In a bounded domain, usually the same boundary conditions are applied for $\boldsymbol{w}$ as they are prescribed for $\boldsymbol{u}$. The differential filter is equipped with homogeneous Neumann boundary conditions, see Remark 7.138. □

Remark 7.146. Analytical results.

- For the Taylor LES model with Smagorinsky sgs term, the existence, uniqueness, and stability of a weak solution for all times was proved in Coletti (1997) genauer under the assumption that $\nu_{\mathrm{T}} \geq \delta^2/6$. This condition means that the Smagorinsky term dominates the Taylor LES model. This relation is not correct since the Taylor LES model is a model for the large scale and cross term, which are $\mathcal{O}^{\mathrm{formal}}\left(\delta^2\right)$ whereas the sgs term is $\mathcal{O}^{\mathrm{formal}}\left(\delta^4\right)$. Under a similar condition, a finite element error analysis was performed in Iliescu et al. (2002).
- The existence and uniqueness of a solution of the rational LES model was studied in Berselli et al. (2002). This model was considered with auxiliary problem but without subgrid scale term, $\nu_T = 0$ in (7.165). In addition, the case of a space-periodic setting was investigated, i.e., $\Omega = (0,L)^3$ and periodic boundary conditions on $\partial\Omega$. Periodic boundary conditions are applied in the auxiliary problem as well. The existence and uniqueness of a solution in an appropriate function space for small time intervals $T = \mathcal{O}\left(\delta^4\right)$ could be proved. It was already mentioned in Remark 7.142 that the rational LES model without model for the sgs term is not robust in long time simulations. The proof uses the Galerkin method in a similar way as described in Section 6.1. A finite element error analysis of the rational LES model is not available.

□

Remark 7.147. Numerical studies. todo □

7.5 Large Eddy Simulation – Approximate Deconvolution Models (ADMs)

Remark 7.148. Basic idea. Approximate deconvolution models (ADM) were introduced in Stolz and Adams (1999); Stolz et al. (2001). Consider the space-averaged Navier–Stokes equations

$$\begin{aligned} \partial_t \overline{\boldsymbol{u}} - 2\nu\nabla\cdot\mathbb{D}(\overline{\boldsymbol{u}}) + \nabla\cdot\left(\overline{\boldsymbol{u}\boldsymbol{u}^T}\right) + \nabla\overline{p} &= \overline{\boldsymbol{f}} \ \text{ in } (0,T]\times\mathbb{R}^d, \\ \nabla\cdot\overline{\boldsymbol{u}} &= 0 \ \text{ in } [0,T]\times\mathbb{R}^d, \end{aligned} \tag{7.166}$$

see (7.26). As already discussed in Remark 7.32, the momentum equation in (7.166) is not yet an equation for $(\overline{\boldsymbol{u}}, \overline{p})$ since the nonlinear convective term still depends on $\boldsymbol{u}$.

Often, the filter is defined by a convolution with an appropriate filter function. For this reason, the filter operator is denoted by G_{conv}, i.e., $\overline{\boldsymbol{u}} =$

$G_{\text{conv}}(\boldsymbol{u})$. If the filter operator is invertible, then $\boldsymbol{u} = G_{\text{deconv}}(\overline{\boldsymbol{u}})$, where $G_{\text{deconv}} = G_{\text{conv}}^{-1}$ is the inverse filter operator or the deconvolution operator. Then, the momentum equation of (7.166) becomes

$$\partial_t \overline{\boldsymbol{u}} - 2\nu\nabla \cdot \mathbb{D}(\overline{\boldsymbol{u}}) + \nabla \cdot \left(\overline{G_{\text{deconv}}(\overline{\boldsymbol{u}})G_{\text{deconv}}(\overline{\boldsymbol{u}})^T}\right) + \nabla \overline{p} = \overline{\boldsymbol{f}}, \quad (7.167)$$

which is an equation for $(\overline{\boldsymbol{u}}, \overline{p})$, involving the filter operator and its inverse.

Even if G_{deconv} exists and would be efficiently computable, (7.167) does not define a turbulence model since the nonlinear term contains still all scales of the flow and it has the same complexity as the nonlinear term of the Navier–Stokes equations. The proposal of Stolz and Adams (1999); Stolz et al. (2001) consists in replacing in (7.167) the deconvolution operator by an approximation of the deconvolution (or of the inverse filter). Denoting the approximate deconvolution of order N by $G_{\text{deconv},N}$ gives the momentum equation

$$\partial_t \overline{\boldsymbol{u}} - 2\nu\nabla \cdot \mathbb{D}(\overline{\boldsymbol{u}}) + \nabla \cdot \left(\overline{G_{\text{deconv},N}(\overline{\boldsymbol{u}})G_{\text{deconv},N}(\overline{\boldsymbol{u}})^T}\right) + \nabla \overline{p} = \overline{\boldsymbol{f}}, \quad (7.168)$$

where the order will be specified more precisely in Remark 7.154. Equation (7.168) is an equation for $(\overline{\boldsymbol{u}}, \overline{p})$ with the known and computable filter operator and approximate deconvolution operator.

Since ADMs compute an approximation on $(\overline{\boldsymbol{u}}, \overline{p})$, the solution will be denoted also in this case by $(\boldsymbol{w}, r)$. Hence, an ADM has the form

$$\begin{aligned} &\partial_t \boldsymbol{w} - 2\nu\nabla \cdot \mathbb{D}(\boldsymbol{w}) \\ +\nabla \cdot \left(\overline{G_{\text{deconv},N}(\boldsymbol{w})G_{\text{deconv},N}(\boldsymbol{w})^T}\right) &+ \nabla r = \overline{\boldsymbol{f}} \text{ in } (0,T] \times \mathbb{R}^d, \\ &\nabla \cdot \boldsymbol{w} = 0 \quad \text{in } [0,T] \times \mathbb{R}^d. \end{aligned} \quad (7.169)$$

□

Remark 7.149. Mathematical literature. There are a number of papers on the analyis and numerical analyis of ADMs. An overview of the available results can be found in the monograph Layton and Rebholz (2012).

Note that the space-average Navier–Stokes equations (7.166), which are the basis of ADMs, were derived with the assumption of the commutation of the filter operator and the differential operators, see Remark 7.28. As it was discussed in Section 7.2, this assumption is usually not satisfied for filters which are defined by convolution. An exception is the case of space-periodic boundary conditions. In ADMs, usually the differential filter, see (7.162) or (7.170) below, is used. The commutation error of the differential filter will be discussed in Remark 7.152 and Example 7.153. Surveying the mathematical literature of ADMs, one finds that exclusively space-periodic boundary conditions are considered. As already mentioned in Remark 7.2, this kind of boundary conditions is not in the focus here. Since a survey of mathematical results is available, Layton and Rebholz (2012), here only some selected topics of ADMs will be presented. □

Example 7.150. Inverse deconvolution operators. In general, the inverse operator for the filter operator does not need to exist. However, for some examples, one can give the explicit inverse operator.

- For the Gaussian filter, a representation of the deconvolution operator is obtained by applying the inverse Fourier transform to (7.150)

$$\boldsymbol{u} = G_{\text{deconv}}\left(\overline{\boldsymbol{u}}\right) = \mathcal{F}^{-1}\left(\mathcal{F}\left(\boldsymbol{u}\right)\right) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}\left(\overline{\boldsymbol{u}}\right)}{\mathcal{F}\left(g_{\text{Gauss}}\right)}\right).$$

- Let the filter be defined by the differential filter

$$-\delta^2 \Delta \overline{\boldsymbol{u}} + \overline{\boldsymbol{u}} = \boldsymbol{u} \quad \text{in } \Omega, \tag{7.170}$$

which was already introduced within the rational LES model in (7.162). This equation has to be equipped with boundary conditions. A possible choice is $\overline{\boldsymbol{u}} = \boldsymbol{u}$ on $\partial\Omega$. Then, it holds

$$\begin{aligned} \overline{\boldsymbol{u}} &= G_{\text{conv}}(\boldsymbol{u}) = \left(-\delta^2\Delta + I\right)^{-1}\boldsymbol{u}, \\ \boldsymbol{u} &= G_{\text{deconv}}\left(\overline{\boldsymbol{u}}\right) = \left(-\delta^2\Delta + I\right)\left(\overline{\boldsymbol{u}}\right). \end{aligned}$$

□

Remark 7.151. On the boundary conditions for the differential filter. Assuming that $\boldsymbol{u}$ possesses Dirichlet boundary conditions, then a straighforward choice consists in using the same conditions for $\overline{\boldsymbol{u}}$. If these conditions are no-slip conditions, then one finds from the definition of the differential filter that not only $\overline{\boldsymbol{u}} = \boldsymbol{0}$ at the boundary but also $\Delta\overline{\boldsymbol{u}} = \boldsymbol{0}$ at the boundary. Thus, $\overline{\boldsymbol{u}}$ is near the boundary (in the layer) close to a harmonic function, e.g., a linear function, which probably does not correctly reflect the physical behavior of the large scales. □

Remark 7.152. The commutation of filtering and spatial derivatives for the differential filter. Let $\boldsymbol{u}$ and $\overline{\boldsymbol{u}}$ be sufficiently smooth, then it follows by differentiating the strong form of the filter equation and using the Theorem of Schwarz that

$$\nabla\boldsymbol{u} = \nabla\left(\overline{\boldsymbol{u}} - \delta^2\Delta\overline{\boldsymbol{u}}\right) = \nabla\overline{\boldsymbol{u}} - \delta^2\Delta\left(\nabla\overline{\boldsymbol{u}}\right).$$

On the other hand, the differential filter, extended to tensor-valued functions, for $\nabla\boldsymbol{u}$ is by definition

$$\nabla\boldsymbol{u} = \overline{\nabla\boldsymbol{u}} - \delta^2\Delta\left(\overline{\nabla\boldsymbol{u}}\right).$$

Thus, $\nabla \overline{\boldsymbol{u}}$ and $\overline{\nabla \boldsymbol{u}}$ satisfy the same elliptic partial differential equation with the same right-hand side. If they would satisfy the same boundary condition, then they are identical and filtering and commutation commute.

In the case of space-periodic boundary conditions for $\boldsymbol{u}$, also space-periodic boundary conditions for $\overline{\boldsymbol{u}}$ will be prescribed. Then, also $\nabla \overline{\boldsymbol{u}}$ and $\overline{\nabla \boldsymbol{u}}$ are space-periodic, i.e., they satisfy the same boundary condition, and thus differentiation and filtering commute.

If Ω is a bounded domain, Dirichlet boundary conditions are prescribed for $\boldsymbol{u}$, and the same boundary conditions are prescribed for each filtered function as for the corresponding unfiltered function, then the boundary conditions are generally not identical, see Example 7.153. □

Example 7.153. Commutation error for the differential filter in a bounded domain. Consider $u(x) = \sin(x) + \delta^2 \sin(x)$ in $(0, \pi)$. Then it is $u(0) = u(\pi) = 0$. Using the same boundary conditions for the filter, one equips the differential filter (7.170) with $\overline{u}\,(0) = \overline{u}\,(\pi) = 0$ and then it follows that $\overline{u}\,(x) = \sin(x)$.

One the one hand, it is $\overline{u}\,'(x) = \cos(x)$ and consequently $\overline{u}\,'(0) = 1$, $\overline{u}\,'(\pi) = -1$. For defining the filter of $u'(x) = \cos(x) + \delta^2 \cos(x)$, one takes the same boundary conditions as the unfiltered function, i.e., $\overline{u'}\,(0) = u'(0) = 1 + \delta^2$ and $\overline{u'}\,(\pi) = u'(\pi) = -1 - \delta^2$. These are not the same boundary vales as for $\overline{u}\,'(x)$ and consequently $\overline{u'}\,(x) \neq \overline{u}\,'(x)$. □

Remark 7.154. Van Cittert approximate deconvolution. The probably most popular approach for defining an approximate deconvolution is the van Cittert approximate deconvolution, van Cittert (1931).

Let $G_{\text{conv}} : L^2(\Omega) \to L^2(\Omega)$ be a convolution operator, i.e., $G_{\text{conv}}(\boldsymbol{u}) = \overline{\boldsymbol{u}}$. Consider the fixed point equation

$$\boldsymbol{u} = \boldsymbol{u} + \left(\overline{\boldsymbol{u}} - G_{\text{conv}}(\boldsymbol{u})\right). \tag{7.171}$$

The N-th order van Cittert deconvolution $G_{\text{deconv},N}\left(\overline{\boldsymbol{u}}\right)$ is defined by applying a fixed point iteration to (7.171) for approximating $\boldsymbol{u}$, starting with $G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right) = \overline{\boldsymbol{u}}$ and performing N steps:

$$\begin{aligned} G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right) &= \overline{\boldsymbol{u}}, \\ G_{\text{deconv},n}\left(\overline{\boldsymbol{u}}\right) &= G_{\text{deconv},n-1}\left(\overline{\boldsymbol{u}}\right) + \left(\overline{\boldsymbol{u}} - G_{\text{conv}}\left(G_{\text{deconv},n-1}\left(\overline{\boldsymbol{u}}\right)\right)\right). \end{aligned} \tag{7.172}$$

□

Remark 7.155. Properties of the van Cittert approximate deconvolution.

- The first members of the family of van Cittert approximate deconvolutions can be derived in a straighforward way from (7.172)

$$
\begin{aligned}
G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right) &= \overline{\boldsymbol{u}}\,, \\
G_{\text{deconv},1}\left(\overline{\boldsymbol{u}}\right) &= G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right) + \left(\overline{\boldsymbol{u}} - G_{\text{conv}}\left(G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right)\right)\right) \\
&= \overline{\boldsymbol{u}} + \left(\overline{\boldsymbol{u}} - G_{\text{conv}}\left(\overline{\boldsymbol{u}}\right)\right) = 2\,\overline{\boldsymbol{u}} - \overline{\overline{\boldsymbol{u}}}\,, \\
G_{\text{deconv},2}\left(\overline{\boldsymbol{u}}\right) &= 2\,\overline{\boldsymbol{u}} - \overline{\overline{\boldsymbol{u}}} + \left(\overline{\boldsymbol{u}} - G_{\text{conv}}\left(2\,\overline{\boldsymbol{u}} - \overline{\overline{\boldsymbol{u}}}\right)\right) \\
&= 2\,\overline{\boldsymbol{u}} - \overline{\overline{\boldsymbol{u}}} + \overline{\boldsymbol{u}} - 2\,\overline{\overline{\boldsymbol{u}}} + \overline{\overline{\overline{\boldsymbol{u}}}} \\
&= 3\,\overline{\boldsymbol{u}} - 3\,\overline{\overline{\boldsymbol{u}}} + \overline{\overline{\overline{\boldsymbol{u}}}}\,.
\end{aligned}
$$

It can be seen that the approximate deconvolution of order N is defined by a sum which involves terms with multiple (at most N) applications of the filter to the function $\boldsymbol{u}$.

- Rewriting the fixed point iteration (7.172), one finds the recursion

$$
G_{\text{deconv},n}\left(\overline{\boldsymbol{u}}\right) = \left(I - G_{\text{conv}}\right) G_{\text{deconv},n-1}\left(\overline{\boldsymbol{u}}\right) + \overline{\boldsymbol{u}}\,, \quad n = 1, \ldots, N.
$$

Straighforward calculations, using $G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right) = \overline{\boldsymbol{u}}$, give

$$
\begin{aligned}
G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right) &= \left(I - G_{\text{conv}}\right)^0 G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right) = \overline{\boldsymbol{u}}\,, \\
G_{\text{deconv},1}\left(\overline{\boldsymbol{u}}\right) &= \left(I - G_{\text{conv}}\right) \overline{\boldsymbol{u}} + \left(I - G_{\text{conv}}\right)^0 \overline{\boldsymbol{u}}\,, \\
G_{\text{deconv},2}\left(\overline{\boldsymbol{u}}\right) &= \left(I - G_{\text{conv}}\right) G_{\text{deconv},1}\left(\overline{\boldsymbol{u}}\right) + \left(I - G_{\text{conv}}\right)^0 G_{\text{deconv},0}\left(\overline{\boldsymbol{u}}\right) \\
&= \left(I - G_{\text{conv}}\right)^2 \overline{\boldsymbol{u}} + \left(I - G_{\text{conv}}\right) \overline{\boldsymbol{u}} + \left(I - G_{\text{conv}}\right)^0 \overline{\boldsymbol{u}}\,,
\end{aligned}
$$

from what one finds by induction that

$$
G_{\text{deconv},N}\left(\overline{\boldsymbol{u}}\right) = \sum_{n=0}^{N} \left(I - G_{\text{conv}}\right)^n \left(\overline{\boldsymbol{u}}\right), \quad 0 \le N < \infty. \tag{7.173}
$$

□

Lemma 7.156. Representation of the error of the approximate deconvolution. *Consider the space-periodic case and the differential filter. Then, it is for any $v \in L^2(\Omega)$*

$$
v - G_{\text{deconv},N} G_{\text{conv}} v = (-1)^{N+1} \delta^{2N+2} \left(\Delta G_{\text{conv}}\right)^{N+1} v. \tag{7.174}
$$

Proof. Using the binomial theorem, one finds that

$$
G_{\text{conv}} \left(I - G_{\text{conv}}\right)^n = G_{\text{conv}} + n G_{\text{conv}}^2 + \ldots + G_{\text{conv}}^{n+1} = \left(I - G_{\text{conv}}\right)^n G_{\text{conv}}, \quad n \ge 0.
$$

From (7.173) it follows that $G_{\text{conv}} G_{\text{deconv},N} = G_{\text{deconv},N} G_{\text{conv}}$. Using the representation (7.173) of the van Cittert approximate deconvolution, one obtains, using the cancelling of terms in telescopic sums,

$$
\begin{aligned}
G_{\text{deconv},N} G_{\text{conv}} &= G_{\text{conv}} G_{\text{deconv},N} = G_{\text{conv}} \sum_{n=0}^{N} (I - G_{\text{conv}})^n \\
&= \sum_{n=0}^{N} (I - G_{\text{conv}})^n - (I - G_{\text{conv}}) \sum_{n=0}^{N} (I - G_{\text{conv}})^n \\
&= \sum_{n=0}^{N} (I - G_{\text{conv}})^n - \sum_{n=1}^{N+1} (I - G_{\text{conv}})^n \\
&= I - (I - G_{\text{conv}})^{N+1}.
\end{aligned}
$$

It follows that

$$v - G_{\text{deconv},N} G_{\text{conv}} v = (I - G_{\text{conv}})^{N+1} v. \tag{7.175}$$

Since for the differential filter it is

$$(-\delta^2 \Delta + I) G_{\text{conv}} = I,$$

one gets

$$I - G_{\text{conv}} = -\delta^2 \Delta G_{\text{conv}}.$$

Inserting this expression into (7.175) gives (7.174). ■

Lemma 7.157. Self-adjointness of the filter operator. *Let the filter be given by convolution* (7.12) *with a symmetric filter function or by the differential filter* (7.170)*, where either homogeneous Dirichlet boundary conditions are used or the space-periodic case is considered. Then it holds*

$$(\,\overline{u}\,, v) = (u, \overline{v}\,) \quad \forall\, u, v \in L^2(\Omega), \tag{7.176}$$

Proof. Filtering with the convolution has to be considered in $\mathbb{R}^d$. Then, one obtains with (7.12), the application of Fubini's theorem, and the symmetry of the filter function

$$
\begin{aligned}
(\,\overline{u}\,, v) &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} g_{\text{fil}}\,(\boldsymbol{x} - \boldsymbol{z})\, u\,(\boldsymbol{z})\ d\boldsymbol{z} \right) v(\boldsymbol{x})\ d\boldsymbol{x} \\
&= \int_{\mathbb{R}^d} u\,(\boldsymbol{z}) \left(\int_{\mathbb{R}^d} g_{\text{fil}}\,(\boldsymbol{x} - \boldsymbol{z})\, v(\boldsymbol{x})\ d\boldsymbol{x} \right)\ d\boldsymbol{z} \\
&= \int_{\mathbb{R}^d} u\,(\boldsymbol{z}) \left(\int_{\mathbb{R}^d} g_{\text{fil}}\,(\boldsymbol{z} - \boldsymbol{x})\, v(\boldsymbol{x})\ d\boldsymbol{x} \right)\ d\boldsymbol{z} = (u, \overline{v}\,).
\end{aligned}
$$

With the assumed boundary conditions, the variational form of the differential filter of u is given by

$$(u, w) = (\,\overline{u}\,, w) + \delta^2 (\nabla\, \overline{u}\,, \nabla w)\,,$$

where the test functions are from $H_0^1(\Omega)$ or the corresponding space with space-periodic boundary conditions. For $v \in L^2(\Omega)$, the function $\overline{v}$ sufficiently smooth such that it can be used as test function, since it also obeys the correct boundary conditions. Choosing $w = \overline{v}$ gives

$$(u, \overline{v}\,) = (\,\overline{u}\,, \overline{v}\,) + \delta^2 (\nabla\, \overline{u}\,, \nabla\, \overline{v}\,)\,.$$

With the same arguments, one can use $\overline{u}$ as test function. Using the definition of the differential filter for v with $w = \overline{u}$ yields

$$(\,\overline{u}\,, v) = (v, \overline{u}\,) = (\,\overline{v}\,, \overline{u}\,) + \delta^2 (\nabla\, \overline{v}\,, \nabla\, \overline{u}\,)\,,$$

which proves the statement for the differential filter. ■

Remark 7.158. On the analysis of ADMs. As already mentioned at the beginning of this section, the analysis for ADMs is performed exclusively for the space-periodic case. To give a flavor of applying the properties of the deconvolution operator in this case, the proof of a stability estimate for the lowest order ADM will be presented below. A short survey of available results will be given in Remark 7.161. □

Remark 7.159. The lowest order ADM. The weak formulation of the lowest order ADM in the space-periodic case with the differential filter has the form: Find $(\boldsymbol{w}, r)$ such that

$$\int_0^T \Big[(\partial_t \boldsymbol{w}, \boldsymbol{v}) + (\nu \nabla \boldsymbol{w}, \nabla \boldsymbol{v}) + \Big(\nabla \cdot \Big(\overline{\boldsymbol{w}\boldsymbol{w}^T}\Big), \boldsymbol{v}\Big) - (\nabla \cdot \boldsymbol{v}, r) + (\nabla \cdot \boldsymbol{w}, q) \Big] \, dt = \int_0^T \big(\overline{\boldsymbol{f}}, \boldsymbol{v}\big) \, dt \quad \forall \, (\boldsymbol{v}, q), \qquad (7.177)$$

where all functions belong to appropriate spaces. □

Lemma 7.160. Stability estimate (energy equality) for the zeroth order ADM. *Let* $\boldsymbol{w}$ *be a sufficiently regular solution of* (7.177)*, then it holds for all* $T > 0$

$$\begin{aligned} &\frac{1}{2} \|\boldsymbol{w}(T)\|^2_{L^2(\Omega)} + \frac{\delta^2}{2} \|\nabla \boldsymbol{w}(T)\|^2_{L^2(\Omega)} + \nu \|\nabla \boldsymbol{w}\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad + \delta^2 \nu \|\Delta \boldsymbol{w}\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad = \frac{1}{2} \|\boldsymbol{w}(0)\|^2_{L^2(\Omega)} + \frac{\delta^2}{2} \|\nabla \boldsymbol{w}(0)\|^2_{L^2(\Omega)} + \int_0^T (\boldsymbol{f}, \boldsymbol{w}) \, dt. \end{aligned} \qquad (7.178)$$

Proof. As always, one has to choose the test function such that the nonlinear convective term vanishes. Using the commutation of filtering and differentiation in the space-periodic case, (7.176), and (1.27) gives

$$\begin{aligned} \Big(\nabla \cdot \Big(\overline{\boldsymbol{w}\boldsymbol{w}^T}\Big), \boldsymbol{v}\Big) &= \Big(\overline{\nabla \cdot (\boldsymbol{w}\boldsymbol{w}^T)}, \boldsymbol{v}\Big) = \big(\nabla \cdot \big(\boldsymbol{w}\boldsymbol{w}^T\big), \overline{\boldsymbol{v}}\,\big) \\ &= \big((\boldsymbol{w} \cdot \nabla)\, \boldsymbol{w}, \overline{\boldsymbol{v}}\,\big) = n_{\mathrm{conv}}\,(\boldsymbol{w}, \boldsymbol{w}, \overline{\boldsymbol{v}}\,)\,. \end{aligned}$$

Hence, one has to choose $\overline{\boldsymbol{v}} = \boldsymbol{w}$ which is equivalent to choosing $\boldsymbol{v} = \big(-\delta^2 \Delta + I\big)\, \boldsymbol{w}$. One has for sufficiently smooth functions

$$\begin{aligned} \nabla \cdot \Delta \boldsymbol{w} &= \partial_x (\partial_{xx} w_1 + \partial_{yy} w_1 + \partial_{zz} w_1) \\ &\quad + \partial_y (\partial_{xx} w_2 + \partial_{yy} w_2 + \partial_{zz} w_2) \\ &\quad + \partial_z (\partial_{xx} w_3 + \partial_{yy} w_3 + \partial_{zz} w_3) \\ &= (\partial_{xx} + \partial_{yy} + \partial_{zz})\,(\partial_x w_1 + \partial_y w_2 + \partial_z w_3) = \Delta\,(\nabla \cdot \boldsymbol{w})\,. \end{aligned} \qquad (7.179)$$

Since $\nabla \cdot \boldsymbol{w} = 0$, see (7.169), it follows that $\nabla \cdot \Delta \boldsymbol{w} = 0$. Inserting this test function in (7.177) yields

$$\int_0^T \left[\left(\partial_t \boldsymbol{w}, \left(-\delta^2\Delta + I\right)\boldsymbol{w}\right) + \left(\nu\nabla\boldsymbol{w}, \nabla\left(\left(-\delta^2\Delta + I\right)\boldsymbol{w}\right)\right)\right]\, dt$$
$$= \int_0^T \left(\overline{\boldsymbol{f}}, \left(-\delta^2\Delta + I\right)\boldsymbol{w}\right)\, dt.$$

Applying integration by parts, where the boundary integrals cancel due to the space-periodic conditions, and using (7.176) gives

$$\int_0^T \left[(\partial_t\boldsymbol{w}, \boldsymbol{w}) + \delta^2\left(\partial_t\nabla\boldsymbol{w}, \nabla\boldsymbol{w}\right) + (\nu\nabla\boldsymbol{w}, \nabla\boldsymbol{w}) + \delta^2\left(\nu\Delta\boldsymbol{w}, \Delta\boldsymbol{w}\right)\right]\, dt$$
$$= \int_0^T \left(\boldsymbol{f}, \overline{\left(-\delta^2\Delta + I\right)\boldsymbol{w}}\right)\, dt = \int_0^T (\boldsymbol{f}, \boldsymbol{w})\, dt. \tag{7.180}$$

Using (6.13) leads to

$$\int_0^T \left[\frac{d}{dt}\frac{1}{2}\|\boldsymbol{w}\|^2_{L^2(\Omega)} + \frac{\delta^2}{2}\frac{d}{dt}\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)} + (\nu\nabla\boldsymbol{w}, \nabla\boldsymbol{w}) + \delta^2\left(\nu\Delta\boldsymbol{w}, \Delta\boldsymbol{w}\right)\right]\, dt$$
$$= \int_0^T (\boldsymbol{f}, \boldsymbol{w})\, dt.$$

Integration in $(0, T)$ gives finally (7.178). ■

Remark 7.161. Short survey of analytical results for ADMs. If not stated otherwise, the results mentioned in this remark are for the space-periodic case, the differential filter, and the van Cittert approximate deconvolution.

- The existence of a weak solution of the zeroth order ADM was proved in Layton and Lewandowski (2003) and its uniqueness was shown in Layton and Lewandowski (2006), see also Layton and Rebholz (2012).
- ADMs of order N were studied in Dunca and Epshteyn (2006). In this paper, the existence and uniqueness of a weak solution was proved. The proof of the existence uses the Galerkin method, which was introduced in Section 6.1. The existence and uniqueness results obtained indicate that ADMs are less complex than the three-dimensional Navier–Stokes equations, compare Remark 7.22.
- Another main topic in Dunca and Epshteyn (2006) is an estimate of the modeling error, i.e., the error between $\overline{\boldsymbol{u}}$ and $\boldsymbol{w}$. It was shown that for sufficiently smooth solutions it holds that $\|\overline{\boldsymbol{u}} - \boldsymbol{w}\|_{L^\infty(0,T;L^2(\Omega))}$ and $\|\nabla(\overline{\boldsymbol{u}} - \boldsymbol{w})\|_{L^2(0,T;L^2(\Omega))}$ are $\mathcal{O}\left(\delta^{N+1}\right)$. The proof uses the error representation (7.174).
- In Stanculescu (2008), a general approximate deconvolution operator $G_{\mathrm{deconv},N}$ is considered in combination with the differential filter. Conditions on $G_{\mathrm{deconv},N}$ were derived which ensure the existence and uniqueness of a weak solution, which guarantee regularity of the weak solution, and which ensure an energy equality. The conditions on $G_{\mathrm{deconv},N}$ are
  - ∘ $G_{\mathrm{deconv},N}$ is a bounded linear operator on $L^2(\Omega)$,
  - ∘ $G_{\mathrm{deconv},N}$ is self-adjoint and positive definite,
  - ∘ $G_{\mathrm{deconv},N}$ commutes with differentiation.

It is pointed out that there are deconvolution operators besides the van Cittert deconvolution which satisfy these conditions and also deconvolution operators which do not.

- convergence for $N \to \infty$, **?**

□

Remark 7.162. Finite element error analyis of ADMs.

- A numerical scheme for the zeroth order approximate deconvolution model is studied in Manica and Merdan (2007). In this paper, the situation of a bounded domain and no-slip boundary conditions is considered. Hence, differentiation and filtering do not commute and the momentum balance has the form

$$\partial_t \boldsymbol{w} - \nu \Delta \boldsymbol{w} + \overline{\nabla \cdot (\boldsymbol{w}\, \boldsymbol{w}^T)} + \overline{\nabla q} = \overline{\boldsymbol{f}}\,.$$

  To handle the nonlinear convective term, it is proposed to use the same kind of test function as in the proof of Lemma 7.160, namely $\left(-\delta^2 \Delta + I\right) \tilde{\boldsymbol{v}}$, such that this term can be rewritten in the form $n_{\mathrm{skew}}(\boldsymbol{w}, \boldsymbol{w}, \tilde{\boldsymbol{v}})$ and it vanishes for $\tilde{\boldsymbol{v}} = \boldsymbol{w}$. However, the use of this kind of test functions leads to a fourth order term, compare the term $\delta^2 \left(\nu \Delta \boldsymbol{w}, \Delta \boldsymbol{w}\right)$ in (7.180). A mixed finite element formulation is considered to handle this situation, which is analyzed in the usual way. A main tool of the analysis is a modified Stokes projection. The optimal choice of δ comes from the properties of the modified Stokes projection and it is $\delta = \mathcal{O}(h)$. work out?
- Galvin, Rebholz, Trenchea 2014

□

7.6 The Leray-α Model

Remark 7.163. Motivation. In Leray (1934), the existence of a weak solution of the Navier–Stokes equations (7.1) (in this paper called turbulent solution) was proved by considering a sequence of simplified problems, where the simplification consisted in replacing in the nonlinear term of the Navier–Stokes equations the convection field by a smooth or regularized velocity field, see Remark 6.11. The case $\Omega = \mathbb{R}^3$ was considered and the regularization was defined by a convolution with a filter function. Then, the behavior was studied for the filter width tending to zero. Based on this idea from the analysis of the Navier–Stokes equations a turbulence model can be proposed, the so-called Leray-α model. □

Remark 7.164. The regularization operator. Since the numerical calculation of convolution operators is expensive and in the case of a bounded domain one has to consider a cut-off of the domain of integration, see Remark 7.138, the regularization operator in the Leray-α model is usually the differential