Using in the definition of the discrete Laplacian as test function the discrete Laplacian itself and applying the inverse inequality gives

$$\left\|\Delta^h \boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 \le \left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)} \left\|\nabla \Delta^h \boldsymbol{w}^h\right\|_{L^2(\Omega)} \le \frac{C_{\text{inv}}}{h} \left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)} \left\|\Delta^h \boldsymbol{w}^h\right\|_{L^2(\Omega)},$$

which gives an inverse estimate for the discrete Laplacian. Inserting this inverse estimate into (7.227) and using the assumption on α yields

$$\left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 \le C \left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}. \tag{7.228}$$

Now, (7.223) is obtained by neglecting the terms with α on the left-hand side of (7.222), estimating the other terms on the left-hand side with (7.226) and (7.228), and bounding the terms from the initial condition with (7.189).

existence and uniqueness, see Leray ■

Remark 7.205. Further results from finite element analysis.

- An error estimate for the continuous-in-time case can be found in Connors (2010). The proof of this estimate follows the standard lines, e.g., as the proof of Theorem 6.46 for the Navier–Stokes equations. It uses some properties of the discrete differential filter which were also used in the proof of Lemma 7.204. The nonlinear convective term is estimated with inequalities that can be found in Section 5.1.2. Under some regularity assumptions, one obtains an estimate for

$$\left\|\left(\boldsymbol{u} - \boldsymbol{w}^h\right)(t)\right\|_{L^2(\Omega)}^2 + \nu \left\|\nabla\left(\boldsymbol{u} - \boldsymbol{w}^h\right)\right\|_{L^2(0,t;L^2(\Omega))}^2$$

  with an factor of size $\exp(C\nu^{-3}t)$, similarly to the Navier–Stokes equations, see (6.39), and the Leray-α model, see (7.200), (7.202). It is assumed that $\alpha = \mathcal{O}(h)$ which leads to second order convergence, as for the Leray-α model, see Remark 7.183.
- A full discretization with the Crank–Nicolson scheme as time integrator was considered in Layton et al. (2010). In this paper, the stability of the finite element solution is proved, but an error analysis is not presented. The error analysis for the fully discrete case seems to be an open problem. but see Miles,Rebholz 2010

□

7.8 Variational Multiscale Methods

7.8.1 The Basic Concept

Remark 7.206. Differences to LES. Similarly to classical LES methods, Variational Multiscale (VMS) methods seek to simulate only large flow structures.

Therefore, these methods are also called VMS-LES. However, there are some fundamental differences between VMS methods and classical LES methods.

The difficulties of the classical LES methods in their mathematical study originate in the definition of the large scales by spatial averaging. As an alternative, VMS methods consider large scales which are defined by projection into appropriate spaces. To this end, a variational formulation of the Navier–Stokes equations is considered, again in contrast to LES methods, whose derivation is based on the strong form of the Navier–Stokes equations, see Sections 7.2.2 and 7.2.3. The consideration of a variational form of the equation and the use of projections for defining the different scales allow to incorporate boundary conditions into the mathematical analysis in a natural way.

There are VMS methods which decompose the flow field into two scales, resolved and unresolved ones, like LES methods, see Remark 7.23. However, the VMS methodology allows also the decomposition into more than two scales. Quite popular are so-called three-scale VMS methods. For these methods, there is another difference to LES methods: the turbulence model does not act directly on all resolved scales but only on the smallest resolved scales, see Remark 7.207.

First ideas of projection-based methods, also for different problems than the Navier–Stokes equations, can be found in Hughes (1995), Guermond (1999a), and Hughes et al. (2000). □

Remark 7.207. Basic approach for an a priorly three-scale VMS method. Consider the Navier–Stokes equations (6.1) in a bounded domain, equipped for simplicity with no-slip conditions, and a decomposition of the flow into three scales, following Collis (2001):

- the large scales $(\overline{\boldsymbol{u}}, \overline{p})$,
- the small resolved scales $(\widehat{\boldsymbol{u}}, \widehat{p})$,
- the unresolved scales $(\boldsymbol{u}', p')$,

with $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widehat{\boldsymbol{u}} + \boldsymbol{u}'$ and $p = \overline{p} + \widehat{p} + p'$.

The starting point of a VMS method is the variational formulation of the Navier–Stokes equations, e.g., like (6.35). This formulation can be written in short form

$$A\left(\boldsymbol{u}; (\boldsymbol{u}, p), (\boldsymbol{v}, q)\right) = F\left(\boldsymbol{v}\right). \tag{7.229}$$

Decomposing the test functions also into three scales, the variational form of the Navier–Stokes equations can be written as a coupled system: Find $\boldsymbol{u} = \overline{\boldsymbol{u}} + \widehat{\boldsymbol{u}} + \boldsymbol{u}' \; : \; [0,T] \to V$, $p = \overline{p} + \widehat{p} + p' \; : \; (0,T] \to Q$ satisfying for all $(\boldsymbol{v}, q) \in V \times Q$ with $\boldsymbol{v} = \overline{\boldsymbol{v}} + \widehat{\boldsymbol{v}} + \boldsymbol{v}'$, $q = \overline{q} + \widehat{q} + q'$

$$\begin{aligned}
A\left(\boldsymbol{u}; (\overline{\boldsymbol{u}}, \overline{p}), (\overline{\boldsymbol{v}}, \overline{q})\right) + A\left(\boldsymbol{u}; (\widehat{\boldsymbol{u}}, \widehat{p}), (\overline{\boldsymbol{v}}, \overline{q})\right) + A\left(\boldsymbol{u}; (\boldsymbol{u}', p'), (\overline{\boldsymbol{v}}, \overline{q})\right) &= F\left(\overline{\boldsymbol{v}}\right), \\
A\left(\boldsymbol{u}; (\overline{\boldsymbol{u}}, \overline{p}), (\widehat{\boldsymbol{v}}, \widehat{q})\right) + A\left(\boldsymbol{u}; (\widehat{\boldsymbol{u}}, \widehat{p}), (\widehat{\boldsymbol{v}}, \widehat{q})\right) + A\left(\boldsymbol{u}; (\boldsymbol{u}', p'), (\widehat{\boldsymbol{v}}, \widehat{q})\right) &= F\left(\widehat{\boldsymbol{v}}\right), \\
A\left(\boldsymbol{u}; (\overline{\boldsymbol{u}}, \overline{p}), (\boldsymbol{v}', q')\right) + A\left(\boldsymbol{u}; (\widehat{\boldsymbol{u}}, \widehat{p}), (\boldsymbol{v}', q')\right) + A\left(\boldsymbol{u}; (\boldsymbol{u}', p'), (\boldsymbol{v}', q')\right) &= F\left(\boldsymbol{v}'\right).
\end{aligned}$$

Here, the linearity of the variational problem with respect to the test function has been used. Now, the basic ideas and assumptions of an a priorly three-scale VMS method are as follows:

- The equation with the test function from the unresolved scales is neglected, i.e., the equations are projected in the space of the resolved scales.
- It is assumed that the direct influence of the unresolved scales on the large scales is negligible, i.e., $A(\boldsymbol{u};(\boldsymbol{u}',p'),(\overline{\boldsymbol{v}},\overline{q}))=0$. The unresolved scales might influence the resolved scales, e.g., by a backscatter of energy. It was mentioned in Remark 7.7 that this backscatter occurs mainly to the next larger eddies, which are represented by the small resolved scales. In this sense, this assumption is reasonable.
- The influence of the unresolved scales onto the small resolved scales is modeled:

$$A(\boldsymbol{u};(\boldsymbol{u}',p'),(\widehat{\boldsymbol{v}},\widehat{q}))\approx T(\boldsymbol{u};(\overline{\boldsymbol{u}},\overline{p}),(\widehat{\boldsymbol{u}},\widehat{p}),(\widehat{\boldsymbol{v}},\widehat{q})).$$

  The choice of the model $T(\boldsymbol{u};(\overline{\boldsymbol{u}},\overline{p}),(\widehat{\boldsymbol{u}},\widehat{p}),(\widehat{\boldsymbol{v}},\widehat{q}))$ may be guided by physical ideas in turbulence modeling, e.g., eddy viscosity models of Smagorinsky type (7.65), for $\overline{\boldsymbol{u}}$ or $\widehat{\boldsymbol{u}}$, are often used. From the numerical point of view, the turbulence model $T(\boldsymbol{u};(\overline{\boldsymbol{u}},\overline{p}),(\widehat{\boldsymbol{u}},\widehat{p}),(\widehat{\boldsymbol{v}},\widehat{q}))$ introduces additional viscosity which acts as stabilization.

Let $\overline{V},\overline{Q}$ be spaces representing the large scales and $\widehat{V},\widehat{Q}$ spaces for the resolved small scales. An abstract a priorly three-scale VMS method reads as a coupled system of the form: Find $(\overline{\boldsymbol{u}},\widehat{\boldsymbol{u}},\overline{p},\widehat{p})\in\overline{V}\times\widehat{V}\times\overline{Q}\times\widehat{Q}$ such that

$$\begin{aligned}
&A(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};(\overline{\boldsymbol{u}},\overline{p}),(\overline{\boldsymbol{v}},\overline{q}))+A(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};(\widehat{\boldsymbol{u}},\widehat{p}),(\overline{\boldsymbol{v}},\overline{q}))=F(\overline{\boldsymbol{v}}),\\
&A(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};(\overline{\boldsymbol{u}},\overline{p}),(\widehat{\boldsymbol{v}},\widehat{q}))+A(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};(\widehat{\boldsymbol{u}},\widehat{p}),(\widehat{\boldsymbol{v}},\widehat{q})) \qquad (7.230)\\
&\qquad +T(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};(\overline{\boldsymbol{u}},\overline{p}),(\widehat{\boldsymbol{u}},\widehat{p}),(\widehat{\boldsymbol{v}},\widehat{q}))=F(\widehat{\boldsymbol{v}}) \qquad (7.231)
\end{aligned}$$

for all $(\overline{\boldsymbol{v}},\widehat{\boldsymbol{v}},\overline{q},\widehat{q})\in\overline{V}\times\widehat{V}\times\overline{Q}\times\widehat{Q}$.

Note that a characteristic feature of an a priorly three-scale VMS method is that the model for the influence of the unresolved scales acts directly only on the small resolved scales. Since the small resolved scales and the large scales are coupled in (7.230), (7.231), the model $T(\boldsymbol{u};(\overline{\boldsymbol{u}},\overline{p}),(\widehat{\boldsymbol{u}},\widehat{p}),(\widehat{\boldsymbol{v}},\widehat{q}))$ influences the large scales indirectly. This situation is in contrast to classical LES models, where the model acts directly on all resolved scales.

To specify a concrete a priorly three-scale VMS method, one has to define the spaces $\overline{V},\widehat{V},\overline{Q},\widehat{Q}$ and a model $T(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};(\overline{\boldsymbol{u}},\overline{p}),(\widehat{\boldsymbol{u}},\widehat{p}),(\widehat{\boldsymbol{v}},\widehat{q}))$. □

Remark 7.208. Principal approaches for choosing appropriate spaces in an a priorly three-scale VMS methods. Concerning finite element methods for discretizing (7.230), (7.231), there are at least two different approaches.

Standard finite element spaces can be used for the large scales $\overline{V}\times\overline{Q}$. The finite element spaces $\widehat{V}\times\widehat{Q}$ need to have a higher resolution since they should represent smaller scales. A proposal consists in using mesh cell bubble functions for this purpose, see Section 7.8.2.1 for details.

The second way for choosing the spaces consists in using a common standard finite element space for all resolved scales and an additional large scale space. Methods of this type will be presented in Sections 7.8.2.2 and 7.8.2.3. □

Remark 7.209. Common goal of an a priorly three-scale VMS methods and the dynamic Smagorinsky model. Using for $T\left(\overline{\boldsymbol{u}}+\widehat{\boldsymbol{u}};\left(\overline{\boldsymbol{u}},\overline{p}\right),\left(\widehat{\boldsymbol{u}},\widehat{p}\right),\left(\widehat{\boldsymbol{v}},\widehat{q}\right)\right)$ a model of Smagorinsky-type, as it is often done, then the principle goal of an a priorly three-scale VMS method and the dynamic Smagorinsky model presented in Remark 7.126 is similar. Based on the experience that the use of the Smagorinsky model (7.65) with a fixed constant C_S as turbulence model introduces too much viscosity, one tries to reduce the influence of this model in accordance to the local flow field. In the dynamic Smagorinsky model, this reduction is done by using a function $C_S\left(t,\boldsymbol{x}\right)$ and adjusting this function appropriately. In an a priorly three-scale VMS method, the reduction is performed by choosing an appropriate small resolved scale space to which the direct influence of the Smagorinsky model is restricted. □

Remark 7.210. Basic approach for an a priorly two-scale VMS method. An a priorly two-scale VMS method uses just a decomposition of the scales in resolved scales $(\overline{\boldsymbol{u}},\overline{p})$ and unresolved scales $(\boldsymbol{u}',p')$ with

$$\boldsymbol{u}=\overline{\boldsymbol{u}}+\boldsymbol{u}',\quad p=\overline{p}+p'.$$

Inserting this decomposition in (7.229), using the same decomposition for the test functions, and neglecting the equation with the test functions from the unresolved scales yields the equation

$$A\left(\boldsymbol{u};\left(\overline{\boldsymbol{u}},\overline{p}\right),\left(\overline{\boldsymbol{v}},\overline{q}\right)\right)+A\left(\boldsymbol{u};\left(\boldsymbol{u}',p'\right),\left(\overline{\boldsymbol{v}},\overline{q}\right)\right)=F\left(\overline{\boldsymbol{v}}\right). \tag{7.232}$$

check Now, one needs a turbulence model for modeling the unresolved scales in (7.232).

There are approaches where the turbulence model introduces a decomposition of the resolved scales into large and small resolved scales, see Section 7.8.2.2. In this way, it is possible that a three-scale VMS method is based on an a priorly two-scale decomposition. □

Remark 7.211. Notation. To be consistent with the presentation of the other turbulence models, the large scales computed with a VMS method are denoted by $(\boldsymbol{w}^h,r^h)$ and the small resolved scales by $\left(\widehat{\boldsymbol{w}}^h,\widehat{r^h}\right)$. □

7.8.2 Realizations of VMS Methods

7.8.2.1 A Three-Scale Bubble VMS Method

Remark 7.212. Realizations of three-scale bubble VMS methods. Bubble VMS methods might be the most direct realization of an a priorly three-scale VMS method as described in Remark 7.207.

The main goal of using bubble functions for approximating the small resolved scales consists in splitting the equation (7.231) for these scales into a number of local problems to obtain an efficient method, see Remark 7.214. This idea was already pointed out in Hughes et al. (2000).

A first realization of this idea can be found in Gravemeier et al. (2004, 2005). In these papers, the velocity and the pressure were approximated with bilinear or trilinear finite elements. Only the velocity space was enriched with bubble functions for the small resolved scales. With this enrichment, the pair of finite element spaces becomes inf-sup stable. The stabilizing effect of bubble functions with respect to the discrete inf-sup condition (2.48) was already seen for the MINI element in Section 2.6.1 or the pair $P_2^{\text{bubble}}/P_1^{\text{disc}}$, see Remark 2.124. The model for the small resolved pressure does not use bubble functions, see Remark 7.216. A main issue in Gravemeier et al. (2004, 2005) was the investigation of the turbulence model applied to the small resolved scales. A realization of a three-scale bubble VMS method with second order velocity and first order pressure, which followed the principal ideas of Gravemeier et al. (2004, 2005), was explored in John and Kindl (2010).

In this section, just one possible approach for a bubble VMS method is sketched. □

Remark 7.213. Bubble VMS method: basic equations. Let the resolved scales $(\boldsymbol{u}^h, p^h)$ be decomposed into large scales $(\overline{\boldsymbol{u}}, \overline{p})$ and small resolved scales $(\widehat{\boldsymbol{u}}, \widehat{p})$.

The equation with the large scale test functions, where the coupling of the large scales and the unresolved scales has been neglected, has the form, compare (7.230),

$$
\begin{aligned}
&\left(\partial_t \boldsymbol{u}^h, \overline{\boldsymbol{v}}\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{u}^h\right), \mathbb{D}\left(\overline{\boldsymbol{v}}\right)\right) \\
&\quad + \left(\left(\boldsymbol{u}^h \cdot \nabla\right) \boldsymbol{u}^h, \overline{\boldsymbol{v}}\right) - \left(\nabla \cdot \overline{\boldsymbol{v}}, p^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, \overline{q}\right) = \left(\boldsymbol{f}, \overline{\boldsymbol{v}}\right).
\end{aligned}
$$

Applying the splitting of the resolved scales yields

$$
\begin{aligned}
&(\partial_t \overline{\boldsymbol{u}}, \overline{\boldsymbol{v}}) + (2\nu\mathbb{D}(\overline{\boldsymbol{u}}), \mathbb{D}(\overline{\boldsymbol{v}})) + ((\overline{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}}) - (\nabla \cdot \overline{\boldsymbol{v}}, \overline{p}) + (\nabla \cdot \overline{\boldsymbol{u}}, \overline{q}) \\
&= (\boldsymbol{f}, \overline{\boldsymbol{v}}) - \Big[(\partial_t \widehat{\boldsymbol{u}}, \overline{\boldsymbol{v}}) + (2\nu\mathbb{D}(\widehat{\boldsymbol{u}}), \mathbb{D}(\overline{\boldsymbol{v}})) \\
&\quad + \left(\left(\boldsymbol{u}^h \cdot \nabla\right)\widehat{\boldsymbol{u}}, \overline{\boldsymbol{v}}\right) + ((\widehat{\boldsymbol{u}} \cdot \nabla)\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}}) - (\nabla \cdot \overline{\boldsymbol{v}}, \widehat{p}) + (\nabla \cdot \widehat{\boldsymbol{u}}, \overline{q}) \Big].
\end{aligned}
\tag{7.233}
$$

Similarly, one obtains an equation for the small resolved scale test functions, where the eddy viscosity model is already included,

$$\begin{aligned}&(\partial_t\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}}) + ((2\nu+\nu_{\mathrm{T}})\,\mathbb{D}\,(\widehat{\boldsymbol{u}})\,,\mathbb{D}\,(\widehat{\boldsymbol{v}}))\\ &\quad+\left(\left(\boldsymbol{u}^h\cdot\nabla\right)\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}}\right) - (\nabla\cdot\widehat{\boldsymbol{v}},\widehat{p}) + (\nabla\cdot\widehat{\boldsymbol{u}},\widehat{q})\\ &= (\boldsymbol{f},\widehat{\boldsymbol{v}}) - \Big[(\partial_t\overline{\boldsymbol{u}},\widehat{\boldsymbol{v}}) + (2\nu\mathbb{D}\,(\overline{\boldsymbol{u}})\,,\mathbb{D}\,(\widehat{\boldsymbol{v}}))\\ &\quad+\left(\left(\boldsymbol{u}^h\cdot\nabla\right)\overline{\boldsymbol{u}},\widehat{\boldsymbol{v}}\right) - (\nabla\cdot\widehat{\boldsymbol{v}},\overline{p}) + (\nabla\cdot\overline{\boldsymbol{u}},\widehat{q})\Big].\end{aligned} \tag{7.234}$$

□

Remark 7.214. Localization of the small resolved scale equation with bubble functions. In a bubble-based finite element VMS method, standard finite element spaces are used for the large scales $\overline{V}\times\overline{Q} = V^h\times Q^h$. The finite element spaces for the small resolved scales require a higher resolution than the finite element spaces for the large scales. This goal can be achieved in various ways: by using higher order finite elements, by refining the given grid, or by combining these approaches. However, the result of all approaches is that the solution of the small resolved scale equations (7.234) would be much more expensive than solving the large scale equations (7.233). This difficulty is circumvented in a bubble-based finite element VMS method by considering (7.234) in a space of bubble functions for the velocity. A bubble function is a function from $H_0^1(\Omega)$ whose support is only one mesh cell and which vanishes on the faces of this mesh cell. With these functions, the solution of (7.234) can be localized. Because of the modeling of the small resolved pressure explained in Remark 7.216, only a bubble space for the small resolved velocity is needed. In practice, this space has to be finite-dimensional and it will be denoted by $\widehat{V}^h_{\mathrm{bub}}$. □

Remark 7.215. An unphysical property introduced by using bubble functions for modeling the small resolved scales. There is a principal question in all bubble-based VMS approaches concerning the physics of the modeling of the small resolved scales. Since these scales are represented by bubble functions, they can move within a mesh cell but they cannot move directly from one mesh cell to another because of the homogeneous Dirichlet boundary conditions on the faces of the mesh cells. The information contained in the small resolved scales can be distributed to other mesh cells only indirectly by the coupling of the small resolved scales to the large scales. This quasi-stationary modeling of the small resolved scales does not reflect the physical reality. However, to the best of our knowledge, there are no numerical studies available which investigate the impact of this unphysical modeling in detail. □

Remark 7.216. Modelling of the small resolved pressure. Likewise as for the two-level residual-based VMS method, see (**??**), it was proposed in Gravemeier et al. (2004, 2005) to model the small resolved scale pressure in the form

$$\widehat{p} = -\sum_{K\in\mathcal{T}^h} \mu_K(\nabla\cdot\overline{\boldsymbol{u}}). \tag{7.235}$$

In this way, the influence of the small resolved scale pressure onto the large scales is not directly taken into account but this influence is modeled. Using (7.235), $\widehat{p}$ vanishes from the small resolved scale equations (7.234). Since the small resolved pressure disappeared, a divergence constraint for the small resolved velocity is not longer needed. Since there is no longer a divergence constraint for $\widehat{\boldsymbol{u}}$, it does not make sense to have a term with this function in the divergence constraint of the large scale equation (7.233). Altogether, all terms in (7.233) and (7.234) coming from the divergence constraint which include small resolved scales will be neglected by setting

$$(\nabla\cdot\widehat{\boldsymbol{u}},\overline{q}) = (\nabla\cdot\overline{\boldsymbol{u}},\widehat{q}) = (\nabla\cdot\widehat{\boldsymbol{u}},\widehat{q}) = 0. \tag{7.236}$$

One obtains from (7.234) a vector-valued equation for $\widehat{\boldsymbol{u}}$. The model (7.235) for the small resolved scale pressure is included into the large scale equation leading to a grad-div stabilization term. □

Remark 7.217. Further simplifications. Usually, some additional simplifying assumptions are made for the terms with the small resolved velocity scales. The equation for the small resolved velocity scales is only solved once in each discrete time, at the beginning giving the solution $\widehat{\boldsymbol{u}}^{(1)}$. Consequently, this equation is linearized and all terms with $\overline{\boldsymbol{u}}$ are treated explicitly. For reasons of efficiency, the gradient form of the viscous term is used in the small resolved scale equation and some right-hand side terms in the large scale equation. In particular, the small resolved scale equation decouples into three scalar equations since the system matrix becomes a block diagonal matrix, see Remarks 3.62 and 3.65.

Since the equation for the small resolved scales is solved only at the beginning of each time step, the temporal derivatives in (7.233) and (7.234) have to be modified. For the large scale equation (7.233) one uses

$$\partial_t\widehat{\boldsymbol{u}} \approx \frac{\widehat{\boldsymbol{u}}_{n+1}-\widehat{\boldsymbol{u}}_n}{\Delta t_{n+1}} \approx \frac{\widehat{\boldsymbol{u}}^{(1)}-\widehat{\boldsymbol{u}}_n}{\Delta t_{n+1}}.$$

In the small resolved scale equations, one assumes that the temporal change in the large scales can be neglected, i.e., that $\partial_t\overline{\boldsymbol{u}} = 0$. □

Remark 7.218. Bubble VMS method with time-dependent small resolved velocity scales. Inserting the models and simplifications from Remarks 7.216 and 7.217 into (7.233) and (7.234) and using the convention for the notation from Remark 7.211 leads to the following system of equations: Find $\overline{\boldsymbol{w}}\ :\ [0,T]\to\overline{V},\ \overline{r}\ :\ (0,T]\to\overline{Q}$ satisfying

$$
\begin{aligned}
&(\partial_t \overline{\boldsymbol{w}}, \overline{\boldsymbol{v}}) + (2\nu \mathbb{D}(\overline{\boldsymbol{w}}), \mathbb{D}(\overline{\boldsymbol{v}})) \\
&\quad + ((\overline{\boldsymbol{w}} \cdot \nabla) \overline{\boldsymbol{w}}, \overline{\boldsymbol{v}}) - (\nabla \cdot \overline{\boldsymbol{v}}, \overline{r}) + (\nabla \cdot \overline{\boldsymbol{w}}, \overline{q}) + \sum_{K \in \mathcal{T}^h} \mu_K (\nabla \cdot \overline{\boldsymbol{w}}, \nabla \cdot \overline{\boldsymbol{v}})_K \\
&= (\boldsymbol{f}, \overline{\boldsymbol{v}}) - \left[\left(\frac{\widehat{\boldsymbol{w}}^{(1)} - \widehat{\boldsymbol{w}}_n}{\Delta t_{n+1}}, \overline{\boldsymbol{v}} \right) + \left(\nu \nabla \widehat{\boldsymbol{w}}^{(1)}, \nabla \overline{\boldsymbol{v}} \right) \right. \qquad (7.237) \\
&\quad \left. \left(\left(\left(\overline{\boldsymbol{w}}_n + \widehat{\boldsymbol{w}}^{(1)} \right) \cdot \nabla \right) \widehat{\boldsymbol{w}}^{(1)}, \overline{\boldsymbol{v}} \right) + \left(\left(\widehat{\boldsymbol{w}}^{(1)} \cdot \nabla \right) \overline{\boldsymbol{w}}_n, \overline{\boldsymbol{v}} \right) \right]
\end{aligned}
$$

for all $(\overline{\boldsymbol{v}}, \overline{p}) \in \overline{V} \times \overline{Q}$. The equation for computing $\widehat{\boldsymbol{w}}^{(1)} : [0, T] \to V_{\text{bub}}^h$ reads as follows

$$
\begin{aligned}
&\left(\partial_t \widehat{\boldsymbol{w}}^{(1)}, \widehat{\boldsymbol{v}} \right) + \left((\nu + \nu_{\text{T}}) \nabla \widehat{\boldsymbol{w}}^{(1)}, \nabla \widehat{\boldsymbol{v}} \right) + \left(\left(\boldsymbol{w}_n^h \cdot \nabla \right) \widehat{\boldsymbol{w}}^{(1)}, \widehat{\boldsymbol{v}} \right) \\
&= (\boldsymbol{f}, \widehat{\boldsymbol{v}}) - \Big[(\nu \nabla \overline{\boldsymbol{w}}_n, \nabla \widehat{\boldsymbol{v}}) + \left(\left(\boldsymbol{w}_n^h \cdot \nabla \right) \overline{\boldsymbol{w}}_n, \widehat{\boldsymbol{v}} \right) \qquad (7.238) \\
&\quad - (\nabla \cdot \widehat{\boldsymbol{v}}, \overline{r}_n) + \sum_{K \in \mathcal{T}^h} \mu_K (\nabla \cdot \overline{\boldsymbol{w}}_n, \nabla \cdot \widehat{\boldsymbol{v}})_K \Big]
\end{aligned}
$$

for all $\widehat{\boldsymbol{v}} \in V_{\text{bub}}^h$. The subscript n refers always to functions computed in the previous discrete time. □

Remark 7.219. Other bubble VMS methods. The way for defining a bubble VMS method described in this section is just one possible approach. Other simplifications are possible, leading to (sligthly) different equations compared with (7.237) and (7.238). For instance, in John and Kindl (2010) a bubble VMS method was studied with quasi-stationary small resolved scales which avoids the storage of the bubble velocity from the previous discrete time, as it is necessary in (7.237), (7.238). □

Remark 7.220. Choice of the turbulence model. The definition of the small resolved scale equation (7.238) requires the choice of the turbulence model ν_{T}. In Gravemeier et al. (2004, 2005), a dynamic Smagorinsky model, see Remark 7.126, was used. The studies in John and Kindl (2010) applied a static Smagorinsky models of the form

$$
\nu_{\text{T}} = C_S h_K^2 \left\| \mathbb{D} \left(\overline{\boldsymbol{w}} + \widehat{\boldsymbol{w}}^{(1)} \right) \right\|_{\text{F}}, \quad \text{and} \quad \nu_{\text{T}} = C_S h_K^2 \left\| \mathbb{D}(\overline{\boldsymbol{w}}) \right\|_{\text{F}}. \qquad (7.239)
$$

□

Remark 7.221. Approximating the small resolved velocity scales with bubble functions. The space $\widehat{V}_{\text{bub}}^h$ has to be specified in order to compute the solution of the bubble equation (7.238). In Gravemeier et al. (2004, 2005); John and Kindl (2010), local grids in each hexahedral mesh cell were used. A typical size of a local grid was $5 \times 5 \times 5$ sub cells. On these local grids, the equation for the bubble functions was discretized, usually with Q_1 finite elements. For the use of the dynamic Smagorinsky model in Gravemeier et al. (2004, 2005), a second local grid was applied which was somewhat finer than the first local grid. □

Remark 7.222. Numerical experience with bubble VMS methods. In John and Kindl (2010), it is mentioned that the application of a bubble finite element VMS method is quite complicated: one has to decide about the simplifying assumptions with respect to the small resolved scales and also the implementation is quite involved. In addition, it turned out that the dominating term of the model is the grad-div term which evolves from modeling the small resolved pressure, see (7.236). Using only this term without modeling the small resolved velocity led to stable simulations. However, applying in addition to the grad-div stabilization also the bubble-based model for the small resolved velocity improved the accuracy of the results. It is also mentioned in John and Kindl (2010) that the rather coarse grids for solving the localized equations (7.238) required to take large values for the parameter C_S of the Smagorinsky models (7.239). Altogether, the use of the bubble VMS method is not recommended in John and Kindl (2010). □

7.8.2.2 A Three-Scale Algebraic Variational Multiscale-Multigrid Method (AVM^3)

Remark 7.223. History. The AVM^3 method was introduced in Gravemeier et al. (2009, 2010) and developed further in Rasthofer and Gravemeier (2013) to AVM^4. □

Remark 7.224. The definition of the small resolved scales. The definition of the small resolved scales uses an idea from algebraic multigrid (AMG) methods. The motivation for this approach comes from the goal to define the scale separation of the resolved scales without introducing another finite element space or another grid.

AMG methods are a proposal for transferring the ideas of geometric multigrid methods, see Remark 8.8, to problems where coarser geometric grids are not available, e.g., see Stüben (2001) for a description and a review. To this end, a multilevel structure is constructed solely based on a matrix, which represents the problem on the given grid. Then, coarser levels, discrete operators on these levels, and transfer operators (restriction and prolongation) are constructed. For the scale separation in AVM^3, only the construction of one coarser level and the corresponding transfer operators is needed.

There are several possibilities for constructing coarser levels in AMG methods. For AVM^3, it is proposed to use the simplest one, namely plain aggregation, see Vaněk et al. (1996). The degrees of freedom on the given grid correspond to the rows of the given matrix A. In Gravemeier et al. (2009, 2010) some root degree of freedom i is chosen and an aggregate is formed from the union of all degrees of freedom j for which the matrix entry a_{ij}

does not vanish. Then, these degrees of freedom are removed from the list, a next root degree of freedom is chosen and this procedure is continued until all degrees of freedom belong to an aggregate. There are also other possibilities for choosing the aggregates, e.g., based on the strength of the coupling in the matrix A, i.e., based on $|a_{ij}|$. The aggregates represent the degrees of freedom on the coarse level. Denoting the fine and the coarse level in terms of the mesh with h of the geometric grid corresponding to the fine level, then the aggregates on the coarse level are usually denoted by $3h$.

Next, operators for the restriction of the residual R_h^{3h} and the prolongation of functions P_{3h}^h have to be defined. To this end, consider the matrix $\tilde{A}$ which differs from A only in the way that essential boundary conditions are replaced with natural boundary conditions. Let $\tilde{A}_0$ be a matrix whose columns span the kernel of $\tilde{A}$, i.e.,

$$\tilde{A}\tilde{A}_0 = 0. \tag{7.240}$$

The matrix on the coarse grid can be defined with the so-called Galerkin projection

$$\tilde{A}^{3h} = R_h^{3h}\tilde{A}P_{3h}^h.$$

Denoting the matrix which spans the kernel of $\tilde{A}^{3h}$ by $\tilde{A}_0^{3h}$, one obtains

$$0 = \tilde{A}^{3h}\tilde{A}_0^{3h} = R_h^{3h}\tilde{A}P_{3h}^h\tilde{A}_0^{3h}.$$

With (7.240) it follows that this equation is satisfied if

$$P_{3h}^h\tilde{A}_0^{3h} = \tilde{A}_0. \tag{7.241}$$

Based on (7.241), the operators P_{3h}^h and $\tilde{A}_0^{3h}$ can be determined simultaneously, see Gravemeier et al. (2009) for details. Finally, one sets

$$R_h^{3h} = \left(P_{3h}^h\right)^T.$$

Note that these operators are linear operators between finite-dimensional spaces and thus they can be represented with matrices. is there a difference between function and residual restriction in AMG?

Now, the operator for defining the large scales is given by

$$S_h^{3h} : V^h \to V^h, \quad \boldsymbol{u}^{3h} = P_{3h}^h R_h^{3h}\boldsymbol{u}^h,$$

i.e., in the first step $\boldsymbol{u}^h$ is restricted to the aggregates and in the second step, the representation of the aggregates in the finite element space is obtained. The small resolved scales are defined by

$$\boldsymbol{u}^h = \boldsymbol{u}^{3h} + \widehat{\boldsymbol{u}}^h \quad \Longleftrightarrow \quad \widehat{\boldsymbol{u}}^h = \boldsymbol{u}^h - \boldsymbol{u}^{3h}.$$

In AVM^3 from Gravemeier et al. (2010), the definition of the aggregates is based on the matrix which contains the complete discretization of the velocity-velocity part of the Navier–Stokes equations, including terms coming from stabilizations. □

Remark 7.225. AVM³. The derivation of this method can be explained by considering first a two-scale decomposition of the velocity and pressure

$$\boldsymbol{u} = \boldsymbol{u}^h + \boldsymbol{u}', \quad p = p^h + p', \tag{7.242}$$

where $(\boldsymbol{u}^h, p^h) \in V^h \times Q^h$ and V^h, Q^h are conforming finite element spaces. Then, the equation with the unresolved test functions is neglected. For the equation with the test functions from the finite element spaces, one obtains, using the decomposition (7.242),

$$\begin{aligned} &\left(\partial_t \boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{u}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) + \left(\left(\boldsymbol{u}^h \cdot \nabla\right)\boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(\nabla \cdot \boldsymbol{u}^h, q^h\right) \\ &\quad - \left(\nabla \cdot \boldsymbol{v}^h, p^h\right) \\ &= \left(\boldsymbol{f}, \boldsymbol{v}^h\right) - \Big[\left(\partial_t \boldsymbol{u}', \boldsymbol{v}^h\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{u}'\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) + \left(\left(\boldsymbol{u}^h \cdot \nabla\right)\boldsymbol{u}', \boldsymbol{v}^h\right) \\ &\quad + \left(\left(\boldsymbol{u}' \cdot \nabla\right)\boldsymbol{u}^h, \boldsymbol{v}^h\right) + \left(\left(\boldsymbol{u}' \cdot \nabla\right)\boldsymbol{u}', \boldsymbol{v}^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, p'\right)\Big] - \left(\nabla \cdot \boldsymbol{u}', q^h\right) \end{aligned} \tag{7.243}$$

Now, the term in the brackets is considered and the test function is split into $\boldsymbol{v}^h = \boldsymbol{v}^{3h} + \widehat{\boldsymbol{v}}^h$ in this term. Then, the assumption for a three-scale VMS method are applied, see Remark 7.207:

- The direct impact of the unresolved scales and the large scales is negligible, i.e., all terms in the brackets with test function $\boldsymbol{v}^{3h}$ are neglected.
- The direct impact of the unresolved scales onto the small resolved scales is modeled with a turbulence model, i.e., all terms in the brackets with test function $\widehat{\boldsymbol{v}}^h$ are modeled. In Gravemeier et al. (2010), a Smagorinsky model of the form

  $$\nabla \cdot \left(C_S h^2 \left\|\mathbb{D}\left(\widehat{\boldsymbol{u}}^h\right)\right\|_{\mathrm{F}} \mathbb{D}\left(\widehat{\boldsymbol{u}}^h\right)\right) = \nabla \cdot \left(\nu_{\mathrm{T}}\left(\widehat{\boldsymbol{u}}^h\right)\mathbb{D}\left(\widehat{\boldsymbol{u}}^h\right)\right) \tag{7.244}$$

  was used.

Thus, the model of the term in the brackets in (7.243) reduces to (7.244). □

Remark 7.226. Realization of AVM³. A realization of AVM³ can be found so far only for the Q_1/Q_1 pair of finite element spaces. To account for the violation of the discrete inf-sup condition in this case, it was proposed in Gravemeier et al. (2010) to include a consistent stabilization which includes the PSPG stabilization, see Section 3.7.3, as model of the last term in (7.243)

$$\left(\nabla \cdot \boldsymbol{w}', q^h\right) \approx \sum_{K \in \mathcal{T}^h} \left(\partial_t \boldsymbol{w}^h - \nu \Delta \boldsymbol{w}^h + \left(\boldsymbol{w}^h \cdot \nabla\right)\boldsymbol{w}^h + \nabla r^h - \boldsymbol{f}, \delta_K^p \nabla q^h\right)_K,$$

where the convention for the notation given in Remark 7.211 was used. Then, the continuous-in-time AVM³ method reads as follows: Find $\boldsymbol{w}^h : [0,T] \to V^h,\ r^h : (0,T] \to Q^h$ satisfying

$$\begin{aligned}
&\left(\partial_t \boldsymbol{w}^h, \boldsymbol{v}^h\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{w}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) + \left(\left(\boldsymbol{w}^h \cdot \nabla\right)\boldsymbol{w}^h, \boldsymbol{v}^h\right) \\
&\quad + \left(\nabla \cdot \boldsymbol{w}^h, q^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, r^h\right) + \left(\nu_T\left(\widehat{\boldsymbol{w}}^h\right)\mathbb{D}\left(\widehat{\boldsymbol{w}}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) \\
&\quad + \sum_{K\in\mathcal{T}^h}\left(\partial_t \boldsymbol{w}^h - \nu\Delta\boldsymbol{w}^h + \left(\boldsymbol{w}^h\cdot\nabla\right)\boldsymbol{w}^h + \nabla r^h, \delta_K^p \nabla q^h\right)_K, \\
&= \left(\boldsymbol{f}, \boldsymbol{v}^h\right) + \sum_{K\in\mathcal{T}^h}\left(\boldsymbol{f}, \delta_K^p\nabla q^h\right)_K, \qquad (7.245)
\end{aligned}$$

where $\widehat{\boldsymbol{w}}^h$ is computed with the help of the AMG approach sketched in Remark 7.224.

Using the short form (7.229), the method AVM^3 can be written as follows

$$\begin{aligned}
&A\left(\boldsymbol{w}^h; \left(\boldsymbol{w}^h, r^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) + \text{PSPG-type stabilization} \qquad (7.246) \\
&\qquad + \left(\nu_T\left(\widehat{\boldsymbol{w}}^h\right)\mathbb{D}\left(\widehat{\boldsymbol{w}}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) = F\left(\boldsymbol{v}^h\right).
\end{aligned}$$

□

AVM^4

7.8.2.3 A Coarse Space Projection-Based VMS Method

Remark 7.227. A coarse space projection-based VMS method. Let $V^h \times Q^h$ be finite element spaces for the velocity and pressure which satisfy the discrete inf-sup stability condition (2.48), let L^H be a finite-dimensional space of symmetric $d\times d$ tensor-valued functions defined on Ω and let $\nu_{\mathrm{T}}\left(\left(\boldsymbol{w}^h, r^h\right), h\right)$ be a non-negative function. Then, the semi-discrete coarse space projection-based VMS method (continuous-in-time) is defined as follows: Find $\boldsymbol{w}^h : [0,T] \to V^h$, $r^h : (0,T] \to Q^h$, and $\mathbb{G}^H : [0,T] \to L^H$ satisfying

$$\begin{aligned}
&\left(\partial_t \boldsymbol{w}^h, \boldsymbol{v}^h\right) + \left(2\nu\mathbb{D}\left(\boldsymbol{w}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) + n(\boldsymbol{w}^h, \boldsymbol{w}^h, \boldsymbol{v}^h) \\
&- \left(\nabla\cdot\boldsymbol{v}^h, r^h\right) + \left(\nu_{\mathrm{T}}\left(\left(\boldsymbol{w}^h, r^h\right), h\right)\left(\mathbb{D}\left(\boldsymbol{w}^h\right) - \mathbb{G}^H\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) = \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle_{V',V} \\
&\left(\nabla\cdot\boldsymbol{w}^h, q^h\right) = 0 \\
&\left(\mathbb{D}\left(\boldsymbol{w}^h\right) - \mathbb{G}^H, \mathbb{L}^H\right) = 0, \qquad (7.247)
\end{aligned}$$

for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$ and $\mathbb{L}^H \in L^H$. The scales are defined by projection in the last equation of (7.247), there are large scales and small resolved scales, and the turbulence model is applied directly only to the small resolved scales, see Remark 7.229 for a detailed discussion of the last two issues.

Using the short form (7.229), the first two equations of (7.247) can be written in the form

$$\begin{aligned}
&A\left(\boldsymbol{w}^h; \left(\boldsymbol{w}^h, r^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \\
&+ \left(\nu_{\mathrm{T}}\left(\left(\boldsymbol{w}^h, r^h\right), h\right)\left(\mathbb{D}\left(\boldsymbol{w}^h\right) - \mathbb{G}^H\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) = F\left(\boldsymbol{v}^h\right). \qquad (7.248)
\end{aligned}$$

Comparing this representation with (7.246) shows that, apart of the PSPG-type stabilization, the coarse space projection-based VMS method and AVM[3] have principally the same form.

The method (7.247) was proposed in John and Kaya (2005) based on ideas from Layton (2002). For applying this method, one has to choose two parameters: the additional viscosity $\nu_{\mathrm{T}}\left(\left(\boldsymbol{w}^h, r^h\right), h\right)$ and the space L^H. □

Remark 7.228. Choice of the additional viscosity. Concerning $\nu_{\mathrm{T}}\left(\left(\boldsymbol{w}^h, r^h\right), h\right)$, numerical studies with the method (7.247) presented in John and Kaya (2005); John and Roland (2007); John and Kindl (2010); **?** used a Smagorinsky models of the form check all

$$\nu_{\mathrm{T}} = C_S \delta^2 \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{\mathrm{F}}, \quad \text{and} \quad \nu_{\mathrm{T}} = C_S \delta^2 \left\|\mathbb{D}\left(\boldsymbol{w}^h\right) - \mathbb{G}^H\right\|_{\mathrm{F}}.$$

□

Remark 7.229. Choice of the large scale projection space. The other parameter in (7.247) is the space of symmetric tensors L^H. The last equation in (7.247) states that the tensor $\mathbb{G}^H$ is just the $L^2(\Omega)$ projection of $\mathbb{D}\left(\boldsymbol{w}^h\right)$ into L^H: $P_{L^H} : L \to L^H,\ \mathbb{D}(\boldsymbol{v}) \to P_{L^H}\mathbb{D}(\boldsymbol{v}) = \mathbb{G}^H$

$$\left(P_{L^H}\mathbb{D}(\boldsymbol{v}) - \mathbb{D}(\boldsymbol{v}), \mathbb{L}^H\right) = 0 \quad \forall\, \mathbb{L}^H \in L^H. \tag{7.249}$$

With this notation, one can reformulate the short form (7.248) as follows: Find $\boldsymbol{w}^h : [0,T] \to V^h,\ r^h : (0,T] \to Q^h$ satisfying

$$\begin{aligned} & A\left(\boldsymbol{w}^h; \left(\boldsymbol{w}^h, r^h\right), \left(\boldsymbol{v}^h, q^h\right)\right) \\ & + \left(\nu_{\mathrm{T}}\left(\left(\boldsymbol{w}^h, r^h\right), h\right)\left(I - P_{L^H}\right)\mathbb{D}\left(\boldsymbol{w}^h\right), \mathbb{D}\left(\boldsymbol{v}^h\right)\right) = \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle_{V',V} \end{aligned} \tag{7.250}$$

for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$.

The space L^H plays the role of a large scale space such that $\left(I - P_{L^H}\right)\mathbb{D}\left(\boldsymbol{w}^h\right)$ represents small resolved scales of $\mathbb{D}\left(\boldsymbol{w}^h\right)$. To avoid a negative additional viscosity, it is required that $L^H \subset \{\mathbb{D}\left(\boldsymbol{v}^h\right) : \boldsymbol{v}^h \in V^h\}$.

Considering the limit cases for L^H gives the following models.

- In the extreme case that both spaces are identical, the second term on the left-hand side of (7.250) vanishes and the Galerkin finite element discretization of the Navier–Stokes equations is recovered.
- If $L^H = \{\mathbb{O}\}$, one obtains an artificial viscosity stabilization of the Navier–Stokes equations with a possible non-linear artificial viscosity. If $\nu_{\mathrm{T}}\left(\left(\boldsymbol{w}^h, r^h\right), h\right)$ is the Smagorinsky eddy viscosity model (7.65), the Smagorinsky LES model is recovered.

Since L^H represents large scales, it has to be in some sense a coarse finite element space. There are essentially two possibilities:

- If V^h is a higher order finite element space, L^H can be defined as low order finite element space on the same grid as V^h. This approach is studied in John and Kaya (2005).

- The second possibility, in particular if V^h is a low order discretization, consists in defining L^H on a coarser grid, see John et al. (2006a) for a study of this approach in the case of convection-dominated convection-diffusion equations.

Altogether, the typical feature of a three-scale VMS method, namely that the turbulence model is applied directly only to the small resolved scales, can be observed very well in the last term on the left-hand side of the first equation of (7.247).

Since $\mathbb{D}\left(bw^h\right)$ is a discontinuous piecewise polynomial tensor, choosing its $L^2(\Omega)$ projection in the same way seems to be natural. Thus, L^H should consist of discontinuous piecewise polynomial tensors. It will be explained in Remark **??** that for the sake of an efficient implementation, this choice is mandatory. □