It is pointed out that there are deconvolution operators besides the van Cittert deconvolution which satisfy these conditions and also deconvolution operators which do not.

- convergence for $N \to \infty$, Berselli and Lewandowski (2012)

□

Remark 7.162. Finite element error analyis of ADMs.

- A numerical scheme for the zeroth order approximate deconvolution model is studied in Manica and Merdan (2007). In this paper, the situation of a bounded domain and no-slip boundary conditions is considered. Hence, differentiation and filtering do not commute and the momentum balance has the form

$$\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + \overline{\nabla \cdot \left(\boldsymbol{w}\, \boldsymbol{w}^T\right)} + \overline{\nabla q} = \overline{\boldsymbol{f}}\,.$$

  To handle the nonlinear convective term, it is proposed to use the same kind of test function as in the proof of Lemma 7.160, namely $\left(-\delta^2 \Delta + I\right) \tilde{\boldsymbol{v}}$, such that this term can be rewritten in the form $n_{\text{skew}}(\boldsymbol{w}, \boldsymbol{w}, \tilde{\boldsymbol{v}})$ and it vanishes for $\tilde{\boldsymbol{v}} = \boldsymbol{w}$. However, the use of this kind of test functions leads to a fourth order term, compare the term $\delta^2 \left(\nu \Delta \boldsymbol{w}, \Delta \boldsymbol{w}\right)$ in (7.180). A mixed finite element formulation is considered to handle this situation, which is analyzed in the usual way. A main tool of the analysis is a modified Stokes projection. The optimal choice of δ comes from the properties of the modified Stokes projection and it is $\delta = \mathcal{O}(h)$. work out?
- Galvin, Rebholz, Trenchea 2014

□

7.6 The Leray-α Model

Remark 7.163. Motivation. In Leray (1934), the existence of a weak solution of the Navier–Stokes equations (7.1) (in this paper called turbulent solution) was proved by considering a sequence of simplified problems, where the simplification consisted in replacing in the nonlinear term of the Navier–Stokes equations the convection field by a smooth or regularized velocity field, see Remark 6.11. The case $\Omega = \mathbb{R}^3$ was considered and the regularization was defined by a convolution with a filter function. Then, the behavior was studied for the filter width tending to zero. Based on this idea from the analysis of the Navier–Stokes equations a turbulence model can be proposed, the so-called Leray-α model. □

Remark 7.164. The regularization operator. Since the numerical calculation of convolution operators is expensive and in the case of a bounded domain one has to consider a cut-off of the domain of integration, see Remark 7.138, the regularization operator in the Leray-α model is usually the differential filter. It was already noted in Remark 7.139 that the differential filter is an approximation of the convolution with the Gaussian filter function. □

Remark 7.165. Transforming an abstract regularized equation into an equation for the large scales. Following Geurts and Holm (2003, 2006), a regularization model can be expressed similarly to the basic form of the equation (7.26) of LES with the sgs tensor given in (7.27). Consider a model of the form

$$\begin{aligned} \partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + (\overline{\boldsymbol{w}} \cdot \nabla)\, \boldsymbol{w} + \nabla r &= \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{w} &= 0, \\ R \overline{\boldsymbol{w}} &= \boldsymbol{w}, \end{aligned} \tag{7.181}$$

where R is some linear regularization operator which is invertible. It will be assumed that the regularization operator and its inverse commute with derivatives.

Using the commutation property of the inverse operator and its linearity yields

$$\nabla \cdot \overline{\boldsymbol{w}} = \nabla \cdot \left(R^{-1} \boldsymbol{w}\right) = R^{-1} \left(\nabla \cdot \boldsymbol{w}\right) = 0.$$

In Geurts and Holm (2003, 2006), the model was equipped from the beginning with the constraint

$$\nabla \cdot \overline{\boldsymbol{w}} = 0.$$

Next, one can write the convective term in the form $\nabla \cdot \left(\boldsymbol{w}\, \overline{\boldsymbol{w}}^T\right)$, see (1.26). Replacing now in the first equation $\boldsymbol{w}$ with $R \overline{\boldsymbol{w}}$ gives

$$\partial_t \left(R \overline{\boldsymbol{w}}\right) - \nu \Delta \left(R \overline{\boldsymbol{w}}\right) + \nabla \cdot \left(\left(R \overline{\boldsymbol{w}}\right) \overline{\boldsymbol{w}}^T\right) + \nabla r = \boldsymbol{f}.$$

With the assumption of the commutation of the regularization operator with all derivatives, one obtains

$$\begin{aligned} & R \left(\partial_t \overline{\boldsymbol{w}}\right) - R \left(\nu \Delta \overline{\boldsymbol{w}}\right) + R \left(\nabla \cdot \left(\overline{\boldsymbol{w}}\, \overline{\boldsymbol{w}}^T\right)\right) + \nabla r \\ & = \boldsymbol{f} - \left(\nabla \cdot \left(\left(R \overline{\boldsymbol{w}}\right) \overline{\boldsymbol{w}}^T\right) - R \left(\nabla \cdot \left(\overline{\boldsymbol{w}}\, \overline{\boldsymbol{w}}^T\right)\right)\right). \end{aligned}$$

The application of the inverse operator, defining $\overline{r} = R^{-1} r$, $\overline{\boldsymbol{f}} = R^{-1} \boldsymbol{f}$, using again the commutation property and the linearity of R^{-1} leads to

$$\begin{aligned} & \partial_t \overline{\boldsymbol{w}} - \nu \Delta \overline{\boldsymbol{w}} + \nabla \cdot \left(\overline{\boldsymbol{w}}\, \overline{\boldsymbol{w}}^T\right) + \nabla \overline{r} \\ & = \overline{\boldsymbol{f}} - \left(\nabla \cdot \left(R^{-1} \left(\left(R \overline{\boldsymbol{w}}\right) \overline{\boldsymbol{w}}^T\right)\right) - \nabla \cdot \left(\overline{\boldsymbol{w}}\, \overline{\boldsymbol{w}}^T\right)\right) \\ & = \overline{\boldsymbol{f}} - \nabla \cdot \left(\overline{\boldsymbol{w}\, \overline{\boldsymbol{w}}^T} - \overline{\boldsymbol{w}}\, \overline{\boldsymbol{w}}^T\right). \end{aligned}$$

In this way, the regularized model (7.181) is written in a similar form as the abstract LES model (7.26), where the sgs stress tensor $\mathbb{T}$ from (7.27) is replaced by an asymmetric tensor which contains in its first term already the filtered velocity field. Note that the asymmetry contradicts the the first property of models for the sgs stress tensor given in Remark 7.64. □

The Continuous Problem

Remark 7.166. The Leray-α model. Let $\Omega = \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with Lipschitz boundary Γ and let $\alpha > 0$ be a constant. Then, the Leray-α model with homogeneous boundary conditions for the velocity is given by

$$\begin{aligned}
\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + (\overline{\boldsymbol{w}} \cdot \nabla)\boldsymbol{w} + \nabla r &= \boldsymbol{f} && \text{in } (0,T] \times \Omega, \\
\nabla \cdot \boldsymbol{w} &= 0 && \text{in } [0,T] \times \Omega, \\
\boldsymbol{w} &= \boldsymbol{0} && \text{on } [0,T] \times \Gamma, \\
\boldsymbol{w}(0,\cdot) &= \boldsymbol{u}_0 && \text{in } \Omega, \\
-\alpha^2 \Delta \overline{\boldsymbol{w}} + \overline{\boldsymbol{w}} &= \boldsymbol{w} && \text{in } (0,T] \times \Omega, \\
\overline{\boldsymbol{w}} &= \boldsymbol{0} && \text{on } (0,T] \times \Gamma, \\
\int_\Omega r \, d\boldsymbol{x} &= 0 && \text{in } [0,T].
\end{aligned} \tag{7.182}$$

The smoothed or regularized velocity field is obtained by the solution of three scalar Helmholtz equations (differential filter).

In the analysis, the case of periodic boundary conditions, see Remark 1.27, is considered, i.e., Ω is a cube and $\boldsymbol{w}$ and $\overline{\boldsymbol{w}}$ are equipped with space-periodic boundary conditions. □

Remark 7.167. The divergence of $\overline{\boldsymbol{w}}$. Assuming sufficient smoothness of $\overline{\boldsymbol{w}}$, taking the divergence of the differential filter, and using (7.179) gives

$$-\alpha^2 \Delta (\nabla \cdot \overline{\boldsymbol{w}}) + \nabla \cdot \overline{\boldsymbol{w}} = \boldsymbol{0} \quad \text{in } (0,T] \times \Omega. \tag{7.183}$$

Equation (7.183) is a Helmholtz equation for $\nabla \cdot \overline{\boldsymbol{w}}$ with homogeneous right-hand side. Thus, its solution depends only on the boundary conditions.

In the case of a bounded domain, boundary values for $\nabla \cdot \overline{\boldsymbol{w}}$ cannot be derived from (7.182). Hence, it is not clear whether or not $\overline{\boldsymbol{w}}$ is divergence-free. From the practical point of view, this issue might not be that important since finite element velocities are usually not (weakly) divergence-free and thus there is no reason why the (discrete) regularization should be divergence-free.

For periodic boundary conditions, it follows from the periodicity of $\overline{\boldsymbol{w}}$ that also $\nabla \cdot \overline{\boldsymbol{w}}$ possesses periodic boundary conditions. Since the Helmholtz problem with periodic boundary conditions has a unique solution, which can be proved, e.g., by an easy extension of (Kreiss and Lorenz, 2004, Lemma 9.1.2), one gets that $\nabla \cdot \overline{\boldsymbol{w}} = 0$ is this solution. Hence, $\overline{\boldsymbol{w}}$ is divergence-free.

In Geurts and Holm (2003, 2006), where the constraint $\nabla \cdot \overline{\boldsymbol{w}} = 0$ was used, an example was considered which possesses periodic boundary conditions in two directions and a free-slip condition in the third direction. However, the third direction is not of importance for the turbulent character of the considered flow. □

Theorem 7.168. Existence and uniqueness of a solution in the space-periodic case. *Let $\Omega = (0, 2\pi L)^3$, $L > 0$, and (7.182) be equipped with periodic boundary conditions*

$$\begin{aligned} \partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + (\overline{\boldsymbol{w}} \cdot \nabla)\boldsymbol{w} + \nabla r &= \boldsymbol{f} && \text{in } (0,T] \times \Omega, \\ \nabla \cdot \boldsymbol{w} &= 0 && \text{in } [0,T] \times \Omega, \\ (\boldsymbol{w}, r) \text{ is periodic} & && \text{on } [0,T] \times \Gamma, \\ \boldsymbol{w}(0,\cdot) &= \boldsymbol{u}_0 && \text{in } \Omega, \\ -\alpha^2 \Delta \overline{\boldsymbol{w}} + \overline{\boldsymbol{w}} &= \boldsymbol{w} && \text{in } (0,T] \times \Omega, \\ \overline{\boldsymbol{w}} \text{ is periodic} & && \text{on } (0,T] \times \Gamma, \\ \int_\Omega r \, d\boldsymbol{x} &= 0 && \text{in } [0,T]. \end{aligned} \tag{7.184}$$

pressure cond. necessary?, periodicity of $\overline{\boldsymbol{w}}$ from periodicity of $\boldsymbol{w}$? Let

$$L^2_{\text{div,per}}(\Omega) = \left\{ \boldsymbol{v} \ : \ \boldsymbol{v} \in L^2(\Omega), \nabla \cdot \boldsymbol{v} = 0, \boldsymbol{v} \text{ is periodic in } \Omega, \int_\Omega \boldsymbol{v} \, d\boldsymbol{x} = 0 \right\},$$

and $V = L^2_{\text{div,per}}(\Omega) \cap H^1(\Omega)$.

If $\boldsymbol{f} \in L^2_{\text{div,per}}(\Omega)$ and $\boldsymbol{u}_0 \in V$, then (7.184) has a unique weak solution which is even a strong solution in $(0,T)$. That means, $\boldsymbol{w}$ satisfies

$$\frac{d}{dt}(\boldsymbol{w}, \boldsymbol{v}) + \nu (\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) + n(\overline{\boldsymbol{w}}, \boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in V,$$

where $\overline{\boldsymbol{w}} = \left(I - \alpha^2 \Delta\right)^{-1} \boldsymbol{w}$ and

$$\boldsymbol{w} \in C([0,T]; V) \cap L^2\left((0,T); V \cap H^2(\Omega)\right), \quad \partial_t \boldsymbol{w} = L^2\left((0,T); L^2_{\text{div,per}}(\Omega)\right).$$

Proof. This theorem is stated in Cheskidov et al. (2005).

The proof for the case $\Omega = \mathbb{R}^3$ follows from the analysis of Leray (1934). For the proof in the periodic case, it is mentioned in Cheskidov et al. (2005) that similar arguments can be applied as in Foias et al. (2002). some details. ■

Remark 7.169. On Theorem 7.168. The existence and uniqueness of a weak solution of (7.184) can be proved even with weaker regularity of the data $\boldsymbol{f}$ and $\boldsymbol{u}_0$, see Cheskidov et al. (2005) for the concrete statement. □

Remark 7.170. Existence of an attractor. It is shown in Cheskidov et al. (2005) that with the regularity assumptions $\boldsymbol{f}, \boldsymbol{u}_0 \in L^2_{\text{div,per}}(\Omega)$, there exists a unique global attractor $\mathcal{A}_{\text{Ler}}$ for the velocity. □

Remark 7.171. Definition of the viscous length scale. For the definition of the viscous scale, one needs an expression for the dissipation of turbulent energy, see (7.3). It is remarked in Cheskidov et al. (2005) that a worst case scenario is given by

$$\varepsilon_{\text{Ler}} = \frac{\nu}{(2\pi L)^3} \sup_{\boldsymbol{u}_0 \in \mathcal{A}_{\text{Ler}}} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \|\nabla \boldsymbol{w}(\tau)\|^2_{L^2(\Omega)} \, d\tau.$$

Then, the viscous length scale (7.3) is defined by

$$\lambda_{\mathrm{Ler}} = \left(\frac{\nu^3}{\varepsilon_{\mathrm{Ler}}}\right)^{1/4}.$$

Since the goal of the model consists in reducing the complexity of the Navier–Stokes equations, it can be expected that $\lambda_{\mathrm{Ler}} > \lambda$ or even $\lambda_{\mathrm{Ler}} \gg \lambda$. □

Remark 7.172. The dimension of the attractor. To estimate the dimension of the global attractor, one linearizes the Leray-α model about a trajectory in the attractor and studies the deviation. The deviation satisfies a first order linear ordinary differential equation. There is a classical solution theory of this type of equations. In particular, it is well known that the norm of the solution (in an appropriate space) can be estimated by the norm of the initial condition times an exponential factor. In order that the deviation tends to zero, the argument of this exponential factor has to be negative. In Cheskidov et al. (2005), this argument is estimated and the condition that it should be negative gives an estimate of the Hausdorff dimension of the attractor

$$d_{\mathrm{H}}(\mathcal{A}_{\mathrm{Ler}-}) \le c \left(\frac{L}{\lambda_{\mathrm{Ler}}}\right)^{12/7} \left(1 + \frac{L}{\alpha}\right)^{9/14} \tag{7.185}$$

with some universal constant, which can be estimated explicitly. □

Remark 7.173. Consequence of estimate (7.185). Already that it is possible to proof the existence and uniqueness of a weak solution, Theorem 7.168, suggests that the Leray-α model is less complex than the Navier–Stokes equations. The estimate of the dimension of the attractor allows to quantify this suggestion to some extent.

The power 12/7 of the attractor of the Leray-α model, for fixed α, is considerably smaller than the power 3 of the invariant bounded set X for the Navier–Stokes equations, see (7.10). In addition, one expects that $\lambda_{\mathrm{Ler}} > \lambda$ or even $\lambda_{\mathrm{Ler}} \gg \lambda$, see Remark 7.171. Altogether, (7.185) indicates that the number of degrees of freedom needed to simulate a flow modeled with the Leray-α model is much smaller than for a flow modeled with the Navier–Stokes equations.

In simulations, α will depend on the mesh width, see Remark 7.183. Inserting the estimate $\lambda_{\mathrm{Ler}} \lesssim \alpha$ into (7.185) gives the estimate of the dimension $12/7 + 9/14 = 33/14 \in (2, 3)$. Also with this estimate, in combination with the expectation $\lambda_{\mathrm{Ler}} > \lambda$ or $\lambda_{\mathrm{Ler}} \gg \lambda$, the Leray-$\alpha$ model is an appropriate candidate for turbulence modeling. □

The Discrete Problem

Remark 7.174. On the application of the differential filter in simulations. In numerical simulations, the differential filter as solution of a Helmholtz equation can be only approximated. For finite element methods, a straightforward approach consists in discretizing the Helmholtz equation in the velocity space, which gives the so-called discrete differential filter. The finite element error analysis requires some estimates of the discrete differential filter, which will be given next.

Since the differential filter requires to solve a Helmholtz equation for each component of the velocity separately, the analysis of the discrete differential filter will be presented for the scalar case. To avoid the introduction of further notations for this case, the continuous space will be denoted by $V = H_0^1(\Omega)$ and the conforming finite element space by $V^h \subset V$. □

Remark 7.175. Differential filter and discrete differential filter. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with Lipschitz boundary Γ and $\alpha > 0$. Then the differential filter of $u \in V$ is the unique solution $\overline{u} \in V$ of

$$\alpha^2 \left(\nabla \overline{u}, \nabla v\right) + \left(\overline{u}, v\right) = (u, v) \quad \forall\, v \in V. \tag{7.186}$$

The discrete differential filter is the solution $\overline{u}^h \in V^h$ of

$$\alpha^2 \left(\nabla \overline{u}^h, \nabla v^h\right) + \left(\overline{u}^h, v^h\right) = \left(u, v^h\right) \quad \forall\, v^h \in V^h. \tag{7.187}$$

□

Lemma 7.176. Properties of the discrete differential filter. *The discrete differential filter is selfadjoint, i.e., it is*

$$\left(\overline{u^h}^h, v^h\right) = \left(u^h, \overline{v^h}^h\right) \quad \forall\, u^h, v^h \in V^h. \tag{7.188}$$

Let $u^h \in V^h$ be a time-dependent function, then it is $\partial_t \overline{u^h}^h = \overline{\partial_t u^h}^h \in V^h$, i.e., the filter and the differentiation in time commute.

Proof. Since $\overline{u^h}^h \in V^h$, one can apply the definition (7.187) of the discrete differential filter with $\overline{u^h}^h$ as test function and obtains

$$\left(\overline{u^h}^h, v^h\right) = \left(\overline{u^h}^h, \overline{v^h}^h\right) + \alpha^2 \left(\nabla \overline{u^h}^h, \nabla \overline{v^h}^h\right).$$

One the other hand, one gets with the test function $\overline{v^h}^h \in V^h$

$$\left(u^h, \overline{v^h}^h\right) = \left(\overline{u^h}^h, \overline{v^h}^h\right) + \alpha^2 \left(\nabla \overline{u^h}^h, \nabla \overline{v^h}^h\right).$$

Combining these two equations proves (7.188).

Let $u^h \in V^h$ be given and let $v^h \in V^h$ be an arbitrary function, both functions might be time-dependent. Note that the filters of u^h, v^h, the temporal derivatives of u^h, v^h and their filters, and the filters of the temporal derivative are contained in V^h as well. Differentiating the definition (7.187) for u^h with respect to time, commuting integration in space and differentiation in time, applying the product rule, and commuting temporal and spatial derivatives gives

$$
\begin{aligned}
&\left(\partial_t u^h, v^h\right) + \left(u^h, \partial_t v^h\right) \\
&= \left(\partial_t \overline{u^h}^h, v^h\right) + \left(\overline{u^h}^h, \partial_t v^h\right) + \alpha^2 \left(\nabla \partial_t \overline{u^h}^h, \nabla v^h\right) + \alpha^2 \left(\nabla \overline{u^h}^h, \nabla \partial_t v^h\right).
\end{aligned}
$$

On the other hand, using the definition (7.187) of the discrete filter, one obtains

$$
\begin{aligned}
&\left(\partial_t u^h, v^h\right) + \left(u^h, \partial_t v^h\right) \\
&= \left(\overline{\partial_t u^h}^h, v^h\right) + \alpha^2 \left(\nabla \overline{\partial_t u^h}^h, \nabla v^h\right) + \left(\overline{u^h}^h, \partial_t v^h\right) + \alpha^2 \left(\nabla \overline{u^h}^h, \nabla \partial_t v^h\right).
\end{aligned}
$$

Combining these equations yields

$$
\left(\partial_t \overline{u^h}^h - \overline{\partial_t u^h}^h, v^h\right) + \alpha^2 \left(\nabla \left(\partial_t \overline{u^h}^h - \overline{\partial_t u^h}^h\right), \nabla v^h\right) = 0 \quad \forall\, v^h \in V^h.
$$

Since the left-hand side of this equation defines an inner product in V^h, the assumptions of the Theorem of Lax–Milgram, Theorem B.3 are satisfied and applying this theorem it follows that $\partial_t \overline{u^h}^h - \overline{\partial_t u^h}^h = 0$ is the unique solution of this equation. ■

Lemma 7.177. Stability of the discrete differential filter. *Let $u \in V$, then it holds*

$$
\left\| \overline{u}^h \right\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}, \tag{7.189}
$$

$$
\left\| \nabla \overline{u}^h \right\|_{L^2(\Omega)} \le \|\nabla u\|_{L^2(\Omega)}. \tag{7.190}
$$

Proof. Inserting $v^h = \overline{u}^h$ in (7.187), applying the Cauchy–Schwarz inequality (A.16), and Young's inequality (A.4) yields

$$
\alpha^2 \left\| \nabla \overline{u}^h \right\|_{L^2(\Omega)}^2 + \left\| \overline{u}^h \right\|_{L^2(\Omega)}^2 \le \|u\|_{L^2(\Omega)} \left\| \overline{u}^h \right\|_{L^2(\Omega)} \le \frac{\|u\|_{L^2(\Omega)}^2}{2} + \frac{\left\| \overline{u}^h \right\|_{L^2(\Omega)}^2}{2}.
$$

Estimating the first term on the left-hand side by zero from below, gives immediately (7.189).

To prove the second estimate, one defines the discrete Laplacian $\Delta^h : V \to V^h$ by

$$
\left(\Delta^h u, v^h\right) = -\left(\nabla u, \nabla v^h\right) \quad \forall\, v^h \in V^h. \tag{7.191}
$$

Applying this definition to the first term of (7.187) gives

$$
\left(\nabla \overline{u}^h, \nabla v^h\right) = -\left(\Delta^h \overline{u}^h, v^h\right).
$$

Inserting now $v^h = \Delta^h \overline{u}^h$ into (7.187) leads to

$$
-\alpha^2 \left\| \Delta^h \overline{u}^h \right\|_{L^2(\Omega)} + \left(\overline{u}^h, \Delta^h \overline{u}^h\right) = \left(u, \Delta^h \overline{u}^h\right),
$$

such that with (7.191)

$$
\alpha^2 \left\| \Delta^h \overline{u}^h \right\|_{L^2(\Omega)} + \left\| \nabla \overline{u}^h \right\|_{L^2(\Omega)} = \left(\nabla u, \nabla \overline{u}^h\right). \tag{7.192}
$$

With the same reasoning as in the first part of the proof, one obtains from here (7.190). ■

Lemma 7.178. Error estimate for the discrete differential filter in terms of u. *Let $u \in V$ with $\Delta u \in L^2(\Omega)$, then it is*

$$\alpha^2 \left\|\nabla \left(u - \overline{u}^h\right)\right\|^2_{L^2(\Omega)} + \left\|u - \overline{u}^h\right\|^2_{L^2(\Omega)} \tag{7.193}$$
$$\leq C \left[\inf_{v^h \in V^h} \left(\alpha^2 \left\|\nabla \left(u - v^h\right)\right\|^2_{L^2(\Omega)} + \left\|u - v^h\right\|^2_{L^2(\Omega)} \right) + \alpha^4 \left\|\Delta u\right\|^2_{L^2(\Omega)} \right],$$

where C does not depend on α and h. Consequently, one has

$$\left\|\nabla \left(u - \overline{u}^h\right)\right\|_{L^2(\Omega)} \leq \frac{C}{\alpha} \left((\alpha + h) \left\|\nabla u\right\|_{L^2(\Omega)} + \alpha^2 \left\|\Delta u\right\|_{L^2(\Omega)} \right), \tag{7.194}$$
$$\left\|u - \overline{u}^h\right\|_{L^2(\Omega)} \leq \left((\alpha + h) \left\|\nabla u\right\|_{L^2(\Omega)} + \alpha^2 \left\|\Delta u\right\|_{L^2(\Omega)} \right). \tag{7.195}$$

Proof. The proof proceeds in principle along the standard lines of proving finite element error estimates. The only non-standard issue is that the continuous function in the error is also on the right-hand side of the discrete problem.

By the regularity assumption, u satisfies

$$\alpha^2 \left(\nabla u, \nabla v^h\right) + \left(u, v^h\right) = -\alpha^2 \left(\Delta u, v^h\right) + \left(u, v^h\right) \quad \forall\, v^h \in V^h.$$

Denoting the error by $e = u - \overline{u}^h$ and subtracting (7.187) from this identity yields an error equation

$$\alpha^2 \left(\nabla e, \nabla v^h\right) + \left(e, v^h\right) = -\alpha^2 \left(\Delta u, v^h\right) \quad \forall\, v^h \in V^h. \tag{7.196}$$

Next, the error is decomposed into $e = \left(u - I^h u\right) - \left(\overline{u}^h - I^h u\right) = \eta - \varphi^h$, where $I^h u \in V^h$ is some arbitrary interpolant. Inserting this decomposition into the error equation and setting $v^h = \varphi^h$ gives

$$\alpha^2 \left\|\nabla \varphi^h\right\|^2_{L^2(\Omega)} + \left\|\varphi^h\right\|^2_{L^2(\Omega)} = \alpha^2 \left(\nabla \eta, \nabla \varphi^h\right) + \left(\eta, \varphi^h\right) + \alpha^2 \left(\Delta u, \varphi^h\right).$$

One obtains, applying the Cauchy–Schwarz inequality (A.16) and Young's inequality (A.4),

$$\begin{aligned}
&\alpha^2 \left\|\nabla \varphi^h\right\|^2_{L^2(\Omega)} + \left\|\varphi^h\right\|^2_{L^2(\Omega)} \\
&\leq \alpha^2 \left\|\nabla \eta\right\|_{L^2(\Omega)} \left\|\nabla \varphi^h\right\|_{L^2(\Omega)} + \left\|\eta\right\|_{L^2(\Omega)} \left\|\varphi^h\right\|_{L^2(\Omega)} + \alpha^2 \left\|\Delta u\right\|_{L^2(\Omega)} \left\|\varphi^h\right\|_{L^2(\Omega)} \\
&\leq \frac{\alpha^2}{2} \left\|\nabla \eta\right\|^2_{L^2(\Omega)} + \frac{\alpha^2}{2} \left\|\nabla \varphi^h\right\|^2_{L^2(\Omega)} + \left\|\eta\right\|^2_{L^2(\Omega)} + \frac{1}{4} \left\|\varphi^h\right\|^2_{L^2(\Omega)} \\
&\quad + \alpha^4 \left\|\Delta u\right\|^2_{L^2(\Omega)} + \frac{1}{4} \left\|\varphi^h\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

The terms with φ^h on the right-hand side can be absorbed from the left-hand side. Finally, an application of the triangle inequality gives

$$\begin{aligned}
&\alpha^2 \left\|\nabla e\right\|^2_{L^2(\Omega)} + \left\|e\right\|^2_{L^2(\Omega)} \\
&\leq 2 \left(\alpha^2 \left\|\nabla \eta\right\|^2_{L^2(\Omega)} + \left\|\eta\right\|^2_{L^2(\Omega)} + \alpha^2 \left\|\nabla \varphi^h\right\|^2_{L^2(\Omega)} + \left\|\varphi^h\right\|^2_{L^2(\Omega)} \right) \\
&\leq C \left(\alpha^2 \left\|\nabla \eta\right\|^2_{L^2(\Omega)} + \left\|\eta\right\|^2_{L^2(\Omega)} + \alpha^4 \left\|\Delta u\right\|^2_{L^2(\Omega)} \right),
\end{aligned}$$

which completes the proof of (7.193), since $I^h u$ was chosen to be arbitrary.

The estimates (7.194) and (7.195) are derived by estimating the best approximation errors in (7.193) with interpolation errors and by applying the interpolation estimate (C.11). ∎

Remark 7.179. Finite element error analysis of the Leray-α model. The available finite element error analysis for the Leray-α model proceeds in the same way as for the Galerkin discretization of the Navier–Stokes equations, see Section 6.3. In particular, the obtained results are qualitatively not better than for the Galerkin discretization in the sense that the constants in the error bounds depend still on $\exp(C\nu^{-3})$, where C depends on norms of the solution of the Navier–Stokes equations. This situation is strange, no better way known? For these reasons, a presentation of a detailed analysis would be just a repetition and it will be omitted here. Instead, only the differences in the finite element error analysis for the continuous-in-time case will be presented. The discussion of the obtained estimates gives a guideline for the asymptotic choice of the parameter α. □

Remark 7.180. Continuous-in-time Leray-α finite element model. Consider inf-sup stable finite element spaces $V^h \subset V$ and $Q^h \subset Q$, then the continuous-in-time Leray-α finite element model reads as follows: Find $\boldsymbol{w}^h : [0,T] \to V^h$ and $r^h : (0,T] \to Q^h$ such that

$$\begin{aligned}\left(\partial_t \boldsymbol{w}^h, \boldsymbol{v}^h\right) + (\nu\nabla\boldsymbol{w}^h, \nabla\boldsymbol{v}^h) + n_{\text{skew}}\left(\overline{\boldsymbol{w}^h}^h, \boldsymbol{w}^h, \boldsymbol{v}^h\right) \\ - \left(\nabla\cdot\boldsymbol{v}^h, r^h\right) + \left(\nabla\cdot\boldsymbol{w}^h, q^h\right) = \langle \boldsymbol{f}, \boldsymbol{v}^h\rangle_{V',V}, \\ \alpha^2\left(\nabla\,\overline{\boldsymbol{w}^h}^h, \nabla\boldsymbol{v}^h\right) + \left(\overline{\boldsymbol{w}^h}^h, \boldsymbol{v}^h\right) = \left(\boldsymbol{w}^h, \boldsymbol{v}^h\right) \end{aligned} \quad (7.197)$$

for all $(\boldsymbol{v}^h, q^h) \in V^h \times Q^h$, $\alpha > 0$, and $\boldsymbol{w}^h(0, \boldsymbol{x}) \in V^h$ is an approximation of $\boldsymbol{u}_0(\boldsymbol{x})$. □

Lemma 7.181. Existence, uniqueness, and stability of the finite element solution. *Let $\boldsymbol{w}_0^h \in V_{\text{div}}^h$ and $\boldsymbol{f} \in L^2(0,t;V')$, then the finite element problem* (7.197) *has a unique solution $\left(\boldsymbol{w}^h, r^h\right) \in V^h \times Q^h$. For all $t \in (0,T]$, the stability estimate*

$$\left\|\boldsymbol{w}^h(t)\right\|_{L^2(\Omega)}^2 + \nu\left\|\nabla\boldsymbol{w}^h\right\|_{L^2(0,t;L^2(\Omega))}^2 \le \left\|\boldsymbol{w}_0^h\right\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\left\|\boldsymbol{f}\right\|_{L^2(0,t;V')}^2 \quad (7.198)$$

holds.

Proof. The existence and uniqueness of a solution can be proved in the same way as it is done in the first step of the Galerkin method for proving the existence of a weak solution, e.g., see Lemma 6.13. check, also paper Leray (1934)

To prove the stability estimate, choose $(\boldsymbol{v}^h, q^h) = (\boldsymbol{w}^h, r^h)$ in (7.197). Using (5.18), the nonlinear convective term vanishes. Now, the proof proceeds analogously as the proof of Lemma 6.13 starting from (6.14). ■

Theorem 7.182. Finite element error estimates for the continuous-in-time Leray-α finite element model. *Let the assumptions of Theorem 6.46 be satisfied and let in addition*

$$\Delta\boldsymbol{u} \in L^4\left(0,T;L^2\left(\Omega\right)\right). \quad (7.199)$$

Then the following error estimate holds for all $t \in (0,T]$

$$
\begin{aligned}
&\left\|\left(\boldsymbol{u}-\boldsymbol{w}^h\right)(t)\right\|_{L^2(\Omega)}^2+\nu\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{w}^h\right)\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
&\le \text{right-hand side of } (6.39) \\
&\quad +\frac{C}{\nu}\exp\left(\frac{C}{\nu^3}\|\nabla\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^4\right)\left[\left(\alpha^{1/2}+\alpha^{-1/2}h\right)^2\|\nabla\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^4\right. \\
&\quad \left.+\alpha^3\|\Delta\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^2\|\nabla\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^4\right].
\end{aligned}
\tag{7.200}
$$

Assuming in addition to (7.199)

$$
\boldsymbol{u}\in L^4\left(0,T;L^\infty(\Omega)\right),\quad \nabla\boldsymbol{u}\in L^4\left(0,T;L^\infty(\Omega)\right),
\tag{7.201}
$$

then one obtains the error estimate for all $t\in(0,T]$

$$
\begin{aligned}
&\left\|\left(\boldsymbol{u}-\boldsymbol{w}^h\right)(t)\right\|_{L^2(\Omega)}^2+\nu\left\|\nabla\left(\boldsymbol{u}-\boldsymbol{w}^h\right)\right\|_{L^2(0,t;L^2(\Omega))}^2 \\
&\le \text{right-hand side of } (6.39) \\
&\quad +\frac{C}{\nu}\exp\left(\frac{C}{\nu^3}\|\nabla\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^4\right)\left[\left((\alpha+h)^2+\alpha^4\right)\right. \\
&\quad \times\left(\|\nabla\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^2+\|\Delta\boldsymbol{u}\|_{L^4(0,t;L^2(\Omega))}^2+\|\boldsymbol{u}\|_{L^4(0,t;L^\infty(\Omega))}^2\right. \\
&\quad \left.\left.+\|\nabla\boldsymbol{u}\|_{L^4(0,t;L^\infty(\Omega))}^2\right)\right].
\end{aligned}
\tag{7.202}
$$

Proof. not presented in course The proof follows exactly the lines of the proof of Theorem 6.46. As mentioned in Remark 7.179, only this part will be presented in detail, which is different.

Using the same notations as in the proof of Theorem 6.46 and considering the decomposition

$$
\boldsymbol{e}(t)=\boldsymbol{u}(t)-\boldsymbol{w}^h(t)=\left(\boldsymbol{u}(t)-I_{\mathrm{St}}^h\boldsymbol{u}(t)\right)+\left(I_{\mathrm{St}}^h\boldsymbol{u}(t)-\boldsymbol{w}^h(t)\right)=\boldsymbol{\eta}(t)-\boldsymbol{\phi}^h(t),
$$

the critical term for the estimate is the difference of the nonlinear convective terms. One gets for the Leray-α model, using (5.18),

$$
\begin{aligned}
& n_{\text{skew}}\left(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\overline{\boldsymbol{w}^h}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)\\
&= n_{\text{skew}}\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) + n_{\text{skew}}\left(\overline{\boldsymbol{u}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\overline{\boldsymbol{w}^h}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)\\
&= n_{\text{skew}}\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) + n_{\text{skew}}\left(\overline{\boldsymbol{\eta}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\overline{\boldsymbol{\phi}^h}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right)\\
&\quad + n_{\text{skew}}\left(\overline{\boldsymbol{w}^h}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\overline{\boldsymbol{w}^h}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)\\
&= n_{\text{skew}}\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) + n_{\text{skew}}\left(\overline{\boldsymbol{\eta}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\overline{\boldsymbol{\phi}^h}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right)\\
&\quad + n_{\text{skew}}\left(\overline{\boldsymbol{w}^h}^h,\boldsymbol{u}-\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)\\
&= n_{\text{skew}}\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) + n_{\text{skew}}\left(\overline{\boldsymbol{\eta}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\overline{\boldsymbol{\phi}^h}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right)\\
&\quad + n_{\text{skew}}\left(\overline{\boldsymbol{w}^h}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h\right).
\end{aligned} \tag{7.203}
$$

The last three terms are estimated the same way as (6.43) – (6.45). Using the stability estimates (7.189) and (7.190) for the discrete filter, one gets even exactly the same estimates as in (6.43) – (6.45).

With the assumptions of Theorem 6.46, one can estimate the first term with (5.30) for $s = 1/2$, Young's inequality (A.4)

$$
\begin{aligned}
& n_{\text{skew}}\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right)\\
&\le C\left\|\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right)\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\\
&\le \frac{C}{\nu}\left\|\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right\|_{L^2(\Omega)}\left\|\nabla\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2 + \frac{\nu}{16}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}
$$

The last term is absorbed in the left-hand side of the differential equation for the error estimate. The error estimate (7.195) gives for the first term an estimate of the form

$$
\begin{aligned}
& \left\|\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right\|_{L^2(\Omega)}\left\|\nabla\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right)\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2\\
&\le \frac{C}{\alpha}\Big((\alpha+h)\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)} + \alpha^2\left\|\Delta\boldsymbol{u}\right\|_{L^2(\Omega)}\Big)^2\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2\\
&\le \frac{C}{\alpha}(\alpha+h)^2\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^4 + C\alpha^3\left\|\Delta\boldsymbol{u}\right\|_{L^2(\Omega)}^2\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality (A.16), one has

$$
\begin{aligned}
& \int_0^t \left\|\Delta\boldsymbol{u}\right\|_{L^2(\Omega)}^2\left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^2\, d\tau\\
&\le \left(\int_0^t \left\|\Delta\boldsymbol{u}\right\|_{L^2(\Omega)}^4\, d\tau\right)^{1/2}\left(\int_0^t \left\|\nabla\boldsymbol{u}\right\|_{L^2(\Omega)}^4\, d\tau\right)^{1/2} < \infty,
\end{aligned} \tag{7.204}
$$

by the regularity assumptions. Now, estimate (7.200) follows in the same way as in Theorem 6.46.

With asumption (7.201), one can estimate the first term of the last estimate in (7.203) with (5.32), Poincaré's inequality (A.17), Young's inequality, and (7.195)

$$
\begin{aligned}
& n_{\text{skew}}\left(\boldsymbol{u}-\overline{\boldsymbol{u}}^h,\boldsymbol{u},\boldsymbol{\phi}^h\right) \\
& \le \frac{1}{2}\left\|\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right\|_{L^2(\Omega)}\left(\|\nabla\boldsymbol{u}\|_{L^\infty(\Omega)}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}+\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\|\boldsymbol{u}\|_{L^\infty(\Omega)}\right) \qquad (7.205)\\
& \le C\left\|\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\left(\|\nabla\boldsymbol{u}\|_{L^\infty(\Omega)}+\|\boldsymbol{u}\|_{L^\infty(\Omega)}\right) \\
& \le \frac{C}{\nu}\left\|\boldsymbol{u}-\overline{\boldsymbol{u}}^h\right\|_{L^2(\Omega)}^2\left(\|\nabla\boldsymbol{u}\|_{L^\infty(\Omega)}+\|\boldsymbol{u}\|_{L^\infty(\Omega)}\right)^2+\frac{\nu}{16}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\
& \le \frac{C}{\nu}\left(\left((\alpha+h)^2\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}^2+\alpha^4\|\Delta\boldsymbol{u}\|_{L^2(\Omega)}^2\right)\right)\left(\|\nabla\boldsymbol{u}\|_{L^\infty(\Omega)}+\|\boldsymbol{u}\|_{L^\infty(\Omega)}\right)^2 \\
& \quad +\frac{\nu}{16}\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}.
\end{aligned}
$$

The last term is absorbed from the left-hand side of the differential inequality. By assumptions (7.199) and (7.201), one shows analogously to (7.204) that all terms are in $L^1(0,T)$ such that Gronwall's lemma can be applied. Then, estimate (7.202) follows again in the same way as in Theorem 6.46. ■

Remark 7.183. Optimal asymptotic choice of α. For both error bounds (7.200) and (7.202), $\alpha = \mathcal{O}(h)$ is the optimal asymptotic choice. In the case of (7.200), this choice follows from equilibrating the terms α and $\alpha^{-1}h^2$. One obtains first order convergence, also for higher order finite elements.

In (7.202), one can choose on the one hand $\alpha = 0$ to get the best error bound. But on the other hand, α should be as large as possible to have a sufficient impact of the turbulence model. The optimal compromise is $\alpha = \mathcal{O}(h)$. Then, one gets the same power for the terms α^2 and h^2 in the parentheses. This power cannot be improved by different choices. Altogether, there is a second order of convergence in this case. □

Remark 7.184. The fully discrete case. A finite element error analysis for the Leray-α finite element model discretized in time with the Crank–Nicolson scheme, see Remark 6.31, can be found in Layton et al. (2008). In fact, this analysis is the special case of $N = 0$ in this paper. The existence of a solution in each discrete time is proved and a stability estimate is derived. The finite element error analysis uses an estimate of type (7.205) for the nonlinear term which is introduced by the turbulence model. As in the continuous-in-time case, an error bound is derived where the constant depends on $\exp(C\nu^{-3})$, where C depends on norms of the solution of the Navier–Stokes equations. The application of the discrete Gronwall lemma, Lemma **??**, gives a very mild time step restriction. Finally, error bounds are proved of second order in α, like in the corresponding situation for the continuous-in-time case, see Remark 7.183, and second order in Δt. check all statements □

Remark 7.185. Leray-α approximate deconvolution model. The idea of applying an approximate deconvolution, see Remark 7.148, can be also applied to the convective field of the Leray-α model, leading to the Leray-α ADM check name

$$
\begin{aligned}
\partial_t\boldsymbol{w}-\nu\Delta\boldsymbol{w}+\left(G_{\text{deconv},N}\left(\overline{\boldsymbol{w}}\right)\cdot\nabla\right)\boldsymbol{w}+\nabla r &= \boldsymbol{f} \ \text{ in } (0,T]\times\Omega,\\
\nabla\cdot\boldsymbol{w} &= 0 \ \text{ in } [0,T]\times\Omega,\\
-\alpha^2\Delta\overline{\boldsymbol{w}}+\overline{\boldsymbol{w}} &= \boldsymbol{w} \text{ in } (0,T]\times\Omega.
\end{aligned}
$$

This model was analyzed in the usual setup for ADMs, i.e., for space-periodic boundary conditions, the differential filter, and the van Cittert approximate deconvolution operator. usual techniques?, overview in Layton and Rebholz (2012) ? □

Remark 7.186. Numerical experience with the Leray-α model. todo □

7.7 The Navier–Stokes-α Model

Remark 7.187. The Navier–Stokes-α model. The Navier–Stokes-α model is given by

$$\begin{aligned} \partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + (\overline{\boldsymbol{w}} \cdot \nabla)\boldsymbol{w} + (\nabla \overline{\boldsymbol{w}})^T \boldsymbol{w} + \nabla r &= \boldsymbol{f} \quad \text{in } (0,T] \times \Omega, \\ \nabla \cdot \overline{\boldsymbol{w}} &= 0 \quad \text{in } [0,T] \times \Omega, \\ \boldsymbol{w}(0,\cdot) &= \boldsymbol{u}_0 \text{ in } \Omega, \\ -\alpha^2 \Delta \overline{\boldsymbol{w}} + \overline{\boldsymbol{w}} &= \boldsymbol{w} \quad \text{in } (0,T] \times \Omega, \\ \int_\Omega r \, d\boldsymbol{x} &= 0 \quad \text{in } [0,T], \end{aligned} \tag{7.206}$$

together with appropriate boundary conditions. The pressure includes some terms which appear in the derivation of this model and it has the form Chen et al. (1998)

$$r = p - \frac{1}{2} \| \overline{\boldsymbol{w}} \|_2^2 - \frac{1}{2}\alpha^2 \left(\nabla \overline{\boldsymbol{w}} : \overline{\boldsymbol{w}}\right).$$

Model (7.206) is also called viscous Camassa–Holm model or isotropic Lagrangian-averaged Navier–Stokes equations.

Using (2.136), the momentum equation can be rewritten, introducing a new pressure, by

$$\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + (\nabla \times \boldsymbol{w}) \times \overline{\boldsymbol{w}} + \nabla \tilde{r} = \boldsymbol{f} \tag{7.207}$$

or

$$\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} - \overline{\boldsymbol{w}} \times (\nabla \times \boldsymbol{w}) + \nabla \tilde{r} = \boldsymbol{f}. \tag{7.208}$$

□

Remark 7.188. The Lagrangian description of a flow field. The description of the flow field in Chapter 1, which finally led to the Navier–Stokes equations, was performed from the point of view of considering a point $(t, \boldsymbol{x})$ in time and space. Then, the flow field was modeled with functions depending on $(t, \boldsymbol{x})$. This approach is called the Eulerian description of the flow field. Alternatively, it is possible to describe the flow from the point of view of a 'fluid particle'. In this case, one follows the motion of that fluid particle through time and space. This approach is called Lagrangian description of a flow field.