) scaled with $\nu^{-1/2}$. However, a dependency of the error on ν can be observed only on coarse grids.

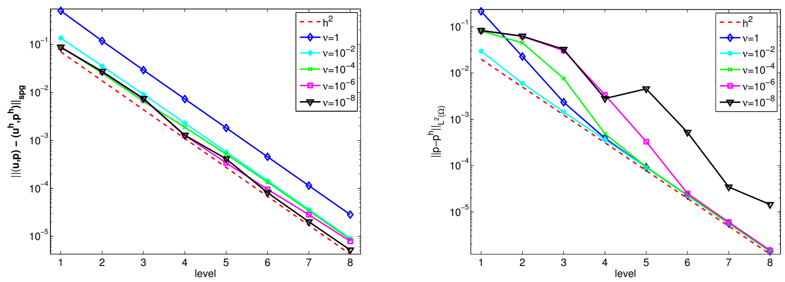

Fig. 4.6 Example 4.42. SUPG/PSPG/grad-div method with Q_2/Q_1, convergence of the errors $\left\|(\boldsymbol{u},p)-(\boldsymbol{u}^h,p^h)\right\|_{\mathrm{spg}}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ for $c=0$ and different values of ν.

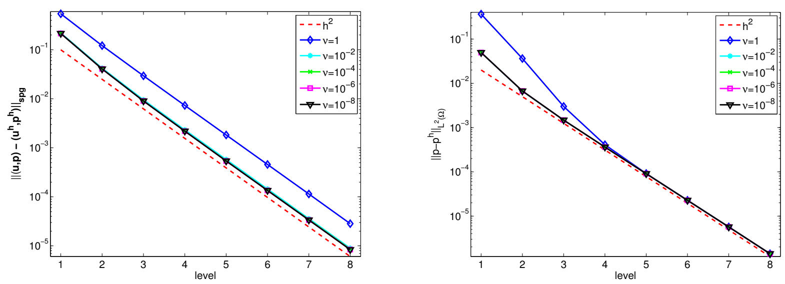

Fig. 4.7 Example 4.42. SUPG/PSPG/grad-div method with Q_2/Q_1, convergence of the errors $\left\|(\boldsymbol{u},p)-(\boldsymbol{u}^h,p^h)\right\|_{\mathrm{spg}}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ for $c=100$ and different values of ν.

P_1/P_1 *finite element.* Remark 4.41 shows that the optimal stabilization parameters for the P_1/P_1 finite element are $\delta_K=\mu_K=\mathcal{O}\left(h_K\right)$ on coarse grids, i.e., if $\nu<h_K$, and $\delta_K=\mathcal{O}\left(h_K^2\right)$ else. For the grad-div parameter, the analysis does not lead to a concrete asymptotic behavior. To prevent a sharp change of this parameter, the results presented in Figures 4.9 – 4.11 were computed with the same grad-div parameter for all situations. The concrete

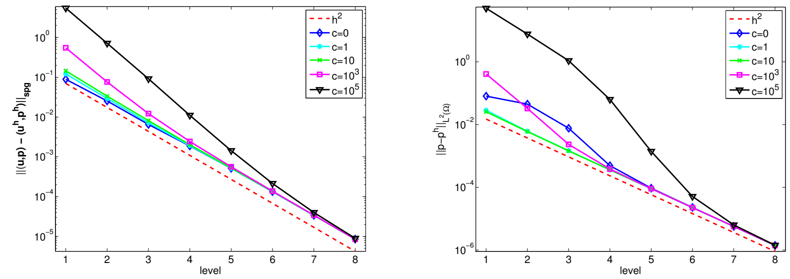

Fig. 4.8 Example 4.42. SUPG/PSPG/grad-div method with Q_2/Q_1, convergence of the errors $\left\|(\boldsymbol{u},p)-(\boldsymbol{u}^h,p^h)\right\|_{\mathrm{spg}}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ for $\nu=10^{-4}$ and different values of c.

choices of the stabilization parameters were

$$\delta_K = \begin{cases} 0.5h_K & \text{if } \nu < h_K, \\ 0.5h_K^2 & \text{else,} \end{cases} \qquad \mu_K = 0.5h_K.$$

The predictions of the numerical analysis are an error reduction in $\|\cdot\|_{\mathrm{spg}}$ of order 1.5 on coarse grids, i.e., if $\nu < h_K$, and a convergence of the error in $\|\cdot\|_{\mathrm{spg}}$ of order 1 on fine grids. Both predictions are supported by the results presented in Figures 4.9 – 4.11. The errors do not depend on the inverse of the viscosity.

Additionally, the error of the pressure in $L^2(\Omega)$ is presented in Figures 4.9 – 4.11. The results with respect to this error do not give a clear picture. Often, the error reduction is better than predicted by the theory. Comparing the errors for $c=0$ and $c=100$ in Figures 4.9 and 4.10, one can see that the reduction of the error is smoother for $c=100$. In Figure 4.11 it can be observed that the pressure error is large for large values of c.

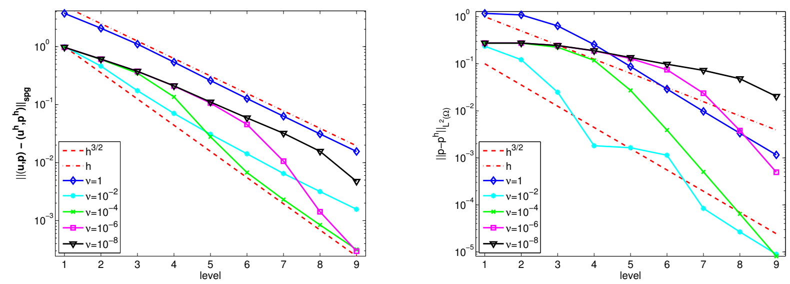

Fig. 4.9 Example 4.42. SUPG/PSPG/grad-div method with P_1/P_1, convergence of the errors $\left\|(\boldsymbol{u},p)-(\boldsymbol{u}^h,p^h)\right\|_{\mathrm{spg}}$ and $\left\|p-p^h\right\|_{L^2(\Omega)}$ for $c=0$ and different values of ν.

□

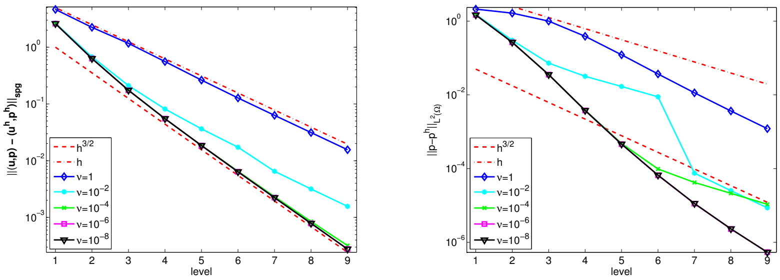

Fig. 4.10 Example 4.42. SUPG/PSPG/grad-div method with P_1/P_1, convergence of the errors $\|(\boldsymbol{u},p)-(\boldsymbol{u}^h,p^h)\|_{\mathrm{spg}}$ and $\|p-p^h\|_{L^2(\Omega)}$ for $c=100$ and different values of ν.

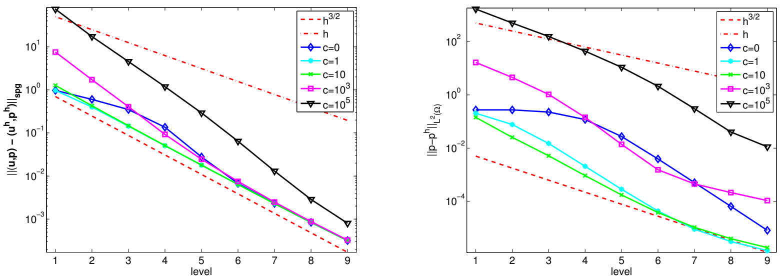

Fig. 4.11 Example 4.42. SUPG/PSPG/grad-div method with P_1/P_1, convergence of the errors $\|(\boldsymbol{u},p)-(\boldsymbol{u}^h,p^h)\|_{\mathrm{spg}}$ and $\|p-p^h\|_{L^2(\Omega)}$ for $\nu=10^{-4}$ and different values of c.

Remark 4.43. Concluding remarks on the SUPG/PSPG/grad-div method.

- In Lube and Rapin (2006), the impact of the polynomial degree of the velocity and pressure finite element spaces on the analytical results is studied. An inf-sup condition similar to (4.53) is proved where the inf-sup constant is independent of the viscosity, the mesh width, and the polynomial degree. Moreover, the polynomial degree enters the definition of the asymptotic optimal stabilization parameters.
- The SUPG/PSPG/grad-div method introduces an artificial nonsymmetric term.
- The SUPG/PSPG/grad-div method introduces a strong coupling of velocity and pressure. There is no physical interpretation for the term $\sum_{K\in\mathcal{T}^h}\delta_K\left\|(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{u}^h+\nabla p^h\right\|^2_{L^2(K)}$.
- Since the residiual contains the right-hand side $\boldsymbol{f}$ of the problem, all residual-based stabilizations introduce a modification of the right-hand side. This modification makes the application of these schemes complicated for time-dependent problems.
- The number of matrix blocks which has to be stored and assembled for the SUPG/PSPG/grad-div method is quite large, see Section 4.6.

□

4.3.3 Other Residual-Based Stabilizations

Remark 4.44. Neglecting the PSPG term for inf-sup stable pairs of finite element spaces. If inf-sup stable pairs of finite element spaces are used, the PSPG term is not necessary since it gives stability if the discrete inf-sup condition (2.45) is not satisfied. The case of inf-sup stable pairs of finite element spaces and the SUPG/grad-div (sg) stabilization was investigated in Gelhard et al. (2005). It has the form: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$A_{\mathrm{sg}}\left(\left(\boldsymbol{u}^h, p^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) = L_{\mathrm{sg}}\left(\boldsymbol{v}^h\right) \quad \forall\ \left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h,$$

with the bilinear form $A_{\mathrm{sg}}\ :\ \left(V \times \tilde{Q}\right) \times \left(V \times \tilde{Q}\right) \to \mathbb{R}$ defined by

$$\begin{aligned}
&A_{\mathrm{sg}}\left(\left(\boldsymbol{u}, p\right), \left(\boldsymbol{v}, q\right)\right) \\
&= \nu\left(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}\right) + \left(\left(\boldsymbol{b} \cdot \nabla\right) \boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}\right) - \left(\nabla \cdot \boldsymbol{v}, p\right) + \left(\nabla \cdot \boldsymbol{u}, q\right) \\
&\quad + \sum_{K \in \mathcal{T}^h} \mu_K \left(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}\right)_K \\
&\quad + \sum_{K \in \mathcal{T}^h} \left(-\nu \Delta \boldsymbol{u} + \left(\boldsymbol{b} \cdot \nabla\right) \boldsymbol{u} + c\boldsymbol{u} + \nabla p, \delta_K^v \left(\boldsymbol{b} \cdot \nabla\right) \boldsymbol{v}\right)_K
\end{aligned}$$

and the linear form $L_{\mathrm{sg}}\ :\ V \to \mathbb{R}$ by

$$L_{\mathrm{sg}}\left(\boldsymbol{v}\right) = \left(\boldsymbol{f}, \boldsymbol{v}\right) + \sum_{K \in \mathcal{T}^h} \left(\boldsymbol{f}, \delta_K^v \left(\boldsymbol{b} \cdot \nabla\right) \boldsymbol{v}\right)_K .$$

This scheme is called reduced stabilized scheme in Gelhard et al. (2005). In this paper, error estimates are proved for families of quasi-uniform triangulations and for discrete pressure spaces satifying $Q^h \in H^1(\Omega)$. The analysis from Gelhard et al. (2005) was refined in Matthies et al. (2009) such that shape regular grids and discontinuous discrete pressure spaces are covered. An important technical tool in this analysis is the use of a quasi-local interpolation operator which preserves the discrete divergence from Girault and Scott (2003). However, this operator requires that the polynomial degree of the discrete velocity space is equal or higher than the dimension of the domain Ω. Hence, the most popular pairs of finite element spaces in three dimensions, which use second order velocity, are not covered by this analysis. The way for performing the error analysis of this method is similar to the analysis presented in Section 4.3.2 for the SUPG/PSPG/grad-div method. The asymptotic optimal choices of the stabilization parameters, derived from the error analysis, are the same for both methods

$$\delta_K = \mathcal{O}\left(h_K^2\right), \quad \mu_K = \mathcal{O}\left(1\right),$$

see Remark 4.41 for the SUPG/PSPG/grad-div method.

In numerical studies in Gelhard et al. (2005), it turned out that the full SUPG/PSPG/grad-div stabilization and the SUPG/grad-div stabilization give almost identical results for inf-sup stable pairs of finite element spaces. In Matthies et al. (2009), extensive and careful numerical studies on the choice of the stabilization parameters are presented. These studies illustrate in particular that different parameters of the same asymptotic type might lead to errors of different orders of magnitude. These studies also come to the conclusion that the use of only the SUPG stabilization might lead to instabilities and at least one of the other stabilizations, PSPG or grad-div, should be added to the SUPG stabilization. □

4.4 Other Stabilized Finite Element Methods

Remark 4.45. Motivation. Other stabilizations than residual-based ones try to avoid the main drawbacks of the latter: the introduction of artificial non-symmetric terms and the strong non-physical coupling of velocity and pressure, see Remark 4.43. In this section, alternative stabilization concepts will be presented and discussed briefly.

A review of stabilization techniques for the Oseen equations can be found in Braack et al. (2007). RELP □

Remark 4.46. The continuous interior penalty (CIP) method. The CIP was already introduced in Douglas and Dupont (1976). Its basic idea consists in increasing the stability of the Galerkin discretization by introducing a least squares control (penalty) of gradient jumps across faces of the mesh cells. Almost three decades later, this approach was applied, e.g., to convection-dominated convection-diffusion equations in Burman and Hansbo (2004) and to the Oseen equations in Burman et al. (2006).

For the Oseen equations, the CIP stabilization, as presented in Burman et al. (2006), reads as follows: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned}
&\left(\nu\nabla\boldsymbol{u}^h,\nabla\boldsymbol{v}^h\right)+\left(\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h+c\boldsymbol{u}^h,\boldsymbol{v}^h\right)-\left(\nabla\cdot\boldsymbol{v}^h,p^h\right)+\left(\nabla\cdot\boldsymbol{u}^h,q^h\right)\\
&\quad+\sum_{K\in\mathcal{T}^h}\delta_0\delta\left(\mathrm{Re}_K\right)h_K^2\\
&\quad\times\sum_{E\in\partial K}\int_E\|\boldsymbol{b}\cdot\boldsymbol{n}_E\|_{L^\infty(E)}\left[\left|\nabla\boldsymbol{u}^h\boldsymbol{n}_K\right|\right]_E\cdot\left[\left|\nabla\boldsymbol{v}^h\boldsymbol{n}_K\right|\right]_E\ d\boldsymbol{s}\\
&\quad+\sum_{K\in\mathcal{T}^h}\delta_0\delta\left(\mathrm{Re}_K\right)\|\boldsymbol{b}\|_{L^\infty(K)}h_K^2\sum_{E\in\partial K}\int_E\left[\left|\nabla\cdot\boldsymbol{u}^h\right|\right]_E\left[\left|\nabla\cdot\boldsymbol{v}^h\right|\right]_E\ d\boldsymbol{s}\\
&\quad+\sum_{K\in\mathcal{T}^h}\delta_0\delta\left(\mathrm{Re}_K\right)\frac{h_K^2}{\|\boldsymbol{b}\|_{L^\infty(K)}}\sum_{E\in\partial K}\int_E\left[\left|\nabla p^h\right|\right]_E\cdot\left[\left|\nabla q^h\right|\right]_E\ d\boldsymbol{s}\\
&=\left(\boldsymbol{f},\boldsymbol{v}^h\right)\quad\forall\ \left(\boldsymbol{v}^h,q^h\right)\in V^h\times Q^h,
\end{aligned}\tag{4.74}$$

with

$$\mathrm{Re}_K=\frac{\|\boldsymbol{b}\|_{L^\infty(K)}h_K}{\nu},\quad\delta\left(\mathrm{Re}_K\right)=\min\left\{1,\mathrm{Re}_K\right\}.$$

The first jump term in (4.74) acts as stabilization of dominating convection, the second jump term gives some control on the violation of the divergence constraint, and the last jump term stabilizes the violation of the discrete inf-sup condition. The stabilization term in (4.74) is symmetric and the right-hand side of the discrete problem is not affected by the stabilization.

The numerical analysis for the case $V^h/Q^h = P_k/P_k$ or $V^h/Q^h = Q_k/Q_k$ was performed in Burman et al. (2006) for weakly imposed Dirichlet boundary conditions. In this approach, the finite element spaces are defined with natural boundary conditions and additional terms defined on the boundary are introduced into the bilinear form which, e.g., penalize the violation of a Dirichlet boundary condition. For large penalty factors, one gets a good approximation of these boundary conditions. Error estimates were derived in a norm which contains contributions from the stabilization terms.

In Braack et al. (2007), a modification of the CIP method is presented which is formulated in terms of sums over the faces

$$\begin{aligned}
&\left(\nu\nabla\boldsymbol{u}^h,\nabla\boldsymbol{v}^h\right)+\left(\left(\boldsymbol{b}\cdot\nabla\right)\boldsymbol{u}^h+c\boldsymbol{u}^h,\boldsymbol{v}^h\right)-\left(\nabla\cdot\boldsymbol{v}^h,p^h\right)+\left(\nabla\cdot\boldsymbol{u}^h,q^h\right)\\
&\qquad+\sum_{E\in\mathcal{E}^h}\int_E\delta_E^{\boldsymbol{u}}\left[\left|\nabla\boldsymbol{u}^h\boldsymbol{n}_K\right|\right]_E\cdot\left[\left|\nabla\boldsymbol{v}^h\boldsymbol{n}_K\right|\right]_E\ d\boldsymbol{s}\\
&\qquad+\sum_{E\in\mathcal{E}^h}\int_E\delta_E^{\mathrm{div}}\left[\left|\nabla\cdot\boldsymbol{u}^h\right|\right]_E\left[\left|\nabla\cdot\boldsymbol{v}^h\right|\right]_E\ d\boldsymbol{s}\\
&\qquad+\sum_{E\in\mathcal{E}^h}\int_E\delta_E^{p}\left[\left|\nabla p^h\cdot\boldsymbol{n}_E\right|\right]_E\cdot\left[\left|\nabla q^h\cdot\boldsymbol{n}_E\right|\right]_E\ d\boldsymbol{s}\\
&\quad=\left(\boldsymbol{f},\boldsymbol{v}^h\right)\quad\forall\ \left(\boldsymbol{v}^h,q^h\right)\in V^h\times Q^h,
\end{aligned}\tag{4.75}$$

where the jumps of the gradient of the pressure are replaced by the jumps of the normal derivatives of the pressure. In this paper, even concrete parameters depending on the polynomial degree k are proposed

$$\begin{aligned}
\delta_E^{\boldsymbol{u}} &= \|\boldsymbol{b}\cdot\boldsymbol{n}_E\|_{L^\infty(E)}\,\frac{h_E^2}{k^\alpha},\\
\delta_E^{\mathrm{div}} &= \|\boldsymbol{b}\|_{L^\infty(E)}\,\frac{h_E^2}{k^\alpha},\\
\delta_E^{p} &= \min\{1,\mathrm{Re}_E\}\,\frac{h_E^2}{\|\boldsymbol{b}\|_{L^\infty(E)}\,k^\alpha},\quad \mathrm{Re}_E=\frac{\|\boldsymbol{b}\|_{L^\infty(E)}\,h_E}{\nu\alpha^{1/2}},
\end{aligned} \tag{4.76}$$

with $\alpha = 7/2$.

The CIP stabilization applied to the Crouzeix–Raviart pair of finite element spaces P_1^{nc}/P_0 of the Oseen equations was analyzed in Burman and Hansbo (2006). □

Remark 4.47. Implementation of the CIP method. The appearance of jumps in the bilinear form in (4.74) couples degrees of freedom which are not coupled in the standard Galerkin discretization and in the velocity-velocity matrix of the residual-based stabilizations. Consider for example P_1 finite elements, two simplices with the common face E, and the two basis functions which take the value 1 in the vertex opposite to the common face and the value 0 in all other vertices. Then, the common support of these basis functions is E and all integrals in the Galerkin finite element method (4.11) where both basis functions are involved vanish. However, the jumps of the derivatives of these basis functions across E do not vanish and consequently, the jump terms in (4.74) couple these functions. Hence, the sparsity pattern of the matrices of the CIP method is denser than for the Galerkin method and for the velocity-velocity matrix of residual-based stabilizations.

The block matrix form of the arising linear system of equations will be discussed in Remark **??**. □

Example 4.48. Analytical example for the CIP method. todo Diplom Umla, EDGE_STAB □

Remark 4.49. Numerical studies for the CIP method. Comprehensive numerical studies of the CIP method applied to the two-dimensional Oseen equations (with $c = 0$) can be found in Umla (2009). In this thesis, the form (4.75) of the CIP method was used and the stabilization parameters (4.76) were scaled with a factor $\delta_0 > 0$. It turned out that the optimal stabilization parameters were generally not given for $\delta_0 = 1$. In some situations, in particular for the finite element pairs P_3/P_3 and Q_3/Q_3, the errors could be reduced by one order of magnitude with the optimal scaling factor compared with $\delta_0 = 1$. Choosing different scaling factors for the different stabilization terms did not lead to notable improvements of the results.

In addition, comparisons with the SUPG/PSPG/grad-div stabilization, see Section 4.3.2, were performed in Umla (2009). Generally, the solutions obtained with both stabilizations were of similar accuracy. But sometimes, the pressure was computed somewhat more accurately with the SUPG/PSPG/grad-div stabilization.

In Umla (2009), the CIP stabilization was applied also to pairs of Taylor–Hood finite element spaces. The observed orders of convergence are the same as for the SUPG/PSPG/grad-div method.

The linear systems of equations in the numerical studies of Umla (2009) were solved with UMFPACK, see Davis (2004b). The computing times for the CIP method were considerably higher than for the SUPG/PSPG/grad-div method, often by a factor of about two. The main reason is the denser sparsity pattern of the matrices coming from the CIP stabilization. □

Remark 4.50. Local projection stabilization (LPS) methods. The main tool of LPS methods is a local projection P^h_{loc} of a finite element space into another finite element space, which is usually a discontinuous space. With this projection, to so-called fluctuation operator $\left(I - P^h_{\mathrm{loc}}\right)$ is defined. Then, a stabilization of the Galerkin finite element discretization is achieved by adding weighted $L^2(\Omega)$ inner products of fluctuations of quantities of interest. insert notations

A short review of the LPS method can be found in Braack and Lube (2009) and a longer presentation in (Roos et al., 2008, Chapter IV.4). □

Remark 4.51. Stabilizing the violation of the discrete inf-sup condition with an LPS method. The LPS method for stabilizing the violation of the discrete inf-sup condition was proposed in Becker and Braack (2001). Applying the general approach of LPS methods, the difference of the gradient of the discrete pressure and a local projection is added to the continuity equation. This idea was already formulated in Codina and Blasco (1997), where a global projection operator was proposed. From the point of view of numerical efficiency, locally computable projections should be preferred.

The numerical analysis of the LPS method requires a number of assumptions, see Becker and Braack (2001); Braack and Lube (2009) for details. The realization of the LPS method as proposed in Becker and Braack (2001) consists in using two triangulations $\mathcal{T}^{2h}$ and $\mathcal{T}^h$ of Ω, where $\mathcal{T}^h$ is a uniform refinement of $\mathcal{T}^{2h}$ which is the so-called two-level method. The mesh cells on the coarse grid are called macro cells $\{M\}$ and each macro cell contains a number of mesh cells on the fine grid. Then, the local $L^2(\Omega)$ projection has the form

$$P_h^{2h} \ : \ L^2(\Omega) \to R^{2h,\mathrm{disc}}, \quad \left(q^h - P_h^{2h} q^h, \psi^h\right) = 0 \quad \forall\, \psi^h \in R^{2h,\mathrm{disc}},$$

where $R^{2h,\mathrm{disc}}$ is a space of discontinuous finite element functions on the coarser grid, e.g., $R^{2h,\mathrm{disc}} = Q_1^{\mathrm{disc}}$ if $V^h = Q_2$. Thus, the LPS term has the form

$$
\begin{aligned}
S_{\mathrm{lps}}^{h}(p,q) &= \left(\delta_{\mathrm{lps},p}^{1/2}\left(I-P_{h}^{2h}\right)\nabla p^{h}, \delta_{\mathrm{lps},p}^{1/2}\left(I-P_{h}^{2h}\right)\nabla q^{h}\right) \\
&= \left(\delta_{\mathrm{lps},p}^{1/2}\kappa_{p}^{h}\left(\nabla p^{h}\right), \delta_{\mathrm{lps},p}^{1/2}\kappa_{p}^{h}\left(\nabla q^{h}\right)\right) \qquad (4.77)
\end{aligned}
$$

for all $p^h, q^h \in Q^h \cap H^1(\Omega)$. Then, the numerical analysis for the LPS method is performed in the norm

$$
\|(\boldsymbol{v},q)\|_{\mathrm{lps}} = \left(\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 + S_{\mathrm{lps}}^{h}(p,p)\right)^{1/2}.
$$

The analysis shows that in the case of using the same finite element spaces for velocity and pressure spaces on the fine grid, then $\delta_{\mathrm{lps},p} = \mathcal{O}\left(h_M^2\right)$ is the asymptotically correct choice, where h_M is the diameter of a macro cell M.

A variant of the LPS method consists in using instead the fluctuation of the gradients of the pressure as in (4.77) the gradient of the fluctuations

$$
S_{\mathrm{lps}}^{h}(p,q) = \left(\delta_{\mathrm{lps},p}^{1/2}\nabla\left(\kappa_{p}^{h}p^{h}\right), \delta_{\mathrm{lps},p}^{1/2}\nabla\left(\kappa_{p}^{h}q^{h}\right)\right).
$$

It is mentioned in Braack and Lube (2009) that this method remains optimal for the Stokes equations but its extension is not optimal for the Oseen equations. □

Remark 4.52. LPS schemes. An LPS scheme for the Oseen equations was studied in Braack and Burman (2006). In this paper, a numerical analysis for low order discretizations was presented. Concretely, the so-called two-level method was considered with the pairs Q_k/Q_k, $k \in \{1,2\}$, for velocity and pressure and a local projection which maps into the space Q_{k-1} on the next coarser grid. In Matthies et al. (2007), an LPS approach was proposed which is based on enrichment and uses only one mesh, the so-called one-level method. A unified finite element error analysis for the one-level and two-level approach and for the case of using the same finite spaces for velocity and pressure is presented in this paper. The case of inf-sup stable finite element spaces is studied in Lube et al. (2008); **?**. check

Using the general approach for constructing LPS schemes leads, see Remark 4.50, to the stabilization term

$$
\begin{aligned}
S_{\mathrm{lps}}^{h}\left((\boldsymbol{u},p),(\boldsymbol{v},q)\right) = &\left(\delta_{\mathrm{lps},\boldsymbol{u}}\kappa_{\boldsymbol{u}}^{h}\left((\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\right), \kappa_{\boldsymbol{u}}^{h}\left((\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\right)\right) \\
&+ \left(\delta_{\mathrm{lps,div}}\kappa_{\mathrm{div}}^{h}\left(\nabla\cdot\boldsymbol{u}\right), \kappa_{\mathrm{div}}^{h}\left(\nabla\cdot\boldsymbol{v}\right)\right) \\
&+ \left(\delta_{\mathrm{lps},p}\kappa_{p}^{h}\left(\nabla p^{h}\right), \kappa_{p}^{h}\left(\nabla q^{h}\right)\right).
\end{aligned}
$$

For practical reasons, all local projections are defined principally in the same way, either with the one-level or the two-level method. A difference might be the actual polynomial order of the projection space. It can be seen that the stabilization term is symmetric and the stabilization does not affect the right-hand side of the equations.

The finite element error analysis is performed in the mesh-dependent norm

$$\|(\boldsymbol{v},q)\|_{\mathrm{lps}} = \Big(\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 + \left\|c^{1/2}\boldsymbol{v}\right\|_{L^2(\Omega)}^2 + \nu \|p\|_{L^2(\Omega)}^2 + \left\|c^{1/2}p\right\|_{L^2(\Omega)}^2 + S_{\mathrm{lps}}^h\left((\boldsymbol{v},p),(\boldsymbol{v},p)\right)\Big)^{1/2}.$$

The weigths of the pressure terms are the same as in the norm $\|\cdot\|_{\mathrm{spg},p}$ used in the SUPG/PSPG/grad-div method, see (4.49). The analysis relies on the construction of an interpolation operator between the finite element spaces and the projection spaces with certain orthogonality properties. A discrete inf-sup condition for the complete bilinear form can be proved. In the case of using the same finite element spaces for velocity and pressure, both of order k, the optimal asymptotic choices of stabilization parameters are

$$\delta_{\mathrm{lps},\boldsymbol{u}} = \mathcal{O}\left(\frac{h_M}{\|\boldsymbol{b}\|_{W^{k,\infty}(M)}}\right), \quad \delta_{\mathrm{lps,div}} = \mathcal{O}\left(h_M\right), \quad \delta_{\mathrm{lps},p} = \mathcal{O}\left(h_M\right),$$

where h_M is the diameter of a macro cell M, see Remark 4.51. With these parameters, one obtains the same order of convergence as for the SUPG/PSPG/grad-div method. However, one should note that only some terms in $\|\cdot\|_{\mathrm{lps}}$ and $\|\cdot\|_{\mathrm{spg},p}$ from (4.49) are the same. □

Remark 4.53. Two-level vs. one-level LPS methods. A discussion of possible choices of finite element spaces for two-level and one-level LPS schemes can be found in Matthies et al. (2007). For a two-level LPS scheme, an example is already given in Remark 4.51. One-level LPS schemes rely on enriching a finite element space of order k with mesh cell bubble functions of a sufficiently large polynomial order. Then, the projection space can be choosen to consists of discontinuous functions of order $k-1$.

The use of a two-level method requires a coarse grid, which is not always available in application. In addition, the matrices possess extended stencils, e.g., compared with the SUPG/PSPG/grad-div method. With the one level method, one obtains a more compact stencil than with the two-level scheme. But on the same grid, the number of degrees of freedom is larger for the one-level method than for other discretizations, like the SUPG/PSPG/grad-div method or the two-level LPS method. □

Remark 4.54. Upwind discretizations for lowest order non-conforming finite elements. In the case of non-conforming discretizations, the SUPG method for stabilizing dominating convection looses the advantage of not enlarging the stencils of the matrices, because a term which includes the jumps of the functions across the faces is needed. This effect was already observed for the Crouzeix–Raviart finite element P_1^{nc} applied in the discretization of scalar convection-diffusion equations in John et al. (1997). For this reason, the SUPG method is not used for non-conforming finite elements and dominating convection is stabilized in different ways. Most popular are upwind methods, which combine the finite element method with ideas from finite volume methods.

Consider a scalar non-conforming space of lowest order, i.e., the Crouzeix–Raviart element P_1^{nc}, see Example B.46, or the Rannacher–Turek element Q_1^{rot}, see Example B.56, and a partition of Ω into mesh cells. For P_1^{nc}, the mesh cells are triangles or tetrahedra and for Q_1^{rot} these are quadrilaterals or hexahedra. Let K_1 and K_2 be two neighboring mesh cells with the common face E_i. Then, exactly one standard basis function $\phi_i(\boldsymbol{x})$ of P_1^{nc} or Q_1^{rot} can be assigned to this face. If $\boldsymbol{m}_{E,i}$ is the barycenter of E_i, then $\phi_i(\boldsymbol{m}_{E,i}) = 1$ for P_1^{nc} and the point value oriented Q_1^{rot} element. In the first step of defining upwind methods, the domain is decomposed into so-called dual domains $\{\omega_{\mathrm{dual},i}\}$. A subdomain $\omega_{\mathrm{dual},i}$ is constructed by taking the barycenters S_{K_1} and S_{K_2} of K_1 and K_2, respectively, and connecting them with the vertices of E_i. The union of the two subdomains obtained in this way is $\omega_{\mathrm{dual},i}$, see Figure 4.12 for the case of a triangular mesh. For faces on the boundary Γ of Ω, the dual domain consists just of one subdomain.

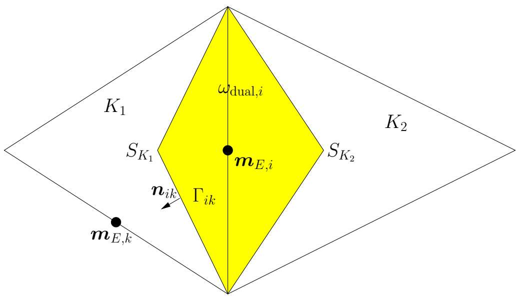

Fig. 4.12 Dual domain for the upwind discretization.

Let Λ_i be the set of all indices $k \neq i$ for which the faces E_k and E_i belong to the same mesh cell. For instance, these are four indices for interior edges in triangular meshes and ten indices for interior faces in hexahedral meshes. Denote the face of $\omega_{\mathrm{dual},i}$ in between E_i and E_k by Γ_{ik} and the corresponding outer unit normal vector by $\boldsymbol{n}_{ik}$, see again Figure 4.12 for an illustration.

Next, an operator L^h is introduced which maps a continuous function $v(\boldsymbol{x})$ onto a function $L^h v(\boldsymbol{x})$ which is constant in each dual domain

$$L^h v(\boldsymbol{x}) = v(\boldsymbol{m}_{E,i}) \quad \forall\, \boldsymbol{x} \in \omega_{\mathrm{dual},i}.$$

To incorporate information about the direction of the convection into the scheme appropriately, the convective term will be approximated with this operator and with integrals on the boundaries of the dual subdomains. One obtains, using the product rule, approximating by applying the operator L^h, utilizing that the approximation is constant in $\omega_{\mathrm{dual},i}$, and applying integration by parts

$$
\begin{aligned}
(\boldsymbol{b}\cdot\nabla u, v) &= \sum_{E_i\in\overline{\mathcal{E}}^h} (\boldsymbol{b}\cdot\nabla u, v)_{\omega_{\mathrm{dual},i}} \\
&= \sum_{E_i\in\overline{\mathcal{E}}^h} \Big[(\nabla\cdot(u\boldsymbol{b}), v)_{\omega_{\mathrm{dual},i}} - (\nabla\cdot\boldsymbol{b}, uv)_{\omega_{\mathrm{dual},i}}\Big] \\
&\approx \sum_{E_i\in\overline{\mathcal{E}}^h} \Big[(\nabla\cdot(u\boldsymbol{b}), L^h v)_{\omega_{\mathrm{dual},i}} - (\nabla\cdot\boldsymbol{b}, L^h(uv))_{\omega_{\mathrm{dual},i}}\Big] \\
&= \sum_{E_i\in\overline{\mathcal{E}}^h} \big((\nabla\cdot(u\boldsymbol{b}), 1)_{\omega_{\mathrm{dual},i}} - u(\boldsymbol{m}_{E,i})(\nabla\cdot\boldsymbol{b}, 1)_{\omega_{\mathrm{dual},i}}\big)\, v(\boldsymbol{m}_{E,i}) \\
&= \sum_{E_i\in\overline{\mathcal{E}}^h} \sum_{k\in\Lambda_i} \big((u\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, 1)_{\Gamma_{ik}} - u(\boldsymbol{m}_{E,i})\,(\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, 1)_{\Gamma_{ik}}\big)\, v(\boldsymbol{m}_{E,i}) \\
&= \sum_{E_i\in\overline{\mathcal{E}}^h} \sum_{k\in\Lambda_i} (\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, u - u(\boldsymbol{m}_{E,i}))_{\Gamma_{ik}}\, v(\boldsymbol{m}_{E,i}). \qquad (4.78)
\end{aligned}
$$

In upwind discretizations, u will be replaced by a convex combination of the form

$$u \approx \lambda_{ik}(\boldsymbol{b})u(\boldsymbol{m}_{E,i}) + (1-\lambda_{ik}(\boldsymbol{b}))\,u(\boldsymbol{m}_{E,k}). \qquad (4.79)$$

The general upwind idea consists in taking the direction of the convection into account in the discretization. Information which comes from where the convection is coming from (upwind) is weighted stronger than information from where the convection is going to (downwind). In this way, an appropriate transport of information is incorporated into the discretization. The boundary integral $(\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, 1)_{\Gamma_{ik}}$ describes the convective flux across Γ_{ik}. The direction of the flux is determined with the sign of $(\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, 1)_{\Gamma_{ik}}$. If $(\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, 1)_{\Gamma_{ik}} < 0$, then the normal and the convection posses different signs, i.e., the convective flux is directed into $\omega_{\mathrm{dual},i}$. That means, the flux occurs from face E_k to face E_i. For this reason, one chooses $\lambda_{ik}\in[0,1/2)$ in (4.79) such that the impact of $u(\boldsymbol{m}_{E,k})$ in the discretization of the convective term in the node $\boldsymbol{m}_{E,i}$ is stronger than the impact of $u(\boldsymbol{m}_{E,i})$. In the case $(\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, 1)_{\Gamma_{ik}} > 0$, one chooses with analogous considerations $\lambda_{ik}\in(1/2,1]$. Thus, one obtains by inserting (4.79) into (4.78) the following discretization of $(\boldsymbol{b}\cdot\nabla u, v)$

$$
\begin{aligned}
&\big(\boldsymbol{b}\cdot\nabla u^h, v^h\big) \approx n^h_{\mathrm{upw}}\big(\boldsymbol{b}, u^h, v^h\big) \\
&= \sum_{E_i\in\overline{\mathcal{E}}^h} \sum_{k\in\Lambda_i} \big(\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, (1-\lambda_{ik}(\boldsymbol{b}))(u^h(\boldsymbol{m}_{E,k}) - u^h(\boldsymbol{m}_{E,i}))\big)_{\Gamma_{ik}}\, v^h(\boldsymbol{m}_{E,i}).
\end{aligned}
$$

Defining

$$\lambda_{ik}(\boldsymbol{b}) = \Phi(t) \quad \text{with} \quad t = \frac{1}{2\nu}\,(\boldsymbol{b}\cdot\boldsymbol{n}_{ik}, 1)_{\Gamma_{ik}}\,,$$

then the function $\Phi(t)$ has to satisfy the following conditions

i) $\Phi(t) = 1-\Phi(-t)$ for all $t>0$ and $0\le\Phi(t)\le 1$ for all $t\in\mathbb{R}$,

ii) $t\left(\Phi(t)-\frac{1}{2}\right)\ge 0$ for all $t\in\mathbb{R}$,

iii) $g(t) = t\Phi(t)$ is Lipschitz continuous in $\mathbb{R}$,
see (Roos et al., 2008, Chapter III.3.1). With the ansatz

$$\Phi(t) = \frac{1}{2} + \operatorname{sgn}(t)\Psi(|t|),$$

one obtains

$$1 - \Phi(-t) = \frac{1}{2} - \operatorname{sgn}(-t)\Psi(|-t|) = \frac{1}{2} + \operatorname{sgn}(t)\Psi(|t|) = \Phi(t),$$

such that the first part of condition i) is satisfied. Some upwind methods which are often used are the simple or standard upwind, given by

$$\Psi(t) = \frac{1}{2},$$

and the Samarskij upwind defined by

$$\Psi(t) = \frac{1}{2}\frac{t}{t+1}.$$

pictures of the upwind functions $\Phi(t)$

For the Oseen equations, the upwind methodology is applied to each component of the term $\left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{u}^h, \boldsymbol{v}^h\right)$.

The convergence of the upwind method applied to the Navier–Stokes equations and error estimates were proved in Schieweck and Tobiska (1989, 1996). An overview of the results can be also found in (Roos et al., 2008, Chapter IV.2). □

example

4.5 The Conservation of Mass

4.5.1 The Grad-Div Stabilization

Remark 4.55. The grad-div stabilization. The grad-div stabilization introduces a penalty term into the Galerkin finite element formulation which penalizes the violation of mass conservation. history

For obtaining an optimal order of convergence, the stabilization parameters have to be chosen appropriately. To highlight the dependency of these parameters not only on the mesh size but also on the data of the problem, the scaled Stokes equations (3.37) will be considered. The weak form of these equations is: Find $(\boldsymbol{u}, p) \in V \times Q$ such that

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V',V} \ \forall\, \boldsymbol{v} \in V, \\ -(\nabla \cdot \boldsymbol{u}, q) &= 0 \qquad\qquad\quad \forall\, q \in Q. \end{aligned} \tag{4.80}$$

In this section, only the case of conforming finite element spaces will be considered. Then, the grad-div stabilization method reads as follows: Find $\left(\boldsymbol{u}^h, p^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned} \nu\left(\nabla \boldsymbol{u}^h, \nabla \boldsymbol{v}^h\right) + \sum_{K\in\mathcal{T}^h} \mu_K \left(\nabla \cdot \boldsymbol{u}^h, \nabla \cdot \boldsymbol{v}^h\right)_K & \\ - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) &= \langle \boldsymbol{f}, \boldsymbol{v}^h \rangle_{V',V} \ \forall\, \boldsymbol{v}^h \in V^h, \\ - \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) &= 0 \qquad\qquad\quad \forall\, q^h \in Q^h, \end{aligned} \tag{4.81}$$

where $\{\mu_K\}$ are the stabilization parameters. Let

$$\mu = \max_{K\in\mathcal{T}^h} \mu_K.$$

The presentation of this topic follows Jenkins et al. (2014). □

Remark 4.56. On the grad-div term. A grad-div term appeared already in the derivation of the Navier–Stokes equations, see (1.18), which reads for constant viscosities

$$\nabla \cdot \left(\left(\zeta - \frac{2\mu}{3}\right) \nabla \cdot \boldsymbol{v}\mathbb{I}\right) = \left(\zeta - \frac{2\mu}{3}\right) \nabla \left(\nabla \cdot \boldsymbol{v}\right).$$

In the process of non-dimensionalization, this term becomes

$$\frac{L}{U^2}\left(\frac{\zeta}{\rho} - \frac{2\nu}{3}\right) \nabla \left(\nabla \cdot \boldsymbol{u}\right).$$

The parameter in front of the grad-div term is just a constant. □

Definition 4.57. Optimal approximation property of a divergence-free subspace. Consider a quasi-uniform family of triangulations $\{\mathcal{T}^h\}_{h>0}$ with characteristic mesh size h and let

$$V^h_{\mathrm{div,div}} = V_{\mathrm{div}} \cap V^h_{\mathrm{div}}.$$

If for all $\boldsymbol{v} \in V_{\mathrm{div}} \cap H^{k+1}(\Omega)$ there exists a sequence of $\left\{\boldsymbol{v}^h\right\} \in V^h_{\mathrm{div,div}}$ with

$$\left\|\nabla\left(\boldsymbol{v} - \boldsymbol{v}^h\right)\right\|_{L^2(\Omega)} \le C_{\mathrm{div}} h^k \left|\boldsymbol{v}\right|_{H^{k+1}(\Omega)},$$

with C_{div} independent of h, then the sequence of spaces $\left\{V^h_{\mathrm{div,div}}\right\}$ is said to possess optimal approximation properties with respect to the space V_{div}. □

Remark 4.58. Existence of divergence-free subspaces with optimal approximation properties. Whether there exist divergence-free subspaces with optimal approximation properties depends on the pair of inf-sup stable finite element spaces and on the underlying triangulation of the domain. Several combinations of pairs and triangulations are known to possess such divergence-free subspaces, e.g., the Taylor–Hood pair of spaces P_k/P_{k-1} with $k \geq d$ on barycentric-refined simplicial grids, see **?**Qin (1994); Zhang (2005), and the MINI element $P_1^{\mathrm{bubble}}/P_1$ on so-called Powell–Sabin grids, see Zhang (2008), or Union Jack (criss-croos) grids, see **?**. explain grids □

Theorem 4.59. Finite element error estimate for the $L^2(\Omega)$ norm of the gradient of the velocity. *Let the assumption of Theorem 3.20 be satisfied, let $(\boldsymbol{u}, p)$ be the solution of (4.80) and let $(\boldsymbol{u}^h, p^h)$ be the solution of (4.81). Then, the error in the $L^2(\Omega)$ norm of the gradient of the velocity is bounded by*

$$\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}^2 \leq \inf_{\boldsymbol{v}^h\in V_{\mathrm{div}}^h}\left(4\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}^2+2\frac{\mu}{\nu}\left\|\nabla\cdot\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2\right) + \frac{2}{\mu\nu}\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}^2. \tag{4.82}$$

Proof. The proof is similar to the proof of Theorem 3.20. It starts with the error decomposition $\boldsymbol{u}-\boldsymbol{u}^h = \left(\boldsymbol{u}-\boldsymbol{v}^h\right)-\left(\boldsymbol{u}^h-\boldsymbol{v}^h\right) = \boldsymbol{\eta}-\boldsymbol{\phi}^h$, where $\boldsymbol{v}^h \in V_{\mathrm{div}}^h$ is arbitrary. First, by the triangle inequality and Young's inequality, one obtains

$$\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}^2 \leq 2\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2+2\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2. \tag{4.83}$$

For any $\boldsymbol{v}^h \in V_{\mathrm{div}}^h$, one concludes by subtracting (4.81) from (4.80) that

$$\nu\left(\nabla\boldsymbol{\phi}^h,\nabla\boldsymbol{v}^h\right)+\mu\left(\nabla\cdot\boldsymbol{\phi}^h,\nabla\cdot\boldsymbol{v}^h\right) = -\nu\left(\nabla\boldsymbol{\eta},\nabla\boldsymbol{v}^h\right)-\mu\left(\nabla\cdot\boldsymbol{\eta},\nabla\cdot\boldsymbol{v}^h\right)+\left(\nabla\cdot\boldsymbol{v}^h,p\right).$$

Choosing $\boldsymbol{v}^h = \boldsymbol{\phi}^h$, and using that $(\nabla\cdot\boldsymbol{\phi}^h, q^h) = 0$ for any $q^h \in Q^h$, the error equation becomes, for any $q^h \in Q^h$,

$$\nu\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\mu\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 = -\nu(\nabla\boldsymbol{\eta},\nabla\boldsymbol{\phi}^h)-\mu(\nabla\cdot\boldsymbol{\eta},\nabla\cdot\boldsymbol{\phi}^h)+(\nabla\cdot\boldsymbol{\phi}^h,p-q^h).$$

Applying the Cauchy–Schwarz inequality (A.8) and Young's inequality (A.4) on the right-hand side, one gets

$$\begin{aligned}&\nu\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\mu\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2\\ &\leq \nu\left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2+\mu\left\|\nabla\cdot\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2+2\left\|p-q^h\right\|_{L^2(\Omega)}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}.\end{aligned} \tag{4.84}$$

The last term on the right-hand side can be estimated by

$$2\left\|p-q^h\right\|_{L^2(\Omega)}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \leq \mu^{-1}\left\|p-q^h\right\|_{L^2(\Omega)}^2+\mu\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2, \tag{4.85}$$

which leads to

$$\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \leq \left\|\nabla\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2+\frac{\mu}{\nu}\left\|\nabla\cdot\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2+\frac{1}{\mu\nu}\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}^2.$$

Finally, (4.83) gives

$$\left\|\nabla \left(\boldsymbol{u}-\nabla \boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}^2 \leq 4\left\|\nabla \boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + 2\frac{\mu}{\nu}\left\|\nabla\cdot\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2 + \frac{2}{\mu\nu}\inf_{q^h\in Q^h}\left\|p-q^h\right\|_{L^2(\Omega)}^2$$

for all $\boldsymbol{v}^h \in V_{\mathrm{div}}^h$, which is just the statement of the theorem. ■

Remark 4.60. On Theorem 4.59. In contrast to the proof of Theorem 3.20, the norm of the divergence of the test function was not estimated by the norm of the gradient, see (3.20). Thus, the best approximation error with respect to the divergence appears in estimate (4.82). Then, the consequences of the error bound (4.82) on the choice of μ can be studied for two different cases. These two cases are characterized by whether or not the divergence-free subspace of the velocity space has optimal approximation properties. □

Corollary 4.61. Application to Taylor–Hood pairs of finite elements. *Consider $V^h/Q^h = P_k/P_{k-1}$, $k \geq 2$, on a family of quasi-uniform meshes and assume that for the solution of* (4.80) *it holds $(\boldsymbol{u}, p) \in H^{k+1}(\Omega)\times H^k(\Omega)$.*

i) If $V_{\mathrm{div,div}}^h$ does not possess optimal approximation properties, then the a-priori estimate (4.82) *has the form*

$$\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}^2 \leq \left(4+\frac{2\mu}{\nu}\right) C_{V_{\mathrm{div}}^h}^2 h^{2k}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}^2 + \frac{2C_{Q^h}^2}{\mu\nu} h^{2k}\left\|p\right\|_{H^k(\Omega)}^2. \tag{4.86}$$

ii) If $V_{\mathrm{div,div}}^h$ has optimal approximation properties, one obtains the a-priori error estimate

$$\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)}^2 \leq \min\left\{\left(4+\frac{2\mu}{\nu}\right) C_{V_{\mathrm{div}}^h}^2, 4C_{V_{\mathrm{div,div}}^h}^2\right\} h^{2k}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}^2 + \frac{2C_{Q^h}^2}{\mu\nu} h^{2k}\left\|p\right\|_{H^k(\Omega)}^2. \tag{4.87}$$

The constants $C_{Q^h}, C_{V_{\mathrm{div,div}}^h}, C_{V_{\mathrm{div}}^h}$ are constants coming from interpolation estimates, where $C_{V_{\mathrm{div,div}}^h}$ and $C_{V_{\mathrm{div}}^h}$ depend on the inverse of β_{is}^h.

Proof. *Case i).* For this case, one can only use (2.39), i.e., that

$$\left\|\nabla\cdot\boldsymbol{v}^h\right\|_{L^2(\Omega)} = \left\|\nabla\cdot\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)} \leq \left\|\nabla(\boldsymbol{u}-\boldsymbol{v}^h)\right\|_{L^2(\Omega)} \tag{4.88}$$

holds in this setting. Then, one applies Lemma 2.51 and the interpolation error estimate (C.11) to prove (4.86).

Case ii). In this case, one gets an additional estimate besides (4.88), since one can choose $\boldsymbol{v}^h \in V_{\mathrm{div,div}}^h$ in (4.82). Note that choosing a special function yields an upper bound of the infimum. Hence, the velocity error term can be also bounded, using Lemma 2.51 and the interpolation error estimate (C.11), by

$$\inf_{\boldsymbol{v}^h\in V_{\mathrm{div}}^h}\left(4\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}^2 + 2\frac{\mu}{\nu}\left\|\nabla\cdot\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2\right) \leq 4C_{V_{\mathrm{div,div}}^h}^2 h^{2k}\left\|\boldsymbol{u}\right\|_{H^{k+1}(\Omega)}^2,$$

since $\left\|\nabla\cdot\boldsymbol{v}^h\right\|_{L^2(\Omega)}^2$ vanishes. Combining both estimates gives (4.87). ■

Remark 4.62. Discussion of the estimates from Corollary 4.61, good choices for the stabilization parameters. The different cases from Corollary 4.61 will be discussed now in more detail.

i) If $V^h_{\text{div,div}}$ does not have optimal approximation properties, one can regard the right-hand side of (4.86) as a function dependent on μ. This function has a minimum which can be determined by elementary calculus, namely by checking the necessary and a sufficient condition for a local minimum. One obtains

$$\mu_{\text{opt}} \approx \frac{C_{Q^h}}{C_{V^h_{\text{div}}}} \frac{\|p\|_{H^k(\Omega)}}{\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}}. \tag{4.89}$$

Hence, with respect to ν and h, the standard parameter choice $\mu = \mathcal{O}(1)$ is recovered. However, it should be emphasized that μ_{opt} from (4.89) may be quite large, whenever the velocity norm is small compared with the pressure norm and that this situation can happen in practice. Moreover, inserting μ_{opt} into the error estimate (4.86) gives

$$\begin{aligned} &\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} && (4.90)\\ &\le 2h^k\left(C^2_{V^h_{\text{div}}}\|\boldsymbol{u}\|^2_{H^{k+1}(\Omega)} + \frac{1}{\nu}C_{V^h_{\text{div}}}C_{Q^h}\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}\|p\|_{H^k(\Omega)}\right)^{1/2}. \end{aligned}$$

This estimate reveals a direct dependency of the error of the gradient of the velocity on the pressure of the form $(\|p\|_{H^k(\Omega)}/\nu)^{1/2}$, even for the best possible stabilization parameter. On the one hand, this dependency on the viscosity is weaker than for the Galerkin discretization where the error bound depends on ν^{-1}, see (3.38), but on the other hand, even if the approximation space V^h_{div} can approximate the velocity solution $\boldsymbol{u}$ perfectly, it is not guaranteed that the grad-div-stabilized solution is a good approximation.

In this case, the grad-div stabilization is therefore able to mitigate the problem of poor mass conservation and the dependency of the velocity error on the pressure, but in general it is not able to solve it.

ii) If $V^h_{\text{div,div}}$ has optimal approximation properties, the right-hand side of estimate (4.87) is not as easy to analyze. Numerical evidence shows, see Jenkins et al. (2014), that, depending on the complexity of the pressure, there may or there may be not an optimal μ, since for $\|p\|_{H^k(\Omega)} \gg \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}$ one has $\mu_{\text{opt}} = \infty$, which is not feasible in practice. Therefore, giving up the idea of finding the optimal μ, one wants to find a good μ, which should not be infinity. One can use the criterion that the contribution of the pressure error equals the maximum possible contribution of the velocity error $4C^2_{V^h_{\text{div,div}}}h^{2k}\|\boldsymbol{u}\|^2_{H^{k+1}(\Omega)}$, which is already asymptotically optimal. This criterion leads to

$$\mu_{\text{good}} \approx \frac{1}{2\nu}\left(\frac{C_{Q^h}}{C_{V^h_{\text{div,div}}}}\frac{\|p\|_{H^k(\Omega)}}{\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}}\right)^2. \quad (4.91)$$

Numerical studies in Jenkins et al. (2014) show that this value gives in fact good results. It is interesting that only in the second case μ_{good} is dependening on ν, which could be observed in the numerical studies of Jenkins et al. (2014) as well. Inserting μ_{good} into (4.87) gives the error estimate

$$\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{u}^h\right)\right\|_{L^2(\Omega)} \le \sqrt{8}C_{V^h_{\text{div,div}}} h^k \|\boldsymbol{u}\|_{H^{k+1}(\Omega)}, \quad (4.92)$$

which does not directly depend on ν and p. But of course $\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}$ might still depend on ν. If $\|\boldsymbol{u}\|_{H^{k+1}(\Omega)}$ does not depend on ν, (4.92) shows that the grad-div stabilization is able to deliver optimal uniform approximations, if there exists a subspace of optimally converging divergence-free finite element functions.

For both cases, one does not observe a dependency on the mesh width h. □

detailed presentation for MINI element

Remark 4.63. On the choice of the stabilization parameters.

- Since $V^h_{\text{div,div}} \subset V^h_{\text{div}}$, it can be expected that $C_{V^h_{\text{div,div}}}$ is larger than $C_{V^h_{\text{div}}}$.
- Letting $\mu \to \infty$ for the P_2/P_1 pair of finite element spaces leads to the Scott–Vogelius pair P_2/P_1^{disc}, see Case et al. (2011).
- The dependency of the stabilization parameter on higher order norms of the solution can be already found in Olshanskii et al. (2009); Heister and Rapin (2013).
- In Jenkins et al. (2014), optimal stabilization parameters on the basis of the $L^2(\Omega)$ error estimate of the pressure were also derived. These parameters differ from the parameters obtained from the estimate of the gradient of the velocity.
- A similar discussion of the results of Corollary 4.61 can be performed for any inf-sup stable conforming pair of finite element spaces. Naturally, results for optimal stabilization parameter μ will vary.

□

some numerical example, with good parameters, focusing on mass conservation

4.6 Implementation of Finite Element Methods

Remark 4.64. The prinicipal approach. The principal approach concerning the construction of finite element spaces is analogously as for the Stokes equations, see Section 3.6. □

Remark 4.65. The Galerkin finite element discretization. The linear saddle point problem for the Galerkin finite element discretization (4.11) has the form

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}, \tag{4.93}$$

where the matrix block A is a diagonal block matrix with the same blocks

$$\begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{11} & 0 \\ 0 & 0 & A_{11} \end{pmatrix}. \tag{4.94}$$

The entries are given by

$$(A_{11})_{ij} = a_{ij} = \sum_{K\in\mathcal{T}^h} \left[\left(\nu\nabla\boldsymbol{\phi}_j^h, \nabla\boldsymbol{\phi}_i^h\right)_K + \left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\phi}_j^h + c\boldsymbol{\phi}_j^h, \boldsymbol{\phi}_i^h\right)_K \right],$$

$i,j = 1,\ldots,3N_v$. Thus, if $\boldsymbol{b} \neq \boldsymbol{0}$, the matrix A is not symmetric. But the matrix structure is the same as for the Stokes problem, see Remark 3.77. □

Remark 4.66. The SUPG term. The SUPG term influences the velocity-velocity coupling, the pressure (ansatz) - velocity (test) coupling, and the right-hand side for the test function of the velocity. One gets the matrix entries

$$\begin{aligned}(A_{11})_{ij} = a_{ij} = \sum_{K\in\mathcal{T}^h} \Big[&\left(\nu\nabla\boldsymbol{\phi}_j^h, \nabla\boldsymbol{\phi}_i^h\right)_K + \left((\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\phi}_j^h + c\boldsymbol{\phi}_j^h, \boldsymbol{\phi}_i^h\right)_K \\ &+ \left(-\nu\Delta\boldsymbol{\phi}_j^h + (\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\phi}_j^h + c\boldsymbol{\phi}_j^h, \delta_K^v\,(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\phi}_i^h\right)_K \Big],\end{aligned}$$

$i,j = 1,\ldots,3N_v$, and

$$(D)_{ij} = d_{ij} = \sum_{K\in\mathcal{T}^h} - \left[\left(\nabla\cdot\boldsymbol{\phi}_i^h, \psi_j^h\right)_K + \left(\nabla\psi_j^h, \delta_K^v\left(\boldsymbol{b}\cdot\nabla\boldsymbol{\phi}_i^h\right)\right)_K \right],$$

$i = 1,\ldots,3N_v$, $j = 1,\ldots,N_p$. The matrix block of the velocity-velocity coupling has still the diagonal form (4.94). The right-hand side becomes

$$(\underline{f})_i = f_i = \sum_{K\in\mathcal{T}^h} \left[\left(\boldsymbol{f}, \boldsymbol{\phi}_i^h\right)_K + \left(\boldsymbol{f}, \delta_K^v\,(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\phi}_i^h\right)_K \right], \quad i = 1,\ldots,3N_v.$$

Altogether, the coupled system has the form

$$\begin{pmatrix} A & D \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix},$$

such that the matrix D has to be stored in addition to A and B. □

Remark 4.67. The PSPG term. The PSPG term introduces a pressure-pressure coupling, it influences the velocity (ansatz) - pressure (test) coupling, and it defines a non-zero right-hand side for the pressure test functions. In detail, the matrix entries are given by

$$\begin{aligned}(B)_{ij} = b_{ij} = \sum_{K\in\mathcal{T}^h} \Big[&- \left(\nabla\cdot\boldsymbol{\phi}_j^h, \psi_i^h\right)_K \\ &+ \left(-\nu\Delta\boldsymbol{\phi}_j^h + (\boldsymbol{b}\cdot\nabla)\,\boldsymbol{\phi}_j^h + c\boldsymbol{\phi}_j^h, \delta_K^p\nabla\psi_i^h\right)_K \Big],\end{aligned}$$

$i = 1,\ldots,N_p$, $j = 1,\ldots,3N_v$, and

$$(C)_{ij} = c_{ij} = -\sum_{E\in\mathcal{E}^h} \gamma_E \left([|\psi_j^h|]_E, [|\psi_i^h|]_E \right)_E - \sum_{K\in\mathcal{T}^h} \left(\nabla\psi_j^h p^h, \delta_K^p \nabla\psi_i^h\right)_K,$$

$i, j = 1,\ldots,N_p$. On the right-hand side, the term

$$(\underline{f_p})_i = -\sum_{K\in\mathcal{T}^h} \left(\boldsymbol{f}, \delta_K^p\nabla\psi_i^h\right)_K, \quad i = 1,\ldots,N_p,$$

appears. The coupled system has the form

$$\begin{pmatrix} A & D \\ B & C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{f_p} \end{pmatrix},$$

where the matrix A is the same matrix as in the Stokes problem (3.90), it is of block-diagonal form (4.94), and the matrix D is the transposed of the matrix B from the Stokes problem (3.91). In summary, the PSPG method requires the additional storage of a velocity-pressure matrix, a pressure-pressure matrix, and a right-hand side for the pressure test functions. □

Remark 4.68. The grad-div term. The grad-div term affects only the velocity-velocity coupling. The matrix entries have the form

$$(A_{kl})_{ij} = \left(\partial_l \phi_j^h, \partial_k \phi_i^h\right), \quad k,l = 1,\ldots,d,\ i,j = 1,\ldots,N_v.$$

such that the matrix A is of the following structure

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix}. \tag{4.95}$$

The symmetry of the off-diagonal blocks of A follows directly from the definition of the grad-div term. □

Remark 4.69. The SUPG/PSPG/grad-div (spg) method. For the SUPG/PSPG/grad-div method one has to add all requirements of the individual stabilization terms. The linear system of equations for this method has the form

$$\begin{pmatrix} A & D \\ B & C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{f_p} \end{pmatrix}, \tag{4.96}$$

where the matrix block A is of structure (4.95). Altogether, the memory requirements of the SUPG/PSPG/grad-div method are much higher compared with the Galerkin method. □

Remark 4.70. The SUPG/grad-div (sg) method for inf-sup stable pairs of finite element spaces. Compared with the SUPG/PSPG/grad-div method, there is no block which couples test and ansatz functions for the pressure such that the coupled system has the form

$$\begin{pmatrix} A & D \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{f_p} \end{pmatrix},$$

where the matrix block A is of structure (4.95). □

Remark 4.71. The CIP method. The CIP method adds stabilization terms which couple either velocity test and ansatz functions or pressure test and ansatz functions. The arising coupled system has the form

$$\begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}.$$

As already discussed in Remark 4.47, the matrices A and C are less sparse than usual finite element matrices. Because of the grad-div term in the CIP method, the block structure of A is as given in (4.95). □

Remark 4.72. The LPS method. For the two-level LPS method, exactly the same remarks hold as for the CIP method. In the one-level LPS method, the sparsity of the matrices is not extended by connecting degrees of freedom which are usually not connected. However, the enrichment of the spaces leads to more degrees of freedom on the same grid compared with the Galerkin method or other stabilized discretizations. check □

Remark 4.73. The upwind method for non-conforming finite element spaces of lowest order. In the upwind method, one obtains the same structure of the coupled system as for the Galerkin method, see (4.93) and (4.94). In addition, the maximal number of entries in each row and column of the matrices is independent of the actual meshes since the degrees of freedom can be assigned to the faces and the number of neighboring faces does not depend on the mesh. For instance, for the Crouzeix–Raviart element P_1^{nc}/P_0 there are at most five entries in each row of A_{11} in two dimensions and seven entries in three dimensions. □