7.3 Large Eddy Simulation – The Smagorinsky Model

Remark 7.63. Motivation and contents of this section. The Smagorinsky model, proposed in Smagorinsky (1963), is one of the most popular turbulence models. From the mathematical point of view, the continuous Smagorinsky model is well understood. Existence and uniqueness of a solution in two and three dimensions can be proved, see Section 7.3.2. These results were obtained in Ladyženskaja (1967) and in the mathematical literature, the continuous model is often called Ladyzhenskaya model. In addition, a finite element error analysis can be performed, see Section 7.3.3. From the computational point of view, the Smagorinsky model is easy to implement, it has low computational cost, and generally it is quite robust in simulations. The last point means that the simulations do not blow up. However, as it shall be discussed in some detail in Section 7.3.4, the Smagorinsky model possesses a number of drawbacks. In particular, the computational results depend on a coefficient, the so-called Smagorinsky coefficient, whose good choice is in general situations an open problem.

In addition, it is part of some other turbulence models, like the projection-based variational multiscale model presented in Section **??**. □

7.3.1 The Model of the SGS Stress Tensor – Eddy Viscosity Models

Remark 7.64. Some properties of the sgs stress tensor. The space-averaged Navier–Stokes equations are not yet closed and the divergence of the subgrid-scale stress tensor

$$\nabla \cdot \mathbb{T} = \nabla \cdot \left(\overline{\boldsymbol{u}\boldsymbol{u}^T} \right) - \nabla \cdot \left(\overline{\boldsymbol{u}}\, \overline{\boldsymbol{u}}^T \right) \tag{7.56}$$

needs to be modelled, see Remark 7.32. Models should posses some important properties of $\mathbb{T}$, as the followings.

- The subgrid stress tensor is symmetric, since $\boldsymbol{u}\boldsymbol{u}^T$ and $\overline{\boldsymbol{u}}\, \overline{\boldsymbol{u}}^T$ are symmetric.
- The subgrid stress tensor is Galilean invariant. Consider two coordinate systems
$$(t, \boldsymbol{x}) \quad \text{and} \quad (\hat{t}, \hat{\boldsymbol{x}}) = (t, \boldsymbol{x} - \boldsymbol{U}t), \tag{7.57}$$
where $\boldsymbol{U}$ is a constant velocity. These systems describe two so-called inertial frames of references. The second frame is moving with constant velocity with respect to the first frame. Galilean invariance means that one gets the same expressions in both frames.
Since $\boldsymbol{u}$ is the velocity (derivative with respect to time) with respect to $\boldsymbol{x}$, it follows from (7.57) by differentiating $\hat{\boldsymbol{x}}$ with respect to time that

$$\hat{\boldsymbol{u}}(\hat{t},\hat{\boldsymbol{x}}) = \boldsymbol{u}(t,\boldsymbol{x}) - \boldsymbol{U}. \tag{7.58}$$

In the special case $\boldsymbol{u} = \boldsymbol{U}$, the coordinate system $(\hat{t},\hat{\boldsymbol{x}})$ moves with the flow. This case is the so-called Lagrangian point of view of a flow field, i.e., one observes the motion of the fluid such that one follows an individual 'fluid particle' as it moves through time and space, see also Remark 7.187. Using the chain rule, (7.58), and (7.57), one obtains for the partial derivative of the velocity

$$\frac{\partial \hat{\boldsymbol{u}}_j}{\partial \hat{x}_i} = \frac{\partial \boldsymbol{u}_j}{\partial t}\frac{\partial t}{\partial \hat{x}_i} + \sum_{k=1}^{d} \frac{\partial \boldsymbol{u}_j}{\partial x_k}\frac{\partial x_k}{\partial \hat{x}_i} = \frac{\partial \boldsymbol{u}_j}{\partial x_i}, \quad i,j = 1,\ldots,d.$$

Since the sgs stress tensor is composed only of partial derivative of the velocity, it is Galilean invariant.

With similar calculations, it can be shown that the Navier–Stokes equations (7.1) are Galilean invariant (with $\hat{\boldsymbol{u}}$ instead of $\boldsymbol{u}$).

Further requirements for satisfactory turbulence models are discussed in (Berselli et al., 2006, Section 6.2). □

Remark 7.65. Modelling of the deviatoric part. For incompressible fluids, the pressure is the trace of the stress tensor multiplied with $-1/3$, see (1.18) and the velocity part of the stress tensor is trace-free. In a similar way, one considers the deviatoric or trace-free sgs stress tensor

$$\mathbb{T} - \frac{\mathbb{T}_{11} + \mathbb{T}_{22} + \mathbb{T}_{33}}{3} \tag{7.59}$$

and the second term is usually added to the filtered pressure $\overline{p}$

$$\overline{p} + \frac{\mathbb{T}_{11} + \mathbb{T}_{22} + \mathbb{T}_{33}}{3}.$$

Hence, it remains to model the deviatoric part (7.59) of the sgs stress tensor.

For simplicity of notation, the modified pressure will be also denoted by $\overline{p}$. □

Remark 7.66. The Boussinesq hypothesis. The starting point of the derivation of the Smagorinsky model is the Boussinesq hypothesis stated in Boussinesq (1877): 'Turbulent fluctuations are dissipative in the mean'. This hypothesis is based upon the resemblance of the elastic collisions of molecules with the interaction of small scales in flows. Expressing this hypothesis for the model of the sgs stress tensor, it has the form

$$\mathbb{T} - \frac{\mathbb{T}_{11} + \mathbb{T}_{22} + \mathbb{T}_{33}}{3} = -\nu_{\mathrm{T}}\mathbb{D}\left(\overline{\boldsymbol{u}}\right), \tag{7.60}$$

where $\nu_{\mathrm{T}} \geq 0$ is called turbulent viscosity or eddy viscosity. From (7.60), one can see that this model introduces a viscous term. Usually, the turbulent viscosity depends on the solution, hence the new viscous term is nonlinear. The term on the right-hand side of (7.60) is often called eddy viscosity model. □

Remark 7.67. Modeling the turbulent viscosity ν_{T}. It is known for the dissipation of turbulent energy that $\varepsilon \sim U^3/L$, see (7.2). For the Kolomogorov scales, one obtains from (7.5) with a direct calculation that $\varepsilon = u_\lambda^3/\lambda$. It is now assumed that not only for the largest and smallest scales a relation of this type holds but for every length scale and the corresponding velocity scale. In particular, it is assumed that

$$\varepsilon \sim \frac{U_{\mathrm{int}}^3}{L_{\mathrm{int}}}, \quad \varepsilon \sim \frac{U_\delta^3}{\delta}, \tag{7.61}$$

where δ is the filter width. The scale L_{int} is the so-called integral length scale, which characterizes the distance over which the small scales are correlated, and U_{int} is the corresponding velocity scale. One obtains directly from (7.61) the relation

$$U_\delta \sim U_{\mathrm{int}} \left(\frac{\delta}{L_{\mathrm{int}}}\right)^{1/3}. \tag{7.62}$$

The goal of the turbulence model is to capture scales of size δ. The Reynolds number of these scales should be 1, i.e., it should hold

$$\mathrm{Re}(\delta) = \frac{\delta U_\delta}{\nu_{\mathrm{T}}} = 1.$$

Hence, one gets with (7.62)

$$\nu_{\mathrm{T}} = U_\delta \delta \sim U_{\mathrm{int}} L_{\mathrm{int}}^{-1/3} \delta^{4/3}. \tag{7.63}$$

The next assumption correlates the integral velocity and length scales. It is assumed that the integral velocity scale depends linearly on the norm of the deformation tensor of the filtered velocity

$$U_{\mathrm{int}} \sim L_{\mathrm{int}} \left\|\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right\|_{\mathrm{F}}.$$

Inserting this assumption into (7.63) and replacing similarity by an unknown coefficient yields

$$\nu_{\mathrm{T}} = C L_{\mathrm{int}}^{2/3} \delta^{4/3} \left\|\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right\|_{\mathrm{F}}.$$

The integral length scale L_{int} is usually hard to determine. Thus, one uses the approximation $L_{\mathrm{int}} \sim \delta$ to get, with a modified constant,

$$\nu_{\mathrm{T}} = C_S \delta^2 \left\|\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right\|_{\mathrm{F}}. \tag{7.64}$$

□

Remark 7.68. The Smagorinsky model. With (7.64), the Smagorinsky model introduces the following additional term to the left-hand side of the momentum equation of the Navier–Stokes equations

$$-\nabla\cdot\left(C_S\delta^2\left\|\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right\|_{\mathrm{F}}\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right), \tag{7.65}$$

where $C_S \geq 0$ is the dimensionless Smagorinsky coefficient and δ is related to the (local) mesh width. In particular in engineering literature, it is common to write the Smagorinsky model in the form

$$-\nabla\cdot\left(2\left(C_S^*\delta\right)^2\left(2\mathbb{D}\left(\overline{\boldsymbol{u}}\right):\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right)^{1/2}\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right).$$

Since the Smagorinsky model contains only the deformation tensor of the velocity, it is a symmetric model. It is also Galilean invariant, since only first order spatial derivatives of the velocity appear, compare Remark 7.64. □

Remark 7.69. The Smagorinsky filter. The use of the Smagorinsky model does not require the specification of a concrete filter. The filter does not appear explicitly, only the filter width. However, it can be shown for homogeneous isotropic turbulence that there is a uniquely implied filter, see (Pope, 2000, Section 13.4.3). □

7.3.2 Existence and Uniqueness of a Solution of the Continuous Smagorinsky Model

Remark 7.70. To the proof of the existence and uniqueness of a solution. The proof of the existence and uniqueness of a weak solution of the continuous Smagorinsky model was given in Ladyženskaja (1967), see also (Ladyzhenskaya, 1969, Supplement 1) for an overview. The presentation in this section follows also (John, 2004, Section 6.1).

The existence of a weak solution is proved with the Galerkin method, which was used in Section 6.1 to show the existence of a weak solution of the Navier–Stokes equations. Because of the nonlinear viscous term of the Smagorinsky model, one needs different spaces than for the Navier–Stokes equations to ensure that all terms are well defined. In fact, the velocity space needed for the Smagorinsky model is of somewhat higher regularity than the space (6.5) used for the Navier–Stokes equations. This additional regularity simplifies the technical tools needed in the proof, in particular the convergence of the nonlinear convective term, compare Lemma 7.83 with Lemma 6.16 and 6.17. The main issue consists in proving the convergence of the nonlinear viscous term needs of the Smagorinsky model, see the considerations from Lemma 7.84 to Lemma 7.90. It can be shown that the Smagorinsky model defines a monotone operator and the theory of such operators can be applied.

Even more important, the higher regularity allows to prove the uniqueness of the weak solution.

Altogether, the main questions concerning the analysis of the Smagorinsky model are answered, in contrast to the Navier–Stokes equations. This situation indicates that the Smagorinsky model is in some sense an easier problem than the Navier–Stokes equations, despite of the additional nonlinear term. In this respect, it behaves exactly the way a turbulence model should do, see Remark 7.22. □

Remark 7.71. Notation. The notation in this section will indicate that the solution of the Smagorinsky model is generally not the solution of the Navier–Stokes equations. To this end, the velocity field will be denoted by $\boldsymbol{w}$ and the pressure by r. □

Remark 7.72. The strong form of the Smagorinsky model. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded domain with Lipschitz boundary Γ. For the sake of simplifying the analysis a little bit, the gradient form of the Smagorinsky model is considered. This model, equipped with homogeneous Dirichlet boundary conditions, has the form

$$\begin{aligned} \partial_t \boldsymbol{w} - \nabla\cdot\left(\left(\nu + \nu_{\mathrm{T}} \left\|\nabla \boldsymbol{w}\right\|_{\mathrm{F}}\right)\nabla \boldsymbol{w}\right) + \left(\boldsymbol{w}\cdot\nabla\right)\boldsymbol{w} + \nabla r &= \boldsymbol{f} \quad \text{in } (0,T]\times\Omega,\\ \nabla\cdot\boldsymbol{w} &= 0 \quad \text{in } [0,T]\times\Omega,\\ \boldsymbol{w} &= \boldsymbol{0} \quad \text{in } [0,T]\times\Gamma,\\ \boldsymbol{w}(0,\cdot) &= \boldsymbol{w}_0 \text{ in } \Omega,\\ \int_\Omega r \, d\boldsymbol{x} &= 0 \quad \text{in } (0,T], \end{aligned} \tag{7.66}$$

with $\nu_{\mathrm{T}} \in \mathbb{R}$, $\nu_{\mathrm{T}} > 0$, $\boldsymbol{f} \in L^2\left(0,T;L^2(\Omega)\right)$, given initial condition $\boldsymbol{w}_0$, and finite final time $T < \infty$. Because a bounded domain with Lipschitz boundary is considered, all Sobolev imbeddings are compact. check in Adams (1975) □

Remark 7.73. Function spaces for the weak formulation. For achieving that all terms in the weak formulation are well defined, the Banach space

$$W^{1,3}_{0,\mathrm{div}}(\Omega) = \{\boldsymbol{v} \in W^{1,3}(\Omega) \;:\; \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\;,\; \nabla\cdot\boldsymbol{v} = 0 \text{ in } \Omega\} \tag{7.67}$$

is used. This space is equipped with the same norm as $W^{1,3}_0(\Omega)$, see Remark A.34. The appropriate velocity space is

$$V = L^3\left(0,T;W^{1,3}_{0,\mathrm{div}}(\Omega)\right) \cap H^1\left(0,T;L^2(\Omega)\right) \tag{7.68}$$

equipped with the norm

$$\|\boldsymbol{v}\|_V = \|\nabla\boldsymbol{v}\|_{L^3(0,T;L^3(\Omega))} + \|\partial_t\boldsymbol{v}\|_{L^2(0,T;L^2(\Omega))}\,.$$

Comparing with the spaces used for the Navier–Stokes equations, see (6.5), one can observe the higher regularity of the spaces for the Smagorinsky model. □

Remark 7.74. Weak formulation. The weak formulation of the Smagorinsky model is obtained in the usual way. The strong formulation (7.66) is multiplied with appropriate test functions, the resulting equations are integrated in Ω, and integration by parts is applied to transfer derivatives from the solution to the test functions. As for the Navier–Stokes equations, the test functions are divergence-free such that the pressure term vanishes.

The weak formulation of the Smagorinsky model reads as follows: Find $\boldsymbol{w} \in V$ such that $\boldsymbol{w}(0, \boldsymbol{x}) = \boldsymbol{w}_0 \in W^{1,3}_{0,\mathrm{div}}(\Omega)$ and for all $\boldsymbol{v} \in V$

$$\int_0^T (\partial_t \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w}, \boldsymbol{v}) + ((\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|_{\mathrm{F}}) \nabla \boldsymbol{w}, \nabla \boldsymbol{v}) \ dt = \int_0^T (\boldsymbol{f}, \boldsymbol{v}) \ dt. \tag{7.69}$$

The function $\boldsymbol{w} \in V$ is called weak solution of the Smagorinsky model. □

Remark 7.75. Preparations for the convergence proof. The existence proof is based on three a priori error estimates which will be given in Lemma 7.76 – 7.78. constants C integrals into norms in time-space ist sonst das $\boldsymbol{x}$ in der Norm? check norm symbols for vectors and tensors □

Lemma 7.76. Stability estimate for $\|\boldsymbol{w}\|_{L^\infty(0,T;L^2(\Omega))}$. *Let $(\boldsymbol{w}, r) \in V \times Q$, with $Q = L^2_0(\Omega)$, $\boldsymbol{f} \in L^2\left(0, T; L^2(\Omega)\right)$, and $\boldsymbol{w}_0 \in W^{1,3}_{0,\mathrm{div}}(\Omega)$. Then, the solution of* (7.66) *satisfies*

$$\|\boldsymbol{w}(T)\|_{L^2(\Omega)} \le \|\boldsymbol{w}_0\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{L^1(0,T;L^2(\Omega))} \quad \forall\, T > 0. \tag{7.70}$$

Proof. Since $(\boldsymbol{w}, r)$ is a solution of (7.66), one obtains by testing the momentum equation of (7.66) with $\boldsymbol{w}$ and integration by parts

$$(\partial_t \boldsymbol{w}, \boldsymbol{w}) + \left((\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|_{\mathrm{F}}) \nabla \boldsymbol{w}, \nabla \boldsymbol{w}\right) + n(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{w}) - (r, \nabla \cdot \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}).$$

Because $\nabla \cdot \boldsymbol{w} = 0$, the convective term and the term with the pressure r vanish, such that

$$\frac{1}{2}\frac{d}{dt}(\boldsymbol{w}, \boldsymbol{w}) + \left((\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|_{\mathrm{F}}) \nabla \boldsymbol{w}, \nabla \boldsymbol{w}\right) = (\boldsymbol{f}, \boldsymbol{w}). \tag{7.71}$$

Since $0 < \nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|_{\mathrm{F}}$, the second term on the right-hand side is non-negative. Neglecting this term, applying the chain rule for the first term of the left-hand side and the Cauchy–Schwarz inequality (A.16) for the right-hand side gives

$$\|\boldsymbol{w}\|_{L^2(\Omega)} \frac{d}{dt} \|\boldsymbol{w}\|_{L^2(\Omega)} \le \|\boldsymbol{f}\|_{L^2(\Omega)} \|\boldsymbol{w}\|_{L^2(\Omega)}.$$

Cancellation of $\|\boldsymbol{w}\|_{L^2(\Omega)}$, integration on $(0, T)$, and noting that for a finite time interval $\boldsymbol{f} \in L^2\left(0, T; L^2(\Omega)\right)$ implies $\boldsymbol{f} \in L^1\left(0, T; L^2(\Omega)\right)$, completes the proof. ■

Lemma 7.77. Stability of $\|\nabla \boldsymbol{w}\|_{L^3(0,T;L^3(\Omega))}$. *With the same assumptions as in Lemma 7.76, it holds for all $T > 0$*

$$\begin{aligned} &\|\boldsymbol{w}(T)\|^2_{L^2(\Omega)} + 2\int_0^T ((\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|_{\mathrm{F}}) \nabla \boldsymbol{w}, \nabla \boldsymbol{w}) \ dt \\ &\le 2\|\boldsymbol{w}_0\|^2_{L^2(\Omega)} + 3\|\boldsymbol{f}\|^2_{L^1(0,T;L^2(\Omega))} = C_1(T). \end{aligned} \tag{7.72}$$

In particular, it is $\|\nabla \boldsymbol{w}\|_{L^3(0,T;L^3(\Omega))} \le \tilde{C}_1(T)$.

Proof. Starting with (7.71), one gets by integration on $(0,T)$

$$\|\boldsymbol{w}(T)\|^2_{L^2(\Omega)} + 2\int_0^T \left(\left(\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|_{\mathrm{F}}\right) \nabla \boldsymbol{w}, \nabla \boldsymbol{w}\right)\, dt \le \|\boldsymbol{w}_0\|^2_{L^2(\Omega)} + 2\int_0^T (\boldsymbol{f}, \boldsymbol{w})\, dt, \tag{7.73}$$

which is an energy inequality like (6.23), here even an energy equality. Applying the Cauchy–Schwarz inequality (A.16) and inequality (7.70), which is valid for all times t, it follows for the second term on the right-hand side that

$$\begin{aligned}
&\int_0^T (\boldsymbol{f}, \boldsymbol{w})\, dt \\
&\le \int_0^T \|\boldsymbol{w}(t)\|_{L^2(\Omega)} \|\boldsymbol{f}(t)\|_{L^2(\Omega)}\, dt \\
&\le \|\boldsymbol{w}_0\|_{L^2(\Omega)} \int_0^T \|\boldsymbol{f}(t)\|_{L^2(\Omega)}\, dt + \int_0^T \|\boldsymbol{f}(t)\|_{L^2(\Omega)} \left(\int_0^t \|\boldsymbol{f}(t')\|_{L^2(\Omega)}\, dt'\right) dt
\end{aligned}$$

Using Young's inequality (A.4) and the non-negativeness of $\|\boldsymbol{f}(t')\|_{L^2(\Omega)}$ yields

$$\begin{aligned}
\int_0^T (\boldsymbol{f}(t,\boldsymbol{x}), \boldsymbol{w}(t,\boldsymbol{x}))\, dt &\le \frac{\|\boldsymbol{w}_0\|^2_{L^2(\Omega)}}{2} + \frac{1}{2}\left(\int_0^T \|\boldsymbol{f}(t)\|_{L^2(\Omega)}\, dt\right)^2 \\
&\quad + \int_0^T \|\boldsymbol{f}(t)\|_{L^2(\Omega)} \left(\int_0^T \|\boldsymbol{f}(t')\|_{L^2(\Omega)}\, dt'\right) dt \\
&= \frac{\|\boldsymbol{w}_0\|^2_{L^2(\Omega)}}{2} + \frac{3}{2}\|\boldsymbol{f}\|^2_{L^1(0,T;L^2)}.
\end{aligned}$$

Inserting this estimate into (7.73) gives (7.72).

It follows that

$$2\nu_{\mathrm{T}} \int_0^T \left(\|\nabla \boldsymbol{w}\|_{\mathrm{F}} \nabla \boldsymbol{w}, \nabla \boldsymbol{w}\right) dt = 2\nu_{\mathrm{T}} \int_0^T \|\nabla \boldsymbol{w}\|^3_{\mathrm{F}}\, d\boldsymbol{x} = 2\nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|^3_{L^3(0,t;L^3(\Omega))} \le C_1(T),$$

which proves the second statement of the lemma. ■

Lemma 7.78. Stability of $\|\boldsymbol{w}\|_{H^1(0,T;L^2(\Omega))}$. *If the assumptions of Lemma 7.76 are valid, then*

$$\|\nabla \boldsymbol{w}(T)\|^3_{L^3(\Omega)} + \frac{3}{2\nu_{\mathrm{T}}} \|\boldsymbol{w}\|^2_{H^1(0,T;L^2(\Omega))} \le C_2(T). \tag{7.74}$$

In particular, it follows that $\|\boldsymbol{w}\|_{H^1(0,T;L^2(\Omega))} \le \tilde{C}_2(T)$.

Proof. To prove (7.74), one starts by testing the momentum equation of (7.66) with $\partial_t \boldsymbol{w}$ and integrating in $(0,T)$. With the chain rule, one gets

$$\frac{1}{3}\int_\Omega \frac{d}{dt} \|\nabla \boldsymbol{w}\|^3_{\mathrm{F}}\, d\boldsymbol{x} = \left(\|\nabla \boldsymbol{w}\|_{\mathrm{F}} \nabla \boldsymbol{w}, \partial_t \nabla \boldsymbol{w}\right) = \left(\|\nabla \boldsymbol{w}\|_{\mathrm{F}} \nabla \boldsymbol{w}, \nabla \partial_t \boldsymbol{w}\right).$$

This relation, integration by parts, using that $\partial_t \boldsymbol{w}$ is weakly divergence-free, and applying formulas of type (6.13) yields

$$\int_0^T \|\partial_t \boldsymbol{w}\|_{L^2(\Omega)}^2 \ dt + \frac{\nu}{2}\left(\nabla \boldsymbol{w}(T), \nabla \boldsymbol{w}(T)\right) + \frac{\nu_{\mathrm{T}}}{3}\int_\Omega \|\nabla \boldsymbol{w}(T)\|_{\mathrm{F}}^3 \ d\boldsymbol{x} \tag{7.75}$$
$$= \frac{\nu}{2}\left(\nabla \boldsymbol{w}_0, \nabla \boldsymbol{w}_0\right) + \frac{\nu_{\mathrm{T}}}{3}\int_\Omega \|\nabla \boldsymbol{w}_0\|_{\mathrm{F}}^3 \ d\boldsymbol{x} - \int_0^T n\left(\boldsymbol{w}, \boldsymbol{w}, \partial_t \boldsymbol{w}\right) \ dt + \int_0^T \left(\boldsymbol{f}, \partial_t \boldsymbol{w}\right) \ dt.$$

First, the convective term will be estimated. Hölder's inequality (A.15) gives

$$\begin{aligned}\int_\Omega \left(\boldsymbol{w}^T \boldsymbol{w}\right)\left(\nabla \boldsymbol{w} : \nabla \boldsymbol{w}\right) \ d\boldsymbol{x} &= \int_\Omega \boldsymbol{w}^2 \left(\nabla \boldsymbol{w}\right)^2 \ d\boldsymbol{x} \\ &\le \left\|\boldsymbol{w}^2\right\|_{L^3(\Omega)} \left\|(\nabla \boldsymbol{w})^2\right\|_{L^{3/2}} \\ &= \|\boldsymbol{w}\|_{L^6(\Omega)}^2 \|\nabla \boldsymbol{w}\|_{L^3(\Omega)}^2 .\end{aligned}$$

With the Sobolev embedding $W^{1,3}(\Omega) \to L^6(\Omega)$, see (A.24), and Poincaré's inequality (A.17) it follows that

$$\int_\Omega \left(\boldsymbol{w}^T \boldsymbol{w}\right)\left(\nabla \boldsymbol{w} : \nabla \boldsymbol{w}\right) \ d\boldsymbol{x} \le C \|\nabla \boldsymbol{w}\|_{L^3(\Omega)}^4 . \tag{7.76}$$

Applying Young's inequality, one gets

$$\begin{aligned}&\left(w_1 \partial_x w_1 + w_2 \partial_y w_1 + w_3 \partial_z w_1\right)^2 \\ &= w_1^2 \left(\partial_x w_1\right)^2 + w_2^2 \left(\partial_y w_1\right)^2 + w_3^2 \left(\partial_z w_1\right)^2 + 2 w_1 w_2 \partial_x w_1 \partial_y w_1 + 2 w_1 w_3 \partial_x w_1 \partial_z w_1 \\ &\quad + 2 w_2 w_3 \partial_y w_1 \partial_z w_1 \\ &\le w_1^2 \left(\partial_x w_1\right)^2 + w_2^2 \left(\partial_y w_1\right)^2 + w_3^2 \left(\partial_z w_1\right)^2 + w_1^2 \left(\partial_y w_1\right)^2 + w_2^2 \left(\partial_x w_1\right)^2 + w_1^2 \left(\partial_z w_1\right)^2 \\ &\quad + w_3^2 \left(\partial_x w_1\right)^2 + w_2^2 \left(\partial_z w_1\right)^2 + w_3^2 \left(\partial_y w_1\right)^2 \\ &= \left(w_1^2 + w_2^2 + w_3^2\right)\left(\left(\partial_x w_1\right)^2 + \left(\partial_y w_1\right)^2 + \left(\partial_x w_1\right)^2\right).\end{aligned}$$

Using now once more Young's inequality and inserting estimate (7.76) leads to

$$\begin{aligned}\int_0^T n\left(\boldsymbol{w}, \boldsymbol{w}, \partial_t \boldsymbol{w}\right) \ dt &= \int_0^T \int_\Omega \left(\boldsymbol{w} \cdot \nabla\right) \boldsymbol{w} \cdot \partial_t \boldsymbol{w} \ d\boldsymbol{x} \ dt \\ &\le \int_0^T \int_\Omega \left(\frac{\left(\partial_t \boldsymbol{w}\right)^2}{4} + \left(\left(\boldsymbol{w} \cdot \nabla\right) \boldsymbol{w}\right)^2\right) \ d\boldsymbol{x} \ dt \\ &\le \int_0^T \int_\Omega \left(\frac{\partial_t \boldsymbol{w}^2}{4} + \left(\boldsymbol{w}^T \boldsymbol{w}\right)\left(\nabla \boldsymbol{w} : \nabla \boldsymbol{w}\right)\right) \ d\boldsymbol{x} \ dt \\ &\le \frac{1}{4} \|\partial_t \boldsymbol{w}\|_{L^2(0,T;L^2(\Omega))}^2 \ dt + C \int_0^T \|\nabla \boldsymbol{w}\|_{L^3(\Omega)}^4 \ dt.\end{aligned}$$

Young's inequality gives also

$$\begin{aligned}\int_0^T \left(\boldsymbol{f}, \partial_t \boldsymbol{w}\right) \ dt &\le \int_0^T \int_\Omega \left(\boldsymbol{f}^2 + \frac{\left(\partial_t \boldsymbol{w}\right)^2}{4}\right) \ d\boldsymbol{x} \ dt \\ &= \|\boldsymbol{f}\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{1}{4} \|\partial_t \boldsymbol{w}\|_{L^2(0,T;L^2(\Omega))}^2 .\end{aligned}$$

Inserting all estimates into (7.75) yields

$$
\begin{aligned}
&\|\partial_t \boldsymbol{w}\|^2_{L^2(0,T;L^2(\Omega))} + \nu \|\nabla \boldsymbol{w}(T)\|^2_{L^2(\Omega)} + \frac{2\nu_{\mathrm{T}}}{3} \|\nabla \boldsymbol{w}(T)\|^3_{L^3(\Omega)} \\
&\le \nu \|\nabla \boldsymbol{w}_0\|^2_{L^2(\Omega)} + \frac{2\nu_{\mathrm{T}}}{3} \|\nabla \boldsymbol{w}_0\|^3_{L^3(\Omega)} + 2\|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad +2C \int_0^T \|\nabla \boldsymbol{w}\|^4_{L^3(\Omega)}\, dt.
\end{aligned} \tag{7.77}
$$

In particular, it follows that

$$
\begin{aligned}
\|\nabla \boldsymbol{w}(T)\|^3_{L^3(\Omega)} \le{}& \frac{3\nu}{2\nu_{\mathrm{T}}} \|\nabla \boldsymbol{w}_0\|^2_{L^2(\Omega)} + \|\nabla \boldsymbol{w}_0\|^3_{L^3(\Omega)} + \frac{3}{\nu_{\mathrm{T}}} \|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega))} \\
&+ \frac{3C}{\nu_{\mathrm{T}}} \int_0^T \|\nabla \boldsymbol{w}\|^4_{L^3(\Omega)}\, dt.
\end{aligned}
$$

The application of Gronwall's lemma, Lemma A.69, gives

$$
\begin{aligned}
&\|\nabla \boldsymbol{w}(T)\|^3_{L^3(\Omega)} \\
&\le \left(\frac{3\nu}{2\nu_{\mathrm{T}}} \|\nabla \boldsymbol{w}_0\|^2_{L^2(\Omega)} + \|\nabla \boldsymbol{w}_0\|^3_{L^3(\Omega)} + \frac{3}{\nu_{\mathrm{T}}} \|\boldsymbol{f}\|^2_{L^2(0,T;L^2(\Omega))} \right) \\
&\quad \times \exp\left(\frac{3C}{\nu_{\mathrm{T}}} \int_0^T \|\nabla \boldsymbol{w}\|_{L^3(\Omega)}\, dt \right).
\end{aligned}
$$

One still has to bound the term in the exponential. Using inequality (A.5) yields

$$
\begin{aligned}
\int_0^T \|\nabla \boldsymbol{w}\|_{L^3(\Omega)}\, dt &= \int_0^T \left(\int_\Omega \sum_{i,j=1}^d \left| \frac{\partial w_i}{\partial x_j} \right|^3 d\boldsymbol{x} \right)^{1/3} dt \\
&\le \int_0^T \left(\int_\Omega \left(\sum_{i,j=1}^d \left| \frac{\partial w_i}{\partial x_j} \right|^2 \right)^{3/2} d\boldsymbol{x} \right)^{1/3} dt.
\end{aligned} \tag{7.78}
$$

On the other hand, by the definition of the Frobenius norm, it is

$$
\begin{aligned}
\int_\Omega \|\nabla \boldsymbol{w}\|_{\mathrm{F}} (\nabla \boldsymbol{w} : \nabla \boldsymbol{w})\, d\boldsymbol{x} &= \int_\Omega (\nabla \boldsymbol{w} : \nabla \boldsymbol{w})^{3/2}\, d\boldsymbol{x} \\
&= \int_\Omega \left(\sum_{i,j=1}^d \left| \frac{\partial w_i}{\partial x_j} \right|^2 \right)^{3/2} d\boldsymbol{x}.
\end{aligned}
$$

Inserting this identity into (7.78), one obtains with Hölder's inequality (A.15) and the a priori estimate (7.72)

$$
\begin{aligned}
\int_0^T \|\nabla \boldsymbol{w}\|_{L^3(\Omega)}\, dt &\le \int_0^T \left(\int_\Omega \|\nabla \boldsymbol{w}\|_{\mathrm{F}} (\nabla \boldsymbol{w} : \nabla \boldsymbol{w})\, d\boldsymbol{x} \right)^{1/3} dt \\
&\le \left(\int_0^T \int_\Omega \|\nabla \boldsymbol{w}\|_{\mathrm{F}} (\nabla \boldsymbol{w} : \nabla \boldsymbol{w})\, d\boldsymbol{x}\, dt \right)^{1/3} \left(\int_0^T dt \right)^{2/3} \\
&\le T^{2/3} \left(\frac{C_1(T)}{2\nu_{\mathrm{T}}} \right)^{1/3},
\end{aligned}
$$

which gives

$$
\|\nabla \boldsymbol{w}(T)\|^3_{L^3(\Omega)} \le C(T).
$$

Together with (7.77), estimate (7.74) follows. ■

Remark 7.79. Setup of the finite-dimensional problem. A sequence $\{\boldsymbol{w}^n\} \subset V$ will be constructed, where the $\boldsymbol{w}^n$ is the unique solution of a Smagorinsky problem in a space with dimension n. It will be shown that a subsequence converges to a solution $\boldsymbol{w} \in V$ of (7.69).

Let $\{\boldsymbol{v}_l^n(\boldsymbol{x})\}_{l=1}^{\infty} \subset W_{0,\mathrm{div}}^{1,3}$ be a sequence of linearly independent functions which are orthonormal with respect to the $L^2(\Omega)$ inner product in Ω and with $\boldsymbol{v}_l^1 = \boldsymbol{w}_0$, $l = 1, \ldots, \infty$. Then, the solution of the Smagorinsky problem in the space spanned by $\{\boldsymbol{v}_l^n(\boldsymbol{x})\}_{l=1}^{n}$ is sought in the form

$$\boldsymbol{w}^n(t, \boldsymbol{x}) = \sum_{l=1}^{n} \alpha_l^n(t)\, \boldsymbol{v}_l^n(\boldsymbol{x}) \tag{7.79}$$

satisfying

$$\alpha_l^n(0) = \begin{cases} 1, & l = 1, \\ 0, & l > 1, \end{cases}$$

and

$$(\partial_t \boldsymbol{w}^n, \boldsymbol{v}_l^n) + ((\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}) \nabla \boldsymbol{w}^n, \nabla \boldsymbol{v}_l^n) + n(\boldsymbol{w}^n, \boldsymbol{w}^n, \boldsymbol{v}_l^n) = (\boldsymbol{f}, \boldsymbol{v}_l^n) \tag{7.80}$$

$l = 1, \ldots, n$. System (7.80) is an autonomous, quasi-linear system of ordinary differential equations with respect to the unknown functions $\alpha_l^n(t)$. □

Lemma 7.80. Existence and uniqueness of a solution of the problem in the finite-dimensional problem space. *System* (7.80) *admits a unique solution for all* $T > 0$*. Moreover, the estimates* (7.70), (7.72), *and* (7.74) *are valid for* $\boldsymbol{w}^n$*, where the right-hand sides do not depend on* n*. Hence* $\boldsymbol{w}^n \in V$.

Proof. Like for the Navier–Stokes equations, the proof is based on the theorem of Carathéodory, see Theorem A.66. The first part can be taken literally from the proof of Lemma 6.10.

One has to show a Lipschitz condition for the right-hand side of

$$\frac{d\alpha_l^n}{dt}(t) = F(\alpha_l^n), \quad t \in (0, T], \tag{7.81}$$

with $F \in L^2(0, T)$. The functions α_l^n appear linearly and quadratically on the right-hand side of (7.81). Hence, the Lipschitz condition is satisfied, since linear and quadratic functions are Lipschitz continuous. Consequently, the local existence and uniqueness of an absolutely continuous solution $\boldsymbol{u}^n(t, \boldsymbol{x})$ in some maximal interval $[0, t_n]$ with $0 < t_n \le T$ can be concluded from the theorem of Carathéodory. If $t_n < T$, then $\boldsymbol{u}^n(t)$ blows up as $t \to t_n$.

Now, stability estimates have be proved which guarantee that this situation cannot happen and therefore $t_n = T$. The boundedness of

$$\sup_{[0,T]} \sum_{l=1}^{n} (\alpha_l^n)^2(t)$$

has to be shown. From the $L^2(\Omega)$ orthonormality of $\{\boldsymbol{v}_l^n(\boldsymbol{x})\}$ it follows that

$$\sup_{[0,T]} \sum_{l=1}^{n} (\alpha_l^n)^2(t) = \sup_{[0,T]} \|\boldsymbol{w}^n(t)\|_{L^2(\Omega)}^2 = \|\boldsymbol{w}\|_{L^\infty(0,T;L^2(\Omega))}^2 .$$

The linear combination of the equations of (7.80) yields

$$(\partial_t \boldsymbol{w}^n, \boldsymbol{w}^n) + \left(\left(\nu + \nu_{\mathrm{T}} \|\boldsymbol{w}^n\|_{\mathrm{F}}\right) \nabla \boldsymbol{w}^n, \nabla \boldsymbol{w}^n\right) = (\boldsymbol{f}, \boldsymbol{w}^n), \tag{7.82}$$

where $n(\boldsymbol{w}^n, \boldsymbol{w}^n, \boldsymbol{w}^n) = \mathbf{0}$ has been used. This equation has the same form as (7.71). Since $\boldsymbol{w}^n$ is defined as a solution of (7.80), it solves also (7.82) such that the techniques used for proving Lemma 7.76 can be applied. Thus, one obtains the estimates

$$\|\boldsymbol{w}^n(t)\|_{L^2(\Omega)} \le \|\boldsymbol{w}_0\|_{L^2(\Omega)} + \left\|\boldsymbol{f}(t')\right\|_{L^1(0,t;L^2(\Omega))}, \quad 0 \le t \le T,$$

uniformly in n, which provides the a priori boundedness which shows that there is no blow-up and thus the existence of a unique solution of (7.80) in $(0,T]$ is proved.

Analogously to the proof of Lemma 7.77, one gets

$$\|\boldsymbol{w}^n(T)\|_{L^2(\Omega)}^2 + 2\int_0^T \left(\left(\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}\right) \nabla \boldsymbol{w}^n, \nabla \boldsymbol{w}^n\right) \, dt \le C_1(T) \tag{7.83}$$

uniformly in n. System (7.80) can be brought also in form (7.75) by multiplication with $d\alpha_l^n(t) / dt$ and summation, such that the estimate

$$\|\nabla \boldsymbol{w}^n(T)\|_{L^3(\Omega)}^3 + \frac{3}{2\nu_{\mathrm{T}}} \|\partial_t \boldsymbol{w}^n\|_{L^2(0,T;L^2(\Omega))}^2 \le C_2(T) \tag{7.84}$$

is valid for $T > 0$ and uniformly in n. From estimates (7.83) and (7.84), one derives that $\boldsymbol{w}^n \in V$, uniformly in n. ■

Lemma 7.81. Existence of a converging subsequence. *There is a function $\boldsymbol{w} \in V$ such that a subsequence of $\{\boldsymbol{w}^n\}_{n=1}^\infty$*

i) converges weakly to $\boldsymbol{w}$ in V,
ii) converges strongly to $\boldsymbol{w}$ in $L^2\left(0,T;L^2(\Omega)\right)$,
iii) converges strongly to $\boldsymbol{w}$ in $L^q(0,T;L^q(\Omega))$ for $q < 4$.
iv) A subsequence of $\{\partial_t \boldsymbol{w}^n\}$ converges weakly to $\partial_t \boldsymbol{w}$ in $L^2\left(0,T;L^2(\Omega)\right)$.
v) There is a subsequence of $\frac{\partial w_j^n}{\partial x_i}$, $i,j = 1,\ldots,d$, which converges weakly to $\frac{\partial w_j}{\partial x_i}$ in $L^3\left(0,T;L^3(\Omega)\right)$.

Proof. For brevity, it will be spoken of the convergence of $\{\boldsymbol{w}^n\} = \{\boldsymbol{w}^n\}_{n=1}^\infty$ instead of the convergence of a subsequence. Besides proving the convergence in the senses given in i) – v), one has to show that all kinds of convergence lead to the same limit $\boldsymbol{w}$.

i) The weak convergence of $\{\boldsymbol{w}^n\}$ to $\boldsymbol{w} \in V$, i.e.,

$$\lim_{n\to\infty} \int_0^T \int_\Omega \boldsymbol{w}^n \boldsymbol{v} \, d\boldsymbol{x} \, dt = \int_0^T \int_\Omega \boldsymbol{w}\boldsymbol{v} \, d\boldsymbol{x} \, dt \quad \forall\, \boldsymbol{v} \in V', \tag{7.85}$$

where V' is the dual space of V, follows from the uniform boundedness of $\{\boldsymbol{w}^n\}$ in the norm $\|\cdot\|_V$ of V, which is a consequence of (7.84), and that every bounded sequence in a reflexive Banach space has a weakly convergent subsequence, see Remark A.52.

ii) Since $V \subset L^2\left(0,T;L^2(\Omega)\right) \subset V'$, (7.85) holds also for all $\boldsymbol{v} \in L^2\left(0,T;L^2(\Omega)\right)$ such that $\{\boldsymbol{w}^n\}$ converges weakly to $\boldsymbol{w}$ in $L^2\left(0,T;L^2(\Omega)\right)$. From the uniform boundedness of $\boldsymbol{w}^n(t,\boldsymbol{x})$ in $W_0^{1,3}(\Omega)$ for every $t \ge 0$, estimate (7.84), and the compact embedding $W_0^{1,3}(\Omega)$ into $L^2(\Omega)$, (A.24) and Theorem A.46 vii), it follows that there is a subsequence of $\{\boldsymbol{w}^n\}$ which converges strongly in $L^2(\Omega)$ to $\tilde{\boldsymbol{w}} \in L^2(\Omega)$. Since this statement is true for all $t \ge 0$, $\{\boldsymbol{w}^n\}$ converges strongly to $\tilde{\boldsymbol{w}} \in L^2\left(0,T;L^2(\Omega)\right)$. Consequently, this subsequence of $\{\boldsymbol{w}^n\}$ converges also weakly to $\tilde{\boldsymbol{w}}$

$$\lim_{n\to\infty}\int_0^T\int_\Omega \boldsymbol{w}^n\boldsymbol{v}\ d\boldsymbol{x}\ dt = \int_0^T\int_\Omega \tilde{\boldsymbol{w}}\boldsymbol{v}\ d\boldsymbol{x}\ dt \quad \forall\ \boldsymbol{v}\in L^2\left(0,T;L^2(\Omega)\right)\subset V'.$$

Since the weak limit is unique, it follows with (7.85) that $\boldsymbol{w} = \tilde{\boldsymbol{w}}$. Hence, (a subsequence of) $\{\boldsymbol{w}^n\}$ converges to $\boldsymbol{w}$ strongly in $L^2\left(0,T;L^2(\Omega)\right)$.

iii) It will be first established that $\{\boldsymbol{w}^n\}$ is uniformly bounded (with respect to n) in $L^4\left(0,T;L^4(\Omega)\right)$. From the Cauchy–Schwarz inequality (A.16), the Sobolev embedding $H^1(\Omega)\to L^6(\Omega)$, see (A.30), and Poincaré's inequality (A.17) it follows that

$$\begin{aligned}\|\boldsymbol{w}^n\|_{L^4(\Omega)}^4 &= \int_\Omega \|\boldsymbol{w}^n\|_2^4\ d\boldsymbol{x} \le \left(\int_\Omega \|\boldsymbol{w}^n\|_2^6\ d\boldsymbol{x}\right)^{1/2}\left(\int_\Omega \|\boldsymbol{w}^n\|_2^2\ d\boldsymbol{x}\right)^{1/2}\\ &\le \|\boldsymbol{w}^n\|_{L^6(\Omega)}^3\|\boldsymbol{w}^n\|_{L^2(\Omega)} \le C\|\boldsymbol{w}^n\|_{H^1(\Omega)}^3\|\boldsymbol{w}^n\|_{L^2(\Omega)}\\ &\le C\|\nabla\boldsymbol{w}^n\|_{L^2(\Omega)}^3\|\boldsymbol{w}^n\|_{L^2(\Omega)}.\end{aligned}$$

One obtains, using (7.83) and (7.84)

$$\begin{aligned}\|\boldsymbol{w}^n\|_{L^4\left(0,T;L^4(\Omega)\right)}^4 &\le C\int_0^T \|\nabla\boldsymbol{w}^n\|_{L^2(\Omega)}^3\|\boldsymbol{w}^n\|_{L^2(\Omega)}\ dt\\ &\le \int_0^T C_2(t)\,C_1(t)^{1/2}\ dt,\end{aligned}$$

uniformly in n. Now, one can prove the strong convergence $\boldsymbol{w}^n\to\boldsymbol{w}$ in $L^q(0,T;L^q(\Omega))$, $q=4-\varepsilon, \varepsilon>0$. The generalized Hölder inequality (5.25) with respect to the time-space norm gives

$$\begin{aligned}&\|\boldsymbol{w}-\boldsymbol{w}^n\|_{L^q(0,T;L^q(\Omega))}^q\\ &= \int_0^T\int_\Omega \|\boldsymbol{w}-\boldsymbol{w}^n\|_2^{2-\varepsilon}\|\boldsymbol{w}-\boldsymbol{w}^n\|_2\|\boldsymbol{w}-\boldsymbol{w}^n\|_2\ d\boldsymbol{x}\ dt\\ &\le \left\|(\boldsymbol{w}-\boldsymbol{w}^n)^{2-\varepsilon}\right\|_{L^2\left(0,T;L^2(\Omega)\right)}\|\boldsymbol{w}-\boldsymbol{w}^n\|_{L^4\left(0,T;L^4(\Omega)\right)}\|\boldsymbol{w}-\boldsymbol{w}^n\|_{L^4\left(0,T;L^4(\Omega)\right)}.\end{aligned}$$

The last two factors are bounded by the triangle inequality and the uniform boundedness of $\{\boldsymbol{w}^n\}$ in $L^4\left(0,T;L^4(\Omega)\right)$. The first term can be written in the form

$$\left\|(\boldsymbol{w}-\boldsymbol{w}^n)^{2-\varepsilon}\right\|_{L^2\left(0,T;L^2(\Omega)\right)}^2 = \int_0^T\int_\Omega \|\boldsymbol{w}-\boldsymbol{w}^n\|_2^{2-2\varepsilon}\|\boldsymbol{w}-\boldsymbol{w}^n\|_2\|\boldsymbol{w}-\boldsymbol{w}^n\|_2\ d\boldsymbol{x}\ dt.$$

Applying the same steps as for the previous estimate gives

$$\|\boldsymbol{w}-\boldsymbol{w}^n\|_{L^q(0,T;L^q(\Omega))}^q \le \left\|(\boldsymbol{w}-\boldsymbol{w}^n)^{2-2\varepsilon}\right\|_{L^2\left(0,T;L^2(\Omega)\right)}\|\boldsymbol{w}-\boldsymbol{w}^n\|_{L^4\left(0,T;L^4(\Omega)\right)}^2.$$

Continuing this approach, i.e., applying several times Hölder's inequality, yields eventually the factor

$$\left\|(\boldsymbol{w}-\boldsymbol{w}^n)^{2-\varepsilon_0}\right\|_{L^2\left(0,T;L^2(\Omega)\right)} \quad \text{with } 2-\varepsilon_0\le 1.$$

If $2-\varepsilon_0=1$, the result of ii) can be applied directly to prove iii). In the case $2-\varepsilon_0<1$, first (A.14) has to be used to obtain

$$\left\|(\boldsymbol{w}-\boldsymbol{w}^n)^{2-\varepsilon_0}\right\|_{L^2\left(0,T;L^2(\Omega)\right)} \le C\left\|\boldsymbol{w}-\boldsymbol{w}^n\right\|_{L^2\left(0,T;L^2(\Omega)\right)},$$

before ii) can be used to prove iii).

iv) For $\boldsymbol{\phi} \in C_0^\infty\left(0,T;L^2(\Omega)\right)$ it follows, using ii), that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega\int_0^T \partial_t \boldsymbol{w}^n\boldsymbol{\phi}\ dt\ d\boldsymbol{x} &= -\lim_{n\to\infty}\int_\Omega\int_0^T \boldsymbol{w}^n\partial_t\boldsymbol{\phi}\ dt\ d\boldsymbol{x}\\ &= -\int_\Omega\int_0^T \boldsymbol{w}\partial_t\boldsymbol{\phi}\ dt\ d\boldsymbol{x} = \int_\Omega\int_0^T \partial_t\boldsymbol{w}\boldsymbol{\phi}\ dt\ d\boldsymbol{x}.\end{aligned}$$

Now, statement iv) is a consequence of the density of $C_0^\infty\left(0,T;L^2(\Omega)\right)$ in $L^2\left(0,T;L^2(\Omega)\right)$, see Theorem A.41.

v) One has to prove

$$\lim_{n\to\infty}\int_0^T\int_\Omega \frac{\partial w_j^n}{\partial x_i}\phi\ d\boldsymbol{x}\ dt = \int_0^T\int_\Omega\frac{\partial w_j}{\partial x_i}\phi\ d\boldsymbol{x}\ dt \quad \forall\ \phi\in L^{3/2}\left(0,T;L^{3/2}(\Omega)\right).$$

It suffices to prove this relation for a dense set in $L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)$, e.g., for functions from the space $C_0\left(0,T;C_0^1(\Omega)\right)$ which is dense in $L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)$, which follows from Theorem A.41. Let $\phi\in C_0\left(0,T;C_0^1(\Omega)\right)$ be arbitrary, then applying ii) and twice integration by parts yields

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\int_\Omega\frac{\partial w_j^n}{\partial x_i}\phi\ d\boldsymbol{x}\ dt &= -\lim_{n\to\infty}\int_0^T\int_\Omega w_j^n\frac{\partial\phi}{\partial x_i}\ d\boldsymbol{x}\ dt\\ &= -\int_0^T\int_\Omega w_j\frac{\partial\phi}{\partial x_i}\ d\boldsymbol{x}\ dt = \int_0^T\int_\Omega\frac{\partial w_j}{\partial x_i}\phi\ d\boldsymbol{x}\ dt.\end{aligned}$$

All boundary integrals vanish since ϕ vanishes on the boundary. Now, the proof concludes by using that Property ii) could be applied since it is $C_0\left(0,T;C_0^1(\Omega)\right)\subset L^2\left(0,T;L^2(\Omega)\right)$. ■

Remark 7.82. Formulation of the finite-dimensional problem with arbitrary test function. In the following, the notation that '$\{\boldsymbol{w}^n\}$ converges' will be used instead of that a subsequence converges.

Let

$$P^n = \left\{\boldsymbol{v} : \boldsymbol{v} = \sum_{l=1}^n \tilde\alpha_l^n(t)\,\boldsymbol{v}_l^n(\boldsymbol{x})\right\},$$

where $\tilde\alpha_l^n(t)$ are absolutely continuous functions of $t\in[0,T]$ with $\tilde\alpha_l^n(t)\in H^1(0,T)$, see Definition A.64. Choosing a fixed function $\boldsymbol{\phi}\in P^n$, it follows from (7.80), by taking a linear combination, that $\boldsymbol{\phi}$ satisfies

$$\begin{aligned}\int_0^T (\partial_t\boldsymbol{w}^n,\boldsymbol{\phi}) + \left(\left(\nu+\nu_{\mathrm{T}}\left\|\nabla\boldsymbol{w}^n\right\|_{\mathrm{F}}\right)\nabla\boldsymbol{w}^n,\nabla\boldsymbol{\phi}\right) + n\left(\boldsymbol{w}^n,\boldsymbol{w}^n,\boldsymbol{\phi}\right)\ dt&\\ = \int_0^T (\boldsymbol{f},\boldsymbol{\phi})\ dt.& \qquad (7.86)\end{aligned}$$

□

Lemma 7.83. Convergence of a bilinear form with the temporal derivative and the nonlinear convective term. *For $\boldsymbol{\phi} \in P^n$ it holds*

$$\lim_{n\to\infty} \int_0^T (\partial_t \boldsymbol{w}^n, \boldsymbol{\phi})\ dt = \int_0^T (\partial_t \boldsymbol{w}, \boldsymbol{\phi})\ dt, \tag{7.87}$$

$$\lim_{n\to\infty} \int_0^T n\,(\boldsymbol{w}^n, \boldsymbol{w}^n, \boldsymbol{\phi})\ dt = \int_0^T n\,(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{\phi})\ dt. \tag{7.88}$$

Proof. (7.87) : This statement follows immediately from Lemma 7.81, iv).
(7.88) : It is

$$\int_0^T n\,(w^n, w^n, \boldsymbol{\phi})\ dt = \sum_{i,j=1}^{d} \int_0^T \int_\Omega w_i^n \frac{\partial w_j^n}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt.$$

Considering an arbitrary term of this sum yields

$$\int_0^T \int_\Omega w_i^n \frac{\partial w_j^n}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt = \int_0^T \int_\Omega w_i \frac{\partial w_j^n}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt + \int_0^T \int_\Omega (w_i^n - w_i) \frac{\partial w_j^n}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt.$$

Since $\frac{\partial w_j^n}{\partial x_i} \in L^3\left(0,T;L^3(\Omega)\right)$ converges weakly to $\frac{\partial w_j}{\partial x_i}$ in this space and it from the Cauchy–Schwarz inequality (A.16) one has that with $w_i, \phi_j \in L^3\left(0,T;L^3(\Omega)\right)$ it follows that $w_i\phi_j \in L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)$, one obtains for the first term with Lemma 7.81 v)

$$\lim_{n\to\infty} \int_0^T \int_\Omega w_i \frac{\partial w_j^n}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt = \int_0^T \int_\Omega w_i \frac{\partial w_j}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt.$$

Applying Hölder's inequality (A.15) to the second term gives

$$\left| \int_0^T \int_\Omega (w_i^n - w_i) \frac{\partial w_j^n}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt \right| \le \left\| (w_i^n - w_i)\,\phi_j \right\|_{L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)} \left\| \frac{\partial w_j^n}{\partial x_i} \right\|_{L^3\left(0,T;L^3(\Omega)\right)}.$$

By the Cauchy–Schwarz inequality it follows that

$$\left\| (w_i^n - w_i)\,\phi_j \right\|_{L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)} \le \left\| w_i^n - w_i \right\|^{3/2}_{L^3\left(0,T;L^3(\Omega)\right)} \left\| \phi_j \right\|^{3/2}_{L^3\left(0,T;L^3(\Omega)\right)}.$$

By Lemma 7.81 iii), one has that w_i^n converges strongly to w_i in $L^3\left(0,T;L^3(\Omega)\right)$. The term $\left\| \frac{\partial w_j^n}{\partial x_i} \right\|_{L^3\left(0,T;L^3(\Omega)\right)}$ is uniformly bounded and $\left\| \phi_j \right\|^{3/2}_{L^3\left(0,T;L^3(\Omega)\right)}$ is just a constant. Altogther, one gets

$$\lim_{n\to\infty} \left| \int_0^T \int_\Omega (w_i^n - w_i) \frac{\partial w_j^n}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt \right| = 0$$

and

$$\lim_{n\to\infty} \int_0^T \int_\Omega w_i^n \frac{\partial w_j^n}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt = \int_0^T \int_\Omega w_i \frac{\partial w_j}{\partial x_i} \phi_j\ d\boldsymbol{x}\ dt.$$

■

Lemma 7.84. The limiting equation. *The limiting equation of* (7.86) *is*

$$\int_0^T (\partial_t \boldsymbol{w}, \boldsymbol{\phi}) + (\boldsymbol{B}, \nabla\boldsymbol{\phi}) + n(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{\phi})\ dt = \int_0^T (\boldsymbol{f}, \boldsymbol{\phi})\ dt \tag{7.89}$$

with $\boldsymbol{B} \in L^{3/2}\left(0, T; L^{3/2}(\Omega)\right)$.

Proof. The limits of the first and the third term were established in Lemma 7.83.

From the choice of $\boldsymbol{\phi}$ it follows that $\nabla\boldsymbol{\phi} \in L^3\left(0, T; L^3(\Omega)\right)$. If one can show now that the sequence $\{\left(\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}\right) \nabla \boldsymbol{w}^n\}$ is uniformly bounded in $L^{3/2}\left(0, T; L^{3/2}(\Omega)\right)$ then there is a subsequence which converges weakly to an operator $\boldsymbol{B} \in L^{3/2}\left(0, T; L^{3/2}(\Omega)\right)$, i.e.,

$$\lim_{n\to\infty} \int_0^T \left(\left(\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}\right) \nabla \boldsymbol{w}^n, \nabla\boldsymbol{\phi}\right)\ dt = \int_0^T (\boldsymbol{B}, \nabla\boldsymbol{\phi})\ dt.$$

Consider the nonlinear viscous term component-by-component. It is

$$\begin{aligned} &\left\| \left(\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}\right) \frac{\partial w_i^n}{\partial x_j} \right\|^{3/2}_{L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)} \\ &= \nu^{3/2} \int_0^T \int_\Omega \left| \frac{\partial w_i^n}{\partial x_j} \right|^{3/2} d\boldsymbol{x}\ dt + \nu_{\mathrm{T}}^{3/2} \int_0^T \int_\Omega \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}^{3/2} \left| \frac{\partial w_i^n}{\partial x_j} \right|^{3/2} d\boldsymbol{x}\ dt, \end{aligned} \tag{7.90}$$

$i, j = 1, \ldots, d$. Using Young's inequality (A.4) with $p = 4/3, q = 4$ gives for the first term

$$\int_0^T \int_\Omega \left| \frac{\partial w_i^n}{\partial x_j} \right|^{3/2} d\boldsymbol{x}\ dt \le \frac{3}{4} \int_0^T \int_\Omega \left| \frac{\partial w_i^n}{\partial x_j} \right|^{2} d\boldsymbol{x}\ dt + \frac{1}{4} \int_0^T \int_\Omega d\boldsymbol{x}\ dt.$$

The right-hand side of this estimate is bounded by (7.83) and since Ω is bounded. For estimating the second term of (7.90), again Young's inequality with $p = 4, q = 4/3$ is used

$$\begin{aligned} &\int_0^T \int_\Omega \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}^{3/2} \left| \frac{\partial w_i^n}{\partial x_j} \right|^{3/2} d\boldsymbol{x}\ dt \\ &= \int_0^T \int_\Omega \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}^{3/4} \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}^{3/4} \left| \frac{\partial w_i^n}{\partial x_j} \right|^{3/2} d\boldsymbol{x}\ dt \\ &\le \frac{1}{4} \int_0^T \int_\Omega \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}^{3}\ d\boldsymbol{x}\ dt + \frac{3}{4} \int_0^T \int_\Omega \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}} \left| \frac{\partial w_i^n}{\partial x_j} \right|^{2} d\boldsymbol{x}\ dt \\ &= \frac{1}{4} \int_0^T \left(\|\nabla \boldsymbol{w}^n\|_{\mathrm{F}} \nabla \boldsymbol{w}^n, \nabla \boldsymbol{w}^n\right)\ dt + \frac{3}{4} \int_0^T \int_\Omega \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}} \left| \frac{\partial w_i^n}{\partial x_j} \right|^{2} d\boldsymbol{x}\ dt. \end{aligned}$$

Both terms in the last line of this inequality posses terms of $\nabla \boldsymbol{w}^n$ to the third power and thus they are bounded uniformly by (7.83).

Hence, (7.90) is bounded uniformly and a subsequence of $\{\left(\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^n\|_{\mathrm{F}}\right) \nabla \boldsymbol{w}^n\}$ converges weakly to some $\boldsymbol{B}(t, \boldsymbol{x})$ from which follows the statement of the lemma. ■

Remark 7.85. On the limiting equation. The limiting equation (7.89) holds for $\boldsymbol{\phi} \in P^n$ for arbitrary n and thus for $\boldsymbol{\phi} \in \cup_{n=1}^{\infty} P^n$. One still has to show that it holds for all test functions from V. □

Lemma 7.86. The limiting equation in V**.** *The limiting equation* (7.89) *is valid for* $\boldsymbol{\phi} \in V$.

Proof. For the proof of this lemma it is refered to (Ladyzhenskaya, 1969, pp. 159). idea of the proof ■

Remark 7.87. The nonlinear viscous operator. The nonlinear viscous operator is defined by $\boldsymbol{A} : L^3(\Omega) \to L^{3/2}(\Omega)$ with

$$\boldsymbol{A}(\nabla \boldsymbol{w}) = \left(\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|_{\mathrm{F}}\right) \nabla \boldsymbol{w}. \tag{7.91}$$

For proving the existence of a weak solution, it will be shown that the nonlinear viscous term defines a so-called monotone operator. □

Lemma 7.88. Strong monotonicity of the nonlinear viscous operator. *For arbitrary functions $\boldsymbol{w}', \boldsymbol{w}'' \in W^{1,3}(\Omega)$ it holds the estimate*

$$\int_\Omega \left(\boldsymbol{A}(\nabla \boldsymbol{w}') - \boldsymbol{A}(\nabla \boldsymbol{w}'')\right) : \left(\nabla \boldsymbol{w}' - \nabla \boldsymbol{w}''\right)\, d\boldsymbol{x} \ge \nu \|\nabla \boldsymbol{w}' - \nabla \boldsymbol{w}''\|_{L^2(\Omega)}^2 \tag{7.92}$$

with the operator $\boldsymbol{A}$ defined in (7.91)*. In addition, it is*

$$\begin{aligned}\int_\Omega \nu_{\mathrm{T}} \left(\|\nabla \boldsymbol{w}'\|_{\mathrm{F}} \nabla \boldsymbol{w}' - \|\nabla \boldsymbol{w}''\|_{\mathrm{F}} \nabla \boldsymbol{w}''\right) : \left(\nabla \boldsymbol{w}' - \nabla \boldsymbol{w}''\right)\, d\boldsymbol{x} \\ \ge \frac{\nu_{\mathrm{T}}}{4} \|\nabla \boldsymbol{w}' - \nabla \boldsymbol{w}''\|_{L^3(\Omega)}^3,\end{aligned} \tag{7.93}$$

i.e., the Smagorinsky term defines a strongly monotone operator from $L^3(\Omega)$ into $L^{3/2}(\Omega)$.

Proof. Let $\boldsymbol{w}', \boldsymbol{w}'' \in C^1(\overline{\Omega})$. With $\boldsymbol{w}^\tau = \tau \boldsymbol{w}' + (1-\tau)\boldsymbol{w}''$, one obtains, applying the fundamental theorem of calculus,

$$\begin{aligned}&\left(\boldsymbol{A}(\nabla \boldsymbol{w}') - \boldsymbol{A}(\nabla \boldsymbol{w}'')\right) : \left(\nabla \boldsymbol{w}' - \nabla \boldsymbol{w}''\right) \\ &= \sum_{i,j=1}^d \left(\boldsymbol{A}_{ij}(\nabla \boldsymbol{w}') - \boldsymbol{A}_{ij}(\nabla \boldsymbol{w}'')\right) \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right) \\ &= \sum_{i,j=1}^d \left(\int_0^1 \frac{d}{d\tau} \boldsymbol{A}_{ij}(\nabla \boldsymbol{w}^\tau)\, d\tau\right) \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right).\end{aligned} \tag{7.94}$$

Applying the product rule, one gets

$$\frac{d}{d\tau} \boldsymbol{A}_{ij}(\nabla \boldsymbol{w}^\tau) = \left(\nu_{\mathrm{T}} \frac{\partial}{\partial \tau} \|\nabla \boldsymbol{w}^\tau\|_{\mathrm{F}}\right) \frac{\partial w_i^\tau}{\partial x_j} + \left(\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^\tau\|_{\mathrm{F}}\right) \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right).$$

With the chain rule, one obtains

$$
\begin{aligned}
\frac{\partial}{\partial \tau}\left\|\nabla \boldsymbol{w}^\tau\right\|_{\mathrm{F}} &= \frac{\partial}{\partial \tau}\left(\sum_{k,l=1}^{d}\left(\tau\frac{\partial w_k'}{\partial x_l}+(1-\tau)\frac{\partial w_k''}{\partial x_l}\right)^2\right)^{1/2} \\
&= \frac{1}{2}\frac{\sum_{k,l=1}^{d} 2\left(\tau\frac{\partial w_k'}{\partial x_l}+(1-\tau)\frac{\partial w_k''}{\partial x_l}\right)\left(\frac{\partial w_k'}{\partial x_l}-\frac{\partial w_k''}{\partial x_l}\right)}{\left(\sum_{k,l=1}^{d}\left(\tau\frac{\partial w_k'}{\partial x_l}+(1-\tau)\frac{\partial w_k''}{\partial x_l}\right)^2\right)^{1/2}} \\
&= \frac{1}{\left\|\nabla \boldsymbol{w}^\tau\right\|_{\mathrm{F}}}\sum_{k,l=1}^{d}\frac{\partial w_k^\tau}{\partial x_l}\left(\frac{\partial w_k'}{\partial x_l}-\frac{\partial w_k''}{\partial x_l}\right).
\end{aligned}
$$

Inserting these expressions into (7.94) yields

$$
\begin{aligned}
&\left(\boldsymbol{A}\left(\nabla \boldsymbol{w}'\right)-\boldsymbol{A}\left(\nabla \boldsymbol{w}''\right)\right):\left(\nabla \boldsymbol{w}'-\nabla \boldsymbol{w}''\right) \\
&= \int_0^1 \sum_{i,j=1}^{d}\left(\nu+\nu_{\mathrm{T}}\left\|\nabla \boldsymbol{w}^\tau\right\|_{\mathrm{F}}\right)\left(\frac{\partial w_i'}{\partial x_j}-\frac{\partial w_i''}{\partial x_j}\right)\left(\frac{\partial w_i'}{\partial x_j}-\frac{\partial w_i''}{\partial x_j}\right)d\tau \qquad (7.95) \\
&\quad + \int_0^1 \nu_{\mathrm{T}}\left\|\nabla \boldsymbol{w}^\tau\right\|_{\mathrm{F}}^{-1}\sum_{i,j,k,l=1}^{d}\frac{\partial w_k^\tau}{\partial x_l}\frac{\partial w_i^\tau}{\partial x_j}\left(\frac{\partial w_k'}{\partial x_l}-\frac{\partial w_k''}{\partial x_l}\right)\left(\frac{\partial w_i'}{\partial x_j}-\frac{\partial w_i''}{\partial x_j}\right)d\tau.
\end{aligned}
$$

The second term is non-negative since

$$
\begin{aligned}
&\sum_{i,j,k,l=1}^{d}\frac{\partial w_k^\tau}{\partial x_l}\frac{\partial w_i^\tau}{\partial x_j}\left(\frac{\partial w_k'}{\partial x_l}-\frac{\partial w_k''}{\partial x_l}\right)\left(\frac{\partial w_i'}{\partial x_j}-\frac{\partial w_i''}{\partial x_j}\right) \\
&= \left(\sum_{i,j=1}^{d}\frac{\partial w_i^\tau}{\partial x_j}\left(\frac{\partial w_i'}{\partial x_j}-\frac{\partial w_i''}{\partial x_j}\right)\right)^2. \qquad (7.96)
\end{aligned}
$$

Thus, this term can be estimated from below by zero.

Estimate (7.92). The first term of (7.95) is estimated by using the non-negativity of the term including the turbulent viscosity

$$
\begin{aligned}
&\int_0^1 \sum_{i,j,=1}^{d}\left(\nu+\nu_{\mathrm{T}}\left\|\nabla \boldsymbol{w}^\tau\right\|_{\mathrm{F}}\right)\left(\frac{\partial w_i'}{\partial x_j}-\frac{\partial w_i''}{\partial x_j}\right)^2 d\tau \\
&\geq \sum_{i,j,=1}^{d}\nu\left(\frac{\partial w_i'}{\partial x_j}-\frac{\partial w_i''}{\partial x_j}\right)^2\int_0^1 d\tau \\
&= \nu\left(\nabla \boldsymbol{w}'-\nabla \boldsymbol{w}''\right):\left(\nabla \boldsymbol{w}'-\nabla \boldsymbol{w}''\right) = \nu\left\|\nabla \boldsymbol{w}'-\nabla \boldsymbol{w}''\right\|_{\mathrm{F}}^2. \qquad (7.97)
\end{aligned}
$$

Estimate (7.93). For this estimate, the term with ν in (7.95) is estimated by zero from below. The estimate of the other term starts by estimating the Frobenius norm from below by the largest term of the sum and using the definition of $\boldsymbol{w}_k^\tau$

$$
\begin{aligned}
&\int_0^1 \sum_{i,j=1}^d \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^\tau\|_{\mathrm{F}} \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right)^2 \, d\tau \\
&\geq \nu_{\mathrm{T}} \int_0^1 \sum_{i,j=1}^d \max_{k,l=1,\ldots,d} \left|\frac{\partial w_k^\tau}{\partial x_l}\right| \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right)^2 \, d\tau \qquad (7.98)\\
&= \nu_{\mathrm{T}} \int_0^1 \sum_{i,j=1}^d \max_{k,l=1,\ldots,d} \left|\tau \frac{\partial w_k'}{\partial x_l} + (1-\tau)\frac{\partial w_k''}{\partial x_l}\right| \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right)^2 \, d\tau.
\end{aligned}
$$

Now, the estimate

$$
\int_0^1 |\tau a + (1-\tau) b| \, d\tau \geq \frac{|a-b|}{4}, \quad a, b \in \mathbb{R}
$$

will be applied. To prove this estimate, one has to distinguish the cases $ab \geq 0$ and $ab < 0$. In the first case, one finds, e.g., for $a, b \leq 0$, with a straightforward calculation of the integral and using the triangle inequality

$$
\begin{aligned}
\int_0^1 |\tau a + (1-\tau) b| \, d\tau &= -\int_0^1 \tau a + (1-\tau) b \, d\tau = -\frac{a+b}{2} = \frac{|a|+|b|}{2} \geq \frac{|a|+|b|}{4} \\
&\geq \frac{|a-b|}{4}.
\end{aligned}
$$

The calculation for $a, b \geq 0$ is analog. In the second case, one finds that the term in the integral, which is linear in τ, has a root at $\tau_0 = -b/(a-b)$. Now, one splits the integral into the integrals on $(0, \tau_0)$ and $(\tau_0, 1)$. Consider, e.g., the case $a > 0$ and $b < 0$, one finds with a direct calculation

$$
\begin{aligned}
\int_0^1 |\tau a + (1-\tau) b| \, d\tau &= -\int_0^{\tau_0} (\tau a + (1-\tau) b) \, d\tau + \int_{\tau_0}^1 (\tau a + (1-\tau) b) \, d\tau \\
&= a\left(\frac{1}{2} - \tau_0^2\right) + b\left(-\frac{1}{2} + (1-\tau_0)^2\right) = \frac{a^2+b^2}{2(a-b)}.
\end{aligned}
$$

Then, dropping a non-negative term yields

$$
\int_0^1 |\tau a + (1-\tau) b| \, d\tau = \frac{1}{2}\frac{a^2+b^2}{a-b} = \frac{1}{4}\frac{(a-b)^2 + (a+b)^2}{a-b} \geq \frac{a-b}{4} = \frac{|a-b|}{4}.
$$

For $a < 0$ and $b > 0$ one gets the same estimate. Inserting this estimate into (7.98) and estimating the maximal value by the absolute value of the individual terms from below gives

$$
\begin{aligned}
&\int_0^1 \sum_{i,j=1}^d \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}^\tau\|_{\mathrm{F}} \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right)^2 \, d\tau \\
&\geq \nu_{\mathrm{T}} \sum_{i,j=1}^d \max_{k,l=1,\ldots,d} \frac{1}{4} \left|\frac{\partial w_k'}{\partial x_l} - \frac{\partial w_k''}{\partial x_l}\right| \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right)^2 \\
&\geq \frac{\nu_{\mathrm{T}}}{4} \sum_{i,j=1}^d \left|\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right| \left(\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right)^2 \\
&= \frac{\nu_{\mathrm{T}}}{4} \sum_{i,j=1}^d \left|\frac{\partial w_i'}{\partial x_j} - \frac{\partial w_i''}{\partial x_j}\right|^3.
\end{aligned}
$$

The proof of both estimates is completed with integration on Ω and using the density of $C^1\left(\overline{\Omega}\right)$ in $W^{1,3}(\Omega)$, which is a consequence of Theorem A.41. ■

Lemma 7.89. Estimate of the difference of the weak equation for two test functions. *Let $\tilde{\boldsymbol{\phi}} \in V$, then it is*

$$-\int_0^T \int_\Omega \left(\partial_t \boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{w} - \boldsymbol{f}\right)\cdot\left(\boldsymbol{w}-\tilde{\boldsymbol{\phi}}\right) + \boldsymbol{A}\left(\nabla\tilde{\boldsymbol{\phi}}\right) : \nabla\left(\boldsymbol{w}-\tilde{\boldsymbol{\phi}}\right)\, d\boldsymbol{x}\, dt \geq 0. \tag{7.99}$$

Proof. To begin with, let $\tilde{\boldsymbol{\phi}} \in P^n$ for an arbitrary n. From (7.92) it follows that

$$\int_0^T \int_\Omega \left(\boldsymbol{A}\left(\nabla \boldsymbol{w}^n\right) - \boldsymbol{A}\left(\nabla\tilde{\boldsymbol{\phi}}\right)\right) : \left(\nabla \boldsymbol{w}^n - \nabla\tilde{\boldsymbol{\phi}}\right)\, d\boldsymbol{x}\, dt \geq 0. \tag{7.100}$$

Since $\boldsymbol{w}^n \in P^n$ and $\tilde{\boldsymbol{\phi}} \in P^n$, one obtains with $\boldsymbol{\phi} = \boldsymbol{w}^n - \tilde{\boldsymbol{\phi}}$ in (7.86)

$$\begin{aligned}
&\int_0^T \int_\Omega \boldsymbol{A}\left(\nabla \boldsymbol{w}^n\right) : \left(\nabla \boldsymbol{w}^n - \nabla\tilde{\boldsymbol{\phi}}\right)\, d\boldsymbol{x}\, dt \\
&= \int_0^T \left(\boldsymbol{f} - \partial_t \boldsymbol{w}^n - \left(\boldsymbol{w}^n\cdot\nabla\right)\boldsymbol{w}^n, \boldsymbol{w}^n - \tilde{\boldsymbol{\phi}}\right)\, dt,
\end{aligned}$$

such that with (7.86)

$$\begin{aligned}
-\int_0^T \int_\Omega \Big[&\left(\partial_t \boldsymbol{w}^n + \left(\boldsymbol{w}^n\cdot\nabla\right)\boldsymbol{w}^n - \boldsymbol{f}\right)\cdot\left(\boldsymbol{w}^n - \tilde{\boldsymbol{\phi}}\right) \\
&+ \boldsymbol{A}\left(\nabla\tilde{\boldsymbol{\phi}}\right) : \left(\nabla \boldsymbol{w}^n - \nabla\tilde{\boldsymbol{\phi}}\right)\Big]\, d\boldsymbol{x}\, dt \geq 0.
\end{aligned} \tag{7.101}$$

It is

$$\int_0^T \left(\boldsymbol{f}, \boldsymbol{w}^n - \tilde{\boldsymbol{\phi}}\right)\, dt = \int_0^T \left(\boldsymbol{f}, \boldsymbol{w} - \tilde{\boldsymbol{\phi}}\right)\, dt + \int_0^T \left(\boldsymbol{f}, \boldsymbol{w}^n - \boldsymbol{w}\right)\, dt.$$

The second term converges to zero since $\boldsymbol{w}^n$ converges to $\boldsymbol{w}$ strongly in $L^2\left(0,T;L^2(\Omega)\right)$ and

$$\left|\int_0^T \left(\boldsymbol{f}, \boldsymbol{w}^n - \boldsymbol{w}\right)\, dt\right| \leq \|\boldsymbol{f}\|_{L^2\left(0,T;L^2(\Omega)\right)} \|\boldsymbol{w}^n - \boldsymbol{w}\|_{L^2\left(0,T;L^2(\Omega)\right)}.$$

The nonlinear term is considered component-by-component

$$\int_0^T \int_\Omega \left(\boldsymbol{w}^n\cdot\nabla\right)\boldsymbol{w}^n\cdot\boldsymbol{w}^n\, d\boldsymbol{x}\, dt = \sum_{k,l=1}^d \int_0^T \int_\Omega w_k^n \frac{\partial w_l^n}{\partial x_k} w_l^n\, d\boldsymbol{x}\, dt.$$

By construction, $\frac{\partial w_l^n}{\partial x_k} \in L^3\left(0,T;L^3(\Omega)\right)$ and it was proved in Lemma 7.81 v) that $\frac{\partial w_l^n}{\partial x_k} \rightharpoonup \frac{\partial w_l}{\partial x_k}$ in $L^3\left(0,T;L^3(\Omega)\right)$. Hence, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^T \int_\Omega w_k^n \frac{\partial w_l^n}{\partial x_k} w_l^n\, d\boldsymbol{x}\, dt \\
&= \lim_{n\to\infty} \int_0^T \int_\Omega w_k \frac{\partial w_l^n}{\partial x_k} w_l\, d\boldsymbol{x}\, dt + \lim_{n\to\infty} \int_0^T \int_\Omega \frac{\partial w_l^n}{\partial x_k}\left(w_k^n w_l^n - w_k w_l\right)\, d\boldsymbol{x}\, dt \\
&= \int_0^T \int_\Omega w_k \frac{\partial w_l}{\partial x_k} w_l\, d\boldsymbol{x}\, dt + \lim_{n\to\infty} \int_0^T \int_\Omega \frac{\partial w_l^n}{\partial x_k}\left(w_k^n w_l^n - w_k w_l\right)\, d\boldsymbol{x}\, dt.
\end{aligned}$$

Since one obtains with Hölder's inequality (A.15)

$$\begin{aligned}&\left|\int_0^T\int_\Omega \frac{\partial w_l^n}{\partial x_k}\left(w_k^n w_l^n - w_k w_l\right)\ d\boldsymbol{x}\ dt\right|\\&\leq \left\|\frac{\partial w_l^n}{\partial x_k}\right\|_{L^3\left(0,T;L^3(\Omega)\right)} \left\|w_k^n w_l^n - w_k w_l\right\|_{L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)},\end{aligned}$$

one has to show that $w_k^n w_l^n \to w_k w_l$ strongly in $L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)$. Using the triangle inequality and the Cauchy–Schwarz inequality (A.16) gives

$$\begin{aligned}&\left\|w_k^n w_l^n - w_k w_l\right\|_{L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)}\\&\leq \left\|\left(w_k^n - w_k\right) w_l^n\right\|_{L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)} + \left\|w_k\left(w_l^n - w_l\right)\right\|_{L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)}\\&\leq \left\|w_k^n - w_k\right\|_{L^3\left(0,T;L^3(\Omega)\right)}\left\|w_l^n\right\|_{L^3\left(0,T;L^3(\Omega)\right)}\\&\quad + \left\|w_l^n - w_l\right\|_{L^3\left(0,T;L^3(\Omega)\right)}\left\|w_l^k\right\|_{L^3\left(0,T;L^3(\Omega)\right)}.\end{aligned}$$

Because $w_k^n \to w_k$ strongly in $L^3\left(0,T;L^3(\Omega)\right)$, see Lemma 7.81 iii), it follows that $w_k^n w_l^n \to w_k w_l$ strongly in $L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)$ and

$$\lim_{n\to\infty}\int_0^T\int_\Omega w_k^n \frac{\partial w_l^n}{\partial x_k} w_l^n\ d\boldsymbol{x}\ dt = \int_0^T\int_\Omega w_k \frac{\partial w_l}{\partial x_k} w_l\ d\boldsymbol{x}\ dt.$$

It was shown in the proof of Lemma 7.84 that $\boldsymbol{A}\left(\nabla\tilde{\boldsymbol{\phi}}\right) \in L^{3/2}\left(0,T;L^{3/2}(\Omega)\right)$. Since $\nabla\boldsymbol{w}^n \in L^3\left(0,T;L^3(\Omega)\right)$ and $\nabla\boldsymbol{w}^n \rightharpoonup \nabla\boldsymbol{w}$ in that space, see Lemma 7.81 v), one gets

$$\lim_{n\to\infty}\int_0^T\int_\Omega \boldsymbol{A}\left(\nabla\tilde{\boldsymbol{\phi}}\right) : \nabla\boldsymbol{w}^n\ d\boldsymbol{x}\ dt = \int_0^T\int_\Omega \boldsymbol{A}\left(\nabla\tilde{\boldsymbol{\phi}}\right) : \nabla\boldsymbol{w}\ d\boldsymbol{x}\ dt.$$

In addition, it is, using the Cauchy–Schwarz inequality

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T\int_\Omega \partial_t\boldsymbol{w}^n\boldsymbol{w}^n\ d\boldsymbol{x}\ dt\\&= \lim_{n\to\infty}\int_0^T\int_\Omega \partial_t\boldsymbol{w}^n\left(\boldsymbol{w}^n - \boldsymbol{w}\right)\ d\boldsymbol{x}\ dt + \lim_{n\to\infty}\int_0^T\int_\Omega \partial_t\boldsymbol{w}^n\boldsymbol{w}\ d\boldsymbol{x}\ dt\\&\leq \lim_{n\to\infty}\left\|\partial_t\boldsymbol{w}^n\right\|_{L^2\left(0,T;L^2(\Omega)\right)}\left\|\boldsymbol{w}^n - \boldsymbol{w}\right\|_{L^2\left(0,T;L^2(\Omega)\right)} + \lim_{n\to\infty}\int_0^T\int_\Omega \partial_t\boldsymbol{w}^n\boldsymbol{w}\ d\boldsymbol{x}\ dt\\&= 0 + \int_0^T\int_\Omega \partial_t\boldsymbol{w}\boldsymbol{w}\ d\boldsymbol{x}\ dt\end{aligned}$$

by Lemma 7.81 ii) and iv).

Thus, inequality (7.99) is proved for any $\tilde{\boldsymbol{\phi}} \in P^n$. But then it is also valid for arbitrary $\tilde{\boldsymbol{\phi}} \in P$ and also for arbitrary $\tilde{\boldsymbol{\phi}} \in V$ by Lemma 7.86. ■

Lemma 7.90. Identifying $\boldsymbol{B}$ and the nonlinear viscous operator. *For all $\boldsymbol{\phi} \in V$ it holds*

$$\int_0^T\int_\Omega \boldsymbol{B} : \nabla\boldsymbol{\phi}\ d\boldsymbol{x}\ dt = \int_0^T\int_\Omega \boldsymbol{A}\left(\nabla\boldsymbol{w}\right) : \nabla\boldsymbol{\phi}\ d\boldsymbol{x}\ dt.$$

Proof. Since (7.89) is valid for all $\boldsymbol{\phi} \in V$, one can choose $\boldsymbol{\phi} = \boldsymbol{w} - \tilde{\boldsymbol{\phi}}$. Adding (7.99) to this equation yields

$$\int_0^T \int_\Omega \left(\boldsymbol{B} - \boldsymbol{A}\left(\nabla \tilde{\boldsymbol{\phi}}\right)\right) : \left(\nabla \boldsymbol{w} - \nabla \tilde{\boldsymbol{\phi}}\right) \, d\boldsymbol{x} \, dt \geq 0.$$

Setting $\tilde{\boldsymbol{\phi}} = \boldsymbol{w} - \varepsilon \widehat{\boldsymbol{\phi}}$ with $\varepsilon > 0$ and $\widehat{\boldsymbol{\phi}} \in V$ arbitrary gives

$$\varepsilon \int_0^T \int_\Omega \left(\boldsymbol{B} - \boldsymbol{A}\left(\nabla \boldsymbol{w} - \varepsilon \nabla \widehat{\boldsymbol{\phi}}\right)\right) : \nabla \widehat{\boldsymbol{\phi}} \, d\boldsymbol{x} \, dt \geq 0.$$

Division by ε, taking the limit $\varepsilon \to 0$, and using the continuity of the Frobenius norm leads to

$$\int_0^T \int_\Omega \left(\boldsymbol{B} - \boldsymbol{A}\left(\nabla \boldsymbol{w}\right)\right) : \nabla \widehat{\boldsymbol{\phi}} \, d\boldsymbol{x} \, dt \geq 0. \tag{7.102}$$

If $\widehat{\boldsymbol{\phi}} \in V$, then also $-\widehat{\boldsymbol{\phi}} \in V$. Thus, if the integral is positive for $\widehat{\boldsymbol{\phi}}$, then it is negative for $-\widehat{\boldsymbol{\phi}}$ which is a contradiction to (7.102), since (7.102) holds for all functions from V. Hence, it follows that

$$\int_0^T \int_\Omega \left(\boldsymbol{B} - \boldsymbol{A}\left(\nabla \boldsymbol{w}\right)\right) : \nabla \widehat{\boldsymbol{\phi}} \, d\boldsymbol{x} \, dt = 0 \quad \forall\, \widehat{\boldsymbol{\phi}} \in V,$$

which is the statement of the lemma. ■

Theorem 7.91. Existence of a weak solution. *Problem* (7.69) *possesses at least one solution* $\boldsymbol{w} \in V$ *for arbitrary* $\boldsymbol{f} \in L^2\left(0, T; L^2(\Omega)\right)$ *and* $\boldsymbol{w}_0 \in W^{1,3}_{0,\mathrm{div}}(\Omega)$.

Proof. It was shown, starting with Lemma 7.81, that there is a $\boldsymbol{w} \in V$ which satisfies the weak formulation of (7.69). Since $\boldsymbol{w}$ is given as a limit of a sequence $\{\boldsymbol{w}^n\}_{n=1}^\infty$ with $\boldsymbol{w}^n(0, \boldsymbol{x}) = \boldsymbol{w}_0(\boldsymbol{x})$ for all n, it follows that $\boldsymbol{w}(0, \boldsymbol{x}) = \boldsymbol{w}_0(\boldsymbol{x})$, such that the initial condition is satisfied, too. ■

Remark 7.92. Generalizations.

- In Ladyženskaja (1967), the existence of a solution has been proved for the weak formulation of the form

$$\int_0^T \left(\partial_t \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla) \boldsymbol{w}, \boldsymbol{v}\right) + \left((\nu + \nu_{\mathrm{T}} \|\nabla \boldsymbol{w}\|_{\mathrm{F}}^{\mu}) \nabla \boldsymbol{w}, \nabla \boldsymbol{v}\right) \, dt = \int_0^T (\boldsymbol{f}, \boldsymbol{v}) \, dt$$

  with $\mu \geq 2/5$. The restriction on μ comes from the application of the Sobolev embedding to obtain inequality (7.76).
- The result of Ladyzhenskaya could be extended in Du and Gunzburger (1991) to $\mu \geq 1/5$ by deriving new stability (a priori) estimates.
- In Świerczewska (2006), the case of a non-constant parameter in the Smagorinsky model was studied. This parameter function was allowed to depend on filtered functions of $\boldsymbol{w}$ and $\mathbb{D}(\boldsymbol{w})$, see Świerczewska (2006) for details. It was assumed that the parameter function is continuous, it is bounded from below by a positive constant, and it is bounded from above. With these conditions, the existence of a weak solution was proved in Świerczewska (2006).

□

Theorem 7.93. Uniqueness of the weak solution. *Under the same assumptions as in Theorem 7.91, the weak solution of* (7.69) *is unique in* V.

Proof. Assume that there are two weak solutions $\boldsymbol{w}', \boldsymbol{w}'' \in V$ of (7.69) and denote $\tilde{\boldsymbol{w}} = \boldsymbol{w}' - \boldsymbol{w}''$. Thus, $\tilde{\boldsymbol{w}} \in V$ and $\tilde{\boldsymbol{w}}(0, \boldsymbol{x}) = \boldsymbol{0}$. Subtracting (7.69) for $\boldsymbol{w} = \boldsymbol{w}', \boldsymbol{v} = \tilde{\boldsymbol{w}}$ and $\boldsymbol{w} = \boldsymbol{w}'', \boldsymbol{v} = \tilde{\boldsymbol{w}}$ gives

$$\begin{aligned} 0 = \int_0^T & (\partial_t \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{w}}) + \left(\boldsymbol{A}\left(\nabla \boldsymbol{w}'\right) - \boldsymbol{A}\left(\nabla \boldsymbol{w}''\right), \nabla \boldsymbol{w}' - \nabla \boldsymbol{w}''\right) \\ & + n\left(\boldsymbol{w}', \boldsymbol{w}', \tilde{\boldsymbol{w}}\right) - n\left(\boldsymbol{w}'', \boldsymbol{w}'', \tilde{\boldsymbol{w}}\right) \, dt. \end{aligned}$$

Adding and subtracting $n(\boldsymbol{w}'', \boldsymbol{w}', \tilde{\boldsymbol{w}})$, using $n(\boldsymbol{w}'', \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{w}}) = 0$, see Lemma 5.10, and (6.13), this equation can be rewritten in the following form

$$0 = \int_0^T \frac{d}{dt} \|\tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 + 2\left(\boldsymbol{A}\left(\nabla \boldsymbol{w}'\right) - \boldsymbol{A}\left(\nabla \boldsymbol{w}''\right), \nabla \boldsymbol{w}' - \nabla \boldsymbol{w}''\right) + 2n\left(\tilde{\boldsymbol{w}}, \boldsymbol{w}', \tilde{\boldsymbol{w}}\right) \, dt.$$

Using the strong monotonicity property (7.93) of $\boldsymbol{A}(\cdot)$, Hölder's inequality (A.15), the Sobolev embedding $H^1(\Omega) \to L^6(\Omega)$, see (A.30), and Poincaré's inequality (A.17) leads to

$$\begin{aligned} & \int_0^T \frac{d}{dt} \|\tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 + 2\nu \|\nabla \tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 + \frac{\nu_T}{2} \|\nabla \tilde{\boldsymbol{w}}\|_{L^3(\Omega)}^3 \, dt \\ & \le -2 \int_0^T n\left(\tilde{\boldsymbol{w}}, \boldsymbol{w}', \tilde{\boldsymbol{w}}\right) \, dt \\ & \le 2 \int_0^T \|\tilde{\boldsymbol{w}}\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{w}'\right\|_{L^3(\Omega)} \|\tilde{\boldsymbol{w}}\|_{L^6(\Omega)} \, dt \\ & \le C \int_0^T \|\tilde{\boldsymbol{w}}\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{w}'\right\|_{L^3(\Omega)} \|\nabla \tilde{\boldsymbol{w}}\|_{L^2(\Omega)} \, dt. \end{aligned} \tag{7.103}$$

The right-hand side can be estimated further by Young's inequality (A.4)

$$\begin{aligned} & \int_0^T \frac{d}{dt} \|\tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 + 2\nu \|\nabla \tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 + \frac{\nu_T}{2} \|\nabla \tilde{\boldsymbol{w}}\|_{L^3(\Omega)}^3 \, dt \\ & \le \int_0^T 2\nu \|\nabla \tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 + \frac{C^2}{8\nu} \|\tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 \left\|\nabla \boldsymbol{w}'\right\|_{L^3(\Omega)}^2 \, dt. \end{aligned} \tag{7.104}$$

Neglecting terms on the left-hand side, integrating in $(0, T)$, and using $\tilde{\boldsymbol{w}}(0) = \boldsymbol{0}$ yields

$$\|\tilde{\boldsymbol{w}}(T)\|_{L^2(\Omega)}^2 \le C \int_0^T \|\tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 \left\|\nabla \boldsymbol{w}'\right\|_{L^3(\Omega)}^2 \, dt. \tag{7.105}$$

Gronwall's lemma (A.41) now gives $\|\tilde{\boldsymbol{w}}(T)\|_{L^2(\Omega)}^2 \le 0$ for all T which proves the uniqueness of the solution in $L^\infty\left(0, T; L^2(\Omega)\right)$. Since it is $H^1\left(0, T; L^2(\Omega)\right) \subset L^\infty\left(0, T; L^2(\Omega)\right)$ by the Sobolev embedding (A.26) in one dimension, $\tilde{\boldsymbol{w}}$ is in the equivalence class of zero in $H^1\left(0, T; L^2(\Omega)\right)$ and hence the solution is unique in $H^1\left(0, T; L^2(\Omega)\right)$.

Applying now the result $\|\tilde{\boldsymbol{w}}(t)\|_{L^2(\Omega)}^2 = 0$ for almost all t, which implies in particular $\frac{d}{dt} \|\tilde{\boldsymbol{w}}\|_{L^2(\Omega)}^2 = 0$ for almost all t, in (7.104) gives

$$\int_0^T \|\nabla \tilde{\boldsymbol{w}}\|_{L^3(\Omega)}^3 \, dt \le 0$$

for almost all T. Hence $\|\nabla\tilde{\boldsymbol{w}}\|_{L^3(0,T;L^3(\Omega))} = 0$, i.e., $\tilde{\boldsymbol{w}}$ is in the equivalence class of zero in $L^3\left(0,T;W^{1,3}_{0,\mathrm{div}}(\Omega)\right)$, which proves the uniqueness of the solution in $L^3\left(0,T;W^{1,3}_{0,\mathrm{div}}(\Omega)\right)$. ■

Theorem 7.94. Stability of the weak solution. *Let the assumptions of Theorem 7.91 be satified and let* $\boldsymbol{w}', \boldsymbol{w}'' \in V$ *be solutions of* (7.69) *with different initial data and different right-hand sides* $\boldsymbol{f}', \boldsymbol{f}''$. *Then it is*

$$\begin{aligned}
&\|\boldsymbol{w}' - \boldsymbol{w}''\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\le \left(\|\boldsymbol{w}'(0) - \boldsymbol{w}''(0)\|^2_{L^2(\Omega)} + \frac{1}{2C_1(T)}\|\boldsymbol{f}' - \boldsymbol{f}''\|^2_{L^2(0,T;L^2(\Omega))}\right) \\
&\quad \times \exp\left(C_2\|\nabla\boldsymbol{w}'\|^2_{L^2(0,T;L^3(\Omega))} + \frac{C_1(T)}{2}T\right),
\end{aligned}$$

with $C_1(T), C_2 > 0$ *and* $C_1(T)$ *can be chosen arbitrarily.*

If $\boldsymbol{f}' = \boldsymbol{f}''$, *then it holds*

$$\begin{aligned}
&\|\boldsymbol{w}' - \boldsymbol{w}''\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\le \|\boldsymbol{w}'(0) - \boldsymbol{w}''(0)\|^2_{L^2(\Omega)} \exp\left(C_2\|\nabla\boldsymbol{w}'\|^2_{L^2(0,T;L^3(\Omega))}\right).
\end{aligned}$$

Proof. The proof starts in the same way as the proof of Theorem 7.93. One obtains instead of (7.105)

$$\begin{aligned}
\|\tilde{\boldsymbol{w}}(T)\|^2_{L^2(\Omega)} \le \|\tilde{\boldsymbol{w}}(0)\|^2_{L^2(\Omega)} &+ \int_0^T \frac{1}{2C_1(T)}\|\boldsymbol{f}' - \boldsymbol{f}''\|^2_{L^2(\Omega)}\ dt \\
&+ \int_0^T \left(\frac{C^2}{4\nu}\|\nabla\boldsymbol{w}'\|^2_{L^3(\Omega)} + \frac{C_1(T)}{2}\right)\|\tilde{\boldsymbol{w}}\|^2_{L^2(\Omega)}\ dt.
\end{aligned}$$

Setting $C^2/(4\nu) = C_2$ and applying Gronwall's lemma (A.40) proves the statement of the theorem. ■

Remark 7.95. Other analytical investigations of the Smagorinsky model.

- Parés (1992) studied the existence and uniqueness of a weak solution of a Smagorinsky model which differs from (7.66) in some aspects. First, the deformation tensor formulation of the viscous term and the deformation tensor formulation of the Smagorinsky model, i.e., (7.65), were considered. Second, homogeneous Dirichlet boundary conditions are prescribed only at a part of the boundary Γ_{nosl} with $\mathrm{meas}(\Gamma_{\mathrm{nosl}}) > 0$. On the rest of the boundary, slip with friction and penetration with resistance boundary conditions are given, e.g., the linear conditions described in Remark 1.25. These boundary conditions lead to an additional term in the weak formulation of the momentum equation of the Smagorinsky model. The existence proof uses the Galerkin method in the same way as described in the present section. In addition, estimates for the additional term coming from the boundary conditions have to be proved.

- The Smagorinsky model can be used to stabilize the dominating convection in the stationary Navier–Stokes equations (5.1). In Parés (1992), the stationary Smagorinsky model is considered with the same features as the time-dependent model. The existence of weak solutions and the uniqueness in the case of small data could be proved.
- In Du and Gunzburger (1991), the stationary Smagorinsky model was studied with gradient formulation of the viscous term, $\|\nabla \boldsymbol{w}\|_{\mathrm{F}}$, and homogeneous Dirichlet boundary conditions. The existence of a unique weak solution for small data was proved.
- Chacón Rebollo and Lewandowski (2014), summarize results for steady-state case in one item, mention non-vanishing for laminar flows

□

7.3.3 Finite Element Error Analysis for the Time-Continuous Case

Remark 7.96. Contents. This section presents a finite element error analysis of the continuous-in-time discretization of the Smagorinsky model from John and Layton (2002), see also (John, 2004, Section 8.1). The error of a finite element discretization of the Smagorinsky model to the continuous Smagorinsky model is analyzed (and not to the Navier–Stokes equations). In this analysis, a generalization of the Smagorinsky model and pairs of inf-sup stable finite element spaces are considered. In addition, other boundary conditions than no-slip conditions are allowed on a part of the boundary. The goal of the analysis consists in deriving uniform estimates, i.e., with right-hand sides which do not depend on the viscosity ν, with as weak assumptions on the regularity of the solution of the continuous Smagorinsky model as possible. In this respect, the obtained estimates are better than for the Navier–Stokes equations, see Theorem 6.46. Besides the existence and uniqueness of a weak solution of the continuous problem, these estimates are another indicator that the complexity of the Smagorinsky model is smaller than of the Navier–Stokes equations, see Remark 7.22. □

The Continuous Problem

Remark 7.97. The strong formulation of the problem. Let Ω be a bounded, simply connected domain in $\mathbb{R}^d$, $d \in \{2,3\}$, with polygonal or polyhedral Lipschitz boundary. Suppose, the boundary $\partial\Omega$ is composed of faces (straight lines or parts of faces) $\Gamma_0, \ldots, \Gamma_J$ with $\operatorname{meas}(\Gamma_0) > 0$.

The Smagorinsky model is considered with slip with linear friction and no penetration boundary conditions, see Remark 1.25,

$$\begin{aligned}
\partial_t \boldsymbol{w} - \nabla \cdot ((2\nu + \nu_{\mathrm{T}}) \mathbb{D}(\boldsymbol{w})) \\
+ (\boldsymbol{w} \cdot \nabla) \boldsymbol{w} + \nabla r &= \boldsymbol{f} \quad \text{in } (0,T] \times \Omega, \\
\nabla \cdot \boldsymbol{w} &= 0 \quad \text{in } [0,T] \times \Omega, \\
\boldsymbol{w} &= \boldsymbol{0} \quad \text{in } [0,T] \times \Gamma_0, \\
\boldsymbol{w} \cdot \boldsymbol{n}_{\partial\Omega} &= 0 \quad \text{in } [0,T] \times \Gamma_j, j = 1, \ldots, J, \\
\boldsymbol{w} \cdot \boldsymbol{\tau}_{j,k} + \beta^{-1} \boldsymbol{n}_{\partial\Omega}^T \mathbb{S}_{\mathrm{sma}}(\boldsymbol{w}, r) \boldsymbol{\tau}_{j,k} &= 0 \quad \text{in } [0,T] \times \Gamma_j, j = 1, \ldots, J, \\
\boldsymbol{w}(0, \cdot) &= \boldsymbol{w}_0 \text{ in } \Omega, \\
\int_\Omega r \, d\boldsymbol{x} &= 0 \quad \text{in } (0,T].
\end{aligned} \tag{7.106}$$

Here, $\{\boldsymbol{\tau}_{j,k}\}_{k=1}^{d-1}$ is an orthonormal system of tangential vectors in each point of Γ_j, $1, \ldots, J$, and

$$\mathbb{S}_{\mathrm{sma}}(\boldsymbol{w}, r) = (2\nu + \nu_{\mathrm{T}}) \mathbb{D}(\boldsymbol{w}) - r\mathbb{I}.$$

The turbulent viscosity is given by

$$\nu_{\mathrm{T}} = \nu_0(\delta) + C_S \delta^2 \|\mathbb{D}(\boldsymbol{w})\|_{\mathrm{F}}, \quad \nu_0(\delta) \geq 0,$$

which is a generalization of (7.64). □

Remark 7.98. The variational formulation. Let

$$\begin{aligned}
V &= \left\{ \boldsymbol{v} \in W^{1,3}(\Omega), \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_0, \boldsymbol{v} \cdot \boldsymbol{n}_{\partial\Omega} = 0 \text{ on } \Gamma_j, j = 1, \ldots, J \right\}, \\
Q &= L_0^2(\Omega).
\end{aligned}$$

To simplify the notation, whenever $\boldsymbol{\tau}_j$ occurs, it will be understood that the term is summed over the two tangential vectors if $d = 3$, i.e.,

$$\|\boldsymbol{v} \cdot \boldsymbol{\tau}_j\|_{L^2(\Gamma_j)}^2 := \|\boldsymbol{v} \cdot \boldsymbol{\tau}_{j,1}\|_{L^2(\Gamma_j)}^2 + \|\boldsymbol{v} \cdot \boldsymbol{\tau}_{j,2}\|_{L^2(\Gamma_j)}^2.$$

The variational formulation of (7.106) is derived in the usual way by multiplying the equations with test functions, integrating on Ω and applying integration by parts. Applying this procedure to the viscous and pressure term gives an integral on the boundary. Using the decomposition

$$\boldsymbol{v} = \sum_{j=1}^{J} (\boldsymbol{v} \cdot \boldsymbol{n}_{\partial\Omega}) \boldsymbol{n}_{\partial\Omega} + (\boldsymbol{v} \cdot \boldsymbol{\tau}_j) \boldsymbol{\tau}_j$$

and the definition of the boundary condition gives for this term

$$\begin{aligned}
&\int_{\partial\Omega} -\boldsymbol{n}_{\partial\Omega}^T \left(\left(2\nu + \nu_{\mathrm{T}}\right)\mathbb{D}\left(\boldsymbol{w}\right) + r\mathbb{I}\right)\boldsymbol{v}\ d\boldsymbol{s} \\
&= \sum_{j=1}^{J} \int_{\Gamma_j} -\boldsymbol{n}_{\partial\Omega}^T \mathbb{S}_{\mathrm{sma}}\left(\boldsymbol{w}, r\right)\boldsymbol{\tau}_j\left(\boldsymbol{v}\cdot\boldsymbol{\tau}_j\right)\ d\boldsymbol{s} \\
&= \sum_{j=1}^{J} \int_{\Gamma_j} \beta\left(\boldsymbol{w}\cdot\boldsymbol{\tau}_j\right)\left(\boldsymbol{v}\cdot\boldsymbol{\tau}_j\right)\ d\boldsymbol{s}.
\end{aligned}$$

The variational problem reads as follows: Find $(\boldsymbol{w}, r) \in V \times Q$ such that for all $t \in (0, T]$ and all $(\boldsymbol{v}, q) \in V \times Q$

$$(\partial_t \boldsymbol{w}, \boldsymbol{v}) + a\left(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}\right) + n_{\mathrm{skew}}\left(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{v}\right) + \left(\nabla\cdot\boldsymbol{w}, q\right) - \left(\nabla\cdot\boldsymbol{v}, r\right) = \left(\boldsymbol{f}, \boldsymbol{v}\right) \tag{7.107}$$

with

$$\begin{aligned}
a\left(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}\right) = &\left(\left(2\nu + \nu_0\left(\delta\right) + C_S\delta^2 \left\|\mathbb{D}\left(\boldsymbol{u}\right)\right\|_{\mathrm{F}}\right)\mathbb{D}\left(\boldsymbol{w}\right), \mathbb{D}\left(\boldsymbol{v}\right)\right) \\
&+\mu\left(\nabla\cdot\boldsymbol{w}, \nabla\cdot\boldsymbol{v}\right) + \sum_{j=1}^{J} \beta\left(\boldsymbol{w}\cdot\boldsymbol{\tau}_j, \boldsymbol{v}\cdot\boldsymbol{\tau}_j\right)_{\Gamma_j},
\end{aligned}$$

where $\mu > 0$ is given, and $\boldsymbol{w}\left(0, \boldsymbol{x}\right) = \boldsymbol{w}_0\left(\boldsymbol{x}\right)$. Note that the skew-symmetric form of the nonlinear convective term is for the continuous problem equivalent to the standard convective form, see Remark 5.8, and that the grad-div stabilization vanishes since $\boldsymbol{w}$ is weakly divergence-free. □

Remark 7.99. Some tools for the analysis. Since $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$ for all $\boldsymbol{v} \in V$, Poincare's inequality (A.17) holds in V. In addition, also Korn's inequality holds in V, i.e.,

$$\|\nabla \boldsymbol{v}\|_{L^3(\Omega)} \le C \|\mathbb{D}\left(\boldsymbol{v}\right)\|_{L^3(\Omega)} \tag{7.108}$$

by taking $|\boldsymbol{v}| = \|\boldsymbol{v}\|_{L^p(\Gamma_0)}$ in (A.18). The dual space V' of V is equipped with the norm

$$\|\boldsymbol{\phi}\|_{V'} := \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{\displaystyle\int_\Omega \boldsymbol{\phi}\cdot\boldsymbol{v}\ d\boldsymbol{x}}{\|\mathbb{D}\left(\boldsymbol{v}\right)\|_{L^3(\Omega)}}. \tag{7.109}$$

Note that $\|\mathbb{D}\left(\boldsymbol{v}\right)\|_{L^3(\Omega)}$ defines a norm in V as a consequence of Poincare's and Korn's inequality. In the case $\partial\Omega = \Gamma_0$, one has $V' = W^{-1,3/2}\left(\Omega\right)$ equipped with the norm (7.109). □

Lemma 7.100. Strong monotonicity of the nonlinear viscous term. *There is a constant $\underline{C}$ such that for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in W^{1,3}\left(\Omega\right)$*

$$\left(\left\|\mathbb{D}\left(\boldsymbol{u}\right)\right\|_{\mathrm{F}}\mathbb{D}\left(\boldsymbol{u}\right) - \left\|\mathbb{D}\left(\boldsymbol{v}\right)\right\|_{\mathrm{F}}\mathbb{D}\left(\boldsymbol{v}\right), \mathbb{D}\left(\boldsymbol{u}-\boldsymbol{v}\right)\right) \ge \underline{C}\left\|\mathbb{D}\left(\boldsymbol{u}-\boldsymbol{v}\right)\right\|_{L^3(\Omega)}^3, \tag{7.110}$$

i.e., the strong monotonicity holds.

Proof. The proof of Lemma 7.88 can be performed also for the deformation tensor instead of the gradient. All partial derivatives have to be replaced by the respective entries of the deformation tensor, e.g., $\frac{\partial \boldsymbol{w}_i'}{\partial x_j}$ by $\mathbb{D}_{ij}(\boldsymbol{w}')$. ■

Lemma 7.101. Norm equivalence for a tensor. *Let $A \in L^3(\Omega)$ with $A(\boldsymbol{x}) \in \mathbb{R}^{d\times d}$ for every $\boldsymbol{x} \in \Omega$, then*

$$\|A\|_{L^3(\Omega)} \le \|\|A\|_{\mathrm{F}}\|_{L^3(\Omega)} \le C(d)\,\|A\|_{L^3(\Omega)}\,. \tag{7.111}$$

Proof. It is

$$\|A\|^3_{L^3(\Omega)} = \int_\Omega \sum_{i,j=1}^d |a_{ij}|^3 \ d\boldsymbol{x}, \quad \|\|A\|_{\mathrm{F}}\|^3_{L^3(\Omega)} = \int_\Omega \left(\sum_{i,j=1}^d a_{ij}^2\right)^{3/2} d\boldsymbol{x}.$$

Since all matrix norms are equivalent, there are constants $0 < C_1(d) < C_2(d)$ such that

$$C_1(d)\left(\sum_{i,j=1}^d |a_{ij}|^3\right)^{1/3} \le \left(\sum_{i,j=1}^d a_{ij}^2\right)^{1/2} \le C_2(d)\left(\sum_{i,j=1}^d |a_{ij}|^3\right)^{1/3}. \tag{7.112}$$

From (A.5), it follows with $p = 3/2$ that one can choose $C_1(d) = 1$. Raising (7.112) to the power 3 and integrating on Ω proves (7.111). ■

Lemma 7.102. Local Lipschitz continuity of the nonlinear viscous term. *There is a constant $\overline{C}$ such that for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in W^{1,3}(\Omega)$*

$$\begin{aligned}&\left(\|\mathbb{D}(\boldsymbol{u})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{u}) - \|\mathbb{D}(\boldsymbol{v})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{w})\right)\\ &\le \overline{C}C_L\,\|\mathbb{D}(\boldsymbol{u}-\boldsymbol{v})\|_{L^3(\Omega)}\,\|\mathbb{D}(\boldsymbol{w})\|_{L^3(\Omega)}\end{aligned} \tag{7.113}$$

with $C_L = \max\left\{\|\mathbb{D}(\boldsymbol{u})\|_{L^3(\Omega)}, \|\mathbb{D}(\boldsymbol{v})\|_{L^3(\Omega)}\right\}$, which is the so-called local Lipschitz continuity.

Proof. Applying Hölder's inequality (A.15) gives

$$\begin{aligned}&\big(\|\mathbb{D}(\boldsymbol{u})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{u}) - \|\mathbb{D}(\boldsymbol{v})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{w})\big)\\ &= \big(\|\mathbb{D}(\boldsymbol{u})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{u}) - \|\mathbb{D}(\boldsymbol{u})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{w})\big) + \big(\|\mathbb{D}(\boldsymbol{u})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{v}) - \|\mathbb{D}(\boldsymbol{v})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{v}), \mathbb{D}(\boldsymbol{w})\big)\\ &\le \big\|\|\mathbb{D}(\boldsymbol{u})\|_{\mathrm{F}}\big\|_{L^3(\Omega)}\,\|\mathbb{D}(\boldsymbol{u}-\boldsymbol{v})\|_{L^3(\Omega)}\,\|\mathbb{D}(\boldsymbol{w})\|_{L^3(\Omega)}\\ &\quad + \big\|\|\mathbb{D}(\boldsymbol{u})\|_{\mathrm{F}} - \|\mathbb{D}(\boldsymbol{v})\|_{\mathrm{F}}\big\|_{L^3(\Omega)}\,\|\mathbb{D}(\boldsymbol{v})\|_{L^3(\Omega)}\,\|\mathbb{D}(\boldsymbol{w})\|_{L^3(\Omega)}\,.\end{aligned} \tag{7.114}$$

Using the triangle inequality and (7.111) yields

$$\big\|\|\mathbb{D}(\boldsymbol{u})\|_{\mathrm{F}} - \|\mathbb{D}(\boldsymbol{v})\|_{\mathrm{F}}\big\|_{L^3(\Omega)} \le \big\|\|\mathbb{D}(\boldsymbol{u}) - \mathbb{D}(\boldsymbol{v})\|_{\mathrm{F}}\big\|_{L^3(\Omega)} \le C\,\|\mathbb{D}(\boldsymbol{u}-\boldsymbol{v})\|_{L^3(\Omega)}\,.$$

Inserting this estimate into (7.114) proves the local Lipschitz continuity. ■

Lemma 7.103. Energy inequality for $\boldsymbol{w}$. *Any solution of* (7.107) *satisfies*

$$\frac{1}{2}\left\|\boldsymbol{w}(T)\right\|_{L^2(\Omega)}^2+\int_0^T\left(\sum_{j=1}^J\beta\left\|\boldsymbol{w}\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2+\left(2\nu+\nu_0(\delta)\right)\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^2(\Omega)}^2\right.$$
$$\left.+\underline{C}C_S\delta^2\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(\Omega)}^3\right)dt$$
$$\leq\frac{1}{2}\left\|\boldsymbol{w}_0\right\|_{L^2(\Omega)}^2+\int_0^T(\boldsymbol{f},\boldsymbol{w})\ dt. \qquad (7.115)$$

Proof. not presented in the course Choosing $(\boldsymbol{v},q)=(\boldsymbol{w},r)$ in (7.107) gives

$$(\partial_t\boldsymbol{w},\boldsymbol{w})+a(\boldsymbol{w},\boldsymbol{w},\boldsymbol{w})+n_{\text{skew}}(\boldsymbol{w},\boldsymbol{w},\boldsymbol{w})+(\nabla\cdot\boldsymbol{w},r)-(\nabla\cdot\boldsymbol{w},r)=(\boldsymbol{f},\boldsymbol{w}).$$

The skew symmetric nonlinear convective term vanishes, see (5.18).

Since every $q\in L^2(\Omega)$ admits a decomposition $q=q_0+C$ with $q_0\in Q$ and C is a constant, see Remark 3.66, it follows that

$$\begin{aligned}(\nabla\cdot\boldsymbol{v},q)&=(\nabla\cdot\boldsymbol{v},q_0+C)=(\nabla\cdot\boldsymbol{v},C)\\&=-\int_\Gamma C\boldsymbol{v}\cdot\boldsymbol{n}_{\partial\Omega}\ d\boldsymbol{s}-(\boldsymbol{v},\nabla C)=0\quad\forall\ \boldsymbol{v}\in V_{\text{div}},q\in L^2(\Omega),\end{aligned} \qquad (7.116)$$

because $\boldsymbol{v}\cdot\boldsymbol{n}_{\partial\Omega}=0$ on the whole boundary. Thus, $\mu(\nabla\cdot\boldsymbol{w},\nabla\cdot\boldsymbol{w})$ vanishes because $\boldsymbol{w}\in V_{\text{div}}$, $\nabla\cdot\boldsymbol{w}\in L^2(\Omega)$, and (7.116).

In addition, it follows from (7.110) that

$$\left(\left\|\mathbb{D}(\boldsymbol{w})\right\|_{\text{F}}\mathbb{D}(\boldsymbol{w}),\mathbb{D}(\boldsymbol{v})\right)\geq\underline{C}\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(\Omega)}^3$$

such that

$$(\partial_t\boldsymbol{w},\boldsymbol{w})+\left(2\nu+\nu_0(\delta)\right)\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^2(\Omega)}^2$$
$$+\underline{C}C_S\delta^2\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(\Omega)}^3+\sum_{j=1}^J\beta\left\|\boldsymbol{w}\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2\leq(\boldsymbol{f},\boldsymbol{w}).$$

Integration on $(0,T)$ gives the statement of the lemma. ■

Lemma 7.104. Stability estimates for $\boldsymbol{w}$ uniformly in ν. *The velocity component $\boldsymbol{w}$ of the solution of* (7.107) *satisfies for* $T>0$

$$\frac{1}{2}\left\|\boldsymbol{w}(T)\right\|_{L^2(\Omega)}^2+\int_0^T\left(\sum_{j=1}^J\beta\left\|\boldsymbol{w}\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2+\left(2\nu+\nu_0(\delta)\right)\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^2(\Omega)}^2\right.$$
$$\left.+\frac{2}{3}\underline{C}C_S\delta^2\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(\Omega)}^3\right)dt \qquad (7.117)$$
$$\leq\frac{1}{2}\left\|\boldsymbol{w}_0\right\|_{L^2(\Omega)}^2+\frac{2}{3}\left(\underline{C}C_S\right)^{-1/2}\delta^{-1}\left(\sup_{\boldsymbol{v}\in L^3(0,T;V)}\frac{\int_0^T(\boldsymbol{f},\boldsymbol{v})\ dt}{\left\|\mathbb{D}(\boldsymbol{v})\right\|_{L^3(0,T;L^3(\Omega))}}\right)^{3/2}$$

and

$$
\begin{aligned}
&\frac{1}{2}\left\|\boldsymbol{w}(T)\right\|_{L^2(\Omega)}^2+\int_0^T e^{T-t}\left(\sum_{j=1}^{J}\beta\left\|\boldsymbol{w}\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2\right.\\
&\qquad\left.+\left(2\nu+\nu_0(\delta)\right)\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^2(\Omega)}^2+\underline{C}C_S\delta^2\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(\Omega)}^3\right)dt\\
&\le\frac{e^T}{2}\left\|\boldsymbol{w}_0\right\|_{L^2(\Omega)}^2+\frac{1}{2}\int_0^T e^{T-t}\left\|\boldsymbol{f}\right\|_{L^2(\Omega)}^2\,dt. \qquad (7.118)
\end{aligned}
$$

Proof. not presented in the course (7.117): This estimate is proved by an application of Young's inequality (A.4) and the first stability estimate (7.115). It is

$$
\begin{aligned}
\int_0^T(\boldsymbol{f},\boldsymbol{w})\,dt&=\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(0,T;L^3(\Omega))}\frac{\int_0^T(\boldsymbol{f},\boldsymbol{w})\,dt}{\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(0,T;L^3(\Omega))}}\\
&\le\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(0,T;L^3(\Omega))}\sup_{\boldsymbol{v}\in L^3(0,T;L^3(\Omega))}\frac{\int_0^T(\boldsymbol{f},\boldsymbol{v})\,dt}{\left\|\mathbb{D}(\boldsymbol{v})\right\|_{L^3(0,T;L^3(\Omega))}}\\
&\le\frac{\underline{C}C_S\delta^2}{3}\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(0,T;L^3(\Omega))}^3\\
&\quad+\frac{2}{3}\left(\underline{C}C_S\right)^{-1/2}\delta^{-1}\left(\sup_{\boldsymbol{v}\in L^3(0,T;L^3(\Omega))}\frac{\int_0^T(\boldsymbol{f},\boldsymbol{v})\,dt}{\left\|\mathbb{D}(\boldsymbol{v})\right\|_{L^3(0,T;L^3(\Omega))}}\right)^{3/2}.
\end{aligned}
$$

Absorbing the first term on the right-hand side into (7.115) gives (7.117).

(7.118): In the same way as in the proof of Lemma 7.103, one obtains with the Cauchy–Schwarz inequality (A.16) and Young's inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^2+\left(2\nu+\nu_0(\delta)\right)\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^2(\Omega)}^2\\
&+\underline{C}C_S\delta^2\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(\Omega)}^3+\sum_{j=1}^{J}\beta\left\|\boldsymbol{w}\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2\le\frac{\left\|\boldsymbol{f}\right\|_{L^2(\Omega)}^2}{2}+\frac{\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^2}{2}.
\end{aligned}
$$

Multiplying this equality with e^{T-t}, integrating on $(0,T)$, and applying integration by parts yields

$$
\begin{aligned}
&\int_0^T e^{T-t}\left(\frac{d}{dt}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^2-\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^2\right)dt\\
&=e^{T-t}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^2\Big|_{t=0}^{t=T}+\int_0^T e^{T-t}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^2\,dt-\int_0^T e^{T-t}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^2\,dt\\
&=\left\|\boldsymbol{w}(T)\right\|_{L^2(\Omega)}^2-e^T\left\|\boldsymbol{w}(0)\right\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Using this equality, (7.118) is proved. ■

Remark 7.105. On the stability estimate (7.117). The right-hand side of the stability estimate (7.117) is independent of ν but it depends on inverse powers of δ. Thus, if $\delta\to 0$, the right-hand side blows up. This behavior is the natural one since otherwise one would find in the limit a uniform stability estimate for the solution of the Navier–Stokes equations. □

Lemma 7.106. Stability of $\partial_t \boldsymbol{w}$ uniformly in ν. *Let $(\boldsymbol{w}, r)$ be a solution of (7.107). Then, there is a constant C independent of ν such that for almost all $t \in [0,T]$*

$$\|\partial_t \boldsymbol{w}\|_{V'} \le C\Big(\|\boldsymbol{w}\|^2_{L^3(\Omega)} + \|r\|_{L^{3/2}(\Omega)} + (2\nu + \nu_0(\delta)) \|\mathbb{D}(\boldsymbol{w})\|_{L^{3/2}(\Omega)} + C_S\delta^2 \|\mathbb{D}(\boldsymbol{w})\|^2_{L^3(\Omega)} + \|\boldsymbol{f}\|_{V'}\Big)$$

and

$$\|\partial_t \boldsymbol{w}\|^{3/2}_{L^{3/2}(0,T;V')} \le C\Big(\|\boldsymbol{w}\|^3_{L^3(0,T;L^3(\Omega))} + \|r\|^{3/2}_{L^{3/2}(0,T;L^{3/2}(\Omega))} + (2\nu + \nu_0(\delta)) \|\mathbb{D}(\boldsymbol{w})\|^{3/2}_{L^{3/2}(0,T;L^{3/2}(\Omega))} + C_S\delta^2 \|\mathbb{D}(\boldsymbol{w})\|^3_{L^3(0,T;L^3(\Omega))} + \|\boldsymbol{f}\|^{3/2}_{L^{3/2}(0,T;V')}\Big).$$

Proof. Multiplying (7.106) with $\boldsymbol{v}$, integrating over Ω, dividing the resulting equation by $\|\mathbb{D}(\boldsymbol{v})\|_{L^3(\Omega)}$, taking the supremum over $\boldsymbol{v} \in V$, and applying the triangle inequality gives

$$\|\partial_t \boldsymbol{w}\|_{V'} \le \left\|\nabla\cdot\left(\boldsymbol{w}\boldsymbol{w}^T\right)\right\|_{V'} + \|\nabla r\|_{V'} + C_S\delta^2 \left\|\nabla\cdot\left(\|\mathbb{D}(\boldsymbol{w})\|_{\mathrm{F}}\,\mathbb{D}(\boldsymbol{w})\right)\right\|_{V'} + (2\nu + \nu_0(\delta)) \|\nabla\cdot\mathbb{D}(\boldsymbol{w})\|_{V'} + \|\boldsymbol{f}\|_{V'}.$$

The definition of the norm (7.109), integration by parts, using $\boldsymbol{v}\cdot\boldsymbol{n}_{\partial\Omega} = 0$ on $\partial\Omega$ for $\boldsymbol{v} \in V$, Hölder's inequality (A.15), estimating the norm of the divergence by the norm of the gradient, and Korn's inequality (7.108) leads to, e.g.,

$$\|\nabla r\|_{V'} = \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{-(\nabla\cdot\boldsymbol{v}, r)}{\|\mathbb{D}(\boldsymbol{v})\|_{L^3(\Omega)}} \le \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{\|r\|_{L^{3/2}(\Omega)}\|\nabla\cdot\boldsymbol{v}\|_{L^3(\Omega)}}{\|\mathbb{D}(\boldsymbol{v})\|_{L^3(\Omega)}} \le C \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{\|r\|_{L^{3/2}(\Omega)}\|\nabla\boldsymbol{v}\|_{L^3(\Omega)}}{\|\mathbb{D}(\boldsymbol{v})\|_{L^3(\Omega)}} \le C\|r\|_{L^{3/2}(\Omega)}.$$

Note that an estimate of the norm of the divergence by the norm of the gradient of form (2.41) can be proved for any $L^p(\Omega)$ norm by using Hölder's inequality instead of the Cauchy–Schwarz inequality in the proof of (2.41). not presented in the course The other terms are estimated in the same way, using the following estimates. With the Cauchy–Schwarz (A.16) inequality, it is

$$\left\|\boldsymbol{w}\boldsymbol{w}^T\right\|_{L^{3/2}(\Omega)} = \left(\int_\Omega \sum_{i,j=1}^d |w_i w_j|^{3/2}\ d\boldsymbol{x}\right)^{2/3} \le \left(\int_\Omega \left(\sum_{i=1}^d |w_i|^3\right)^{1/2}\left(\sum_{i=j}^d |w_j|^3\right)^{1/2} d\boldsymbol{x}\right)^{2/3} = \left(\int_\Omega \sum_{i=1}^d |w_i|^3\ d\boldsymbol{x}\right)^{2/3} = \|\boldsymbol{w}\|^2_{L^3(\Omega)}.$$

Using Hölder's inequality and (7.111), one finds

$$\begin{aligned}\left\| \|\mathbb{D}(\boldsymbol{w})\|_{\mathrm{F}} \mathbb{D}(\boldsymbol{w}) \right\|_{L^{3/2}(\Omega)} &= \left(\int_\Omega \sum_{i,j=1}^{d} \left| \|\mathbb{D}(\boldsymbol{w})\|_{\mathrm{F}} \mathbb{D}_{ij}(\boldsymbol{w}) \right|^{3/2} \, d\boldsymbol{x} \right)^{2/3} \\ &\le C \left(\int_\Omega \|\mathbb{D}(\boldsymbol{w})\|_{\mathrm{F}}^{3/2} \left(\sum_{i,j=1}^{d} |\mathbb{D}_{ij}(\boldsymbol{w})|^2 \right)^{3/4} d\boldsymbol{x} \right)^{2/3} \\ &= C \left(\int_\Omega \|\mathbb{D}(\boldsymbol{w})\|_{\mathrm{F}}^{3/2} \|\mathbb{D}(\boldsymbol{w})\|_{\mathrm{F}}^{3/2} \, d\boldsymbol{x} \right)^{2/3} \\ &= C \left\| \|\mathbb{D}(\boldsymbol{w})\|_{\mathrm{F}} \right\|_{L^3(\Omega)}^2 \le C \|\mathbb{D}(\boldsymbol{w})\|_{L^3(\Omega)}^2 .\end{aligned}$$

The second inequality stated in the lemma follows by raising both sides of the first estimate to the power $3/2$ and integrating in time and absorbing some powers of coefficients and parameters into the constant. ■

The Finite Element Problem

Remark 7.107. Finite element spaces. The discrete problem is defined in finite element spaces V^h and Q^h with $V^h \subset V$ and $Q^h \subset Q$. It will be assumed that the spaces V^h and Q^h satisfy the discrete inf-sup condition

$$\inf_{\substack{q^h \in Q^h \\ q^h \neq 0}} \sup_{\substack{\boldsymbol{v}^h \in V^h \\ \boldsymbol{v}^h \neq \boldsymbol{0}}} \frac{\left(\nabla \cdot \boldsymbol{v}^h, q^h\right)}{\left\|q^h\right\|_{L^2(\Omega)} \left(\left\|\mathbb{D}\left(\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}^2 + \sum_{j=1}^{J} \left\|\boldsymbol{v}^h \cdot \boldsymbol{\tau}_j\right\|_{H^{1/2}(\Gamma_j)}^2 \right)^{1/2}} \ge \beta_{\mathrm{is,sma}}^h, \tag{7.119}$$

where $\beta_{\mathrm{is,sma}} > 0$ is independent of h. is this needed? □

Lemma 7.108. The standard discrete inf-sup condition implies condition (7.119)**.** *If* $\left(V^h, Q^h\right)$ *satisfies the discrete inf-sup condition* (2.48) *then* (7.119) *holds.*

Proof. By trace theorem, Grisvard (1992), genauer and the Poincaré inequality (A.17), it follows for any $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$, $\left(\boldsymbol{v}^h, q^h\right) \neq (\boldsymbol{0}, 0)$, that

$$\begin{aligned}&\frac{\left(\nabla \cdot \boldsymbol{v}^h, q^h\right)}{\left\|q^h\right\|_{L^2(\Omega)} \left[\left\|\mathbb{D}\left(\boldsymbol{v}^h\right)\right\|_{L^2(\Omega)}^2 + \sum_{j=1}^{J} \left\|\boldsymbol{v}^h \cdot \boldsymbol{\tau}_j\right\|_{H^{1/2}(\Gamma_j)}^2 \right]^{1/2}} \\ &\ge \frac{\left(\nabla \cdot \boldsymbol{v}^h, q^h\right)}{\left\|q^h\right\|_{L^2(\Omega)} \left(\left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)}^2 + C \left\|\boldsymbol{v}^h\right\|_{H^1(\Omega)}^2 \right)^{1/2}} \\ &\ge C \frac{\left(\nabla \cdot \boldsymbol{v}^h, q^h\right)}{\left\|q^h\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)}},\end{aligned}$$

which gives the statement of the lemma. ■

Remark 7.109. The continuous-in-time finite element problem. The continuous-in-time finite element problem reads as follows: Find $\left(\boldsymbol{w}^h, r^h\right) \in V^h \times Q^h$ such that

$$\begin{aligned} \left(\partial_t \boldsymbol{w}^h, \boldsymbol{v}^h\right) + a\left(\boldsymbol{w}^h, \boldsymbol{w}^h, \boldsymbol{v}^h\right) + n_{\text{skew}}\left(\boldsymbol{w}^h, \boldsymbol{w}^h, \boldsymbol{v}^h\right) \\ + \left(\nabla \cdot \boldsymbol{w}^h, q^h\right) - \left(\nabla \cdot \boldsymbol{v}^h, r^h\right) = \left(\boldsymbol{f}, \boldsymbol{v}^h\right) \end{aligned} \tag{7.120}$$

for all $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$ where $\boldsymbol{w}^h(0, \boldsymbol{x})$ is an approximation to $\boldsymbol{w}_0(\boldsymbol{x})$. □

Lemma 7.110. Energy inequality for the finite element solution. *A solution of* (7.120) *satisfies*

$$\begin{aligned} &\frac{1}{2}\left\|\boldsymbol{w}^h(T)\right\|_{L^2(\Omega)}^2 + \int_0^T \left(\mu \left\|\nabla \cdot \boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 + \sum_{j=1}^J \beta \left\|\boldsymbol{w}^h \cdot \boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2 \right. \\ &\quad \left. + \left(2\nu + \nu_0(\delta)\right)\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^2(\Omega)}^2 + \underline{C} C_S \delta^2 \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^3 \right) dt \\ &\le \frac{1}{2}\left\|\boldsymbol{w}_0\right\|_{L^2(\Omega)}^2 + \int_0^T \left(\boldsymbol{f}, \boldsymbol{w}^h\right) dt. \end{aligned}$$

Proof. The proof proceeds in the same way as the proof of Lemma 7.103. ■

Theorem 7.111. Stability estimates for $\boldsymbol{w}^h$ uniformly in ν. *The velocity component $\boldsymbol{w}^h$ of the solution of* (7.120) *satisfies for $T > 0$*

$$\begin{aligned} &\frac{1}{2}\left\|\boldsymbol{w}^h(T)\right\|_{L^2(\Omega)}^2 + \int_0^T \left(\mu \left\|\nabla \cdot \boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 + \sum_{j=1}^J \beta \left\|\boldsymbol{w}^h \cdot \boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2 \right. \\ &\quad \left. + \left(2\nu + \nu_0(\delta)\right)\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^2(\Omega)}^2 + \frac{2}{3}\underline{C} C_S \delta^2 \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^3 \right) dt \qquad (7.121) \\ &\le \frac{1}{2}\left\|\boldsymbol{w}_0^h\right\|_{L^2(\Omega)}^2 + \frac{2}{3}\left(\underline{C} C_S\right)^{-1/2} \delta^{-1} \left(\sup_{\boldsymbol{v} \in L^3(0,T;V)} \frac{\int_0^T (\boldsymbol{f}, \boldsymbol{v})\, dt}{\|\mathbb{D}(\boldsymbol{v})\|_{L^3(0,T;L^3(\Omega))}}\right)^{3/2}, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\left\|\boldsymbol{w}^h(T)\right\|_{L^2(\Omega)}^2 + \int_0^T e^{T-t}\left(\mu \left\|\nabla \cdot \boldsymbol{w}^h\right\|_{L^2(\Omega)}^2 + \sum_{j=1}^J \beta \left\|\boldsymbol{w}^h \cdot \boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2 \right. \\ &\quad \left. + \left(2\nu + \nu_0(\delta)\right)\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^2(\Omega)}^2 + \underline{C} C_S \delta^2 \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^3 \right) dt \\ &\le \frac{e^T}{2}\left\|\boldsymbol{w}_0^h\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\int_0^T e^{T-t}\left\|\boldsymbol{f}\right\|_{L^2(\Omega)}^2 dt. \qquad (7.122) \end{aligned}$$

Proof. The proof is performed in the same way as the proof of Lemma 7.104. ■

Remark 7.112. Goal and summary of the finite element error analysis. The goal of the analysis is to estimate the error $\left\|\boldsymbol{w} - \boldsymbol{w}^h\right\|$ in appropriate norms under consideration of the following aspects:

- The error estimate should be independent of ν.
- The assumption on the regularity of the solution of the variational problem (7.107) should be as weak as possible.

The following results will be presented.

- First, the natural regularity

$$\nabla \boldsymbol{w} \in L^3\left(0, T; L^3\left(\Omega\right)\right), \tag{7.123}$$

  for the variational formulation of the Smagorinsky model will be assumed. This regularity ensures that the Smagorinsky term is well defined, see (7.68). A finite element error analysis, using a standard approach sketched in Remark 7.113, can be performed for the case $\nu_0\left(\delta\right) > 0$. The error estimate is given in Theorem 7.119.
- Second, the case $\nu_0\left(\delta\right) = 0$ is discussed for the regularity assumption (7.123), see Remark 7.120. It will be shown that Gronwall's lemma cannot be applied. Thus, a uniform estimate for this case is open.
- Finally, the situation of assuming higher regularity of $\boldsymbol{w}$ uniformly in ν and $\nu_0\left(\delta\right) \geq 0$ is studied and a uniform estimate is presented in Theorem 7.121.

□

Remark 7.113. Outline of the proof. Since the error analysis involves a lot of technical details, a plan of the proof is given in advance.

1. The variational formulation (7.107) and the discrete problem (7.120) with arbitrary test functions $\left(\boldsymbol{v}^h, q^h\right) \in V^h \times Q^h$ are subtracted to derive the error equation (7.127).
2. One chooses an arbitrary $\tilde{\boldsymbol{w}}^h \in V_{\mathrm{div}}^h$ and splits the error $\boldsymbol{e} = \left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right) - \left(\boldsymbol{w}^h - \tilde{\boldsymbol{w}}^h\right) = \boldsymbol{\eta} - \boldsymbol{\phi}^h$. Then, one takes $\boldsymbol{\phi}^h$ as test function in (7.127) to derive (7.128).
3. The left-hand side of the error equation (7.128) is estimated from below by the strong monotonicity (7.110) and the right-hand side of (7.128) from above by the local Lipschitz continuity (7.113) of the Smagorinsky term.
4. Estimating the other terms, in particular the nonlinear convective terms, one derives a differential inequality of the form

$$\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \tilde{g}\left(\boldsymbol{\phi}^h\right) \leq g\left(\boldsymbol{\eta}\right) + \lambda\left(t, \boldsymbol{w}, \boldsymbol{w}^h\right)\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \tag{7.124}$$

   with $0 \leq \tilde{g}\left(\boldsymbol{\phi}^h\right), g\left(\boldsymbol{\eta}\right), \lambda\left(t, \boldsymbol{w}, \boldsymbol{w}^h\right)$ and $\tilde{g}\left(\boldsymbol{\phi}^h\right) \in L^1(0, T)$.

5. The term $g(\boldsymbol{\eta})$ is bounded from above uniformly in ν by the stability estimates already proved in this section. The bounds have to be in $L^1(0,T)$ uniformly in ν.
6. One shows that $\lambda\left(t,\boldsymbol{w},\boldsymbol{w}^h\right) \in L^1(0,T)$ uniformly in ν by the stability estimates.
7. The application of Gronwall's lemma, Lemma A.71, to (7.124) gives an error estimate for $\boldsymbol{\phi}^h$.
8. The error estimate for $\boldsymbol{w}-\boldsymbol{w}^h$ is obtained by the triangle inequality. The terms containing $\boldsymbol{\eta}$ are bounded independently of ν by best approximation error estimates of the finite element spaces.

□

Lemma 7.114. Differential inequality for performing the error estimates. *Let $\boldsymbol{e}=\boldsymbol{w}-\boldsymbol{w}^h$ denote the error, let $\tilde{\boldsymbol{w}}^h \in V_{\mathrm{div}}^h$ be arbitrary, and consider the decomposition*

$$\boldsymbol{e}=\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)-\left(\boldsymbol{w}^h-\tilde{\boldsymbol{w}}^h\right)=\boldsymbol{\eta}-\boldsymbol{\phi}^h. \tag{7.125}$$

If $\mu>0$, then it holds the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\frac{\mu}{2}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\frac{2\nu+\nu_0(\delta)}{2}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2\\
&\quad+\frac{\underline{C}C_S\delta^2}{3}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3+\sum_{j=1}^{J}\frac{\beta}{2}\left\|\boldsymbol{\phi}^h\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2\\
&\le \frac{2}{3\left(\underline{C}C_S\right)^{1/2}\delta}\left\|\partial_t\boldsymbol{\eta}\right\|_{V'}^{3/2}+\mu\left\|\nabla\cdot\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2+\frac{2\nu+\nu_0(\delta)}{2}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^2(\Omega)}^2\\
&\quad+\frac{2\overline{C}^{3/2}C_L^{3/2}C_S\delta^2}{3\underline{C}^{1/2}}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3(\Omega)}^{3/2}+\sum_{j=1}^{J}\frac{\beta}{2}\left\|\boldsymbol{\eta}\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2 \qquad (7.126)\\
&\quad+\left|n_{\mathrm{skew}}\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{\phi}^h\right)-n_{\mathrm{skew}}\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)\right|+\frac{1}{\mu}\left\|r-q^h\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

Proof. not presented in course By the choice of $\tilde{\boldsymbol{w}}^h$ it follows that $\boldsymbol{\phi}^h \in V_{\mathrm{div}}^h$. Subtracting (7.120) from (7.107) gives for all $\boldsymbol{v}^h \in V_{\mathrm{div}}^h$ and $q^h \in Q_h$

$$\begin{aligned}
&\left(\partial_t\boldsymbol{e},\boldsymbol{v}^h\right)+a\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}^h\right)-a\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{v}^h\right)+n_{\mathrm{skew}}\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{v}_h\right)\\
&\qquad-n_{\mathrm{skew}}\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{v}_h\right)-\left(r-q^h,\nabla\cdot\boldsymbol{v}^h\right)=0.
\end{aligned} \tag{7.127}$$

One obtains for $\boldsymbol{v}^h=\boldsymbol{\phi}^h$

$$\begin{aligned}
&\left(\partial_t\boldsymbol{\phi}^h,\boldsymbol{\phi}^h\right)+a\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)-a\left(\tilde{\boldsymbol{w}}^h,\tilde{\boldsymbol{w}}^h,\boldsymbol{\phi}^h\right)\\
&=\left(\partial_t\boldsymbol{\eta},\boldsymbol{\phi}^h\right)+a\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{\phi}^h\right)-a\left(\tilde{\boldsymbol{w}}^h,\tilde{\boldsymbol{w}}^h,\boldsymbol{\phi}^h\right)+n_{\mathrm{skew}}\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{\phi}^h\right)\\
&\quad-n_{\mathrm{skew}}\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)-\left(r-q^h,\nabla\cdot\boldsymbol{\phi}^h\right).
\end{aligned} \tag{7.128}$$

The monotonicity of $a(\cdot,\cdot,\cdot)$, (7.110), implies

$$
\begin{aligned}
& a\left(\boldsymbol{w}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right) - a\left(\tilde{\boldsymbol{w}}^h, \tilde{\boldsymbol{w}}^h, \boldsymbol{\phi}^h\right) \\
& \geq \mu \left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + (2\nu + \nu_0(\delta)) \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2 + \underline{C} C_S \delta^2 \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3 \\
& \quad + \sum_{j=1}^{J} \beta \left\|\boldsymbol{\phi}^h \cdot \boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2 .
\end{aligned}
$$

For estimating the right-hand side of (7.128), one has to get norms of $\boldsymbol{\phi}^h$ which can be absorbed from the left-hand side or $\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2$ for the application of Gronwall's lemma, and the norms of $\boldsymbol{\eta}$ have to be bounded by the stability estimates given at the beginning of this section. The local Lipschitz continuity of the trilinear form, (7.113), gives the estimate

$$
\begin{aligned}
& a\left(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{\phi}^h\right) - a\left(\tilde{\boldsymbol{w}}^h, \tilde{\boldsymbol{w}}^h, \boldsymbol{\phi}^h\right) \\
& \leq \mu \left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \|\nabla \cdot \boldsymbol{\eta}\|_{L^2(\Omega)} + (2\nu + \nu_0(\delta)) \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)} \|\mathbb{D}(\boldsymbol{\eta})\|_{L^2(\Omega)} \\
& \quad + \overline{C} C_L C_S \delta^2 \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)} \|\mathbb{D}(\boldsymbol{\eta})\|_{L^3(\Omega)} + \sum_{j=1}^{J} \beta \left\|\boldsymbol{\phi}^h \cdot \boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)} \|\boldsymbol{\eta} \cdot \boldsymbol{\tau}_j\|_{L^2(\Gamma_j)}
\end{aligned}
$$

with

$$
C_L = \max\left\{\|\mathbb{D}(\boldsymbol{w})\|_{L^3(\Omega)}, \left\|\mathbb{D}\left(\tilde{\boldsymbol{w}}^h\right)\right\|_{L^3(\Omega)}\right\}. \tag{7.129}
$$

The terms on the right-hand side are estimated further by Young's inequality (A.4) and the definition of the norm in V', (7.109),

$$
\begin{aligned}
& \overline{C} C_L C_S \delta^2 \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)} \|\mathbb{D}(\boldsymbol{\eta})\|_{L^3(\Omega)} \\
& \leq \frac{\underline{C} C_S \delta^2}{3} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3 + \frac{2\overline{C}^{3/2} C_L^{3/2} C_S \delta^2}{3\underline{C}^{1/2}} \|\mathbb{D}(\boldsymbol{\eta})\|_{L^3(\Omega)}^{3/2},
\end{aligned}
$$

$$
\mu \left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \|\nabla \cdot \boldsymbol{\eta}\|_{L^2(\Omega)} \leq \frac{\mu}{4} \left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \mu \|\nabla \cdot \boldsymbol{\eta}\|_{L^2(\Omega)}^2,
$$

$$
\begin{aligned}
\left(\partial_t \boldsymbol{\eta}, \boldsymbol{\phi}^h\right) & \leq \|\partial_t \boldsymbol{\eta}\|_{V'} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)} \\
& \leq \frac{2}{3} \left(\underline{C} C_S \delta^2\right)^{-1/2} \|\partial_t \boldsymbol{\eta}\|_{V'}^{3/2} + \frac{\underline{C} C_S \delta^2}{3} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3 .
\end{aligned}
$$

Young's inequality is applied to the other terms with $p = q = 2$ and $t = 1$.

Inserting these estimates into (7.128), using the Cauchy–Schwarz inequality (A.16), and collecting terms gives in the case $\mu > 0$ estimate (7.126). ■

Lemma 7.115. Estimate of the nonlinear convective term. *It holds*

$$\begin{aligned}
&\left|n_{\text{skew}}\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{\phi}^h\right)-n_{\text{skew}}\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)\right|\\
&\le\frac{1}{4}\left\|\nabla\boldsymbol{\eta}\right\|_{L^3(\Omega)}^2+\frac{\varepsilon_1}{6}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3+\frac{C}{3\varepsilon_1^{1/2}}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^{3/2}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^{3/2}\\
&\quad+\frac{1}{4}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^2+\frac{\varepsilon_1}{6}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3+\frac{C}{3\varepsilon_1^{1/2}}\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^{3/2}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^{3/2}\\
&\quad+\frac{\nu_0(\delta)}{3}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2+\frac{\mu}{8}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2\\
&\quad+\left[\frac{1}{4}\left\|\boldsymbol{w}\right\|_{L^6(\Omega)}^2+\frac{1}{4}\left\|\nabla\boldsymbol{w}^h\right\|_{L^3(\Omega)}^2+\frac{C}{\nu_0(\delta)}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2\right.\\
&\quad\left.+\frac{C}{\nu_0(\delta)^{1/2}\mu^{3/2}}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^3\right]\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2. \qquad (7.130)
\end{aligned}$$

Proof. Again, the guidlines for deriving estimates are to get norms of $\boldsymbol{\phi}^h$ which can be absorbed from the left-hand side of (7.128) or to obtain $\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2$ for the application of Gronwall's lemma. The norms of $\boldsymbol{\eta}$ have to be bounded by the stability estimates given at the beginning of this section.

A straightforward calculation gives the decomposition, see (5.55),

$$\begin{aligned}
&n_{\text{skew}}\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{\phi}^h\right)-n_{\text{skew}}\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right) \qquad (7.131)\\
&=n_{\text{skew}}\left(\boldsymbol{w},\boldsymbol{\eta},\boldsymbol{\phi}^h\right)+n_{\text{skew}}\left(\boldsymbol{\eta},\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)-n_{\text{skew}}\left(\boldsymbol{\phi}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right),
\end{aligned}$$

where $n_{\text{skew}}\left(\boldsymbol{w},\boldsymbol{\phi}^h,\boldsymbol{\phi}^h\right)=0$ has been used, see (5.18).

The first term of (7.131) is bounded using Hölder's inequality (A.15), Korn's inequality (7.108), and Young's inequality (A.4)

$$\begin{aligned}
&\left|n_{\text{skew}}\left(\boldsymbol{w},\boldsymbol{\eta},\boldsymbol{\phi}^h\right)\right|\\
&=\frac{1}{2}\left|n_{\text{conv}}n\left(\boldsymbol{w},\boldsymbol{\eta},\boldsymbol{\phi}^h\right)-n_{\text{conv}}\left(\boldsymbol{w},\boldsymbol{\phi}^h,\boldsymbol{\eta}\right)\right|\\
&\le\frac{1}{2}\left(\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^3(\Omega)}\left\|\boldsymbol{w}\right\|_{L^6(\Omega)}+\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^3(\Omega)}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}\right)\\
&\le\frac{1}{2}\left(\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\left\|\nabla\boldsymbol{\eta}\right\|_{L^3(\Omega)}\left\|\boldsymbol{w}\right\|_{L^6(\Omega)}+C\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}\right)\\
&\le\frac{1}{4}\left\|\boldsymbol{w}\right\|_{L^6(\Omega)}^2\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\frac{1}{4}\left\|\nabla\boldsymbol{\eta}\right\|_{L^3(\Omega)}^2+\frac{\varepsilon_1}{6}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3 \qquad (7.132)\\
&\quad+\frac{C}{3\varepsilon_1^{1/2}}\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^{3/2}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^{3/2}.
\end{aligned}$$

In the same way, one obtains

$$\begin{aligned}
n_{\text{skew}}\left(\boldsymbol{\eta},\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)&\le\frac{1}{4}\left\|\nabla\boldsymbol{w}^h\right\|_{L^3(\Omega)}^2\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\frac{1}{4}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^2 \qquad (7.133)\\
&\quad+\frac{\varepsilon_1}{6}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3+\frac{C}{3\varepsilon_1^{1/2}}\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^{3/2}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^{3/2}.
\end{aligned}$$

The estimate of the third term starts with applying (5.15) and Hölder's inequality such that

$$
\begin{aligned}
n_{\text{skew}}\left(\boldsymbol{\phi}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right) &= \frac{1}{2} n_{\text{conv}}\left(\boldsymbol{\phi}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right) - \frac{1}{2} n_{\text{conv}}\left(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h, \boldsymbol{w}^h\right) \\
&= n_{\text{conv}}\left(\boldsymbol{\phi}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right) + \frac{1}{2}\left(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \boldsymbol{w}^h\right) \qquad (7.134)\\
&\le \left\|\nabla \boldsymbol{w}^h\right\|_{L^3(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{L^3(\Omega)}^2 + \frac{1}{2}\left|\left(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \boldsymbol{w}^h\right)\right|.
\end{aligned}
$$

In this estimate, the maximal regularity is used for $\nabla \boldsymbol{w}^h$ and both factors $\boldsymbol{\phi}^h$ are treated the same way. By the Sobolev embedding $H^{1/2}(\Omega) \to L^3(\Omega)$, see (A.23), the interpolation theorem (A.21) and Poincare's inequality (A.17), one obtains

$$
\begin{aligned}
\left\|\boldsymbol{\phi}^h\right\|_{L^3(\Omega)}^2 &\le C \left\|\boldsymbol{\phi}^h\right\|_{H^{1/2}(\Omega)}^2 \le C \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{H^1(\Omega)} \\
&\le C \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}.
\end{aligned}
$$

Inserting this estimate into the previous estimate and applying Korn's and Young's inequalities gives

$$
\begin{aligned}
\left|n_{\text{skew}}\left(\boldsymbol{\phi}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right)\right| \le{}& \frac{\nu_0(\delta)}{6} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2 + \frac{3C}{2\nu_0(\delta)} \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2 \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \\
&+ \frac{1}{2}\left|\left(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \boldsymbol{w}^h\right)\right|.
\end{aligned}
$$

not presented in course The last term of the right-hand side of this inequality is estimated by Hölder's and by Young's inequality leading to

$$
\begin{aligned}
\left|\left(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \boldsymbol{w}^h\right)\right| &\le \left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{L^{18/7}(\Omega)} \left\|\boldsymbol{w}^h\right\|_{L^9(\Omega)} \\
&\le \frac{\mu}{4} \left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{1}{\mu} \left\|\boldsymbol{\phi}^h\right\|_{L^{18/7}(\Omega)}^2 \left\|\boldsymbol{w}^h\right\|_{L^9(\Omega)}^2.
\end{aligned}
$$

The Sobolev embedding theorem $W^{1,3}(\Omega) \to L^9(\Omega)$, see (A.24), implies together with Poincare's and Korn's inequality

$$
\left\|\boldsymbol{w}^h\right\|_{L^9(\Omega)}^2 \le C \left\|\boldsymbol{w}^h\right\|_{W^{1,3}(\Omega)}^2 \le C \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3}^2.
$$

The Sobolev embedding theorem implies also $H^{1/3}(\Omega) \to L^{18/7}(\Omega)$, see (A.23). With the interpolation theorem (A.21), Poincare's and Korn's inequality it follows that

$$
\begin{aligned}
\left\|\boldsymbol{\phi}^h\right\|_{L^{18/7}(\Omega)}^2 &\le C \left\|\boldsymbol{\phi}^h\right\|_{H^{1/3}(\Omega)}^2 \le C \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^{4/3} \left\|\boldsymbol{\phi}^h\right\|_{H^1(\Omega)}^{2/3} \\
&\le C \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^{4/3} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^{2/3}.
\end{aligned}
$$

These bounds and Young's inequality yields

$$
\begin{aligned}
n_{\text{skew}}\left(\boldsymbol{\phi}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right) \le{}& \frac{\nu_0(\delta)}{6} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2 + \frac{C}{\nu_0(\delta)} \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2 \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \\
&+ \frac{\mu}{8} \left\|\nabla \cdot \boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{\nu_0(\delta)}{6} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2 \\
&+ \frac{C}{\nu_0(\delta)^{1/2} \mu^{3/2}} \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3}^3 \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Combining this estimate with (7.132) and (7.133) gives (7.130). ■

Remark 7.116. On inequalities (7.126) *and* (7.130). The right-hand sides of these inequalities have to be estimated further. Because of the appearance of $\left\|\phi^h\right\|^2_{L^2(\Omega)}$ on the right-hand side of (7.130), Gronwall's lemma will be applied. In Lemmas 7.117 and 7.118 it will be shown that the terms on the right-hand side are sufficiently regular. The proofs use the stability estimates for $\boldsymbol{w}$ and $\boldsymbol{w}^h$. □

Lemma 7.117. Regularity in time of the term in the brackets in (7.130)**.** *Assume* $\mu > 0$ *and* $\nu_0(\delta) > 0$ *for* $\delta > 0$. *Let*

$$\lambda(t) := \frac{1}{4}\|\boldsymbol{w}\|^2_{L^6(\Omega)} + \frac{1}{4}\left\|\nabla \boldsymbol{w}^h\right\|^2_{L^3(\Omega)} + \frac{C}{\nu_0(\delta)}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|^2_{L^3(\Omega)}$$
$$+C\nu_0(\delta)^{-1/2}\mu^{-3/2}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|^3_{L^3(\Omega)},$$

then there is a constant $C_1(\delta)$ *independent of* ν *and* h *such that for* $0 < T < \infty$

$$\|\lambda(t)\|_{L^1(0,T)} \le C_1(\delta).$$

Proof. By the Sobolev embedding $W^{1,3}(\Omega) \to L^6(\Omega)$, see (A.24), Poincaré's and Korn's inequality, one has $\|\boldsymbol{w}\|_{L^6(\Omega)} \le C\|\mathbb{D}(\boldsymbol{w})\|_{L^3(\Omega)}$ which is bounded uniformly in ν by (7.117) and (7.118). By the stability estimates (7.121) and (7.122) it follows that $\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)} \in L^3(0,T)$ uniformly in ν and h. Since $L^3(0,T) \subset L^2(0,T)$, it is also $\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)} \in L^2(0,T)$, such that $\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|^3_{L^3(\Omega)} \in L^1(0,T)$ and $\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|^2_{L^3(\Omega)} \in L^1(0,T)$ uniformly in ν and h. ■

Lemma 7.118. Regularity in time of the other terms from the right-hand side of (7.130)**.** *Under the assumptions of Lemma 7.117 there is a constant* $C_2(\delta)$ *independent of* ν *and* h *such that for* $T \in (0,\infty)$

$$\left(\left\|\boldsymbol{w}^h\right\|^{3/2}_{L^2(\Omega)} + \|\boldsymbol{w}\|^{3/2}_{L^2(\Omega)}\right) \le C_2(\delta).$$

Proof. not presented in course The statement of the lemma follows for $\|\boldsymbol{w}\|_{L^2(\Omega)}$ by the stability estimates (7.117) and (7.118) since

$$\int_0^T \|\boldsymbol{w}\|^{3/2}_{L^2(\Omega)}\,dt \le \|\boldsymbol{w}\|^{3/2}_{L^\infty(0,T;L^2(\Omega))} T \le C(\delta).$$

Similarly, this property follows for $\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}$ from (7.121) and (7.122). ■

Theorem 7.119. Uniform finite element error estimate for the natural regularity of the solution and $\nu_0(\delta) > 0$**.** *Assume that* $\nabla \boldsymbol{w} \in L^3\left(0,T;L^3(\Omega)\right)$, $\partial_t \boldsymbol{w} \in L^{3/2}(0,T;V')$, $r \in L^2\left(0,T;L^2(\Omega)\right)$, $\mu > 0$, *and* $\nu_0(\delta) > 0$. *Then, the error* $\boldsymbol{w} - \boldsymbol{w}^h$ *satisfies for* $0 < T < \infty$

$$
\begin{aligned}
&\left\|\boldsymbol{w}-\boldsymbol{w}^h\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\mu\left\|\nabla\cdot\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\right\|^2_{L^2(0,T;L^2(\Omega))}\\
&\quad+\left(\nu+C\nu_0\left(\delta\right)\right)\left\|\mathbb{D}\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\right\|^2_{L^2(0,T;L^2(\Omega))}\\
&\quad+\delta^2\left\|\mathbb{D}\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\right\|^3_{L^3(0,T;L^3(\Omega))}+\sum_{j=1}^J\beta\left\|\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\cdot\boldsymbol{\tau}_j\right\|^2_{L^2(0,T;L^2(\Gamma_j))}\\
&\le C\exp\left(C_1(\delta)\right)\left\|\left(\boldsymbol{w}-\boldsymbol{w}^h\right)(0)\right\|^2_{L^2(\Omega)}\\
&\quad+C\inf_{\tilde{\boldsymbol{w}}^h\in V^h_{\mathrm{div}},q^h\in Q^h}\mathcal{F}\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h,r-q^h,\delta\right)
\end{aligned}
$$

with

$$
\begin{aligned}
&\mathcal{F}\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h,r-q^h,\delta\right)\\
&=\left\|\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right\|^2_{L^\infty(0,T;L^2(\Omega))}+\delta^2\left\|\mathbb{D}\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)\right\|^3_{L^3(0,T;L^3(\Omega))}\\
&\quad+\exp\left(C_1(\delta)\right)\Bigg[\left\|\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)(0)\right\|^2_{L^2(\Omega)}+\delta^{-1}\left\|\partial_t\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)\right\|^{3/2}_{L^{3/2}(0,T;V')}\\
&\quad+\mu\left\|\nabla\cdot\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)\right\|^2_{L^2(0,T;L^2(\Omega))}+\left(2\nu+\nu_0\left(\delta\right)\right)\left\|\mathbb{D}\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)\right\|^2_{L^2(0,T;L^2(\Omega))}\\
&\quad+C\left(\delta\right)\left\|\mathbb{D}\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)\right\|^{3/2}_{L^3(0,T;L^3(\Omega))}+\sum_{j=1}^J\beta\left\|\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)\cdot\boldsymbol{\tau}_j\right\|^2_{L^2(0,T;L^2(\Gamma_j))}\\
&\quad+\left\|\nabla\left(\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right)\right\|^2_{L^2(0,T;L^3(\Omega))}+C_2\left(\delta\right)\left\|\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right\|^{3/2}_{L^{3/2}(0,T;L^6(\Omega))}\\
&\quad+\left\|\boldsymbol{w}-\tilde{\boldsymbol{w}}^h\right\|^2_{L^2(0,T;L^6(\Omega))}+\frac{1}{\mu}\left\|r-q^h\right\|^2_{L^2(0,T;L^2(\Omega))}\Bigg]
\end{aligned}
$$

and $C_1(\delta)$ *and* $C_2(\delta)$ *defined in Lemma 7.117 and 7.118.*

Proof. not presented in course The bound (7.130). is substituted into (7.126) yielding the differential inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\frac{3\mu}{8}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\left(\nu+\frac{\nu_0(\delta)}{6}\right)\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2\\
&\quad+\left(\frac{\underline{C}C_S\delta^2}{3}-\frac{\varepsilon_1}{3}\right)\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3+\sum_{j=1}^{J}\frac{\beta}{2}\left\|\boldsymbol{\phi}^h\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2\\
&\le\Bigg[\frac{2}{3\left(\underline{C}C_S\right)^{1/2}\delta}\left\|\partial_t\boldsymbol{\eta}\right\|_{V'}^{3/2}+\mu\left\|\nabla\cdot\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2+\frac{2\nu+\nu_0(\delta)}{2}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^2(\Omega)}^2\\
&\quad+\frac{2\overline{C}^{3/2}C_L^{3/2}C_S\delta^2}{3\underline{C}^{1/2}}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3(\Omega)}^{3/2}+\sum_{j=1}^{J}\frac{\beta}{2}\left\|\boldsymbol{\eta}\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2+\frac{1}{4}\left\|\nabla\boldsymbol{\eta}\right\|_{L^3(\Omega)}^2\\
&\quad+\frac{C}{3\varepsilon_1^{1/2}}\left(\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^{3/2}+\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^{3/2}\right)\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^{3/2}+\frac{1}{4}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^2\\
&\quad+\frac{1}{\mu}\left\|r-q^h\right\|_{L^2(\Omega)}^2\Bigg]+\Bigg[\frac{1}{4}\left\|\boldsymbol{w}\right\|_{L^6(\Omega)}^2+\frac{1}{4}\left\|\nabla\boldsymbol{w}^h\right\|_{L^3(\Omega)}^2+\frac{C}{\nu_0(\delta)}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2\\
&\quad+\frac{C}{\nu_0(\delta)^{1/2}\mu^{3/2}}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^3\Bigg]\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{7.135}
$$

Picking $\varepsilon_1=\underline{C}C_S\delta^2/2$ such that

$$
\frac{\underline{C}C_S\delta^2}{3}-\frac{\varepsilon_1}{3}=\frac{\underline{C}C_S\delta^2}{6}\quad\Longrightarrow\quad\frac{1}{\varepsilon_1^{1/2}}=\frac{C}{\delta},
$$

and multiplying with 2 yields

$$
\begin{aligned}
&\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\frac{3\mu}{4}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2+\left(2\nu+\frac{\nu_0(\delta)}{3}\right)\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2\\
&\qquad+\frac{\underline{C}C_S\delta^2}{3}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3+\sum_{j=1}^{J}\beta\left\|\boldsymbol{\phi}^h\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2
\end{aligned}
$$

$$
\begin{aligned}
&\le\Bigg[\frac{4}{3\left(\underline{C}C_S\right)^{1/2}\delta}\left\|\partial_t\boldsymbol{\eta}\right\|_{V'}^{3/2}+2\mu\left\|\nabla\cdot\boldsymbol{\eta}\right\|_{L^2(\Omega)}^2+\left(2\nu+\nu_0(\delta)\right)\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^2(\Omega)}^2\\
&\quad+\frac{4\overline{C}^{3/2}C_L^{3/2}C_S\delta^2}{3\underline{C}^{1/2}}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3(\Omega)}^{3/2}+\sum_{j=1}^{J}\beta\left\|\boldsymbol{\eta}\cdot\boldsymbol{\tau}_j\right\|_{L^2(\Gamma_j)}^2+\frac{1}{2}\left\|\nabla\boldsymbol{\eta}\right\|_{L^3(\Omega)}^2\\
&\quad+C\delta^{-1}\left(\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^{3/2}+\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^{3/2}\right)\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^{3/2}+\frac{1}{2}\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^2\\
&\quad+\frac{2}{\mu}\left\|r-q^h\right\|_{L^2(\Omega)}^2\Bigg]+\Bigg[\frac{1}{2}\left\|\boldsymbol{w}\right\|_{L^6(\Omega)}^2+\frac{1}{2}\left\|\nabla\boldsymbol{w}^h\right\|_{L^3(\Omega)}^2\\
&\quad+\frac{C}{\nu_0(\delta)}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2+C\nu_0(\delta)^{-1/2}\mu^{-3/2}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^3\Bigg]\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Before the application of Gronwall's lemma, Lemma A.69, one has to check that all functions are sufficiently smooth in time. All terms which involve only norms of $\boldsymbol{\eta}$ and derivatives of $\boldsymbol{\eta}$ are in $L^1(0,T)$. The other term in the first bracket is shown to be in $L^1(0,T)$ in Lemma 7.118. Finally, the term in the second bracket is also in $L^1(0,T)$ by Lemma 7.117. The application of Gronwall's lemma in form (A.47) gives for almost all $t\in[0,T]$

$$
\begin{aligned}
&\left\|\boldsymbol{\phi}^h(t)\right\|_{L^2(\Omega)}^2+\mu\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2\left(0,t;L^2(\Omega)\right)}^2+\left(\nu+C\nu_0(\delta)\right)\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2\left(0,t;L^2(\Omega)\right)}^2\\
&\quad+\delta^2\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3\left(0,t;L^3(\Omega)\right)}^3+\sum_{j=1}^{J}\beta\left\|\boldsymbol{\phi}^h\cdot\boldsymbol{\tau}_j\right\|_{L^2\left(0,t;L^2(\Gamma_j)\right)}^2\\
&\le C\exp\Big(\left\|\lambda(t)\right\|_{L^1(0,t)}\Big)\left\|\boldsymbol{\phi}^h(0)\right\|_{L^2(\Omega)}^2\\
&\quad+C\exp\Big(\left\|\lambda(t)\right\|_{L^1(0,t)}\Big)\Bigg[\delta^{-1}\left\|\partial_t\boldsymbol{\eta}\right\|_{L^{3/2}(0,t;V')}^{3/2}+\mu\left\|\nabla\cdot\boldsymbol{\eta}\right\|_{L^2\left(0,t;L^2(\Omega)\right)}^2\\
&\quad+\left(2\nu+\nu_0(\delta)\right)\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^2\left(0,t;L^2(\Omega)\right)}^2+\delta^2\int_0^t C_L^{3/2}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3(\Omega)}^{3/2}\,dt'\\
&\quad+\sum_{j=1}^{J}\beta\left\|\boldsymbol{\eta}\cdot\boldsymbol{\tau}_j\right\|_{L^2\left(0,t;L^2(\Gamma_j)\right)}^2+\left\|\nabla\boldsymbol{\eta}\right\|_{L^2\left(0,t;L^3(\Omega)\right)}^2\\
&\quad+\delta^{-1}\int_0^t\Big(\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^{3/2}+\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^{3/2}\Big)\left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^{3/2}\,dt'\\
&\quad+\left\|\boldsymbol{\eta}\right\|_{L^2\left(0,t;L^6(\Omega)\right)}^2+\frac{1}{\mu}\left\|r-q^h\right\|_{L^2\left(0,t;L^2(\Omega)\right)}^2\Bigg].
\end{aligned}
$$

Using the definition of C_L in (7.129), Lemma 7.100, the Cauchy–Schwarz inequality (A.16) in $L^2(0,T)$, and the stability estimates (7.117) and (7.118), one gets

$$
\begin{aligned}
&\int_0^t C_L^{3/2}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3(\Omega)}^{3/2}\,dt'\\
&\le\int_0^t\Big(\max\Big\{\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(\Omega)},\left\|\mathbb{D}\left(\tilde{\boldsymbol{w}}^h\right)\right\|_{L^3(\Omega)}\Big\}\Big)^{3/2}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3(\Omega)}^{3/2}\,dt'\\
&\le C\int_0^t\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3(\Omega)}^{3/2}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3(\Omega)}^{3/2}\,dt'\\
&\le C\left\|\mathbb{D}(\boldsymbol{w})\right\|_{L^3\left(0,t;L^3(\Omega)\right)}^{3/2}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3\left(0,t;L^3(\Omega)\right)}^{3/2}\\
&\le C(\delta)\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3\left(0,t;L^3(\Omega)\right)}^{3/2}.
\end{aligned}
$$

All constants which does not depend on the problem or the discretization are collected into the generic constants C. Using the definition of $C_2(\delta)$, Lemma 7.118, and applying the essential supremum on $(0,T)$ on both sides of the inequality gives

$$
\begin{aligned}
&\left\|\boldsymbol{\phi}^h\right\|^2_{L^\infty\left(0,T;L^2(\Omega)\right)} + \mu\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)}\\
&\quad + \left(\nu + C\nu_0\left(\delta\right)\right)\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \delta^2\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|^3_{L^3\left(0,T;L^3(\Omega)\right)}\\
&\quad + \sum_{j=1}^{J}\beta\left\|\boldsymbol{\phi}^h\cdot\boldsymbol{\tau}_j\right\|^2_{L^2\left(0,T;L^2(\Gamma_j)\right)}\\
&\le C\exp\left(\left\|\lambda\left(t\right)\right\|_{L^1(0,T)}\right)\left\|\boldsymbol{\phi}^h\left(0\right)\right\|^2_{L^2(\Omega)}\\
&\quad + C\exp\left(\left\|\lambda\left(t\right)\right\|_{L^1(0,T)}\right)\Bigg[\delta^{-1}\left\|\partial_t\boldsymbol{\eta}\right\|^{3/2}_{L^{3/2}(0,T;V')} + \mu\left\|\nabla\cdot\boldsymbol{\eta}\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)}\\
&\quad + \left(2\nu + \nu_0\left(\delta\right)\right)\left\|\mathbb{D}\left(\boldsymbol{\eta}\right)\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + C\left(\delta\right)\left\|\mathbb{D}\left(\boldsymbol{\eta}\right)\right\|^{3/2}_{L^3\left(0,T;L^3(\Omega)\right)}\\
&\quad + \sum_{j=1}^{J}\beta\left\|\boldsymbol{\eta}\cdot\boldsymbol{\tau}_j\right\|^2_{L^2\left(0,T;L^2(\Gamma_j)\right)} + \left\|\nabla\boldsymbol{\eta}\right\|^2_{L^2\left(0,T;L^3(\Omega)\right)}\\
&\quad + C_2\left(\delta\right)\left\|\boldsymbol{\eta}\right\|^{3/2}_{L^{3/2}\left(0,T;L^6(\Omega)\right)} + \left\|\boldsymbol{\eta}\right\|^2_{L^2\left(0,T;L^6(\Omega)\right)} + \frac{1}{\mu}\left\|r - q^h\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)}\Bigg].
\end{aligned}
$$

The triangle inequality implies

$$
\begin{aligned}
&\left\|\boldsymbol{w} - \boldsymbol{w}^h\right\|^2_{L^\infty\left(0,T;L^2(\Omega)\right)} + \mu\left\|\nabla\cdot\left(\boldsymbol{w} - \boldsymbol{w}^h\right)\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)}\\
&\quad + \left(\nu + C\nu_0\left(\delta\right)\right)\left\|\mathbb{D}\left(\boldsymbol{w} - \boldsymbol{w}^h\right)\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)}\\
&\quad + \delta^2\left\|\mathbb{D}\left(\boldsymbol{w} - \boldsymbol{w}^h\right)\right\|^3_{L^3\left(0,T;L^3(\Omega)\right)} + \sum_{j=1}^{J}\beta\left\|\left(\boldsymbol{w} - \boldsymbol{w}^h\right)\cdot\boldsymbol{\tau}_j\right\|^2_{L^2\left(0,T;L^2(\Gamma_j)\right)}\\
&\le C\Bigg[\left\|\boldsymbol{\eta}\right\|^2_{L^\infty\left(0,T;L^2(\Omega)\right)} + \mu\left\|\nabla\cdot\boldsymbol{\eta}\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)}\\
&\quad + \left(\nu + C\nu_0\left(\delta\right)\right)\left\|\mathbb{D}\left(\boldsymbol{\eta}\right)\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \delta^2\left\|\mathbb{D}\left(\boldsymbol{\eta}\right)\right\|^3_{L^3\left(0,T;L^3(\Omega)\right)}\\
&\quad + \sum_{j=1}^{J}\beta\left\|\boldsymbol{\eta}\cdot\boldsymbol{\tau}_j\right\|^2_{L^2\left(0,T;L^2(\Gamma_j)\right)} + \left\|\boldsymbol{\phi}^h\right\|^2_{L^\infty\left(0,T;L^2(\Omega)\right)}\\
&\quad + \mu\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)} + \left(\nu + C\nu_0\left(\delta\right)\right)\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|^2_{L^2\left(0,T;L^2(\Omega)\right)}\\
&\quad + \delta^2\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|^3_{L^3\left(0,T;L^3(\Omega)\right)} + \sum_{j=1}^{J}\beta\left\|\boldsymbol{\phi}^h\cdot\boldsymbol{\tau}_j\right\|^2_{L^2\left(0,T;L^2(\Gamma_j)\right)}\Bigg].
\end{aligned}
$$

With the estimate for $\boldsymbol{\phi}^h$ and since $\exp\left(\left\|\lambda\left(t\right)\right\|_{L^1(0,T)}\right) \ge 1$, one gets

$$
\begin{aligned}
&\left\|\boldsymbol{w}-\boldsymbol{w}^h\right\|_{L^\infty(0,T;L^2(\Omega))}^2+\mu\left\|\nabla\cdot\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\right\|_{L^2(0,T;L^2(\Omega))}^2\\
&\quad+\left(\nu+C\nu_0(\delta)\right)\left\|\mathbb{D}\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\right\|_{L^2(0,T;L^2(\Omega))}^2\\
&\quad+\delta^2\left\|\mathbb{D}\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\right\|_{L^3(0,T;L^3(\Omega))}^3+\sum_{j=1}^J\beta\left\|\left(\boldsymbol{w}-\boldsymbol{w}^h\right)\cdot\boldsymbol{\tau}_j\right\|_{L^2(0,T;L^2(\Gamma_j))}^2\\
&\le C\exp\left(\|\lambda(t)\|_{L^1(0,T)}\right)\left\|\boldsymbol{\phi}^h(0)\right\|_{L^2(\Omega)}^2\\
&\quad+C\Bigg\{\|\boldsymbol{\eta}\|_{L^\infty(0,T;L^2(\Omega))}^2+\delta^2\|\mathbb{D}(\boldsymbol{\eta})\|_{L^3(0,T;L^3(\Omega))}^3\\
&\quad+\exp\left(\|\lambda(t)\|_{L^1(0,T)}\right)\Bigg[\delta^{-1}\|\partial_t\boldsymbol{\eta}\|_{L^{3/2}(0,T;V')}^{3/2}+\mu\|\nabla\cdot\boldsymbol{\eta}\|_{L^2(0,T;L^2(\Omega))}^2\\
&\quad+\left(2\nu+\nu_0(\delta)\right)\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2(0,T;L^2(\Omega))}^2+C(\delta)\|\mathbb{D}(\boldsymbol{\eta})\|_{L^3(0,T;L^3(\Omega))}^{3/2}\\
&\quad+\sum_{j=1}^J\beta\|\boldsymbol{\eta}\cdot\boldsymbol{\tau}_j\|_{L^2(0,T;L^2(\Gamma_j))}^2+\|\nabla\boldsymbol{\eta}\|_{L^2(0,T;L^3(\Omega))}^2\\
&\quad+C_2(\delta)\|\boldsymbol{\eta}\|_{L^{3/2}(0,T;L^6(\Omega))}^{3/2}+\|\boldsymbol{\eta}\|_{L^2(0,T;L^6(\Omega))}^2+\frac{1}{\mu}\left\|r-q^h\right\|_{L^2(0,T;L^2(\Omega))}^2\Bigg]\Bigg\}.
\end{aligned}
$$

Applying the triangle inequality

$$
\left\|\boldsymbol{\phi}^h(0)\right\|_{L^2(\Omega)}^2\le C\left(\left\|\left(\boldsymbol{w}-\boldsymbol{w}^h\right)(0)\right\|_{L^2(\Omega)}^2+\|\boldsymbol{\eta}(0)\|_{L^2(\Omega)}^2\right),
$$

and taking the infimum over $\tilde{\boldsymbol{w}}^h$ completes the proof of Theorem 7.119. ■

Remark 7.120. The case $\nabla\boldsymbol{w}\in L^3\left(0,T;L^3(\Omega)\right)$ *and* $\nu_0(\delta)=0$. The result proved in Theorem 7.119 requires that $\nu_0(\delta)>0$, i.e., there is some artificial viscosity in addition to the Smagorinsky model. In practice, the Smagorinsky model is used without artificial viscosity such that an error analysis for the case $\nu_0(\delta)=0$ is of much interest. However, so far uniform finite element error estimates for the case $\nabla\boldsymbol{w}\in L^3\left(0,T;L^3(\Omega)\right)$ and $\nu_0(\delta)=0$ are not known. This remark explains why a straightforward approach for deriving an error estimate fails.

Let $\nu_0(\delta)=0$ and $\nabla\boldsymbol{w}\in L^3\left(0,T;L^3(\Omega)\right)$. The key of the error analysis is the estimate of the nonlinear convective term. Using the decomposition (7.131), the first and the second term are estimated as before, see (7.132) and (7.133). The critical term is the last one. As in estimate (7.134), the maximal regularity will be used for $\nabla\boldsymbol{w}^h$. However, the two factors $\boldsymbol{\phi}^h$ will be treated in a different way, having mind that $\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2$ might be a good term on the right-hand side for the application of Gronwall's lemma. Estimating this term by Hölder's inequality (A.15), the Sobolev embedding $W^{1,3}(\Omega)\to L^6(\Omega)$, see (A.24), Korn's inequality (7.108), and Young's inequality (A.4) yields

$$
\begin{aligned}
& n_{\text{skew}} \left(\boldsymbol{\phi}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right) \\
& = \frac{1}{2} \left(n_{\text{conv}} \left(\boldsymbol{\phi}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right) - n_{\text{conv}} \left(\boldsymbol{\phi}^h, \boldsymbol{\phi}^h, \boldsymbol{w}^h\right)\right) \\
& \le \frac{1}{2} \Bigg(\left\|\nabla \boldsymbol{w}^h\right\|_{L^3(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{L^6(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\
& \quad + \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^3(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \left\|\boldsymbol{w}^h\right\|_{L^6(\Omega)} \Bigg) \\
& \le C \left\|\nabla \boldsymbol{w}^h\right\|_{L^3(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{W^{1,3}(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\
& \quad + C \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)} \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \left\|\boldsymbol{w}^h\right\|_{W^{1,3}(\Omega)} \\
& \le \frac{\varepsilon_1}{6} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3 + \frac{C}{\varepsilon_1^{1/2}} \left\|\nabla \boldsymbol{w}^h\right\|_{L^3(\Omega)}^{3/2} \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^{3/2} \\
& \quad + \frac{\varepsilon_1}{6} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3 + \frac{C}{\varepsilon_1^{1/2}} \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^{3/2} \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^{3/2},
\end{aligned}
\tag{7.136}
$$

where in the last step Poincare's inequality (A.17) and Korn's inequality were used to obtain

$$
\left\|\boldsymbol{\phi}^h\right\|_{W^{1,3}(\Omega)} \le C \left\|\nabla \boldsymbol{\phi}^h\right\|_{L^3(\Omega)} \le C \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}.
$$

Collecting the terms of (7.132), (7.133), and (7.136) yields

$$
\begin{aligned}
& \left| n_{\text{skew}} \left(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{\phi}^h\right) - n_{\text{skew}} \left(\boldsymbol{w}^h, \boldsymbol{w}^h, \boldsymbol{\phi}^h\right)\right| \\
& \le \left[\frac{1}{4} \left\|\nabla \boldsymbol{\eta}\right\|_{L^3(\Omega)}^2 + \frac{C}{\varepsilon_1^{1/2}} \left(\left\|\boldsymbol{w}\right\|_{L^2(\Omega)}^{3/2} + \left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^{3/2}\right) \left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^{3/2} + \frac{1}{4} \left\|\boldsymbol{\eta}\right\|_{L^6(\Omega)}^2\right] \\
& \quad + \frac{2\varepsilon_1}{3} \left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3 + \frac{C}{\varepsilon_1^{1/2}} \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^{3/2} \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^{3/2} \\
& \quad + \left[\frac{1}{4} \left\|\boldsymbol{w}\right\|_{L^6}^2 + \frac{1}{4} \left\|\nabla \boldsymbol{w}^h\right\|_{L^3}^2\right] \left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{7.137}
$$

Choosing now

$$
\frac{2\varepsilon_1}{3} = \frac{1}{6} \underline{C} C_S \delta^2 \quad \Longrightarrow \quad \varepsilon_1 = \mathcal{O}\left(\delta^2\right), \; \frac{1}{\varepsilon_1^{1/2}} = \mathcal{O}\left(\delta^{-1}\right),
$$

one obtains with (7.126) the differential inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|^2_{L^2(\Omega)} + \frac{\mu}{2}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|^2_{L^2(\Omega)} + \nu\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{\underline{C}C_S\delta^2}{6}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|^3_{L^3(\Omega)} + \sum_{j=1}^{J}\frac{\beta}{2}\left\|\boldsymbol{\phi}^h\cdot\boldsymbol{\tau}_j\right\|^2_{L^2(\Gamma_j)} \\
&\le \Bigg[\frac{2}{3\left(\underline{C}C_S\right)^{1/2}\delta}\left\|\partial_t\boldsymbol{\eta}\right\|^{3/2}_{V'} + \mu\left\|\nabla\cdot\boldsymbol{\eta}\right\|^2_{L^2(\Omega)} + \nu\left\|\mathbb{D}\left(\boldsymbol{\eta}\right)\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{2\overline{C}^{3/2}C_L^{3/2}C_S\delta^2}{3\underline{C}^{1/2}}\left\|\mathbb{D}\left(\boldsymbol{\eta}\right)\right\|^{3/2}_{L^3(\Omega)} + \sum_{j=1}^{J}\frac{\beta}{2}\left\|\boldsymbol{\eta}\cdot\boldsymbol{\tau}_j\right\|^2_{L^2(\Gamma_j)} + \frac{1}{4}\left\|\nabla\boldsymbol{\eta}\right\|^2_{L^3(\Omega)} \\
&\quad + C\delta^{-1}\left(\left\|\boldsymbol{w}\right\|^{3/2}_{L^2(\Omega)} + \left\|\boldsymbol{w}^h\right\|^{3/2}_{L^2(\Omega)}\right)\left\|\boldsymbol{\eta}\right\|^{3/2}_{L^6(\Omega)} + \frac{1}{4}\left\|\boldsymbol{\eta}\right\|^2_{L^6(\Omega)} \\
&\quad + \frac{1}{\mu}\left\|r - q^h\right\|^2_{L^2(\Omega)}\Bigg] + C\delta^{-1}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|^{3/2}_{L^3(\Omega)}\left\|\boldsymbol{\phi}^h\right\|^{3/2}_{L^2(\Omega)} \\
&\quad + \left[\frac{1}{4}\left\|\boldsymbol{w}\right\|^2_{L^6} + \frac{1}{4}\left\|\nabla\boldsymbol{w}^h\right\|^2_{L^3}\right]\left\|\boldsymbol{\phi}^h\right\|^2_{L^2(\Omega)}. \qquad (7.138)
\end{aligned}
$$

The last step of the proof would be the application of Gronwall's lemma, Lemma A.71, for $f(t) = \left\|\boldsymbol{\phi}^h(t)\right\|^2_{L^2(\Omega)}$. However, the term with $\left\|\boldsymbol{\phi}^h\right\|^{3/2}_{L^2(\Omega)} = \left(\left\|\boldsymbol{\phi}^h\right\|^2_{L^2(\Omega)}\right)^{3/4}$ in the right-hand side of (7.138) does not fit into the basic inequality (A.45) of Lemma A.71. The power 3/4 is too small. Thus, Gronwall's lemma cannot be applied and this approach fails.

- Also other attempts for deriving a uniform error estimate in the case $\nabla\boldsymbol{w} \in L^3\left(0,T;L^3(\Omega)\right)$ and $\nu_0(\delta) = 0$ failed so far, see John and Layton (2002) and (John, 2004, Section 8.1.4).
- It would be possible to prove a finite element error estimate which is not uniform, i.e., where the constant depends on ν, see (John, 2004, Section 8.1.7) for a discussion of this topic.
- A uniform error estimate in the case $\nu_0(\delta) \ge 0$ can be achieved if a higher regularity of the solution is assumed, see Theorem 7.121.

Maybe, the deeper reason for the failure is that one cannot exclude that the Smagorinsky model with $\nu_0(\delta) = 0$ vanishes in some points $(t, \boldsymbol{x})$ or even in some regions. Thus, from the analytical point of view, there is no uniform positive bound form below for the additional viscosity and maybe one cannot expect to obtain different estimates than for the Navier–Stokes equations since the proof uses only global estimates. □

Theorem 7.121. Uniform finite element error estimate for the case $\nu_0(\delta) \ge 0$ and higher regularity of $\boldsymbol{w}$ uniformly in ν. *Suppose $\nu_0(\delta) \ge 0, \mu > 0$, and $\boldsymbol{w} \in L^2\left(0,T;W^{1,\infty}(\Omega)\right)$, $\nabla\boldsymbol{w} \in L^4\left(0,T;L^3(\Omega)\right)$, both uniformly in ν. Let*

$$a(t) := \frac{3}{4} + \|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)} + \left(\frac{1}{4} + \frac{1}{4\mu}\right) \|\boldsymbol{w}\|^2_{L^\infty(\Omega)} + \frac{1}{2} \|\nabla \boldsymbol{w}\|^2_{L^\infty(\Omega)},$$

then there is a $C_3 = C_3(\boldsymbol{w})$ *such that*

$$\|a(t)\|_{L^1(0,T)} \le C_3(\boldsymbol{w}).$$

Let $C_4 = C_4(\delta)$ *be such that* $\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(0,T;L^3)} \le C_4(\delta)$. *Then, the error* $\boldsymbol{w} - \boldsymbol{w}^h$ *satisfies*

$$\begin{aligned}
&\left\|\boldsymbol{w} - \boldsymbol{w}^h\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \delta^2 \left\|\mathbb{D}\left(\boldsymbol{w} - \boldsymbol{w}^h\right)\right\|^3_{L^3(0,T;L^3(\Omega))} \\
&\quad + \mu \left\|\nabla \cdot \left(\boldsymbol{w} - \boldsymbol{w}^h\right)\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \left(\nu + C\nu_0(\delta)\right) \left\|\mathbb{D}\left(\boldsymbol{w} - \boldsymbol{w}^h\right)\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \sum_{j=1}^{J} \beta \left\|\left(\boldsymbol{w} - \boldsymbol{w}^h\right) \cdot \hat{\boldsymbol{\tau}}_j\right\|^2_{L^2(0,T;L^2(\Gamma_j))} \\
&\le C \exp\left(C_3(\boldsymbol{w})\right) \left\|\left(\boldsymbol{w} - \boldsymbol{w}^h\right)(0)\right\|^2_{L^2(\Omega)} \\
&\quad + C \inf_{\tilde{\boldsymbol{w}}^h \in V^h_{\mathrm{div}}, q^h \in Q^h} \mathcal{F}\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h, r - q^h, \delta\right)
\end{aligned} \tag{7.139}$$

with

$$\begin{aligned}
&\mathcal{F}\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h, r - q^h, \delta\right) \\
&= \left\|\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \delta^2 \left\|\mathbb{D}\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right)\right\|^3_{L^3(0,T;L^3(\Omega))} \\
&\quad + \exp\left(C_3(\boldsymbol{w})\right) \Bigg[\left\|\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right)(0)\right\|^2_{L^2(\Omega)} + \delta^{-1} \left\|\partial_t \left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right)\right\|^{3/2}_{L^{3/2}(0,T;V')} \\
&\quad + \left(2\nu + \nu_0(\delta)\right) \left\|\mathbb{D}\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right)\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \sum_{j=1}^{J} \beta \left\|\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right) \cdot \hat{\boldsymbol{\tau}}_j\right\|^2_{L^2(0,T;L^2(\Gamma_j))} + C(\delta) \left\|\mathbb{D}\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right)\right\|^{3/2}_{L^3(0,T;L^3(\Omega))} \\
&\quad + \mu^{-1} \left\|r - q^h\right\|^2_{L^2(0,T;L^2(\Omega))} + \left(\frac{1}{4} + \mu\right) \left\|\nabla \cdot \left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right)\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + \left\|\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right\|^2_{L^2(0,T,L^2(\Omega))} \\
&\quad + C_4(\delta) \left(\left\|\mathbb{D}\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right)\right\|^2_{L^{18/5}(0,T;L^3(\Omega))} + \left\|\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right\|^2_{L^6(0,T;L^6(\Omega))} \right) \Bigg].
\end{aligned} \tag{7.140}$$

Proof. not presented in course First note that with the Gagliardo–Nirenberg inequality (A.20) and Korn's inequality (A.18), it follows that

$$\begin{aligned}\|\boldsymbol{w}\|_{L^6(0,T;L^6(\Omega))}^6 &= \int_0^T \|\boldsymbol{w}\|_{L^6(\Omega)}^6 \, dt \le C\int_0^T \|\nabla \boldsymbol{w}\|_{L^3(\Omega)}^4 \|\boldsymbol{w}\|_{L^2(\Omega)}^2 \, dt \\ &\le C\|\boldsymbol{w}\|_{L^\infty(0,T;L^2(\Omega))}^2 \int_0^T \|\mathbb{D}(\boldsymbol{w})\|_{L^3(\Omega)}^4 \, dt \\ &= C\|\boldsymbol{w}\|_{L^\infty(0,T;L^2(\Omega))}^2 \|\mathbb{D}(\boldsymbol{w})\|_{L^4(0,T;L^3(\Omega))}^4 \le C < \infty\end{aligned}$$

by the regularity assumptions and stability estimates (7.117) and (7.118). Note also that $\nabla \boldsymbol{w} \in L^4\left(0,T;L^3(\Omega)\right)$ uniformly in ν implies $\nabla \boldsymbol{w} \in L^{18/5}\left(0,T;L^3(\Omega)\right)$ uniformly in ν since the time interval is bounded.

The proof is based on the differential inequality (7.126). The non-linear convective term is decomposed in the form 5.56

$$\begin{aligned}&\left|n_{\text{skew}}\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)\right| \\ &= \left|n_{\text{skew}}\left(\boldsymbol{\eta}-\boldsymbol{\phi}^h,\boldsymbol{w},\boldsymbol{\phi}^h\right) + n_{\text{skew}}\left(\boldsymbol{w}^h,\boldsymbol{\eta}-\boldsymbol{\phi}^h,\boldsymbol{\phi}^h\right)\right| \\ &= \left|n_{\text{skew}}\left(\boldsymbol{\eta},\boldsymbol{w},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\boldsymbol{\phi}^h,\boldsymbol{w},\boldsymbol{\phi}^h\right) + n_{\text{skew}}\left(\boldsymbol{w}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h\right)\right|,\end{aligned}$$

where $n_{\text{skew}}\left(\boldsymbol{w}^h,\boldsymbol{\phi}^h,\boldsymbol{\phi}^h\right)=0$, see (5.18), was used. With this decomposition, the critical term is $n_{\text{skew}}\left(\boldsymbol{\phi}^h,\boldsymbol{w},\boldsymbol{\phi}^h\right)$ and not $n_{\text{skew}}\left(\boldsymbol{\phi}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)$ as in (7.131). For estimating the critical term, the assumptions on $\boldsymbol{w}$ can be used now. Note that one can assume regularity for the solution of the continuous problem but not for the finite element solution.

The individual terms of the right-hand side are first transformed with (5.15) and then estimated by Hölder's inequality (A.15) and by Young's inequality (A.4)

$$\begin{aligned}&\left|n_{\text{skew}}\left(\boldsymbol{\eta},\boldsymbol{w},\boldsymbol{\phi}^h\right)\right| = \left|n_{\text{conv}}\left(\boldsymbol{\eta},\boldsymbol{w},\boldsymbol{\phi}^h\right) + \frac{1}{2}\left(\nabla\cdot\boldsymbol{\eta},\boldsymbol{\phi}^h\cdot\boldsymbol{w}\right)\right| \\ &\le \|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)}\|\boldsymbol{\eta}\|_{L^2(\Omega)}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} + \frac{1}{2}\|\boldsymbol{w}\|_{L^\infty(\Omega)}\|\nabla\cdot\boldsymbol{\eta}\|_{L^2(\Omega)}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\|\boldsymbol{\eta}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)}^2\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{1}{4}\|\nabla\cdot\boldsymbol{\eta}\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{4}\|\boldsymbol{w}\|_{L^\infty(\Omega)}^2\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2,\end{aligned}$$

$$\begin{aligned}&\left|n_{\text{skew}}\left(\boldsymbol{\phi}^h,\boldsymbol{w},\boldsymbol{\phi}^h\right)\right| = \left|n_{\text{conv}}\left(\boldsymbol{\phi}^h,\boldsymbol{w},\boldsymbol{\phi}^h\right) + \frac{1}{2}\left(\nabla\cdot\boldsymbol{\phi}^h,\boldsymbol{w}\cdot\boldsymbol{\phi}^h\right)\right| \\ &\le \|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\boldsymbol{w}\|_{L^\infty(\Omega)}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\ &\le \|\nabla \boldsymbol{w}\|_{L^\infty(\Omega)}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{\mu}{4}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{1}{4\mu}\|\boldsymbol{w}\|_{L^\infty(\Omega)}^2\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2,\end{aligned}$$

$$\begin{aligned}&\left|n_{\text{skew}}\left(\boldsymbol{w}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h\right)\right| = \left|n_{\text{conv}}\left(\boldsymbol{w}^h,\boldsymbol{\eta},\boldsymbol{\phi}^h\right) + \frac{1}{2}\left(\nabla\cdot\boldsymbol{w}^h,\boldsymbol{\eta}\cdot\boldsymbol{\phi}^h\right)\right| \\ &\le \left\|\boldsymbol{w}^h\right\|_{L^6(\Omega)}\|\nabla\boldsymbol{\eta}\|_{L^3(\Omega)}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} + \frac{1}{2}\left\|\nabla\cdot\boldsymbol{w}^h\right\|_{L^3(\Omega)}\|\boldsymbol{\eta}\|_{L^6(\Omega)}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\left\|\boldsymbol{w}^h\right\|_{L^6(\Omega)}^2\|\nabla\boldsymbol{\eta}\|_{L^3(\Omega)}^2 + C\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2\|\boldsymbol{\eta}\|_{L^6(\Omega)}^2 + \frac{3}{4}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2,\end{aligned}$$

where Korn's inequality (7.108) was used in the last line. The term $\left\|\boldsymbol{w}^h\right\|_{L^6(\Omega)}$ is bounded using the Gagliardo–Nirenberg inequality (A.20), Korn's inequality, and the uniform boundedness of $\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}$

$$\left\|\boldsymbol{w}^h\right\|_{L^6(\Omega)}^2 \leq C\left\|\boldsymbol{w}^h\right\|_{L^2(\Omega)}^{2/3}\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^{4/3} \leq C\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^{4/3}.$$

Collecting all estimates yields

$$\begin{aligned}
&\left|n_{\text{skew}}\left(\boldsymbol{w},\boldsymbol{w},\boldsymbol{\phi}^h\right) - n_{\text{skew}}\left(\boldsymbol{w}^h,\boldsymbol{w}^h,\boldsymbol{\phi}^h\right)\right| \\
&\leq \frac{1}{2}\|\boldsymbol{\eta}\|_{L^2(\Omega)}^2 + \frac{1}{4}\|\nabla\cdot\boldsymbol{\eta}\|_{L^2(\Omega)}^2 + C\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^{4/3}\|\nabla\boldsymbol{\eta}\|_{L^3(\Omega)}^2 \\
&\quad + C\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2\|\boldsymbol{\eta}\|_{L^6(\Omega)}^2 + \frac{\mu}{4}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \\
&\quad + \left(\frac{3}{4} + \frac{1}{2}\|\nabla\boldsymbol{w}\|_{L^\infty(\Omega)}^2 + \frac{1}{4}\|\boldsymbol{w}\|_{L^\infty(\Omega)}^2 + \|\nabla\boldsymbol{w}\|_{L^\infty(\Omega)} + \frac{1}{4\mu}\|\boldsymbol{w}\|_{L^\infty(\Omega)}^2\right)\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

This bound is inserted in the right-hand side of (7.126) giving

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 + \frac{\underline{C}C_S\delta^2}{3}\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^3(\Omega)}^3 + \frac{\mu}{2}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \\
&\quad + \frac{1}{2}\left(2\nu + \nu_0(\delta)\right)\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2(\Omega)}^2 + \sum_{j=1}^J\frac{\beta}{2}\left\|\boldsymbol{\phi}^h\cdot\hat{\boldsymbol{\tau}}_j\right\|_{L^2(\Gamma_j)}^2 \\
&\leq \Bigg[\frac{2}{3\left(\underline{C}C_S\right)^{1/2}\delta}\|\partial_t\boldsymbol{\eta}\|_{V'}^{3/2} + \frac{1}{2}\left(2\nu + \nu_0(\delta)\right)\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2(\Omega)}^2 \\
&\quad + \sum_{j=1}^J\frac{\beta}{2}\|\boldsymbol{\eta}\cdot\hat{\boldsymbol{\tau}}_j\|_{L^2(\Gamma_j)}^2 + \frac{2}{3}\underline{C}^{-1/2}C_S\overline{C}^{3/2}C_L^{3/2}\delta^2\|\mathbb{D}(\boldsymbol{\eta})\|_{L^3(\Omega)}^{3/2} \\
&\quad + \mu^{-1}\left\|r - q^h\right\|_{L^2(\Omega)}^2 + \mu\|\nabla\cdot\boldsymbol{\eta}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\boldsymbol{\eta}\|_{L^2(\Omega)}^2 + \frac{1}{4}\|\nabla\cdot\boldsymbol{\eta}\|_{L^2(\Omega)}^2 \\
&\quad + C\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^{4/3}\|\nabla\boldsymbol{\eta}\|_{L^3(\Omega)}^2 + C\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2\|\boldsymbol{\eta}\|_{L^6(\Omega)}^2\Bigg] + \frac{\mu}{4}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2 \\
&\quad + \left(\frac{3}{4} + \|\nabla\boldsymbol{w}\|_{L^\infty(\Omega)} + \left(\frac{1}{4} + \frac{1}{4\mu}\right)\|\boldsymbol{w}\|_{L^\infty(\Omega)}^2 + \frac{1}{2}\|\nabla\boldsymbol{w}\|_{L^\infty(\Omega)}^2\right)\left\|\boldsymbol{\phi}^h\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

For the application of Gronwall's lemma, Lemma A.71, one needs that

$$\frac{3}{4} + \|\nabla\boldsymbol{w}\|_{L^\infty(\Omega)} + \left(\frac{1}{4} + \frac{1}{4\mu}\right)\|\boldsymbol{w}\|_{L^\infty(\Omega)}^2 + \frac{1}{2}\|\nabla\boldsymbol{w}\|_{L^\infty(\Omega)}^2 \in L^1(0,T),$$

in other words $\boldsymbol{w} \in L^2\left(0,T;W^{1,\infty}(\Omega)\right)$. The term on the right-hand side of this inequality containing $C_L^{3/2}$ is treated as in the proof of Theorem 7.119. To obtain finally a uniform error estimate, one has also to verify that the terms containing $\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}$ are bounded uniformly in ν. To this end, one gets with Hölder's inequality

$$\begin{aligned}
&\int_0^T\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^{4/3}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^3(\Omega)}^2\ dt \\
&\leq \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^{4q/3}(0,T;L^3(\Omega))}^{4/3}\|\mathbb{D}(\boldsymbol{\eta})\|_{L^{2q'}(0,T;L^3(\Omega))}^2,
\end{aligned}$$

where $q^{-1} + q'^{-1} = 1$. From the stability estimates (7.121) and (7.122) it follows that one has to take q such that $4q/3 \le 3$. Accordingly, one can choose $q = 9/4, q' = 9/5$. Inserting this choice gives

$$\begin{aligned}&\int_0^T \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^{4/3} \|\mathbb{D}(\boldsymbol{\eta})\|_{L^3(\Omega)}^2 \ dt \\ &\le C\left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(0,T;L^3(\Omega))}^{4/3} \|\mathbb{D}(\boldsymbol{\eta})\|_{L^{18/5}(0,T;L^3(\Omega))}^2 \\ &\le CC_4(\delta)\|\mathbb{D}(\boldsymbol{\eta})\|_{L^{18/5}(0,T;L^3(\Omega))}^2 .\end{aligned}$$

Similarly, for the conjugate exponents $q = 3/2,\ q' = 3$, one obtains

$$\begin{aligned}\int_0^T \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(\Omega)}^2 \|\boldsymbol{\eta}\|_{L^6(\Omega)}^2 \ dt &\le \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^{2q}(0,T;L^3(\Omega))}^2 \|\boldsymbol{\eta}\|_{L^{2q'}(0,T;L^6(\Omega))}^2 \\ &\le \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{L^3(0,T;L^3(\Omega))}^2 \|\boldsymbol{\eta}\|_{L^6(0,T;L^6(\Omega))}^2 \\ &\le C_4(\delta)\|\boldsymbol{\eta}\|_{L^6(0,T;L^6(\Omega))}^2 .\end{aligned}$$

The stated error estimate now follows from Gronwall's inequality (A.47) and the triangle inequality as in the proof of Theorem 7.119. ■

Remark 7.122. Interpretation of the error estimates and numerical studies. The estimates given in Theorems 7.119 and 7.121 show that the error between the solution of the continuous and the discrete Smagorinsky model in different norms is bounded independently of ν for fixed filter width δ. In this case, the order of convergence is related to the best approximation errors of the finite element spaces in several norms. The best approximation error of V_{div}^h can be estimated by the best approximation error of V^h, see Lemma 2.55 for the $L^2(\Omega)$ norm of the gradient. For instance, considering an error which is squared on the left-hand side of estimate (7.139), then the order of convergence is bounded, e.g., by the best approximation error $\left\|\mathbb{D}\left(\boldsymbol{w} - \tilde{\boldsymbol{w}}^h\right)\right\|_{L^3(0,T;L^3(\Omega))}^{3/4}$ in expression (7.140). Numerical studies can be found in John and Layton (2002) and (John, 2004, Section 8.1.8). These studies show the independency of the errors on ν and an even higher order of convergence than predicted by the error bounds. □

Remark 7.123. Analysis of finite element discretizations for the stationary Smagorinsky model. Finite element methods for the stationary Smagorinsky model were studied in Du and Gunzburger (1990). It was proved that the discrete solution converges to the solution of the continuous problem under minimal regularity assumptions on this solution. In addition, an optimal order finite element error estimate for $\left\|\boldsymbol{w} - \boldsymbol{w}^h\right\|_{H^1(\Omega)}$ is given. check also Chacón Rebollo and Lewandowski (2014) □

7.3.4 Variants for Reducing Some Drawbacks of the Smagorinsky Model

Remark 7.124. Drawbacks of the Smagorinsky model in numerical simulations. The advantages of the Smagorinsky model were already mentioned in Remark 7.63: easiness of implementation, robustness, and low costs. However, this model possesses in its application for flow simulations also a number of drawbacks, e.g., see Zang et al. (1993). check paper, further references?

The easiest way, which is in fact quite popular, consists in choosing the Smagorinsky coefficient C_S in (7.64) a priori as a constant. However, it is well known that it is generally not possible to represent the large scales of turbulent flows correctly with a single constant. In addition, a reasonable good choice of C_S depends on many aspects, e.g., the flow problem or the discretization. For a concrete simulation, it is usually not clear what are good values for C_S. Numerical simulations with the Smagorinsky model which can be found in the literature use typically a Smagorinsky constant of size $C_S \in [0.01, 0.1]$, e.g., see (Sagaut, 2006, p. 95) or Piomelli (1999). check,new book of Sagaut For the case of homogeneous isotropic turbulence, in **?** the constant $C_S =$?? was derived. check also other papers Inappropriate choices of C_S might give very bad computational results.

The Smagorinsky model introduces generally too much viscosity into the flow simulations. This bevavior can be observed in particular near solid walls with no-slip boundary conditions.

More drawbacks of the Smagorinsky model arise from the fact that $C_S\delta^2 \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{\mathrm{F}} \geq 0$. Thus, backscatter of energy is prevented. Usually it is $C_S\delta^2 \left\|\mathbb{D}\left(\boldsymbol{w}^h\right)\right\|_{\mathrm{F}} > 0$, even for laminar flows or in subregions where the studied flow field is laminar. Hence, the Smagorinsky model generally does not vanish for laminar flows and thus it introduces also in simulations of such flows unneccessary viscosity. □

Remark 7.125. Contents of this section. This section describes two approaches which are used in practice for reducing the drawbacks of the Smagorinsky model. In the dynamic Smagorinsky model, Remark 7.126, the Smagorinsky parameter C_S is computed a posteriori as a function in space and time. To reduce the introduction of model viscosity near solid walls, there is a proposal to decrease C_S, see Remark 7.127, the so-called van Driest damping. □

Remark 7.126. The dynamic Smagorinsky model. The Smagorinsky model (7.64) contains the parameter C_S. As already noted, a good choice of C_S depends on the concrete flow problem and it is in general a priori hardly to achieve. It is even desireable to choose C_S in a different way in different flow regions. In particular, the impact of the Smagorinsky model should be small in subregions with laminar flows such that there small values of C_S are preferable.

An approach which determines values for C_S as a function of space and time was proposed in Germano et al. (1991). This proposal was modified in

Lilly (1992) to the form presented here. It is called dynamic Smagorinsky model or dynamic subgrid scale model.

The dynamic Smagorinsky model starts with introducing a second filter, a so-called test filter denoted by a hat, with $\widehat{\delta} > \delta$. Then, the space-averaged Navier–Stokes equations (7.26) and (7.27)

$$\partial_t \overline{\boldsymbol{u}} - 2\nu\nabla\cdot\mathbb{D}\left(\overline{\boldsymbol{u}}\right) + \nabla\cdot\left(\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T\right) + \nabla\cdot\mathbb{T} + \nabla\overline{p} = \overline{\boldsymbol{f}},$$

are filtered once more with the test filter. Assuming that differentiation and filtering commute yields

$$\begin{aligned}\partial_t \widehat{\overline{\boldsymbol{u}}} - 2\nu\nabla\cdot\mathbb{D}\left(\widehat{\overline{\boldsymbol{u}}}\right) + \nabla\cdot\left(\widehat{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T}\right) + \nabla\cdot\widehat{\mathbb{T}} + \nabla\widehat{\overline{p}} &= \widehat{\overline{\boldsymbol{f}}} \quad \text{in } (0,T]\times\Omega,\\ \nabla\cdot\widehat{\overline{\boldsymbol{u}}} &= 0 \quad \text{in } [0,T]\times\Omega.\end{aligned}$$

A direct calculation gives for

$$\mathbb{K} = \widehat{\overline{\boldsymbol{u}\boldsymbol{u}^T}} - \widehat{\overline{\boldsymbol{u}}}\,\widehat{\overline{\boldsymbol{u}}}^T$$

that

$$\mathbb{K} - \widehat{\mathbb{T}} = \widehat{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} - \widehat{\overline{\boldsymbol{u}}}\,\widehat{\overline{\boldsymbol{u}}}^T. \tag{7.141}$$

Now, one makes the ansatz of the Smagorinsky model (7.60), (7.64) for both tensors with the the same parameter $C_S(t,\boldsymbol{x})$

$$\begin{aligned}\mathbb{T}(t,\boldsymbol{x}) - \frac{\operatorname{trace}(\mathbb{T})}{3}\mathbb{I} &= -C_S(t,\boldsymbol{x})\,\delta^2\left\|\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right\|_{\mathrm{F}}\mathbb{D}\left(\overline{\boldsymbol{u}}\right),\\ \mathbb{K}(t,\boldsymbol{x}) - \frac{\operatorname{trace}(\mathbb{K})}{3}\mathbb{I} &= -C_S(t,\boldsymbol{x})\,\widehat{\delta}^2\left\|\mathbb{D}\left(\widehat{\overline{\boldsymbol{u}}}\right)\right\|_{\mathrm{F}}\mathbb{D}\left(\widehat{\overline{\boldsymbol{u}}}\right)\end{aligned}$$

Inserting this ansatz into (7.141) yields

$$\begin{aligned}\mathbf{0} &= -\widehat{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} + \mathbb{K}(t,\boldsymbol{x}) - \mathbb{T}(t,\boldsymbol{x})\\ &= \widehat{\overline{\boldsymbol{u}}}\,\widehat{\overline{\boldsymbol{u}}}^T + -\widehat{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} + \widehat{\overline{\boldsymbol{u}}}\,\widehat{\overline{\boldsymbol{u}}}^T + \frac{1}{3}\left(\operatorname{trace}(\mathbb{K}) - \widehat{\operatorname{trace}(\mathbb{T})}\right)\mathbb{I} \qquad (7.142)\\ &\quad + \left(C_S(t,\boldsymbol{x})\,\delta^2\,\widehat{\left\|\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\right\|_{\mathrm{F}}\mathbb{D}\left(\overline{\boldsymbol{u}}\right)}\right) - C_S(t,\boldsymbol{x})\,\widehat{\delta}^2\left\|\mathbb{D}\left(\widehat{\overline{\boldsymbol{u}}}\right)\right\|_{\mathrm{F}}\mathbb{D}\left(\widehat{\overline{\boldsymbol{u}}}\right).\end{aligned}$$

From the linearity of the filter (7.23), the linearity of the trace operator, and (7.141) it follows that

$$\begin{aligned}\operatorname{trace}(\mathbb{K}) - \widehat{\operatorname{trace}(\mathbb{T})} &= \operatorname{trace}(\mathbb{K}) - \operatorname{trace}\left(\widehat{\mathbb{T}}\right) = \operatorname{trace}\left(\mathbb{K} - \widehat{\mathbb{T}}\right)\\ &= \operatorname{trace}\left(\widehat{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} - \widehat{\overline{\boldsymbol{u}}}\,\widehat{\overline{\boldsymbol{u}}}^T\right).\end{aligned} \tag{7.143}$$

wider hat possible? In order to obtain an equation for $C_S(t,\boldsymbol{x})$, one approximates

$$\left(C_S\,(t,\boldsymbol{x})\,\delta^2\,\widehat{\|\mathbb{D}\,(\,\overline{\boldsymbol{u}}\,)\|_{\mathrm{F}}\,\mathbb{D}\,(\,\overline{\boldsymbol{u}}\,)}\right) \approx C_S\,(t,\boldsymbol{x})\,\delta^2\left(\widehat{\|\mathbb{D}\,(\,\overline{\boldsymbol{u}}\,)\|_{\mathrm{F}}\,\mathbb{D}\,(\,\overline{\boldsymbol{u}}\,)}\right). \quad (7.144)$$

If C_S depends only on t but not on $\boldsymbol{x}$, one has an equality instead of an approximation. Inserting (7.144) and (7.143) into (7.142) gives

$$\begin{aligned}
\mathbf{0} \approx &- \widehat{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} + \widehat{\overline{\boldsymbol{u}}}\ \widehat{\overline{\boldsymbol{u}}}^{\,T} + \frac{1}{3}\mathrm{trace}\left(\widehat{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} - \widehat{\overline{\boldsymbol{u}}}\ \widehat{\overline{\boldsymbol{u}}}^{\,T}\right)\mathbb{I} \\
&+C_S\,(t,\boldsymbol{x})\left(\delta^2\left(\widehat{\|\mathbb{D}\,(\,\overline{\boldsymbol{u}}\,)\|_{\mathrm{F}}\,\mathbb{D}\,(\,\overline{\boldsymbol{u}}\,)}\right) - \widehat{\delta}^{\,2}\left\|\mathbb{D}\left(\,\widehat{\overline{\boldsymbol{u}}}\,\right)\right\|_{\mathrm{F}}\mathbb{D}\left(\,\widehat{\overline{\boldsymbol{u}}}\,\right)\right) \\
=:&\ \mathbb{L} + C_S\mathbb{M}. \qquad (7.145)
\end{aligned}$$

Equations for $C_S\,(t,\boldsymbol{x})$ are obtained by replacing the approximation sign in (7.145) with the equal sign. Then, there are $d\,(d+1)\,/2$ equations to determine a scaler value for given t and $\boldsymbol{x}$. Because of the divergence constraint, the traces of the deformation tensors vanish, such that only $d\,(d+1)\,/2-1$ equations are linearly independent. In Lilly (1992) it was proposed to determine the parameter $C_S\,(t,\boldsymbol{x})$ by the least squares method, i.e., to find $C_S\,(t,\boldsymbol{x})$ such that $\|\mathbb{L}+C_S\,(t,\boldsymbol{x})\,\mathbb{M}\|_{\mathrm{F}}^2$ is minimized. Evaluating the necessary condition for a minimum gives

$$\begin{aligned}
0 &= \frac{d}{dC_S}\,\|\mathbb{L}+C_S\mathbb{M}\|_{\mathrm{F}}^2 = \frac{d}{dC_S}\sum_{i,j=0}^{d}\left(\mathbb{L}_{ij}+C_S\mathbb{M}_{ij}\right)^2 \\
&= 2\sum_{i,j=0}^{d}\left(\mathbb{L}_{ij}+C_S\mathbb{M}_{ij}\right)\mathbb{M}_{ij} = 2\sum_{i,j=0}^{d}\mathbb{L}_{ij}\mathbb{M}_{ij} + 2C_S\sum_{i,j=0}^{d}\mathbb{M}_{ij}\mathbb{M}_{ij} \\
&= 2\mathbb{L}:\mathbb{M} + 2C_S\mathbb{M}:\mathbb{M}.
\end{aligned}$$

It follows that

$$C_S\,(t,\boldsymbol{x}) = -\frac{\mathbb{L}:\mathbb{M}}{\mathbb{M}:\mathbb{M}}\,(t,\boldsymbol{x})\,. \qquad (7.146)$$

In practical computations, the test filter can be applied by solving the space-averaged Navier–Stokes equations on a coarse grid. If the next coarser grid of the current grid is used, then $\widehat{\delta} = 2\delta$.

The dynamic subgrid scale model can predict negative values for $C_S\,(t,\boldsymbol{x})$. In this way, backscatter of energy is possible, in contrast to the Smagorinsky model, see Remark 7.7. However, experience shows that $C_S\,(t,\boldsymbol{x})$ can vary strongly in space and time and it might contain negative values with a very large amplitude. These properties may strongly destabilize numerical simulations. In practice, the nominator and denominator of (7.146) are averaged, often in time, to compute a smoother function $C_S\,(t,\boldsymbol{x})$, e.g., see (Lesieur, 1997, p. 405), Breuer (1998), or (Sagaut, 2006, Section 5.3.3). check □

Remark 7.127. Van Driest damping. As already mentioned in Remark 7.124, the Smagorinsky model introduces too much viscosity in particular near solid walls. The application of a van Driest damping is a proposal to reduce this viscosity. citation The van Driest damping changes the eddy viscosity of the Smagorinsky model (7.64) in the viscous sublayer, see Remark 7.13, to (Pope, 2000, p. 599)

$$\nu_{\mathrm{T}} = C_S\delta^2 \left(1 - \exp\left(-\frac{y^+}{A^+}\right)\right)^2 \|\mathbb{D}\left(\overline{\boldsymbol{u}}\right)\|_{\mathrm{F}}\,, \quad \text{if } y^+ < 5,$$

with $A^+ = 26$. Vreman'04 PoF 16, 3670-3681, Verstappen JSC'11 □

7.4 Large Eddy Simulation – Models Based On Approximations in Wave Number Space

Remark 7.128. The basic approach. Inserting the decomposition (7.11) and applying the linearity (7.23) of the filter yields for filtered nonlinear convective term, which is unknown,

$$\overline{\boldsymbol{u}\boldsymbol{u}^T} = \overline{\overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}^T} + \overline{\overline{\boldsymbol{u}}\,\boldsymbol{u}'^T} + \overline{\boldsymbol{u}'\,\overline{\boldsymbol{u}}^T} + \overline{\boldsymbol{u}'\boldsymbol{u}'^T}\,. \tag{7.147}$$

The first term in (7.147) is called large scale advective term. It describes the convection of the large eddies driven by themselves. The second and third term are the so-called cross terms describing the interaction of the large scale and subgrid scale components. The last tensor is the subgrid scale (sgs) term which describes how the small eddies extract energy from the flow.

Models that are base on approximations in wave number space consider each term on the right-hand side of (7.147) separately. Each term is transformed to the Fourier or wave number space and then an approximation is applied, see Section 7.4.1. It turns out that with this approach the sgs term is modelled with the zero tensor. Numerical simulations show that this model is not sufficient, e.g., see (John, 2004, Section 10.3.3). Section 7.4.2 presents some proposals for modeling the sgs term. The final models will be presented and discussed in Section 7.4.3. □

7.4.1 Modelling of the Large Scale and Cross Terms

Remark 7.129. Assumptions. Let $\Omega = \mathbb{R}^d$ and let the averaging be performed with the Gaussian filter

$$\overline{\boldsymbol{u}}\,(t, \boldsymbol{x}) = g_{\mathrm{Gauss}} * \boldsymbol{u}\,(t, \boldsymbol{x})$$

where g_{Gauss} is defined in (7.17) and δ is a constant. □