This model was analyzed in the usual setup for ADMs, i.e., for space-periodic boundary conditions, the differential filter, and the van Cittert approximate deconvolution operator. usual techniques?, overview in Layton and Rebholz (2012) ? □

Remark 7.186. Numerical experience with the Leray-α model. todo □

7.7 The Navier–Stokes-α Model

Remark 7.187. The Navier–Stokes-α model. The Navier–Stokes-α model is given by

$$\begin{aligned} \partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + (\overline{\boldsymbol{w}} \cdot \nabla)\boldsymbol{w} + (\nabla \overline{\boldsymbol{w}})^T \boldsymbol{w} + \nabla r &= \boldsymbol{f} \quad \text{in } (0,T] \times \Omega, \\ \nabla \cdot \overline{\boldsymbol{w}} &= 0 \quad \text{in } [0,T] \times \Omega, \\ \boldsymbol{w}(0,\cdot) &= \boldsymbol{u}_0 \text{ in } \Omega, \\ -\alpha^2 \Delta \overline{\boldsymbol{w}} + \overline{\boldsymbol{w}} &= \boldsymbol{w} \quad \text{in } (0,T] \times \Omega, \\ \int_\Omega r \, d\boldsymbol{x} &= 0 \quad \text{in } [0,T], \end{aligned} \tag{7.206}$$

together with appropriate boundary conditions. The pressure includes some terms which appear in the derivation of this model and it has the form Chen et al. (1998)

$$r = p - \frac{1}{2} \| \overline{\boldsymbol{w}} \|_2^2 - \frac{1}{2}\alpha^2 \left(\nabla \overline{\boldsymbol{w}} : \overline{\boldsymbol{w}}\right).$$

Model (7.206) is also called viscous Camassa–Holm model or isotropic Lagrangian-averaged Navier–Stokes equations.

Using (2.136), the momentum equation can be rewritten, introducing a new pressure, by

$$\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + (\nabla \times \boldsymbol{w}) \times \overline{\boldsymbol{w}} + \nabla \tilde{r} = \boldsymbol{f} \tag{7.207}$$

or

$$\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} - \overline{\boldsymbol{w}} \times (\nabla \times \boldsymbol{w}) + \nabla \tilde{r} = \boldsymbol{f}. \tag{7.208}$$

□

Remark 7.188. The Lagrangian description of a flow field. The description of the flow field in Chapter 1, which finally led to the Navier–Stokes equations, was performed from the point of view of considering a point $(t, \boldsymbol{x})$ in time and space. Then, the flow field was modeled with functions depending on $(t, \boldsymbol{x})$. This approach is called the Eulerian description of the flow field. Alternatively, it is possible to describe the flow from the point of view of a 'fluid particle'. In this case, one follows the motion of that fluid particle through time and space. This approach is called Lagrangian description of a flow field.

The Eulerian description can be thought of sitting at the banks of a river and describing the flow of the river from this point. For the Lagrangian description, one sits in a boat and describes the flow by following the river with the boat. □

Remark 7.189. Sketch of the derivation. Although the Navier–Stokes-α model has a convective term with regularized velocity, its derivation is not based on regularization. One can find in the literature two ways for deriving (7.206).

One way, whose details can be found in Chen et al. (1999b), considers a Lagrangian functional comprised of the kinetic energy and the incompressibility constraint

$$\int \left[\frac{1}{2} \left\| \boldsymbol{u}(t, \boldsymbol{x}) \right\|_2^2 + p\left(\boldsymbol{X}(t, \boldsymbol{x}), t\right) \left(\det\left(\nabla \boldsymbol{X}\left(t, \boldsymbol{x}\right)\right) - 1\right) \right] \, d\boldsymbol{x}$$

with $\boldsymbol{u}(t, \boldsymbol{x}) = \partial_t \boldsymbol{X}(t, \boldsymbol{x})$ and $\boldsymbol{X}(t, \boldsymbol{x})$ is the Lagrangian trajectory. The incompressibility constraint leads finally to the requirement $\det\left(\nabla \boldsymbol{X}\left(t, \boldsymbol{x}\right)\right) = 1$. Then, the Lagrangian trajectory is augmented with fluctuations. This step resembles the decomposition (7.11) of the flow field: the Lagrangian trajectory represents the mean flow field and the fluctuations the small scales. The trajectory with fluctuations is inserted into the Lagrangian functional. As a next crucial step, the velocity field and the pressure in this functional are approximated with a linear Taylor series expansion. This step assumes that the fluctuations are sufficiently small. Then, the Lagrangian functional is averaged and minimized. The optimality conditions are derived by computing its variational derivatives. In this way, one obtains a similar model to (7.206), but without viscous term and with a more complicated relation between $\overline{\boldsymbol{w}}$ and $\boldsymbol{w}$. Adding the viscous term, to use this equation as a turbulence model for incompressible flow problems, was proposed in Chen et al. (1998, 1999b,a). If the fluctuations are assumed to be isotropic, one obtains the equation for $\overline{\boldsymbol{w}}$ given in (7.206). If they are in addition homogeneous, then α is a constant. This derivation generalizes a one-dimensional shallow water model from Camassa and Holm (1993) to d dimensions. Therefore, (7.206) is called also viscous Camassa–Holm model.

A different derivation of the model (7.206) can be found in Marsden and Shkoller (2001, 2003). There, a so-called Lagrangian average is applied over a set of solutions of the Euler equations with initial data in some ball. The viscous term is treated via stochastic variations. It is noted that in the case of a bounded domain the definition of the viscous term should include a projection onto divergence-free vector fields. □

Remark 7.190. The divergence constraint. If $\boldsymbol{w}$ and $\overline{\boldsymbol{w}}$ are sufficiently smooth, one obtains from the definition of $\overline{\boldsymbol{w}}$ in (7.206), $\nabla \cdot \overline{\boldsymbol{w}} = 0$, and a calculation like in (7.179)

$$\nabla \cdot \boldsymbol{w} = \nabla \cdot \overline{\boldsymbol{w}} - \alpha^2 \nabla \cdot \Delta \overline{\boldsymbol{w}} = \nabla \cdot \overline{\boldsymbol{w}} - \alpha^2 \Delta \left(\nabla \cdot \overline{\boldsymbol{w}}\right) = 0.$$

Hence, $\boldsymbol{w}$ is also divergence-free. □

Remark 7.191. Analysis of the Navier–Stokes-α model in turbulent channel and pipe flows. The Navier–Stokes-α model for turbulent channel and pipe flows was studied analytically in Chen et al. (1998, 1999a,b). It was found that the analytical steady-state solution of the Navier–Stokes-α model with constant α shows a good agreement with available mean velocities and Reynolds stresses genauer, notation introduced? away from the boundary of a distance of order α. Note that the Navier–Stokes-α model was derived with the assumption of homogeneous and isotropic fluctuations, see Remark 7.189, which is usually not satisfied close to the boundary. Scaling arguments suggest that near the boundary α should decrease as the Reynolds number increases. The decrease of α, and with that the smaller influence of the turbulence model, resembles the reduction of the eddy viscosity in the Smagorinsky model by applying the van Driest damping, see Remark 7.127. Away from the boundary, the scaling arguments imply that α is independent of the Reynolds number. good values for α? □

Remark 7.192. Well-posedness of the Navier–Stokes-α model. One can find results on the existence and uniqueness of a solution of the Navier–Stokes-α model in Foias et al. (2002) for the case of periodic boundary conditions and in Marsden and Shkoller (2001) for the case of a bounded domain with no-slip boundary conditions. In Marsden and Shkoller (2001), first local well-posedness is proved by using Banach's fixed point theorem. Then, global well-posedness is obtained by proving appropriate stability (a priori) estimates. The analysis in Foias et al. (2002) uses the Galerkin method presented in Section 6.1. It will be sketched here. □

Remark 7.193. The Navier–Stokes-α model in a periodic domain. not presented in the course until remark about the attractor. Let $\Omega = (0, L)^3$, then the Navier–Stokes-α model with rotational form of the convective term has the form, see (7.208)

$$\begin{aligned}
\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} - \overline{\boldsymbol{w}} \times (\nabla \times \boldsymbol{w}) + \nabla r = \boldsymbol{f} \quad & \text{in } (0,T] \times \Omega, \\
(\boldsymbol{w}, r) \text{ is periodic} \quad & \text{on } [0,T] \times \Gamma, \\
-\alpha^2 \Delta \overline{\boldsymbol{w}} + \overline{\boldsymbol{w}} = \boldsymbol{w} \quad & \text{in } (0,T] \times \Omega, \\
\nabla \cdot \overline{\boldsymbol{w}} = 0 \quad & \text{in } [0,T] \times \Omega, \\
\overline{\boldsymbol{w}}(0,\cdot) = \overline{\boldsymbol{u}_0} \quad & \text{in } \Omega, \\
\overline{\boldsymbol{w}} \text{ is periodic} \quad & \text{on } (0,T] \times \Gamma, \\
\int_\Omega r \, d\boldsymbol{x} = 0 \quad & \text{in } [0,T],
\end{aligned} \tag{7.209}$$

where for simplicity of notation the tilde is omitted for the pressure. periodicity of $\overline{\boldsymbol{w}}$ from periodicity of $\boldsymbol{w}$? Note that it does not matter if the initial condition for $\boldsymbol{w}$ or $\overline{\boldsymbol{w}}$ is prescribed. If one is known, the other one can be computed. □

Remark 7.194. Assumptions and consequences. In the analysis, it will be assumed that the right-hand side does not depend on time, i.e., $\boldsymbol{f}(t,\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$. Further, it will be assumed that the right-hand side and the initial condition have zero mean, i.e.,

$$\int_\Omega \overline{\boldsymbol{w}}\,(0,\boldsymbol{x})\ d\boldsymbol{x} = \int_\Omega \boldsymbol{f}(\boldsymbol{x})\ d\boldsymbol{x} = 0. \tag{7.210}$$

Integration by parts gives

$$\begin{aligned}
\int_\Omega \nabla r\ d\boldsymbol{x} &= -\int_{\partial\Omega} r\boldsymbol{n}\ d\boldsymbol{s} \\
&= \int_{x=0\cap\partial\Omega} r(-\boldsymbol{e}_1)\ d\boldsymbol{s} + \int_{x=L\cap\partial\Omega} r\boldsymbol{e}_1\ d\boldsymbol{s} \\
&\quad + \int_{y=0\cap\partial\Omega} r(-\boldsymbol{e}_2)\ d\boldsymbol{s} + \int_{y=L\cap\partial\Omega} r\boldsymbol{e}_2\ d\boldsymbol{s} \\
&\quad + \int_{z=0\cap\partial\Omega} r(-\boldsymbol{e}_3)\ d\boldsymbol{s} + \int_{z=L\cap\partial\Omega} r\boldsymbol{e}_3\ d\boldsymbol{s} = 0
\end{aligned}$$

since r is periodic. With the same argument, one can derive

$$\int_\Omega \Delta\boldsymbol{w}\ d\boldsymbol{x} = \int_\Omega \overline{\boldsymbol{w}} \times (\nabla\times\boldsymbol{w})\ d\boldsymbol{x} = \int_\Omega \Delta\overline{\boldsymbol{w}}\ d\boldsymbol{x} = 0,$$

such that one finds from (7.209) and (7.210)

$$\begin{aligned}
0 &= \int_\Omega \boldsymbol{f}\ d\boldsymbol{x} = \int_\Omega \partial_t\boldsymbol{w}\ d\boldsymbol{x} = \frac{d}{dt}\int_\Omega \boldsymbol{w}\ d\boldsymbol{x} \\
&= \frac{d}{dt}\left(-\alpha^2\int_\Omega \Delta\overline{\boldsymbol{w}}\ d\boldsymbol{x} + \int_\Omega \overline{\boldsymbol{w}}\ d\boldsymbol{x}\right) = \frac{d}{dt}\int_\Omega \overline{\boldsymbol{w}}\ d\boldsymbol{x}.
\end{aligned}$$

Thus, the last integral is a constant with respect to time and since this constant is zero at the initial time, see (7.210), it follows that

$$\int_\Omega \overline{\boldsymbol{w}}\ d\boldsymbol{x} = 0 \quad \forall\, t \ge 0.$$

□

Remark 7.195. Setup for the analysis. Let

$$\begin{aligned}
\mathcal{V} = \Bigg\{ \boldsymbol{v}\ &: \boldsymbol{v} \text{ is a trigonometric polynomial on } \Omega \text{ with} \\
&\nabla\cdot\boldsymbol{v} = 0, \int_\Omega \boldsymbol{v}\ d\boldsymbol{x} = 0 \Bigg\}.
\end{aligned}$$

In comparison with the situation in the non-periodic case, the condition with the vanishing mean value appears. Nevertheless, the same notations will be used here, namely $L^2_{\text{div}}(\Omega)$ is the completion of $\mathcal{V}$ in $L^2(\Omega)$ and V_{div} is the completion in $H^1(\Omega)$.

The Helmholtz projector $P_{\text{helm}} : L^2_0(\Omega) \to L^2_{\text{div}}$ is given in Definition 2.160. Finally, $A = -P_{\text{helm}}\Delta$ is the Stokes operator with domain $D(A) = H^2(\Omega) \cap V_{\text{div}}$.

Now, (7.209) can be written in operator form in the divergence-free subspace as follows

$$\begin{aligned} \frac{d}{dt}\boldsymbol{w} + \nu A\boldsymbol{w} + N(\overline{\boldsymbol{w}}, \boldsymbol{w}) &= P_{\text{helm}}\boldsymbol{f}, \\ \alpha^2 A\overline{\boldsymbol{w}} + \overline{\boldsymbol{w}} &= \boldsymbol{w}, \\ \overline{\boldsymbol{w}}(0) &= \overline{\boldsymbol{u}_0}, \end{aligned} \tag{7.211}$$

with $N(\overline{\boldsymbol{w}}, \boldsymbol{w}) = -P_{\text{helm}}(\overline{\boldsymbol{w}} \times (\nabla \times \boldsymbol{w}))$. □

Remark 7.196. On the Stokes operator. In the case of periodic boundary conditions, the restriction of A to $D(A)$ is a selfadjoint operator with compact inverse. reference One can apply the theorem of Hilbert–Schmidt reference to get a set of eigenfunctions of the Stokes operator, which are orthonormal in $L^2(\Omega)$, with corresponding eigenvalues. The eigenfunctions form an orthonormal basis of V_{div} and the eigenvalues are positive. □

Lemma 7.197. Norm estimates. *Let λ_1 be the smallest eigenvalue of the Stokes operator, then there holds the Poincaré-type inequality*

$$\|\boldsymbol{v}\|^2_{L^2(\Omega)} \le \lambda_1^{-1} \|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in D(A). \tag{7.212}$$

For the Stokes operator, the following norm equivalence is valid

$$\|A\boldsymbol{v}\|_{L^2(\Omega)} \le \|\boldsymbol{v}\|_{H^2(\Omega)} \le C\,\|A\boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in D(A). \tag{7.213}$$

Proof. Let $\{\boldsymbol{v}_l\}_{l=1}^\infty$ be the basis of orthonormal eigenfunctions of the Stokes operator and let $\boldsymbol{v} = \sum_{i=l}^\infty v_l \boldsymbol{v}_l \in D(A)$. Using an argument as in (7.179) shows that $\Delta\boldsymbol{v}$ is divergence-free, hence $-\Delta\boldsymbol{v} = A\boldsymbol{v}$. Then, one gets with integration by parts, the argument from the previous sentence, the definition of the eigenvalues, the orthonormality of the eigenfunctions, the positivity of the eigenvalues, and once more the orthonormality of the eigenfunctions

$$\begin{aligned} \|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)} &= (\nabla\boldsymbol{v}, \nabla\boldsymbol{v}) = -(\boldsymbol{v}, \Delta\boldsymbol{v}) = (\boldsymbol{v}, A\boldsymbol{v}) \\ &= \left(\sum_{i=l}^\infty v_l\boldsymbol{v}_l, A\sum_{i=l}^\infty v_l\boldsymbol{v}_l\right) = \left(\sum_{i=l}^\infty v_l\boldsymbol{v}_l, \sum_{i=l}^\infty v_l\lambda_l\boldsymbol{v}_l\right) = \int_\Omega \sum_{i=l}^\infty v_l^2\lambda_l\boldsymbol{v}_l\cdot\boldsymbol{v}_l\, d\boldsymbol{x} \\ &\ge \lambda_1 \int_\Omega \sum_{i=l}^\infty v_l^2\boldsymbol{v}_l\cdot\boldsymbol{v}_l\, d\boldsymbol{x} = \lambda_1\left(\sum_{i=l}^\infty v_l\boldsymbol{v}_l, \sum_{i=l}^\infty v_l\boldsymbol{v}_l\right) = \lambda_1\|\boldsymbol{v}\|^2_{L^2(\Omega)}. \end{aligned}$$

The inequality on the left-hand side of (7.213) follows from

$$\|P_{\text{helm}}\Delta\boldsymbol{v}\|_{L^2(\Omega)} \le \|\Delta\boldsymbol{v}\|_{L^2(\Omega)},$$

see (2.142) and the fact that the $H^2(\Omega)$ norm has more terms than $\|\Delta\boldsymbol{v}\|_{L^2(\Omega)}$. The right-hand side is obtained form the Poincaré inequality, reference because the mean value of $\boldsymbol{v} \in D(A)$ vanishes and the mean values of all derivatives vanish by applying integration by parts, see Remark 7.194, i.e., one gets

$$\|\boldsymbol{v}\|_{H^2(\Omega)} \le C\,\|\Delta\boldsymbol{v}\|_{L^2(\Omega)}.$$

Since $-\Delta\boldsymbol{v} = A\boldsymbol{v}$, the right-hand side of (7.213) is proved. ■

Lemma 7.198. Estimate of the convective term. *Let $\boldsymbol{u} \in V_{\text{div}}$, $\boldsymbol{v} \in L^2_{\text{div}}(\Omega)$, and $\boldsymbol{w} \in D(A)$, then it is*

$$\begin{aligned}&\left|\left(-P_{\text{helm}}\left(\boldsymbol{u}\times(\nabla\times\boldsymbol{v})\right),\boldsymbol{w}\right)\right|\\ &\le C\Big(\|\boldsymbol{w}\|^{1/2}_{L^2(\Omega)}\|\nabla\boldsymbol{w}\|^{1/2}_{L^2(\Omega)}\|A\boldsymbol{u}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)}\\ &\quad+\|\nabla\boldsymbol{u}\|^{1/2}_{L^2(\Omega)}\|A\boldsymbol{u}\|^{1/2}_{L^2(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)}\Big).\end{aligned} \tag{7.214}$$

Proof. Since the Helmholtz projection is just the $L^2(\Omega)$ projection into $L^2_{\text{div}}(\Omega)$ and in particular $\boldsymbol{w} \in L^2_{\text{div}}(\Omega)$ by the definition of $D(A)$, one gets

$$\left(-P_{\text{helm}}\left(\boldsymbol{u}\times(\nabla\times\boldsymbol{v})\right),\boldsymbol{w}\right) = \left(-\boldsymbol{u}\times(\nabla\times\boldsymbol{v}),\boldsymbol{w}\right) = \left((\nabla\times\boldsymbol{v})\times\boldsymbol{u},\boldsymbol{w}\right).$$

Now, equality (5.10) can be derived also in the case of periodic boundary conditions. The integrals on the boundary, which appear in this derivation, will vanish because all functions are periodic and thus the integrals on opposite faces sum up to zero. Hence, one gets

$$\left(-P_{\text{helm}}\left(\boldsymbol{u}\times(\nabla\times\boldsymbol{w})\right),\boldsymbol{u}\right) = n_{\text{conv}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) - n_{\text{conv}}(\boldsymbol{w},\boldsymbol{v},\boldsymbol{u}). \tag{7.215}$$

Also in the periodic case, the convective term is skew-symmetric, i.e., (5.17) holds. This property is derived as in Remark 5.8, where the boundary integral vanishes again because of the periodicity of the functions. With the triangle inequality, the skew-symmetry of the convective term, and Hölder's inequality (A.15), one gets

$$\begin{aligned}&\left|\left(-P_{\text{helm}}\left(\boldsymbol{u}\times(\nabla\times\boldsymbol{v})\right),\boldsymbol{w}\right)\right|\\ &\le |n_{\text{conv}}(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| + |n_{\text{conv}}(\boldsymbol{w},\boldsymbol{v},\boldsymbol{u})|\\ &\le |n_{\text{conv}}(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})| + |n_{\text{conv}}(\boldsymbol{w},\boldsymbol{u},\boldsymbol{v})|\\ &\le \|\boldsymbol{u}\|_{L^\infty(\Omega)}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)} + \|\boldsymbol{w}\|_{L^3(\Omega)}\|\nabla\boldsymbol{u}\|_{L^6(\Omega)}\|\boldsymbol{v}\|_{L^2(\Omega)}.\end{aligned} \tag{7.216}$$

With the Sobolev imbedding (A.23), the interpolation theorem in Sobolev spaces (A.21), and Poincaré's inequality todo, one obtains

$$\|\boldsymbol{w}\|_{L^3(\Omega)} \le C\,\|\boldsymbol{w}\|_{H^{1/2}(\Omega)} \le C\,\|\boldsymbol{w}\|^{1/2}_{L^2(\Omega)}\|\nabla\boldsymbol{w}\|^{1/2}_{L^2(\Omega)}.$$

From the Sobolev imbedding (A.22) with $m = 1, p = 2, j = 1$, and with the norm equivalence (7.213), it follows that

$$\|\nabla\boldsymbol{u}\|_{L^6(\Omega)} \le C\,\|\boldsymbol{u}\|_{H^2(\Omega)} \le C\,\|A\boldsymbol{u}\|_{L^2(\Omega)}.$$

Using Agmon's inequality reference in the book by Foias, Poincar'e's inequality, and the equivalence (7.213) yields

$$\|\boldsymbol{u}\|_{L^\infty(\Omega)} \le C\,\|\boldsymbol{u}\|_{H^1(\Omega)}^{1/2}\,\|\boldsymbol{u}\|_{H^2(\Omega)}^{1/2} \le C\,\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^{1/2}\,\|A\boldsymbol{u}\|_{L^2(\Omega)}^{1/2}.$$

Inserting all estimates into (7.216) gives the statement of the lemma. ■

Theorem 7.199. Existence and uniqueness of a solution in the space-periodic case. *Let* $\boldsymbol{f} \in L^2_{\text{div}}(\Omega)$ *and* $\overline{\boldsymbol{u}_0} \in V_{\text{div}}$. *Then there is for any* $T > 0$ *a unique solution of* (7.211)

$$\boldsymbol{w} \in L^\infty_{\text{loc}}\left((0,T], H^3(\Omega)\right).$$

Proof. The proof utilizes the Galerkin method presented in Section 6.1. It consists of four parts:

1. Show the well-posedness of the problem in a finite-dimensional subspace.
2. Prove stability estimates in $H^1(\Omega), H^2(\Omega)$, and $H^3(\Omega)$.
3. Pass to the limit with the dimension and show the convergence of a subsequence.
4. Prove uniqueness of the solution.

The first three steps will be only sketched here.

1. Show the well-posedness of the problem in a finite-dimensional subspace. Analogously to the proof of Lemma 6.13, a problem in a finite-dimensional space is considered. The space is spanned by the eigenfunctions of the Stokes operator. The existence and uniqueness of an absolutely continuous solution in $[0,T]$ is proved with the theorem of Carathéodory, see Theorem A.66.

2. Prove stability estimates in $H^1(\Omega), H^2(\Omega)$*, and* $H^3(\Omega)$. The derivation of these estimates is somewhat longer. It uses standard estimates like Hölder's and Young's inequality, the Gronwall lemma, and estimates for the Stokes operator A and the operator for the nonlinear term N. Let $\overline{\boldsymbol{w}}^{\,n}(t)$ be the solution of the problem in the subspace with dimension n. Then, one obtains for all $t \in [0,T]$, e.g.,

$$\begin{aligned}\|\,\overline{\boldsymbol{w}}^{\,n}(t)\|_{L_2(\Omega)}^2 + \alpha^2\,\|\nabla\,\overline{\boldsymbol{w}}^{\,n}(t)\|_{L_2(\Omega)}^2 &\le C_1,\\ \|\nabla\,\overline{\boldsymbol{w}}^{\,n}(t)\|_{L_2(\Omega)}^2 + \alpha^2\,\|A\,\overline{\boldsymbol{w}}^{\,n}(t)\|_{L_2(\Omega)}^2 &\le C_2(t).\end{aligned}$$

Note that there is the norm equivalence of $\|A\,\overline{\boldsymbol{w}}^{\,n}(t)\|_{L_2(\Omega)}$ and the $H^2(\Omega)$ norm, see (7.213).

3. Pass to the limit with the dimension and show the convergence of a subsequences. This part of the proof is performed like the proofs of Corollary 6.14 and Lemma 6.17, using the stability estimates from the previous part and the theorem of Lions–Aubin, (Girault and Raviart, 1979, pp. 153)). Because of the higher regularity of the solution proved in the second step, in comparison with the solution of the Navier–Stokes equations, the proof of the convergence of the nonlinear convective term is simpler than in Section 6.1.

4. Prove uniqueness of the solution. Assume that there are two solutions $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ of (7.211) to the same data $\boldsymbol{f}$ and $\boldsymbol{u}_0$. Denoting $\boldsymbol{w}_{21} = \boldsymbol{w}_2 - \boldsymbol{w}_1$ and correspondingly $\overline{\boldsymbol{w}_{21}} = \overline{\boldsymbol{w}_2} - \overline{\boldsymbol{w}_1}$, one obtains by subtracting (7.211) for $\boldsymbol{w}_1$ from (7.211) for $\boldsymbol{w}_2$ and expanding with $N(\overline{\boldsymbol{w}_1}, \boldsymbol{w}_2) - N(\overline{\boldsymbol{w}_1}, \boldsymbol{w}_2)$

$$\begin{aligned}0 &= \frac{d}{dt}\boldsymbol{w}_{21} + \nu A\boldsymbol{w}_{21} + N\left(\overline{\boldsymbol{w}_2}, \boldsymbol{w}_2\right) - N\left(\overline{\boldsymbol{w}_1}, \boldsymbol{w}_1\right)\\ &= \frac{d}{dt}\boldsymbol{w}_{21} + \nu A\boldsymbol{w}_{21} + N\left(\overline{\boldsymbol{w}_{21}}, \boldsymbol{w}_2\right) - N\left(\overline{\boldsymbol{w}_1}, \boldsymbol{w}_{21}\right).\end{aligned} \tag{7.217}$$

The next step consists in testing (7.217) with $\overline{\boldsymbol{w}_{21}}$. One obtains for the first term on the right-hand side with the definition of the $\overline{\boldsymbol{w}_{21}}$, integration by parts, and relations of the form (6.13)

$$
\begin{aligned}
\left(\frac{d}{dt}\boldsymbol{w}_{21}, \overline{\boldsymbol{w}_{21}}\right) &= \frac{d}{dt}\left(\alpha^2 A\,\overline{\boldsymbol{w}_{21}} + \overline{\boldsymbol{w}_{21}}, \overline{\boldsymbol{w}_{21}}\right) \\
&= \alpha^2 \frac{d}{dt}\left(\nabla\overline{\boldsymbol{w}_{21}}, \nabla\overline{\boldsymbol{w}_{21}}\right) + \frac{d}{dt}\left(\overline{\boldsymbol{w}_{21}}, \overline{\boldsymbol{w}_{21}}\right) \\
&= \frac{1}{2}\frac{d}{dt}\left(\|\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \alpha^2\left\|\overline{\nabla\boldsymbol{w}_{21}}\right\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

For the viscous term, one gets with applying repeatedly integration by parts

$$
\begin{aligned}
\left(\nu A\boldsymbol{w}_{21}, \overline{\boldsymbol{w}_{21}}\right) &= \alpha^2\left(\nu A\left(A\,\overline{\boldsymbol{w}_{21}}\right), \overline{\boldsymbol{w}_{21}}\right) + \left(\nu A\,\overline{\boldsymbol{w}_{21}}, \overline{\boldsymbol{w}_{21}}\right) \\
&= \alpha^2\left(\nu A\,\overline{\boldsymbol{w}_{21}}, A\,\overline{\boldsymbol{w}_{21}}\right) + \left(\nu\nabla\overline{\boldsymbol{w}_{21}}, \nabla\overline{\boldsymbol{w}_{21}}\right) \\
&= \nu\left(\alpha^2\|A\,\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \left\|\overline{\nabla\boldsymbol{w}_{21}}\right\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

One finds with (7.215) that

$$
\begin{aligned}
\left(N\left(\overline{\boldsymbol{w}_{21}}, \boldsymbol{w}_2\right), \overline{\boldsymbol{w}_{21}}\right) &= \left(-P_{\mathrm{helm}}\left(\overline{\boldsymbol{w}_{21}} \times (\nabla\times\boldsymbol{w}_2)\right), \overline{\boldsymbol{w}_{21}}\right) \\
&= n_{\mathrm{conv}}\left(\overline{\boldsymbol{w}_{21}}, \boldsymbol{w}_2, \overline{\boldsymbol{w}_{21}}\right) - n_{\mathrm{conv}}\left(\overline{\boldsymbol{w}_{21}}, \boldsymbol{w}_2, \overline{\boldsymbol{w}_{21}}\right) = 0.
\end{aligned}
$$

Inserting all identities into (7.217), applying estimate (7.214), collecting all norms of $\boldsymbol{w}_1$, which are known to be finite, into the constant, using (7.212), and Young's inequality (A.4) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \alpha^2\left\|\overline{\nabla\boldsymbol{w}_{21}}\right\|^2_{L^2(\Omega)}\right) + \nu\left(\alpha^2\|A\,\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \left\|\overline{\nabla\boldsymbol{w}_{21}}\right\|^2_{L^2(\Omega)}\right) \\
&\quad\le \left|\left(-N\left(\overline{\boldsymbol{w}_1}, \boldsymbol{w}_{21}\right), \overline{\boldsymbol{w}_{21}}\right)\right| \\
&\quad= \left|\left(-P_{\mathrm{helm}}\left(\boldsymbol{w}_1 \times (\nabla\times\boldsymbol{w}_{21})\right), \overline{\boldsymbol{w}_{21}}\right)\right| \\
&\quad\le C\|\boldsymbol{w}_{21}\|_{L^2(\Omega)}\left(\|\overline{\boldsymbol{w}_{21}}\|^{1/2}_{L^2(\Omega)}\|\nabla\overline{\boldsymbol{w}_{21}}\|^{1/2}_{L^2(\Omega)}\|A\,\overline{\boldsymbol{w}_1}\|_{L^2(\Omega)}\right. \\
&\qquad\left. + \|\nabla\overline{\boldsymbol{w}_1}\|^{1/2}_{L^2(\Omega)}\|A\,\overline{\boldsymbol{w}_1}\|^{1/2}_{L^2(\Omega)}\|\nabla\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}\right) \\
&\quad= C\|\boldsymbol{w}_{21}\|_{L^2(\Omega)}\left(\|\overline{\boldsymbol{w}_{21}}\|^{1/2}_{L^2(\Omega)}\|\nabla\overline{\boldsymbol{w}_{21}}\|^{1/2}_{L^2(\Omega)} + \|\nabla\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}\right) \\
&\quad\le C\left(1 + \lambda_1^{-1/2}\right)\|\boldsymbol{w}_{21}\|_{L^2(\Omega)}\|\nabla\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)} \\
&\quad\le \frac{CC_0}{\nu}\|\nabla\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \frac{\nu}{2C_0}\|\boldsymbol{w}_{21}\|^2_{L^2(\Omega)} \qquad (7.218)
\end{aligned}
$$

with $C_0 = 2\alpha^2 + \lambda_1^{-1}$. For the last term, one obtains with the definition of $\overline{\boldsymbol{w}_{21}}$, integration by parts, and (7.212)

$$
\begin{aligned}
\frac{\nu}{2C_0}\|\boldsymbol{w}_{21}\|^2_{L^2(\Omega)} &= \frac{\nu}{2C_0}\left\|\alpha^2 A\,\overline{\boldsymbol{w}_{21}} + \overline{\boldsymbol{w}_{21}}\right\|^2_{L^2(\Omega)} \\
&= \frac{\nu}{2C_0}\left(\|\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \alpha^4\|A\,\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + 2\alpha^2\left(A\,\overline{\boldsymbol{w}_{21}}, \overline{\boldsymbol{w}_{21}}\right)\right) \\
&= \frac{\nu}{2C_0}\left(\frac{1}{\lambda_1}\|\nabla\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \alpha^4\|A\,\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + 2\alpha^2\|\nabla\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)}\right) \\
&\le \frac{\nu}{2C_0}\left(2\alpha^2 + \frac{1}{\lambda_1}\right)\left(\|\nabla\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \alpha^2\|A\,\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)}\right) \\
&= \frac{\nu}{2}\left(\|\nabla\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)} + \alpha^2\|A\,\overline{\boldsymbol{w}_{21}}\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

This estimate is inserted into (7.218). Then, this term is absorbed by the left-hand side. Neglecting the arising non-negative term on the left-hand side and neglecting also the dependency of the constant on C_0 and ν, which is not of importance here, one gets

$$\begin{aligned}\frac{d}{dt}\left(\|\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2 + \alpha^2 \|\overline{\nabla \boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2\right) &\le C\|\nabla\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2 \\ &= C\alpha^{-2}\alpha^2\|\nabla\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2 \\ &\le C\alpha^{-2}\left(\|\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2 + \alpha^2\|\nabla\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2\right).\end{aligned}$$

Applying Gronwall's lemma, Lemma A.70, yields for almost all $t \in [0, T]$

$$\begin{aligned}&\left(\|\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2 + \alpha^2\|\nabla\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2\right)(t) \\ &\le C\left(\|\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2 + \alpha^2\|\nabla\overline{\boldsymbol{w}_{21}}\|_{L^2(\Omega)}^2\right)(0) = 0,\end{aligned}$$

since $\|\overline{\boldsymbol{w}_{21}}(0)\|_{L^2(\Omega)} = \|\nabla\overline{\boldsymbol{w}_{21}}(0)\|_{L^2(\Omega)} = 0$ because $\overline{\boldsymbol{w}_1}$ and $\overline{\boldsymbol{w}_2}$ have the same initial data. Therefore, $\overline{\boldsymbol{w}_1} = \overline{\boldsymbol{w}_2}$ in the sense of $L^2_{\text{div}}(\Omega)$ and V_{div} for almost all $t \in [0, T]$, and with that also $\boldsymbol{w}_1 = \boldsymbol{w}_2$. ■

Remark 7.200. The Hausdorff dimension of the global attractor. In Foias et al. (2002), an estimate for the Hausdorff dimension of the global attractor $\mathcal{A}_{\text{NS}-\alpha}$ of the Navier–Stokes-α model in the case of periodic boundary conditions was derived. Let λ_α be the length scale for which there is a balance of the mean rates of nonlinear transport of energy and viscous dissipation of energy in the Navier–Stokes-α model. This scale depends on α, it is not smaller than the Kolmogorov scale λ, usually it is (much) larger. Then, the estimate proved in Foias et al. (2002) has the form

$$d_{\text{H}}\left(\mathcal{A}_{\text{NS}-\alpha}\right) = \mathcal{O}\left(\left(\frac{L}{\lambda_\alpha}\right)^3\right).$$

Since usually $\lambda_\alpha \gg \lambda$, this dimension is asymptotically smaller than the dimension (7.10) for the Navier–Stokes equations. Besides that one can prove the existence and uniqueness of a weak solution, also this results indicates that the Navier–Stokes-α model is less complex than the Navier–Stokes equations. □

Remark 7.201. Convergence to a weak solution of the Navier–Stokes equations as $\alpha \to 0$. Similarly as for the Leray-α model, one can show that a subsequence of $\{\boldsymbol{w}_\alpha\}_{\alpha>0}$ converges to a weak solution of the Navier–Stokes equations as $\alpha \to 0$. The proof of this statement, in the case of periodic boundary conditions, can be found in Foias et al. (2002). In the proof, one shows the uniform (with respect to α) boundedness of some norms from which it follows the weak convergence of a subsequence in the corresponding spaces. Further bounds imply, together with Aubin's compactness theorem, reference (Girault and Raviart, 1979, pp. 153)?? the strong convergence in appropriate spaces. □

Remark 7.202. The finite element problem. For sufficiently smooth functions, it is known that from $\nabla \cdot \overline{\boldsymbol{w}}$ in Ω it follows that $\nabla \cdot \boldsymbol{w} = 0$ in Ω, see Remark 7.190. However, finite element functions are not sufficiently smooth, e.g., the strong form of the Laplacian is not well defined, they are only discretely divergence-free, and one has to apply a discrete filter. For these reasons, finite element formulations have been studied in the literature where the discrete divergence constraint was posed the for discrete velocity and the discretely filtered discrete velocity separately.

In Connors (2010), a finite element error analysis for the Navier–Stokes-α model of the following form is presented

$$\begin{aligned}\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + (\nabla \times \boldsymbol{w}) \times \overline{\boldsymbol{w}} + \nabla r = \boldsymbol{f} &\quad \text{in } (0,T] \times \Omega, \\ \nabla \cdot \boldsymbol{w} = 0 &\quad \text{in } [0,T] \times \Omega, \\ \boldsymbol{w} = \boldsymbol{0} &\quad \text{in } [0,T] \times \Gamma, \\ -\alpha^2 \Delta \overline{\boldsymbol{w}} + \overline{\boldsymbol{w}} + \nabla \tilde{r} = \boldsymbol{w} &\quad \text{in } (0,T] \times \Omega, \\ \nabla \cdot \overline{\boldsymbol{w}} = 0 &\quad \text{in } [0,T] \times \Omega, \\ \overline{\boldsymbol{w}} = \boldsymbol{0} &\quad \text{in } (0,T] \times \Gamma, \\ \boldsymbol{w}(0,\cdot) = \boldsymbol{u}_0 &\quad \text{in } \Omega, \\ \int_\Omega r \, d\boldsymbol{x} = \int_\Omega \tilde{r} \, d\boldsymbol{x} = 0 &\quad \text{in } [0,T].\end{aligned} \tag{7.219}$$

In (7.219), Ω is a bounded domain in $\mathbb{R}^d$, $d \in \{2,3\}$, with polyhedral Lipschitz boundary Γ and the form (7.207) of the momentum equation is used. Since there is a separate divergence constraint for the filtered velocity, one needs also a Lagrangian multiplier in the corresponding equation.

The weak form of (7.219) is derived by multiplying the equations with appropriate test functions, integrating the equations in Ω, and applying integration by parts. The correct function spaces with respect to the spatial variable are $V = H_0^1(\Omega)$ and $Q = L_0^2(\Omega)$. In Connors (2010), the continuous-in-time case is considered for conforming finite element spaces: Find $\left(\boldsymbol{w}^h, r^h\right) : [0,T] \to V^h \times Q^h$, $\left(\overline{\boldsymbol{w}^h}^h, \tilde{r}^h\right) : [0,T] \to V^h \times Q^h$ with $V^h \subset V$, $Q^h \subset Q$ and

$$\begin{aligned}\left(\partial_t \boldsymbol{w}^h, \boldsymbol{v}^h\right) + \left(\nu \nabla \boldsymbol{w}^h, \nabla \boldsymbol{v}^h\right) + n_{\text{rot}}\left(\boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h, \boldsymbol{v}^h\right) & \\ - \left(\nabla \cdot \boldsymbol{v}^h, r^h\right) = \left(\boldsymbol{f}, \boldsymbol{v}^h\right) &\quad \forall\, \boldsymbol{v}^h \in V^h, \\ \left(\nabla \cdot \boldsymbol{w}^h, q^h\right) = 0 &\quad \forall\, q^h \in Q^h \\ \alpha^2\left(\overline{\boldsymbol{w}^h}^h, \nabla \tilde{\boldsymbol{v}}^h\right) + \left(\overline{\boldsymbol{w}^h}^h, \tilde{\boldsymbol{v}}^h\right) - \left(\nabla \cdot \tilde{\boldsymbol{v}}^h, \tilde{r}^h\right) = \left(\boldsymbol{w}^h, \tilde{\boldsymbol{v}}^h\right) &\quad \forall\, \tilde{\boldsymbol{v}}^h \in V^h, \\ \left(\nabla \cdot \overline{\boldsymbol{w}^h}^h, \tilde{q}^h\right) = 0 &\quad \forall\, \tilde{q}^h \in Q^h,\end{aligned} \tag{7.220}$$

and $\boldsymbol{w}(0,\cdot) = \boldsymbol{u}_0$. The rotational form of the convective term is defined in (5.14). In Connors (2010), an additional grad-div stabilization term was included into the first equation of (7.220) but not to the equation for the discrete filter.

The functions $\boldsymbol{w}^h$ and $\overline{\boldsymbol{w}^h}^h$ are contained in V_{div}^h. Considering only velocity test functions from this space, problem (7.220) can be restricted to V_{div}^h: Find $\boldsymbol{w}^h \;:\; [0,T] \to V_{\text{div}}^h$, $\overline{\boldsymbol{w}^h}^h \;:\; [0,T] \to V_{\text{div}}^h$ such that

$$\begin{aligned}\left(\partial_t \boldsymbol{w}^h, \boldsymbol{v}^h\right) + \left(\nu \nabla \boldsymbol{w}^h, \nabla \boldsymbol{v}^h\right) + n_{\text{rot}}\left(\boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h, \boldsymbol{v}^h\right) &= \left(\boldsymbol{f}, \boldsymbol{v}^h\right), \quad (7.221)\\ \alpha^2\left(\nabla \overline{\boldsymbol{w}^h}^h, \nabla \tilde{\boldsymbol{v}}^h\right) + \left(\overline{\boldsymbol{w}^h}^h, \tilde{\boldsymbol{v}}^h\right) &= \left(\boldsymbol{w}^h, \tilde{\boldsymbol{v}}^h\right),\end{aligned}$$

for all $\boldsymbol{v}^h, \tilde{\boldsymbol{v}}^h \in V_{\text{div}}^h$. □

Remark 7.203. Properties of the discrete differential filter. In comparison with the discrete differential filter for the Leray-α model, see (7.187), the discretely filtered finite element velocity for the Navier–Stokes-α model has to satisfy the discrete divergence constraint. Thus, the corresponding equation is defined in V_{div}^h and not in V^h as for the Leray-α model. Inspecting the proofs of Lemmas 7.176 – 7.178 shows that one can obtain the same results for the discrete differential filter of the Navier–Stokes-α model, where always V^h has to be replaced by V_{div}^h in the formulas and also in the definition of the discrete Laplacian. □

Lemma 7.204. Existence, uniqueness, and stability of the finite element solution. *Let $\boldsymbol{w}_0^h \in V_{\text{div}}^h$ and $\boldsymbol{f} \in L^2(0,t;V')$, then the finite element problem* (7.221) *possesses a unique solution. The following stability estimate holds for the discretly filtered finite element velocity field*

$$\begin{aligned}&\left\|\overline{\boldsymbol{w}^h}^h(t)\right\|_{L^2(\Omega)}^2 + \alpha^2\left\|\nabla \overline{\boldsymbol{w}^h}^h(t)\right\|_{L^2(\Omega)}^2\\ &\quad+\nu\left(\left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(0,t;L^2(\Omega))}^2 + 2\alpha^2\left\|\Delta^h\left(\nabla \overline{\boldsymbol{w}^h}^h\right)\right\|_{L^2(0,t;L^2(\Omega))}^2\right)\\ &\le \left\|\overline{\boldsymbol{w}^h}^h(0)\right\|_{L^2(\Omega)}^2 + \alpha^2\left\|\nabla \overline{\boldsymbol{w}^h}^h(0)\right\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\left\|\boldsymbol{f}\right\|_{L^2(0,t;V')}^2. \qquad (7.222)\end{aligned}$$

If $\alpha \le Ch$, then there holds for the unfiltered finite element velocity field

$$\begin{aligned}&\left\|\boldsymbol{w}^h(t)\right\|_{L^2(\Omega)}^2 + \nu\left\|\nabla \boldsymbol{w}^h\right\|_{L^2(0,t;L^2(\Omega))}^2 \qquad (7.223)\\ &\le C\left(\left\|\boldsymbol{w}^h(0)\right\|_{L^2(\Omega)}^2 + \alpha^2\left\|\nabla \boldsymbol{w}^h(0)\right\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\left\|\boldsymbol{f}\right\|_{L^2(0,t;V')}^2\right),\end{aligned}$$

where the constant depends on the constant of the inverse estimate (C.29)*.*

Proof. As usual, stability estimates are derived by using appropriate test functions in the equation. Considering (7.221), then the test function has to be chosen such that the nonlinear convective term vanishes, i.e., one has to choose $\boldsymbol{v}^h = \overline{\boldsymbol{w}^h}^h$ which gives the desired result, see (5.18), and leads to the equation

$$\left(\partial_t \boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h\right) + \left(\nu \nabla \boldsymbol{w}^h, \nabla \overline{\boldsymbol{w}^h}^h\right) = \left(\boldsymbol{f}, \overline{\boldsymbol{w}^h}^h\right). \qquad (7.224)$$

With the selfadjointness property (7.188) and the commutation of filtering and temporal derivative, Lemma 7.188, one obtains

$$\left(\partial_t \boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h\right) = \left(\overline{\partial_t \boldsymbol{w}^h}^h, \boldsymbol{w}^h\right) = \left(\partial_t \overline{\boldsymbol{w}^h}^h, \boldsymbol{w}^h\right),$$

from what

$$\left(\partial_t \boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h\right) = \frac{1}{2}\left(\partial_t \boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h\right) + \frac{1}{2}\left(\boldsymbol{w}^h, \partial_t \overline{\boldsymbol{w}^h}^h\right)$$

follows. Then, the product rule shows that

$$\frac{1}{2}\frac{d}{dt}\left(\boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h\right) = \frac{1}{2}\left(\partial_t \boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h\right) + \frac{1}{2}\left(\boldsymbol{w}^h, \partial_t \overline{\boldsymbol{w}^h}^h\right) = \left(\partial_t \boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h\right).$$

Inserting this expression into (7.224) gives

$$\frac{1}{2}\frac{d}{dt}\left(\boldsymbol{w}^h, \overline{\boldsymbol{w}^h}^h\right) + \left(\nu \nabla \boldsymbol{w}^h, \nabla \overline{\boldsymbol{w}^h}^h\right) = \left(\boldsymbol{f}, \overline{\boldsymbol{w}^h}^h\right). \tag{7.225}$$

Using now the definition of the discrete filter (7.221) and (7.192) on the left-hand side, and the estimate for the duality pairing and Young's inequality (A.4) on the right-hand side yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}^2 + \alpha^2\left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}^2\right) \\ &+\nu\left(\left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}^2 + \alpha^2\left\|\Delta^h\left(\nabla \overline{\boldsymbol{w}^h}^h\right)\right\|_{L^2(\Omega)}^2\right) \le \|\boldsymbol{f}\|_{V'}\left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)} \\ &\qquad \le \frac{1}{2\nu}\|\boldsymbol{f}\|_{V'}^2 + \frac{\nu}{2}\left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}^2. \end{aligned}$$

The last term can be absorbed from the left-hand side. Multiplying the resulting estimate by two and integrating in $(0,T)$ gives (7.222).

To prove (7.223), norms of $\overline{\boldsymbol{w}^h}^h$ will be bounded from below with norms of $\boldsymbol{w}^h$. Inserting $\tilde{\boldsymbol{v}}^h = \boldsymbol{w}^h \in V_{\mathrm{div}}^h$ into the definition of the discrete filter (7.221), applying the Cauchy–Schwarz inequality (A.16) and the inverse inequality (C.29) gives

$$\begin{aligned} \left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 &= \left(\overline{\boldsymbol{w}^h}^h, \boldsymbol{w}^h\right) + \alpha^2\left(\nabla \overline{\boldsymbol{w}^h}^h, \nabla \boldsymbol{w}^h\right) \\ &\le \left\|\overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)} + \alpha^2\left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}\left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)} \\ &\le \left\|\overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)} + \frac{\alpha^2 C_{\mathrm{inv}}^2}{h^2}\left\|\overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}. \end{aligned}$$

Using the condition on α gives

$$\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 \le C\left\|\overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}. \tag{7.226}$$

Choosing in (7.221) $\tilde{\boldsymbol{v}}^h$ to be the discrete Laplacian $\Delta^h \boldsymbol{w}^h \in V_{\mathrm{div}}^h$, see Remark 7.203, and applying the definition (7.191) of the discrete Laplacian yields with the same tools the estimate

$$\begin{aligned} \left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 &= \left(\nabla \overline{\boldsymbol{w}^h}^h, \nabla \boldsymbol{w}^h\right) + \alpha^2\left(\Delta^h \overline{\boldsymbol{w}^h}^h, \Delta^h \boldsymbol{w}^h\right) \qquad (7.227) \\ &\le \left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}\left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)} + \alpha^2\left\|\Delta^h \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}\left\|\Delta^h \boldsymbol{w}^h\right\|_{L^2(\Omega)}. \end{aligned}$$

Using in the definition of the discrete Laplacian as test function the discrete Laplacian itself and applying the inverse inequality gives

$$\left\|\Delta^h \boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 \le \left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)} \left\|\nabla \Delta^h \boldsymbol{w}^h\right\|_{L^2(\Omega)} \le \frac{C_{\text{inv}}}{h} \left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)} \left\|\Delta^h \boldsymbol{w}^h\right\|_{L^2(\Omega)},$$

which gives an inverse estimate for the discrete Laplacian. Inserting this inverse estimate into (7.227) and using the assumption on α yields

$$\left\|\nabla \boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 \le C \left\|\nabla \overline{\boldsymbol{w}^h}^h\right\|_{L^2(\Omega)}. \tag{7.228}$$

Now, (7.223) is obtained by neglecting the terms with α on the left-hand side of (7.222), estimating the other terms on the left-hand side with (7.226) and (7.228), and bounding the terms from the initial condition with (7.189).

existence and uniqueness, see Leray ■

Remark 7.205. Further results from finite element analysis.

- An error estimate for the continuous-in-time case can be found in Connors (2010). The proof of this estimate follows the standard lines, e.g., as the proof of Theorem 6.46 for the Navier–Stokes equations. It uses some properties of the discrete differential filter which were also used in the proof of Lemma 7.204. The nonlinear convective term is estimated with inequalities that can be found in Section 5.1.2. Under some regularity assumptions, one obtains an estimate for

$$\left\|\left(\boldsymbol{u} - \boldsymbol{w}^h\right)(t)\right\|_{L^2(\Omega)}^2 + \nu \left\|\nabla\left(\boldsymbol{u} - \boldsymbol{w}^h\right)\right\|_{L^2(0,t;L^2(\Omega))}^2$$

  with an factor of size $\exp(C\nu^{-3}t)$, similarly to the Navier–Stokes equations, see (6.39), and the Leray-α model, see (7.200), (7.202). It is assumed that $\alpha = \mathcal{O}(h)$ which leads to second order convergence, as for the Leray-α model, see Remark 7.183.
- A full discretization with the Crank–Nicolson scheme as time integrator was considered in Layton et al. (2010). In this paper, the stability of the finite element solution is proved, but an error analysis is not presented. The error analysis for the fully discrete case seems to be an open problem. but see Miles,Rebholz 2010

□

7.8 Variational Multiscale Methods

7.8.1 The Basic Concept

Remark 7.206. Differences to LES. Similarly to classical LES methods, Variational Multiscale (VMS) methods seek to simulate only large flow structures.