Remark 7.127. Van Driest damping. As already mentioned in Remark 7.124, the Smagorinsky model introduces too much viscosity in particular near solid walls. The application of a van Driest damping is a proposal to reduce this viscosity. citation The van Driest damping changes the eddy viscosity of the Smagorinsky model (7.64) in the viscous sublayer, see Remark 7.13, to (Pope, 2000, p. 599)

$$\nu_{\mathrm{T}} = C_S \delta^2 \left(1 - \exp\left(-\frac{y^+}{A^+}\right)\right)^2 \|\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\|_{\mathrm{F}}, \quad \text{if } y^+ < 5,$$

with $A^+ = 26$. Vreman'04 PoF 16, 3670-3681, Verstappen JSC'11 □

7.4 Large Eddy Simulation – Models Based On Approximations in Wave Number Space

Remark 7.128. The basic approach. Inserting the decomposition (7.11) and applying the linearity (7.23) of the filter yields for filtered nonlinear convective term, which is unknown,

$$\overline{\boldsymbol{u}\boldsymbol{u}^T} = \overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} + \overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T} + \overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T} + \overline{\boldsymbol{u}'\boldsymbol{u}'^T}. \tag{7.147}$$

The first term in (7.147) is called large scale advective term. It describes the convection of the large eddies driven by themselves. The second and third term are the so-called cross terms describing the interaction of the large scale and subgrid scale components. The last tensor is the subgrid scale (sgs) term which describes how the small eddies extract energy from the flow.

Models that are base on approximations in wave number space consider each term on the right-hand side of (7.147) separately. Each term is transformed to the Fourier or wave number space and then an approximation is applied, see Section 7.4.1. It turns out that with this approach the sgs term is modelled with the zero tensor. Numerical simulations show that this model is not sufficient, e.g., see (John, 2004, Section 10.3.3). Section 7.4.2 presents some proposals for modeling the sgs term. The final models will be presented and discussed in Section 7.4.3. □

7.4.1 Modelling of the Large Scale and Cross Terms

Remark 7.129. Assumptions. Let $\Omega = \mathbb{R}^d$ and let the averaging be performed with the Gaussian filter

$$\overline{\boldsymbol{u}}\,(t, \boldsymbol{x}) = g_{\mathrm{Gauss}} * \boldsymbol{u}\,(t, \boldsymbol{x})$$

where g_{Gauss} is defined in (7.17) and δ is a constant. □

Remark 7.130. Principal approach. The model of the large scale and cross terms is obtained in five steps:

1. compute the Fourier transform,
2. replace $\mathcal{F}(\boldsymbol{u}')$ by a function of $\mathcal{F}(\overline{\boldsymbol{u}})$ if necessary,
3. approximate the Fourier transform of the Gaussian filter with a simpler function,
4. neglect all terms which are in a certain sense of higher order in δ,
5. compute the inverse Fourier transform.

There are two approaches in the literature which differ in the third point. The first approach, see Leonard (1975) or Clark et al. (1979), approximates $\mathcal{F}(g_{\text{Gauss}})$ by a Taylor polynomial, Remark 7.132, whereas the second approach from Galdi and Layton (2000) uses a rational approximation, see Remark 7.135. The first approach gives the Taylor LES model and the second one the rational LES model. □

Remark 7.131. Step 1 and 2. Using (A.8), the Fourier transform of the large scale term is

$$\mathcal{F}\left(\overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T}\right) = \mathcal{F}(g_{\text{Gauss}})\,\mathcal{F}\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right), \tag{7.148}$$

and the Fourier transforms of the cross terms are

$$\begin{aligned}\mathcal{F}\left(\overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T}\right) &= \mathcal{F}(g_{\text{Gauss}})\,\mathcal{F}\left(\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T\right)\\ &= \mathcal{F}(g_{\text{Gauss}})\left(\mathcal{F}(\overline{\boldsymbol{u}}) * \mathcal{F}(\boldsymbol{u}')^T\right),\\ \mathcal{F}\left(\overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T}\right) &= \mathcal{F}(g_{\text{Gauss}})\left(\mathcal{F}(\boldsymbol{u}') * \mathcal{F}(\overline{\boldsymbol{u}})^T\right).\end{aligned} \tag{7.149}$$

Since $\mathcal{F}(g_{\text{Gauss}}) \neq 0$, one obtains

$$\mathcal{F}(\boldsymbol{u}) = \frac{\mathcal{F}(g_{\text{Gauss}})\,\mathcal{F}(\boldsymbol{u})}{\mathcal{F}(g_{\text{Gauss}})} = \frac{\mathcal{F}(\overline{\boldsymbol{u}})}{\mathcal{F}(g_{\text{Gauss}})}. \tag{7.150}$$

Inserting the decomposition $\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}'$ on the left-hand side and rearranging terms yields

$$\mathcal{F}(\boldsymbol{u}') = \left(\frac{1}{\mathcal{F}(g_{\text{Gauss}})} - 1\right)\mathcal{F}(\overline{\boldsymbol{u}}). \tag{7.151}$$

Thus, the Fourier transform of $\boldsymbol{u}'$ can be represented with the Fourier transforms of $\boldsymbol{u}$ and of the Gaussian filter. For this representation it is important that $\mathcal{F}(g_{\text{Gauss}}) \neq 0$. Inserting (7.151) into (7.149) gives

$$\begin{aligned}\mathcal{F}\left(\overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T}\right) &= \mathcal{F}(g_{\text{Gauss}})\left(\mathcal{F}(\overline{\boldsymbol{u}}) * \left(\frac{1}{\mathcal{F}(g_{\text{Gauss}})} - 1\right)\mathcal{F}(\overline{\boldsymbol{u}})^T\right),\\ \mathcal{F}\left(\overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T}\right) &= \mathcal{F}(g_{\text{Gauss}})\left(\left(\frac{1}{\mathcal{F}(g_{\text{Gauss}})} - 1\right)\mathcal{F}(\overline{\boldsymbol{u}}) * \mathcal{F}(\overline{\boldsymbol{u}})^T\right).\end{aligned} \tag{7.152}$$

Although on the right-hand sides $\boldsymbol{u}'$ does not appear, there is no reduction of the complexity of the terms so far since still equality holds. □

Remark 7.132. Step 3 with Taylor polynomial approximation. The Fourier transform of the Gaussian filter is an exponential, see (7.18), and the Taylor series expansion for the exponential has the form

$$e^{ax} = 1 + ax + \mathcal{O}\left(x^2\right).$$

Applying this expansion to $\mathcal{F}\left(g_{\text{Gauss}}\right)$ with respect to δ and for fixed $\boldsymbol{y}$ gives

$$\mathcal{F}\left(g_{\text{Gauss}}\right)(\delta, \boldsymbol{y}) = 1 - \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 + \mathcal{O}\left(\delta^4\right). \tag{7.153}$$

Since of $1/\mathcal{F}\left(g_{\text{Gauss}}\right)$ is an exponential as well, one obtains the expansion

$$\frac{1}{\mathcal{F}\left(g_{\text{Gauss}}\right)}(\delta, \boldsymbol{y}) = 1 + \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 + \mathcal{O}\left(\delta^4\right). \tag{7.154}$$

Now, $\mathcal{F}\left(g_{\text{Gauss}}\right)$ and $1/\mathcal{F}\left(g_{\text{Gauss}}\right)$ are approximated in (7.148) and (7.152) by quadratic polynomials which are obtained by neglecting the terms of $\mathcal{O}\left(\delta^4\right)$ in (7.153) and (7.154), see Figure 7.3 for the one-dimensional situation. It can be seen that the polynomial approximation of $\mathcal{F}\left(g_{\text{Gauss}}\right)$ is a good approximation only for small wave numbers and it is completely wrong for high wave numbers. Consequently, the most important property of a filter function, the damping of the high wave number components of $(\boldsymbol{u}, p)$, is not preserved by its Taylor polynomial approximation !

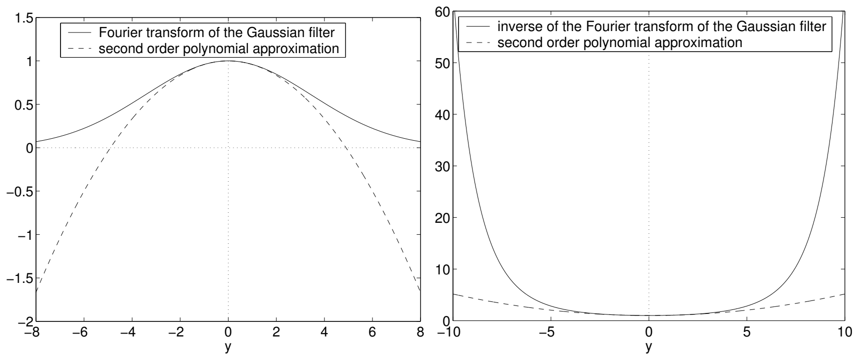

Fig. 7.3 $\mathcal{F}\left(g_{\text{Gauss}}\right)$ (left) and $1/\mathcal{F}\left(g_{\text{Gauss}}\right)$ (right) with their polynomial approximations, $\delta = 1$. new pics

Inserting (7.153) and (7.154) into (7.148) and (7.152) gives

$$\mathcal{F}\left(\overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T}\right) = \left(1 - \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 + \mathcal{O}\left(\delta^4\right)\right)\mathcal{F}\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right), \tag{7.155}$$

$$\mathcal{F}\left(\overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T}\right) = \left(1 - \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 + \mathcal{O}\left(\delta^4\right)\right) \tag{7.156}$$

$$\times\left[\mathcal{F}\left(\overline{\boldsymbol{u}}\right) * \left(\frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 + \mathcal{O}\left(\delta^4\right)\right)\mathcal{F}\left(\overline{\boldsymbol{u}}\right)^T\right],$$

$$\mathcal{F}\left(\overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T}\right) = \left(1 - \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 + \mathcal{O}\left(\delta^4\right)\right)$$

$$\times\left[\left(\frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 + \mathcal{O}\left(\delta^4\right)\right)\mathcal{F}\left(\overline{\boldsymbol{u}}\right) * \mathcal{F}\left(\overline{\boldsymbol{u}}\right)^T\right].$$

□

Remark 7.133. Step 4 with Taylor polynomial approximation. Now, the expressions obtained in Step 3 are simplified using properties of the Fourier transform. All terms which has as factor the fourth or a higher power of δ are neglected. However, the other factors of these terms depend on $\overline{\boldsymbol{u}}$ which in turn depends on δ in some unknown way. That means, the neglected terms are only formally of fourth order in δ. One gets with (A.9) for the large scale advective term from (7.155)

$$\begin{aligned}\mathcal{F}\left(\overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T}\right) &= \mathcal{F}\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) - \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2\mathcal{F}\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) + \mathcal{O}^{\mathrm{formal}}\left(\delta^4\right)\\ &= \mathcal{F}\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) + \frac{\delta^2}{24}\mathcal{F}\left(\Delta\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right)\right) + \mathcal{O}^{\mathrm{formal}}\left(\delta^4\right).\end{aligned}$$

For the first cross term (7.156), one obtains with (A.9) and (A.8)

$$\begin{aligned}\mathcal{F}\left(\overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T}\right) &= \left(1 - \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2\right)\left(\mathcal{F}\left(\overline{\boldsymbol{u}}\right) * \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2\mathcal{F}\left(\overline{\boldsymbol{u}}\right)^T\right) + \mathcal{O}^{\mathrm{formal}}\left(\delta^4\right)\\ &= \left(\mathcal{F}\left(\overline{\boldsymbol{u}}\right) * \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2\mathcal{F}\left(\overline{\boldsymbol{u}}\right)^T\right) + \mathcal{O}^{\mathrm{formal}}\left(\delta^4\right)\\ &= -\frac{\delta^2}{24}\left(\mathcal{F}\left(\overline{\boldsymbol{u}}\right) * \mathcal{F}\left(\Delta\overline{\boldsymbol{u}}\right)^T\right) + \mathcal{O}^{\mathrm{formal}}\left(\delta^4\right)\\ &= -\frac{\delta^2}{24}\mathcal{F}\left(\overline{\boldsymbol{u}}\,\Delta\left(\overline{\boldsymbol{u}}\right)^T\right) + \mathcal{O}^{\mathrm{formal}}\left(\delta^4\right).\end{aligned}$$

In the same way, one gets for the other cross term

$$\mathcal{F}\left(\overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T}\right) = -\frac{\delta^2}{24}\mathcal{F}\left(\Delta\left(\overline{\boldsymbol{u}}\right)\overline{\boldsymbol{u}}^T\right) + \mathcal{O}^{\mathrm{formal}}\left(\delta^4\right).$$

□

Remark 7.134. Step 5 with Taylor polynomial approximation – the Taylor LES model. The final approximation of the individual terms is computed by applying the inverse Fourier transform

$$\begin{aligned}
\overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} &= \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T + \frac{\delta^2}{24}\Delta\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) + \mathcal{O}^{\text{formal}}\left(\delta^4\right), \\
\overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T} &= -\frac{\delta^2}{24}\overline{\boldsymbol{u}}\,\Delta\left(\overline{\boldsymbol{u}}\right)^T + \mathcal{O}^{\text{formal}}\left(\delta^4\right), \\
\overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T} &= -\frac{\delta^2}{24}\Delta\left(\overline{\boldsymbol{u}}\right)\overline{\boldsymbol{u}}^T + \mathcal{O}^{\text{formal}}\left(\delta^4\right).
\end{aligned}$$

In this way, the approximation of the large scale and cross terms of the so-called Taylor LES model reads as follows

$$\begin{aligned}
&\overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} + \overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T} + \overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T} \\
&\approx \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T + \frac{\delta^2}{24}\left(\Delta\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) - \overline{\boldsymbol{u}}\,\Delta\left(\overline{\boldsymbol{u}}\right)^T - \Delta\left(\overline{\boldsymbol{u}}\right)\overline{\boldsymbol{u}}^T\right) \\
&= \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T + \frac{\delta^2}{12}\nabla\overline{\boldsymbol{u}}\,\nabla\overline{\boldsymbol{u}}^T, \qquad (7.157)
\end{aligned}$$

where

$$\Delta\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) - \overline{\boldsymbol{u}}\,\Delta\left(\overline{\boldsymbol{u}}\right)^T - \Delta\left(\overline{\boldsymbol{u}}\right)\overline{\boldsymbol{u}}^T = 2\nabla\overline{\boldsymbol{u}}\,\nabla\overline{\boldsymbol{u}}^T \qquad (7.158)$$

was used. Equality (7.158) can be checked by a direct calculation, e.g., considering just an entry of the tensor and a second order derivative gives with the product rule

$$\begin{aligned}
\partial_{xx}\left(\overline{u}_i\overline{u}_j\right) &= \partial_x\left(\partial_x\overline{u}_i\overline{u}_j + \overline{u}_i\partial_x\overline{u}_j\right) \\
&= \partial_{xx}\overline{u}_i\overline{u}_j + 2\partial_x\overline{u}_i\partial_x\overline{u}_j + \overline{u}_i\partial_{xx}\overline{u}_j.
\end{aligned}$$

□

Remark 7.135. Step 3 with rational approximation. Based on the observation that the Fourier transform of the Gaussian filter is approximated very badly with the Taylor polynomial approximation for large wave numbers, it was proposed in Galdi and Layton (2000) to use a second order rational approximation of the exponential of the form

$$e^{ax} = \frac{1}{1+ax} + \mathcal{O}\left(a^2x^2\right).$$

Applying this subdiagonal Padé approximation to $\mathcal{F}\left(g_{\text{Gauss}}\right)$ gives

$$\mathcal{F}\left(g_{\text{Gauss}}\right)(\delta, \boldsymbol{y}) = \frac{1}{1 + \dfrac{\|\boldsymbol{y}\|_2^2}{24}\delta^2} + \mathcal{O}\left(\delta^4\right) \tag{7.159}$$

and transforming this formula to $1/\mathcal{F}\left(g_{\text{Gauss}}\right)$ yields

$$\frac{1}{\mathcal{F}\left(g_{\text{Gauss}}\right)}(\delta, \boldsymbol{y}) = 1 + \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 + \mathcal{O}^{\text{formal}}\left(\delta^4\right). \tag{7.160}$$

The last term in (7.160) is actually $\mathcal{O}\left(\delta^4\right)/\mathcal{F}\left(g_{\text{Gauss}}\right)$ such that it is only formally of fourth order. The rational approximations of $\mathcal{F}\left(g_{\text{Gauss}}\right)$ and $1/\mathcal{F}\left(g_{\text{Gauss}}\right)$ are obtained by neglecting all (formal) fourth order terms in (7.159) and (7.160). The behaviour of $\mathcal{F}\left(g_{\text{Gauss}}\right)$ for high wave numbers is much better approximated than with the Taylor polynomial, see Figure 7.4 for a one-dimensional sketch. The approximation of $1/\mathcal{F}\left(g_{\text{Gauss}}\right)$ is the same as in the Taylor polynomial case. □

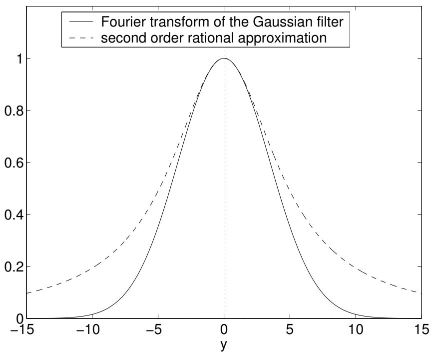

Fig. 7.4 $\mathcal{F}\left(g_{\text{Gauss}}\right)$ with its second order rational approximations, $\delta = 1$

Remark 7.136. Steps 4 and 5 with rational approximation. The derivation of the model continues now in the same way as in the Taylor polynomial case. Inserting (7.159) and (7.160) into (7.148) and (7.152), simplifying the arising terms using properties of the Fourier transform like (A.8), (A.9), (A.11), neglecting all terms which are formally of fourth order with respect to δ and applying the inverse Fourier transform gives

$$
\begin{aligned}
\overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} &= \left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) + \mathcal{O}^{\text{formal}}\left(\delta^4\right), \\
\overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T} &= -\frac{\delta^2}{24}\left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \left(\overline{\boldsymbol{u}}\,\Delta\left(\overline{\boldsymbol{u}}\right)^T\right) + \mathcal{O}^{\text{formal}}\left(\delta^4\right), \\
\overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T} &= -\frac{\delta^2}{24}\left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \left(\Delta\left(\overline{\boldsymbol{u}}\right)\overline{\boldsymbol{u}}^T\right) + \mathcal{O}^{\text{formal}}\left(\delta^4\right).
\end{aligned}
$$

□

Remark 7.137. The (second order) rational LES model. The approximation of the large scale and the cross terms for the so-called (second order) rational LES model has the form, using (7.158) in the derivation,

$$
\begin{aligned}
&\overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} + \overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T} + \overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T} \\
&\approx \left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T - \frac{\delta^2}{24}\left(\overline{\boldsymbol{u}}\,\Delta\left(\overline{\boldsymbol{u}}\right)^T + \Delta\left(\overline{\boldsymbol{u}}\right)\overline{\boldsymbol{u}}^T\right)\right] \\
&= \left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T - \frac{\delta^2}{24}\Delta\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) + \frac{\delta^2}{12}\nabla\overline{\boldsymbol{u}}\,\nabla\overline{\boldsymbol{u}}^T\right] \\
&= \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T + \frac{\delta^2}{12}\left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \nabla\overline{\boldsymbol{u}}\,\nabla\overline{\boldsymbol{u}}^T. \qquad (7.161)
\end{aligned}
$$

In (Galdi and Layton, 2000, Formula (2.10)), there is a misprint (minus sign instead of plus sign).

The operator $\left(I - \frac{\delta^2}{24}\Delta\right)^{-1}$ describes an elliptic, second order problem which has to be solved

$$
-\frac{\delta^2}{24}\Delta\overline{\mathbb{U}} + \overline{\mathbb{U}} = \nabla\overline{\boldsymbol{u}}\,\nabla\overline{\boldsymbol{u}}^T. \qquad (7.162)
$$

This problem is a Helmholtz equation. It is called differential filter in turbulence modeling. In connection with the rational LES model, it is usually called auxiliary problem. reference to analysis of this filter in ADM and Leray model

Some properties of the differential filter will be discussed in Section 7.5 and of the Galerkin finite element discretization of the differential filter in Section 7.6, starting from Remark 7.174. □

Remark 7.138. The differential filter in a bounded domain. If Ω is a bounded domain, which is usually the case in computations, the differential filter has to be equipped with boundary conditions on $\partial\Omega$. In Galdi and Layton (2000) it is proposed to use homogeneous Neumann boundary conditions. The only exception is the case that periodic boundary conditions are prescribed for the flow problem at some part of the boundary. Then, the differential filter is equipped also with periodic boundary conditions at those parts of the boundary.

This kind of boundary conditions were used for the simulations, e.g., in Iliescu et al. (2003); John et al. (2010). These simulations showed that the use of homogeneous Neumann boundary conditions gives generally reasonable results. □

Remark 7.139. The differential filter is an approximation of the convolution. A direct calculation, using the rational approximation (7.159) of $\mathcal{F}(g_{\text{Gauss}})$ and the property (A.11) of the Fourier transform, gives

$$\mathcal{F}(g_{\text{Gauss}} * \boldsymbol{u}) = \mathcal{F}(g_{\text{Gauss}})\,\mathcal{F}(\boldsymbol{u}) \approx \frac{1}{1+\frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2}\mathcal{F}(\boldsymbol{u}) = \mathcal{F}\left(\left(I - \frac{\delta^2}{24}\Delta\right)^{-1}\boldsymbol{u}\right),$$

from what it follows that

$$g_{\text{Gauss}} * \boldsymbol{u} \approx \left(I - \frac{\delta^2}{24}\Delta\right)^{-1}\boldsymbol{u}.$$

Thus, the differential filter is an approximation of the convolution operator with the Gaussian filter.

This link suggests that the rational LES model can be defined with a convolution instead of the auxiliary problem

$$\overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} + \overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T} + \overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T} \approx \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T + \frac{\delta^2}{12}g_{\text{Gauss}} * \left(\nabla\overline{\boldsymbol{u}}\nabla\overline{\boldsymbol{u}}^T\right).$$

This model is called rational LES model with convolution. However, it is practically not used, partly because of the overhead for approximating the convolution operator, see (John, 2004, Section 7.8). □

Remark 7.140. A fourth order rational approximation. The second order polynomial and rational approximations model the sgs term $\overline{\boldsymbol{u}'\boldsymbol{u}'^T}$ by $\mathbf{0}$, see Remark 7.142. Using a fourth order rational approximation, one gets a non-trivial model for the sgs term, see (John, 2004, Section 4.2.3). However, the arising approximation of the sgs tensor involves the solution of a fourth order partial differential equation and it involves some higher order terms which are hard to approximate. Altogether, the fourth order rational LES model is not used. Berselli, Iliescu □

7.4.2 Models for the Subgrid Scale Term

Remark 7.141. On the sgs term. The sgs term $\overline{\boldsymbol{u}'\boldsymbol{u}'^T}$ is considered to possess a great influence on the formation of turbulence. Thus, its modelling is of great importance. □

Remark 7.142. The second order Fourier transform approach. If the sgs term is modelled with the second order approaches which were used for the large scale term and the cross terms, Section 7.4.1, one obtains with the second order polynomial approximation of the Fourier transform of the Gaussian filter

$$\overline{\boldsymbol{u}'\boldsymbol{u}'^T} = \frac{\delta^4}{576}\left(\Delta\overline{\boldsymbol{u}}\,\Delta\overline{\boldsymbol{u}}^T\right) + \mathcal{O}^{\text{formal}}\left(\delta^6\right)$$

and with the second order rational approximation

$$\overline{\boldsymbol{u}'\boldsymbol{u}'^T} = \frac{\delta^4}{576}\left(I - \frac{\delta^2}{24}\Delta\right)^{-1}\left(\Delta\overline{\boldsymbol{u}}\,\Delta\overline{\boldsymbol{u}}^T\right) + \mathcal{O}^{\text{formal}}\left(\delta^6\right).$$

Both approximations are formally of fourth order in δ and therefore they will be neglected in these approaches. That means, one obtains the approximation

$$\overline{\boldsymbol{u}'\boldsymbol{u}'^T} \approx \boldsymbol{0},$$

which proves to be not robust for long time simulations, see (John, 2004, Section 10.3.3) for a numerical example. □

Remark 7.143. The Smagorinsky model. A popular approach consists in using the Smagorinsky model (7.65) for approximating the sgs term. However, with the Fourier space approach, Remark 7.142, one obtains that the sgs term is formally of fourth order in δ whereas the Smagorinsky model is formally of second order in δ. There is a contradiction, at least formally. It should be pointed out again that in the formal higher order terms there is some dependence on δ which is not known.

In the rational LES models, the Smagorinsky model is used only as a model of the sgs term. Thus, its influence should be kept smaller than in computations with the pure Smagorinsky model, which can be achieved by choosing a smaller constant C_S for the rational LES model than for the pure Smagorinsky model. □

Remark 7.144. Models based on physical arguments. In Iliescu and Layton (1998), several eddy viscosity models for modelling the sgs term based on physical arguments were derived. One of these models has the form

$$\nu_{\text{T}} = C_S\delta \,\|\,\overline{\boldsymbol{u}} - g_{\text{Gauss}} * \overline{\boldsymbol{u}}\,\|_2\,. \tag{7.163}$$

Using (A.8) gives

$$\begin{aligned}\overline{\boldsymbol{u}} - g_{\text{Gauss}} * \overline{\boldsymbol{u}} &= \mathcal{F}^{-1}\left(\mathcal{F}\left(\overline{\boldsymbol{u}} - g_{\text{Gauss}} * \overline{\boldsymbol{u}}\right)\right) \\ &= \mathcal{F}^{-1}\left(\mathcal{F}\left(\overline{\boldsymbol{u}}\right) - \mathcal{F}\left(g_{\text{Gauss}}\right)\mathcal{F}\left(\overline{\boldsymbol{u}}\right)\right) \\ &= \mathcal{F}^{-1}\left(\left(1 - \mathcal{F}\left(g_{\text{Gauss}}\right)\right)\mathcal{F}\left(\overline{\boldsymbol{u}}\right)\right).\end{aligned}$$

For the polynomial approximation (7.153) it is obvious that the term in parentheses is $\mathcal{O}\left(\delta^2\right)$. Using the rational approximation (7.159) yields

$$1 - \mathcal{F}\left(g_{\text{Gauss}}\right) = \frac{1 + \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2 - 1}{1 + \frac{\|\boldsymbol{y}\|_2^2}{24}\delta^2} + \mathcal{O}\left(\delta^4\right) = \mathcal{O}\left(\delta^2\right).$$

Altogether, model (7.163) is of third order in δ.

In numerical simulation, model (7.163) is usually applied in such a way that the convolution operator is approximated by a second order partial differential operator, see Remark 7.139, i.e.,

$$\nu_{\mathrm{T}} = C_S\delta \left\| \overline{\boldsymbol{u}} - \left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \overline{\boldsymbol{u}} \right\|_2 . \tag{7.164}$$

□

7.4.3 The Resulting Models

Remark 7.145. Models based on approximations in wave number space. The models derived in Sections 7.4.1 and 7.4.2 can be written concisely in the following strong form

$$\begin{aligned}\partial_t \boldsymbol{w} - \nabla\cdot\left((2\nu + \nu_T)\,\mathbb{D}\left(\boldsymbol{w}\right)\right) + \left(\boldsymbol{w}\cdot\nabla\right)\boldsymbol{w} & \\ +\nabla r + \nabla\cdot\frac{\delta^2}{12}\left(A_{\text{wave}}\left(\nabla\boldsymbol{w}\nabla\boldsymbol{w}^T\right)\right) &= \overline{\boldsymbol{f}} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{w} &= 0 \quad \text{in } [0,T]\times\Omega, \\ \boldsymbol{w}\left(0,\cdot\right) &= \boldsymbol{w}_0 \text{ in } \Omega,\end{aligned} \tag{7.165}$$

where $(\boldsymbol{w}, r)$ should be an approximation to $(\overline{\boldsymbol{u}}, \overline{p})$. The operator A depends on the approximation of the Fourier transform of the Gaussian filter:

- $A_{\text{wave}} = I$ for the Taylor LES model (7.157),
- $A_{\text{wave}} = \left(I - \delta^2/\left(24\right)\Delta\right)^{-1}$ for the second order rational LES model with auxiliary problem (7.161),
- $A_{\text{wave}} = g_{\text{Gauss}}*$ for the second order rational LES model with convolution (7.163).

Possible choices for the turbulent viscosity ν_T are the Smagorinsky model (7.64) and the Iliescu–Layton model (7.164).

In a bounded domain, usually the same boundary conditions are applied for $\boldsymbol{w}$ as they are prescribed for $\boldsymbol{u}$. The differential filter is equipped with homogeneous Neumann boundary conditions, see Remark 7.138. □

Remark 7.146. Analytical results.

- For the Taylor LES model with Smagorinsky sgs term, the existence, uniqueness, and stability of a weak solution for all times was proved in Coletti (1997) genauer under the assumption that $\nu_{\mathrm{T}} \geq \delta^2/6$. This condition means that the Smagorinsky term dominates the Taylor LES model. This relation is not correct since the Taylor LES model is a model for the large scale and cross term, which are $\mathcal{O}^{\mathrm{formal}}\left(\delta^2\right)$ whereas the sgs term is $\mathcal{O}^{\mathrm{formal}}\left(\delta^4\right)$. Under a similar condition, a finite element error analysis was performed in Iliescu et al. (2002).
- The existence and uniqueness of a solution of the rational LES model was studied in Berselli et al. (2002). This model was considered with auxiliary problem but without subgrid scale term, $\nu_T = 0$ in (7.165). In addition, the case of a space-periodic setting was investigated, i.e., $\Omega = (0, L)^3$ and periodic boundary conditions on $\partial\Omega$. Periodic boundary conditions are applied in the auxiliary problem as well. The existence and uniqueness of a solution in an appropriate function space for small time intervals $T = \mathcal{O}\left(\delta^4\right)$ could be proved. It was already mentioned in Remark 7.142 that the rational LES model without model for the sgs term is not robust in long time simulations. The proof uses the Galerkin method in a similar way as described in Section 6.1. A finite element error analysis of the rational LES model is not available.

□

Remark 7.147. Numerical studies. todo □

7.5 Large Eddy Simulation – Approximate Deconvolution Models (ADMs)

Remark 7.148. Basic idea. Approximate deconvolution models (ADM) were introduced in Stolz and Adams (1999); Stolz et al. (2001). Consider the space-averaged Navier–Stokes equations

$$\begin{aligned} \partial_t \overline{\boldsymbol{u}} - 2\nu\nabla\cdot\mathbb{D}\left(\overline{\boldsymbol{u}}\right) + \nabla\cdot\left(\overline{\boldsymbol{u}\boldsymbol{u}^T}\right) + \nabla\overline{p} &= \overline{\boldsymbol{f}} \ \text{ in } (0,T]\times\mathbb{R}^d, \\ \nabla\cdot\overline{\boldsymbol{u}} &= 0 \ \text{ in } [0,T]\times\mathbb{R}^d, \end{aligned} \tag{7.166}$$

see (7.26). As already discussed in Remark 7.32, the momentum equation in (7.166) is not yet an equation for $(\overline{\boldsymbol{u}}, \overline{p})$ since the nonlinear convective term still depends on $\boldsymbol{u}$.

Often, the filter is defined by a convolution with an appropriate filter function. For this reason, the filter operator is denoted by G_{conv}, i.e., $\overline{\boldsymbol{u}} =$