Another indicator, coming from finite element error analysis, is that one can prove error estimates which depend not on the Reynolds number but on some reduced Reynolds number, e.g., see todo. □

7.2 Large Eddy Simulation – The Concept of Space Averaging

7.2.1 The Basic Concept of LES, Space Averaging, Convolution with Filters

Remark 7.23. The basic idea of large eddy simulation. In the case of turbulent flows, only in some sense large scales can be represented on given grids and these scales are the only ones which can be simulated, see Remark 7.11 for a discussion of this issue. There are different concepts of defining scales to be large. In the case of Large Eddy Simulation (LES), the large scales are defined by an average in space. The space averaging is usually given by a convolution with a filter function.

In this way, one obtains a decomposition

$$\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}', \quad p = \overline{p} + p', \tag{7.11}$$

of the solution of the Navier–Stokes equations. In (7.11), $(\overline{\boldsymbol{u}}, \overline{p})$ are the large scales, and $(\boldsymbol{u}', p')$ are the small scales or unresolved scales or subgrid scales or fluctuations. □

Remark 7.24. Space averaging with convolution, filter function, wave numbers. The space average of a sufficiently smooth function $v(t, \boldsymbol{x})$ defined in $\mathbb{R}^d$ given by a convolution with an appropriate filter function $g_{\text{fil}}(\boldsymbol{x})$ is defined by

$$\begin{aligned} \overline{v}\,(t, \boldsymbol{x}) = (g_{\text{fil}} * v)\,(t, \boldsymbol{x}) &= \int_{\mathbb{R}^d} g_{\text{fil}}\,(\boldsymbol{x} - \boldsymbol{z})\, v\,(t, \boldsymbol{z})\;\; d\boldsymbol{z} \\ &= \int_{\mathbb{R}^d} g_{\text{fil}}\,(\boldsymbol{z})\, v\,(t, \boldsymbol{x} - \boldsymbol{z})\;\; d\boldsymbol{z}. \end{aligned} \tag{7.12}$$

Applying the Fourier transform to (7.12) gives

$$\mathcal{F}\,(\,\overline{v}\,)\,(t, \boldsymbol{k}) = \left(\mathcal{F}\,(g_{\text{fil}})\,\mathcal{F}\,(v)\right)(t, \boldsymbol{k})\,, \tag{7.13}$$

where $\boldsymbol{k}$ is the dual variable or wave number. If $\mathcal{F}\,(g_{\text{fil}})\,(t, \boldsymbol{k}) = 0$ for $\|\boldsymbol{k}\|_2 > k_c$, where k_c is a cut-off wave number, then all high wave number components of $v\,(t, \boldsymbol{x})$ are filtered out by convolving v with g_{fil}. This situation is the ideal situation. For practical applications, it is sufficient that the high wave numbers are damped sufficiently fast. Thus, the most essential requirement on a filter function is that its Fourier transform decreases rapidly for high wave numbers.

It will be assumed that the filter function $g_{\mathrm{fil}}(\boldsymbol{x})$ can be represented as tensor product of one-dimensional filter functions

$$g_{\mathrm{fil}}(\boldsymbol{x}) = \prod_{i=1}^{d} g_{\mathrm{fil},i}(x_i). \tag{7.14}$$

Since all factors have different variables, the Fourier transform of the filter function is

$$\mathcal{F}(g_{\mathrm{fil}})(\boldsymbol{k}) = \prod_{i=1}^{d} \mathcal{F}(g_{\mathrm{fil},i})(k_i).$$

The filtering effect in the definition of $\overline{v}$ is often described by a positive constant δ, called the characteristic filter width or averaging radius or scale of the filter. The filter width is a measure of the size of the eddies which are damped out, which are all eddies with size less than $\mathcal{O}(\delta)$. It is clear that the smaller δ is the larger becomes the cut-off wave number k_c and the less eddies are filtered out. □

Example 7.25. The Gaussian filter. The Gaussian filter is one of the most common filters applied in LES. Its filter function is given by

$$g_{\mathrm{fil},i}(x_i) = g_{\mathrm{Gauss},i}(x_i) = \sqrt{\frac{6}{\pi\delta^2}} \exp\left(-\frac{6}{\delta^2} x_i^2.\right)$$

It is

$$\int_0^\infty \exp(-ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \quad a > 0,$$

and because of symmetry

$$\int_{-\infty}^\infty \exp(-ax^2)\,dx = \sqrt{\frac{\pi}{a}}, \quad a > 0. \tag{7.15}$$

Then, one obtains for the Fourier transform (A.6)

$$
\begin{aligned}
&\mathcal{F}\left(g_{\mathrm{Gauss},j}\right)(k_j) \\
&= \sqrt{\frac{6}{\pi\delta^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{6}{\delta^2}x_j^2 - ik_jx_j\right)\, dx_j \\
&= \sqrt{\frac{6}{\pi\delta^2}} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\sqrt{6}}{\delta}x_j + \frac{ik_j\delta}{2\sqrt{6}}\right)^2 + \frac{i^2k^2\delta^2}{24}\right)\, dx_j \\
&= \sqrt{\frac{6}{\pi\delta^2}} \exp\left(-\frac{k_j^2\delta^2}{24}\right) \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\sqrt{6}}{\delta}x_j + \frac{ik_j\delta}{2\sqrt{6}}\right)^2\right)\, dx_j \\
&= \sqrt{\frac{6}{\pi\delta^2}} \frac{\delta}{\sqrt{6}} \exp\left(-\frac{\delta^2}{24}k_j^2\right) \int_{-\infty}^{\infty} \exp\left(-y^2\right)\, dy \\
&= \exp\left(-\frac{\delta^2}{24}k_j^2\right), \quad j = 1, \ldots, d.
\end{aligned}
\tag{7.16}
$$

The constant is chosen such that the filtering effect of the Gaussian filter and the filtering effect of a certain discrete approximation in a model problem are equilibrated, see (Aldama, 1990, Section 3.8) for a detailed discussion. In addition, the integral of the filter function is normalized, see (7.20). The Gaussian filter function and its Fourier transform are of the same form.

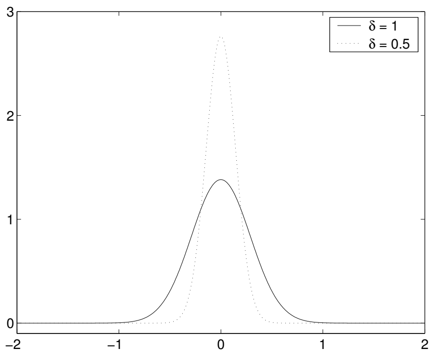

Fig. 7.2 The Gaussian filter in one dimension for different δ.new pic

The Gaussian filter g_{Gauss} in $\mathbb{R}^d$ is given by

$$
g_{\mathrm{Gauss}}(\boldsymbol{x}) = \left(\frac{6}{\delta^2\pi}\right)^{d/2} \exp\left(-\frac{6}{\delta^2}\|\boldsymbol{x}\|_2^2\right), \tag{7.17}
$$

see Figure 7.2. Its Fourier transform is

$$
\mathcal{F}\left(g_{\mathrm{Gauss}}\right)(\boldsymbol{k}) = \exp\left(-\frac{\delta^2}{24}\|\boldsymbol{k}\|_2^2\right). \tag{7.18}
$$

For convenience of notations, the Gaussian filter with a scalar argument x is understood to be

$$g_{\text{Gauss}}(x) := \left(\frac{6}{\delta^2\pi}\right)^{\frac{d}{2}} \exp\left(-\frac{6x^2}{\delta^2}\right).$$

The Gaussian filter has the following properties which can directly follow from properties of the exponential or which can be verified by direct calculations, see also Figure 7.2:

- regularity: Since the exponential is infinitely often differentiable, it follows that
$$g_{\text{Gauss}} \in C^\infty\left(\mathbb{R}^d\right), \quad \mathcal{F}\left(g_{\text{Gauss}}\right) \in C^\infty\left(\mathbb{R}^d\right).$$
- positivity: The positivity of the exponential and the fact that (7.17) and (7.18) take their maximal value at the origin yields
$$0 < g_{\text{Gauss}}(\boldsymbol{x}) \leq \left(\frac{6}{\delta^2\pi}\right)^{\frac{d}{2}}, \quad 0 < \mathcal{F}\left(g_{\text{Gauss}}\right)(\boldsymbol{k}) \leq 1. \tag{7.19}$$
- integrability: From (7.15), one finds
$$\begin{aligned}\|g_{\text{Gauss}}\|_{L^p(\mathbb{R}^d)} &= \left(\frac{6}{\delta^2\pi}\right)^{d/2} \left(\int_{\mathbb{R}^d} \exp\left(-p\frac{6}{\delta^2}\|\boldsymbol{x}\|_2^2\right)\, dx\right)^{1/p} \\ &= \left(\frac{6}{\delta^2\pi}\right)^{d/2} \left(\frac{\pi\delta^2}{6p}\right)^{d/(2p)}.\end{aligned}$$
Together with (7.19), one gets
$$\|g_{\text{Gauss}}\|_{L^p(\mathbb{R}^d)} < \infty,\ p \in [1,\infty],, \quad \|g_{\text{Gauss}}\|_{L^1(\mathbb{R}^d)} = 1. \tag{7.20}$$
- symmetry: Since the functions (7.17) and (7.18) depend only on the norms of $\boldsymbol{x}$ and $\boldsymbol{k}$, it follows that
$$g_{\text{Gauss}}(\boldsymbol{x}) = g_{\text{Gauss}}(-\boldsymbol{x}), \quad \mathcal{F}\left(g_{\text{Gauss}}\right)(\boldsymbol{k}) = \mathcal{F}\left(g_{\text{Gauss}}\right)(-\boldsymbol{k}). \tag{7.21}$$
- monotonicity: From the monotonicity of the exponential, one obtains
$$g_{\text{Gauss}}(\boldsymbol{x}) \geq g_{\text{Gauss}}(\boldsymbol{y}) \quad \text{if} \quad \|\boldsymbol{x}\|_2 \leq \|\boldsymbol{y}\|_2. \tag{7.22}$$

□

Lemma 7.26. Further properties of the Gaussian filter.
i) Let $\varphi \in L^p\left(\mathbb{R}^d\right)$, then for $1 \leq p < \infty$

$$\lim_{\delta\to 0} \|g_{\text{Gauss}} * \varphi - \varphi\|_{L^p(\mathbb{R}^d)} = 0.$$

ii) Let $\varphi \in L^\infty\left(\mathbb{R}^d\right)$. If φ is uniformly continuous on a set ω, then there is a pointwise convergence $g_{\text{Gauss}} * \varphi \to \varphi$ uniformly on ω as $\delta \to 0$.
iii) If $\varphi \in C_0^\infty\left(\mathbb{R}^d\right)$, then it is for $1 \le p < \infty$, $0 \le r < \infty$

$$\lim_{\delta\to 0} \|g_{\text{Gauss}} * \varphi - \varphi\|_{W^{r,p}(\mathbb{R}^d)} = 0.$$

all things needed?

Proof. The proof of the first two statements can be found, e.g., in (Folland, 1995, Theorem 0.13). The third statement is an immediate consequence of the first one, since $D^{\boldsymbol{\alpha}}\varphi \in L^p\left(\mathbb{R}^d\right)$ for all $0 \le |\boldsymbol{\alpha}| \le r$. □ ■

Example 7.27. The box filter. Another popular filter is the box filter or top hat filter. For the filter width δ it is given in one dimension by

$$g_{\text{box}} = \begin{cases} \frac{1}{\delta} & \text{for } x \in \left[-\frac{\delta}{2}, \frac{\delta}{2}\right], \\ 0 & \text{else,} \end{cases}$$

see Figure **??**. todo For multiple dimensions, it is defined by a tensor product as in (7.14). Main properties of the box filter are as follows:

- it is piecewise constant,
- it is non-negative,
- it has compact support: $\operatorname{supp}(g_{\text{box}}) = [-\delta/2, \delta/2]$,
- $g_{\text{box}} \in L^p(\mathbb{R})$ for $p \in [1,\infty]$, $\|g_{\text{box}}\|_{L^1(\mathbb{R})} = 1$.
- For the Fourier transform (A.6), one obtains with the fundamental theorem of calculus and with Euler's formula for the exponential with complex argument

$$\begin{aligned}
&\mathcal{F}\left(g_{\text{box},j}\right)(k_j) \\
&= \int_{-\delta/2}^{\delta/2} \frac{1}{\delta} \exp\left(-ik_jx_j\right)\,dx_j \\
&= \frac{1}{\delta}\left(-\frac{1}{ik_j}\right)\left(\exp\left(-\frac{ik_j\delta}{2}\right) - \exp\left(\frac{ik_j\delta}{2}\right)\right) \\
&= -\frac{1}{ik_j\delta}\left(\cos\left(\frac{k_j\delta}{2}\right) - i\sin\left(\frac{k_j\delta}{2}\right) - \cos\left(\frac{k_j\delta}{2}\right) - i\sin\left(\frac{k_j\delta}{2}\right)\right) \\
&= \frac{2}{k_j\delta}\sin\left(\frac{k_j\delta}{2}\right).
\end{aligned}$$

Thus, the Fourier transform of the box filter is a damped sine function. This function has negative values.

□

7.2.2 The Space-Averaged Navier–Stokes Equations in the Case $\Omega = \mathbb{R}^d$

Remark 7.28. Steps for deriving LES models. To compute the space-averaged velocity $\overline{\boldsymbol{u}}$ and pressure $\overline{p}$, equations for these quantities are needed. These equations have to be derived from the governing equations for $\boldsymbol{u}$ and p, i.e., from the Navier–Stokes equations. The first step consists in applying the filter which defines $(\overline{\boldsymbol{u}}, \overline{p})$ also to the Navier–Stokes equations (1.24). Thus, LES modelling is based on the strong form of the Navier–Stokes equations, which is in contrast to other turbulence models, which are based on the variational form of the Navier–Stokes equations, e.g., see Section **??**. After having applied the filter to (1.24), the basic equation for LES, the space-averaged Navier–Stokes equations, are obtained with the assumption that convolution and differentiation commute. However, due to the nonlinear term of the Navier–Stokes equations, an additional modelling step is necessary to derive equations for $(\overline{\boldsymbol{u}}, \overline{p})$ from the space-averaged Navier–Stokes equations.

The assumption that convolution and differentiation commute will be studied in the remainder of Section 7.2. The additional modelling step is the topic of Sections 7.3 and 7.4. □

Remark 7.29. Assumptions on the filter. It will be assumed that the filter operator which defines $(\overline{\boldsymbol{u}}, \overline{p})$ has the following properties:

- The filter is a linear operator

$$\overline{\boldsymbol{u} + \lambda \boldsymbol{v}} = \overline{\boldsymbol{u}} + \lambda \overline{\boldsymbol{v}}. \qquad (7.23)$$

- Derivatives (first and second order in space, first order in time) and averages commute, e.g.,

$$\overline{\partial_{x_i} \boldsymbol{u}} = \partial_{x_i} \overline{\boldsymbol{u}}, \quad i = 1, \ldots, d, \qquad (7.24)$$

  and

$$\overline{\partial_t \boldsymbol{u}} = \partial_t \overline{\boldsymbol{u}}. \qquad (7.25)$$

  It is known already from classical calculus that the commutation of differentiation and integration of a parametric function requires sufficient smoothness of this function with respect to the parameter.

□

Remark 7.30. Filtering with convolution in the case $\Omega = \mathbb{R}^d$. Let $\Omega = \mathbb{R}^d$ and consider as filter operator the convolution with a filter function given in (7.12). Note that this filter operator is only defined in $\mathbb{R}^d$. The linearity (7.23) follows from the linearity of the integral operator. The commutation with respect to the temporal derivative (7.25) follows from the fact that integration with respect to a certain variable and differentiation with respect to another variable can be interchanged. For property (7.24), it will be assumed in addition that the filter width δ is constant. Then, it is known that reference

$$\partial_{x_i} \int_{\mathbb{R}^d} g_{\text{fil}}\,(\boldsymbol{x}-\boldsymbol{z})\, v\,(t,\boldsymbol{z})\; d\boldsymbol{z} = \int_{\mathbb{R}^d} g_{\text{fil}}\,(\boldsymbol{x}-\boldsymbol{z})\, \partial_{z_i} v\,(t,\boldsymbol{z})\; d\boldsymbol{z}, \quad i=1,\ldots,d,$$

for sufficiently smooth functions g_{fil} and v. □

Remark 7.31. The space-averaged Navier–Stokes equations in the case $\Omega = \mathbb{R}^d$. Applying a filter with the properties (7.23) and (7.24) to the Navier–Stokes equations (1.24) with sufficiently smooth functions and the initial condition $\boldsymbol{u}_0$ gives the space-averaged Navier–Stokes equations (or Reynolds equations)

$$\begin{aligned} \partial_t\, \overline{\boldsymbol{u}} - 2\nu\nabla\cdot\mathbb{D}\,(\,\overline{\boldsymbol{u}}\,) + \nabla\cdot\left(\overline{\boldsymbol{u}\boldsymbol{u}^T}\right) + \nabla\,\overline{p} &= \overline{\boldsymbol{f}} \quad \text{in } (0,T]\times\mathbb{R}^d,\\ \nabla\cdot\,\overline{\boldsymbol{u}} &= 0 \quad \text{in } [0,T]\times\mathbb{R}^d,\\ \overline{\boldsymbol{u}}\,(0,\cdot) &= \overline{\boldsymbol{u}}_{\,0} \text{ in } \mathbb{R}^d. \end{aligned} \tag{7.26}$$

□

Remark 7.32. The filter of the nonlinear term and the closure problem. The space-averaged Navier–Stokes equations 7.26 are not yet equations for $(\,\overline{\boldsymbol{u}}\,,\,\overline{p}\,)$ because the tensor $\overline{\boldsymbol{u}\boldsymbol{u}^T}$ is not expressed in terms of $(\,\overline{\boldsymbol{u}}\,,\,\overline{p}\,)$. Since the dyadic product of a d-dimensional vector with itself is a symmetric tensor, $\overline{\boldsymbol{u}\boldsymbol{u}^T}$ is symmetric, too. Thus, on the one hand there are $(d+1)$ equations in (7.26) and on the other hand there are $(d+1)$ unknown space-averaged values and $d\,(d+1)\,/2$ unknown quantities in $\overline{\boldsymbol{u}\boldsymbol{u}^T}$. Hence, a closure problem arises. Writing the nonlinear term in the form

$$\nabla\cdot\left(\overline{\boldsymbol{u}\boldsymbol{u}^T}\right) = \nabla\cdot\left(\,\overline{\boldsymbol{u}}\;\overline{\boldsymbol{u}}^{\,T}\right) + \nabla\cdot\left(\overline{\boldsymbol{u}\boldsymbol{u}^T} - \overline{\boldsymbol{u}}\;\overline{\boldsymbol{u}}^{\,T}\right) = \nabla\cdot\left(\,\overline{\boldsymbol{u}}\;\overline{\boldsymbol{u}}^{\,T}\right) + \nabla\cdot\mathbb{T}, \tag{7.27}$$

where $\mathbb{T}$ is called subgrid-scale (sgs) stress tensor Meneveau and Katz (2000), subgrid tensor, (Sagaut, 2006, pp. 49), or residual-stress tensor, (Pope, 2000, p. 581), or Reynolds stress tensor. A model of the subgrid scale tensor is needed for closing the equations. Modelling $\mathbb{T}$ in terms of $(\,\overline{\boldsymbol{u}}\,,\,\overline{p}\,)$ is the main issue in LES. This topic will be addressed in Sections 7.3 and 7.4. □

Remark 7.33. Averaging in time. The concept of LES models does not include averaging in time. Temporal averaging is used in the derivation of Reynolds Averaged Navier–Stokes (RANS) models, e.g., see (Ferziger and Perić, 1999, Section 9.4) for more details. □

7.2.3 The Space-Averaged Navier–Stokes Equations in a Bounded Domain

Remark 7.34. The space-averaged Navier–Stokes equations in bounded domains in practical computations. Usually, (7.26) is also used in practical computations in bounded domains Ω, i.e., $\mathbb{R}^d$ is simply replaced by Ω, also in the definition (7.12) of the filter. However, if Ω is a bounded domain, the commutation of filtering and differentiation requires special attention from the mathematical point of view. One has to extend all functions of the Navier–Stokes equations consistently from Ω to $\mathbb{R}^d$. In general, these extensions are not sufficiently smooth for the simple commutation of differentiation and integration to be valid. An extra term occurs and omitting this term leads to a so-called commutation error. This type of commutation error will be studied in the following.

The presentation of this topic is based on Dunca et al. (2004). □

Remark 7.35. Setup of the problem. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \in \{2,3\}$, with Lipschitz boundary $\partial\Omega$ and $(d-1)$ dimensional measure $|\partial\Omega| < \infty$. The incompressible Navier–Stokes equations with homogeneous Dirichlet boundary conditions

$$
\begin{aligned}
\partial_t \boldsymbol{u} - 2\nu\nabla\cdot\mathbb{D}(\boldsymbol{u}) + \nabla\cdot\left(\boldsymbol{u}\boldsymbol{u}^T\right) + \nabla p &= \boldsymbol{f} \quad \text{in } (0,T]\times\Omega,\\
\nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } [0,T]\times\Omega,\\
\boldsymbol{u} &= \boldsymbol{0} \quad \text{on } [0,T]\times\partial\Omega,\\
\boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_0 \text{ in } \Omega,\\
\int_\Omega p\, d\boldsymbol{x} &= 0 \quad \text{in } (0,T],
\end{aligned}
\tag{7.28}
$$

will be considered.

The derivation of the space-averaged Navier–Stokes equations and the analysis of the commutation error require that (7.28) possesses a unique solution $(\boldsymbol{u},p)$ which is sufficiently regular, such that the normal stress has a well defined trace on the $\partial\Omega$ which belongs to some Lebesgue space defined on $\partial\Omega$. The stress tensor $\mathbb{S}$ is given in (1.32) and the normal stress or Cauchy stress vector on $\partial\Omega$ is defined by $\mathbb{S}\boldsymbol{n}$. Concretely, it will be assumed that

$$
\begin{aligned}
\boldsymbol{u} \in H^2(\Omega)\cap V,\ p \in H^1(\Omega)\cap Q \quad &\text{for a.e. } t\in[0,T],\\
\boldsymbol{u} \in H^1((0,T)) \quad &\text{for a.e. } \boldsymbol{x}\in\overline{\Omega}.
\end{aligned}
\tag{7.29}
$$

In order to apply a convolution operator to (7.28), one has to extend first all functions outside the domain. These functions will satisfy the Navier–Stokes equations in a distributional sense. Then, the convolution operator can be applied, filtering and differentiation commute, and the space-averaged Navier–Stokes equations are obtained. □

Lemma 7.36. Regularity of the normal stress and estimate of its $L^p(\partial\Omega)$ norm. *If (7.29) holds then $\mathbb{S}\boldsymbol{n} \in H^{1/2}(\partial\Omega)$. In particular, for al-*

most every $t \in [0,T]$, $\mathbb{S}\boldsymbol{n} \in L^q(\partial\Omega)$ *with*

$$\begin{cases} q \in [1,\infty) & \text{if } d = 2, \\ q \in [1,4] & \text{if } d = 3. \end{cases}$$

Proof. By the trace theorem reference, it follows that $\nabla \boldsymbol{u} \in H^{1/2}(\partial\Omega)$ and $p \in H^{1/2}(\partial\Omega)$, such that $\mathbb{S}\boldsymbol{n}\ H^{1/2}(\partial\Omega)$. Then, the statement of the lemma follows from embedding theorems, e.g., see (Galdi, 1994, Chapter II, Theorem 3.1). more details, new book ∎

Remark 7.37. Extensions of the functions to $\mathbb{R}^d$. For deriving an equation to which the convolution with a filter function can be applied, $\boldsymbol{f}$ has to be extended off Ω and then $(\boldsymbol{u}, p)$ must be extended compatibly with the extension of $\boldsymbol{f}$ such that the extended functions satisfy a kind of Navier–Stokes equations. Since the right-hand side is data, the extension of $\boldsymbol{f}$ has to be known. For $\overline{\boldsymbol{f}}$ to be easily to compute, $\boldsymbol{f}$ is extended by $\boldsymbol{0}$ off Ω. Thus, $(\boldsymbol{u}, p)$ can be extended by zero off Ω, too. This extension is reasonable since $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$. Thus, one defines

$$\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{u}_0 = \boldsymbol{0}, \quad p = 0, \quad \boldsymbol{f} = \boldsymbol{0} \quad \text{if } \boldsymbol{x} \notin \overline{\Omega}.$$

The extended functions posses the following regularities

$$\begin{aligned} \boldsymbol{u} \in H_0^1\left(\mathbb{R}^d\right),\ p \in L_0^2\left(\mathbb{R}^d\right) &\quad \text{for a.e. } t \in [0,T], \\ \boldsymbol{u} \in H^1\left((0,T)\right) &\quad \text{for a.e. } \boldsymbol{x} \in \mathbb{R}^d\ . \end{aligned} \tag{7.30}$$

From (7.29) and (7.30), it follows that the first order weak derivatives of the extended velocity $\partial_t \boldsymbol{u}$, $\nabla \boldsymbol{u}$,$\nabla \cdot \boldsymbol{u}$, and $\nabla \cdot \left(\boldsymbol{u}\boldsymbol{u}^T\right)$ are well defined on $\mathbb{R}^d$, taking their indicated values in Ω and being identically zero off Ω.

An extension of $\boldsymbol{u}$ off Ω as a function in $H^2\left(\mathbb{R}^d\right)$ exists but it is unknown, in particular since $\boldsymbol{u}$ is not known away from the boundary. Using this extension, instead of $\boldsymbol{u} \equiv \boldsymbol{0}$ on $\mathbb{R}^d \setminus \Omega$, would lead to an unknown extension of $\boldsymbol{f}$ and hence $\overline{\boldsymbol{f}}$ would not be known in the space-averaged momentum equation. □

Remark 7.38. Extension of the pressure term and the viscous term. Since $\boldsymbol{u} \notin H^2\left(\mathbb{R}^d\right)$, $p \notin H^1\left(\mathbb{R}^d\right)$, the terms $\nabla \cdot \mathbb{D}(\boldsymbol{u})$ and ∇p must be defined in the sense of distributions. To this end, let $\varphi \in C_0^\infty\left(\mathbb{R}^d\right)$. Since $p \equiv 0$ on $\mathbb{R}^d \setminus \Omega$ for all times, one obtains

$$\begin{aligned} (\partial_i p)(\varphi)(t) &= -\int_{\mathbb{R}^d} p(t,\boldsymbol{x})\,\partial_i\varphi(\boldsymbol{x})\ d\boldsymbol{x} \\ &= \int_\Omega \varphi(\boldsymbol{x})\,\partial_i p(t,\boldsymbol{x})\ d\boldsymbol{x} - \int_{\partial\Omega} p(t,\boldsymbol{s})\,\varphi(\boldsymbol{s})\,\boldsymbol{e}_i\cdot\boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s}, \end{aligned}$$

$i = 1, \ldots, d$, or in compact form

$$
\begin{aligned}
(\nabla p)(\varphi)(t) &= -\int_{\mathbb{R}^d} p(t,\boldsymbol{x})\,\nabla\varphi(\boldsymbol{x})\ d\boldsymbol{x} \qquad (7.31)\\
&= \int_{\Omega} \varphi(\boldsymbol{x})\,\nabla p(t,\boldsymbol{x})\ d\boldsymbol{x} - \int_{\partial\Omega} p(t,\boldsymbol{s})\,\varphi(\boldsymbol{s})\,\boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s}.
\end{aligned}
$$

In the same way, one derives

$$
\begin{aligned}
&\nabla\cdot\mathbb{D}(\boldsymbol{u})(\varphi)(t)\\
&= -\int_{\mathbb{R}^d} \mathbb{D}(\boldsymbol{u})(t,\boldsymbol{x})\,\nabla\varphi(\boldsymbol{x})\ d\boldsymbol{x} \qquad (7.32)\\
&= \int_{\Omega} \varphi(\boldsymbol{x})\,\nabla\cdot\mathbb{D}(\boldsymbol{u})(t,\boldsymbol{x})\ d\boldsymbol{x} - \int_{\partial\Omega} \varphi(\boldsymbol{s})\,\mathbb{D}(\boldsymbol{u})(t,\boldsymbol{s})\,\boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s}.
\end{aligned}
$$

Both distributions define continuous linear functionals on $C_0^\infty(\mathbb{R}^d)$ such that they have compact support. Note that from the regularity assumption (7.29) it follows that the traces of p and $\mathbb{D}(\boldsymbol{u})$ on $\partial\Omega$ are well defined. □

Remark 7.39. The distributional form of the momentum equation. From (7.31) and (7.32), it follows that the extended functions $(\boldsymbol{u},p)$ satisfy the following distributional form of the momentum equation

$$
\begin{aligned}
&\left(\partial_t\boldsymbol{u} - 2\nu\nabla\cdot\mathbb{D}(\boldsymbol{u}) + \nabla\cdot\left(\boldsymbol{u}\boldsymbol{u}^T\right) + \nabla p\right)(\varphi)(t) \qquad (7.33)\\
&= \int_{\Omega} \boldsymbol{f}(t,\boldsymbol{x})\varphi(\boldsymbol{x})\ d\boldsymbol{x} + \int_{\partial\Omega} \mathbb{S}(t,\boldsymbol{s})\,\boldsymbol{n}(\boldsymbol{s})\,\varphi(\boldsymbol{s})\ d\boldsymbol{s}, \quad \forall\,\varphi\in C_0^\infty\left(\mathbb{R}^d\right).
\end{aligned}
$$

□

Remark 7.40. The space-averaged Navier–Stokes equations in a bounded domain. The space-averaged Navier–Stokes equations are now derived by convolving (7.33) with a filter function $g_{\text{fil}}(\boldsymbol{x})\in C^\infty\left(\mathbb{R}^d\right)$. Applying the convolution with g_{fil} to (7.33) gives a function in $C^\infty\left(\mathbb{R}^d\right)$ and moreover convolution and differentiation commute on $\mathbb{R}^d$, see (Hörmander, 1990, Theorem 4.1.1). One obtains for the left-hand side of (7.33)

$$
\begin{aligned}
&g_{\text{fil}} * \left[\left(\partial_t\boldsymbol{u} - 2\nu\nabla\cdot\mathbb{D}(\boldsymbol{u}) + \nabla\cdot\left(\boldsymbol{u}\boldsymbol{u}^T\right) + \nabla p\right)(\varphi)\right]\\
&= \partial_t\,\overline{\boldsymbol{u}} - 2\nu\nabla\cdot\mathbb{D}(\overline{\boldsymbol{u}}) + \nabla\cdot\left(\overline{\boldsymbol{u}\boldsymbol{u}^T}\right) + \nabla\,\overline{p} \qquad (7.34)
\end{aligned}
$$

in $(0,T]\times\mathbb{R}^d$.

The filter of the viscous term and the pressure term will be studied in more detail. Let $H(\varphi)$ be a distribution with compact support which has the form

$$
H(\varphi) = -\int_{\mathbb{R}^d} f(\boldsymbol{x})\,D^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x})\ d\boldsymbol{x},
$$

where $D^{\boldsymbol{\alpha}}$ is the derivative of φ with the multi-index $\boldsymbol{\alpha}$. Then, $H * g_{\text{fil}} \in C^\infty\left(\mathbb{R}^d\right)$, see (Rudin, 1991, Theorem 6.35), where

$$\overline{H}\ (\boldsymbol{x}) = (H * g_{\text{fil}})(\boldsymbol{x}) = H\left(g_{\text{fil}}(\boldsymbol{x} - \cdot)\right) = -\int_{\mathbb{R}^d} f(\boldsymbol{y})\, D^{\boldsymbol{\alpha}} g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{y})\ d\boldsymbol{y}. \tag{7.35}$$

Applying (7.35) to (7.31) yields

$$\begin{aligned}\nabla\, \overline{p}\ (t, \boldsymbol{x}) &= g_{\text{fil}} * ((\nabla p)(\varphi))(t, \boldsymbol{x}) \\ &= -\int_{\mathbb{R}^d} p(t, \boldsymbol{y})\, \nabla g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{y})\ d\boldsymbol{y} \qquad (7.36)\\ &= \int_{\Omega} \nabla p(t, \boldsymbol{y})\, g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{y})\ d\boldsymbol{y} - \int_{\partial\Omega} g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{s})\, p(t, \boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s}.\end{aligned}$$

Convolving (7.32) in the same way gives

$$\begin{aligned}&\nabla \cdot \mathbb{D}(\,\overline{\boldsymbol{u}}\,)(t, \boldsymbol{x}) \qquad (7.37)\\ &= \int_{\Omega} \nabla \cdot \mathbb{D}(\boldsymbol{u})(t, \boldsymbol{y})\, g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{y})\ d\boldsymbol{y} - \int_{\partial\Omega} g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{s})\, \mathbb{D}(\boldsymbol{u})(t, \boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s}.\end{aligned}$$

Combining (7.34), (7.36), and (7.37) leads to the space-averaged momentum equation

$$\begin{aligned}&\partial_t\, \overline{\boldsymbol{u}}\, - 2\nu \nabla \cdot \mathbb{D}(\,\overline{\boldsymbol{u}}\,) + \nabla \cdot \left(\overline{\boldsymbol{u}\boldsymbol{u}^T}\right) + \nabla\, \overline{p} \\ &= \overline{\boldsymbol{f}}\, + \int_{\partial\Omega} g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{s})\, \mathbb{S}(t, \boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s} \quad \text{in } (0, T] \times \mathbb{R}^d \qquad (7.38)\end{aligned}$$

with

$$\overline{\boldsymbol{f}}\ (t, \boldsymbol{x}) = \int_{\Omega} \left(\partial_t \boldsymbol{u} - 2\nu\nabla \cdot \mathbb{D}(\boldsymbol{u}) + \nabla \cdot \left(\boldsymbol{u}\boldsymbol{u}^T\right) + \nabla p\right)(t, \boldsymbol{y}) g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{y})\ d\boldsymbol{y}. \tag{7.39}$$

Since $\boldsymbol{f}$ vanishes outside Ω for $t \in [0, T]$, (7.39) has the same form which is obtained if $\boldsymbol{f}$ is filtered directly. This correspondence has to be expected from a consistent extension of the functions. □

Definition 7.41. Commutation error. Let $g_{\text{fil}} \in C^\infty\left(\mathbb{R}^d\right)$ be a filter function with filter width δ. The commutation error $A_\delta(\mathbb{S})$ in the space-averaged Navier–Stokes equations is defined to be

$$A_\delta(\mathbb{S})(t, \boldsymbol{x}) = \int_{\partial\Omega} g_{\text{fil}}(\boldsymbol{x} - \boldsymbol{s})\, \mathbb{S}(t, \boldsymbol{s})\, \boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s}.$$

□

Remark 7.42. On the commutation error.

- The analysis of the commutation error will be performed for the Gaussian filter g_{Gauss} defined in Example 7.25.

- In the analysis of the commutation error, an arbitrary but fixed time $t \in (0, T]$ will be considered such that the dependency of $A_\delta(\mathbb{S})$ on the time can be neglected.
- If the viscous term in the Navier–Stokes equations is written as $\nu\Delta\boldsymbol{u}$ instead of $2\nu\nabla\cdot\mathbb{D}(\boldsymbol{u})$, the resulting space-averaged momentum equation is given by replacing $2\nu\mathbb{D}(\overline{\boldsymbol{u}})$ in (7.38) by $\nu\nabla\overline{\boldsymbol{u}}$ and $2\nu\mathbb{D}(\boldsymbol{u})(t,\boldsymbol{s})$ by $\nu\nabla\boldsymbol{u}(t,\boldsymbol{s})$ in the stress tensor.
- The space-averaged Navier–Stokes equations arising from the Navier–Stokes equations in a bounded domain thus posses the extra boundary integral, $A_\delta(\mathbb{S})$. Omitting this integral results in committing a commutation error. Including this integral in the space-averaged momentum equation introduces a new modelling question since it depends on the unknown normal stress on $\partial\Omega$ of $(\boldsymbol{u}, p)$ but not of $(\overline{\boldsymbol{u}}, \overline{p})$.

□

7.2.4 Analysis of the Commutation Error for the Gaussian Filter

Remark 7.43. The term to be studied. In view of Definition 7.41, Lemma 7.36, and Remark 7.42, one has to study terms of the form

$$\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\,d\boldsymbol{s} \qquad (7.40)$$

with $\psi \in L^q(\partial\Omega)$, $1 \le q \le \infty$. □

Lemma 7.44. Expression (7.40) **defines a function in** $L^p(\mathbb{R}^d)$**,** $p \in [1,\infty]$**.** *Let* $\psi \in L^q(\partial\Omega)$*,* $1 \le q \le \infty$*, then* (7.40) *belongs to* $L^p(\mathbb{R}^d)$*,* $1 \le p \le \infty$*.*

Proof. i) $p=\infty, q>1$. By the Hölder inequality (A.11), one obtains with $r^{-1}+q^{-1}=1$, $q>1$,

$$\begin{aligned}
&\left|\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\;d\boldsymbol{s}\right| \\
&\le \left(\int_{\partial\Omega} g_{\text{Gauss}}^r(\boldsymbol{x}-\boldsymbol{s})\;d\boldsymbol{s}\right)^{1/r}\|\psi\|_{L^q(\partial\Omega)} \\
&= \left(\int_{\partial\Omega}\left(\frac{6}{\delta^2\pi}\right)^{rd/2}\exp\left(-\frac{6r}{\delta^2}\|\boldsymbol{x}-\boldsymbol{s}\|_2^2\right)\;d\boldsymbol{s}\right)^{1/r}\|\psi\|_{L^q(\partial\Omega)}\,. \qquad (7.41)
\end{aligned}$$

By the triangle inequality and Young's inequality (A.4), it follows that

$$\|\boldsymbol{x}\|_2^2 \le 2\|\boldsymbol{x}-\boldsymbol{s}\|_2^2 + 2\|\boldsymbol{s}\|_2^2 \quad\Longleftrightarrow\quad 2\|\boldsymbol{x}-\boldsymbol{s}\|_2^2 \ge \|\boldsymbol{x}\|_2^2 - 2\|\boldsymbol{s}\|_2^2\,,$$

which implies, together with the monotonicity of the exponential, that

$$\exp\left(-\frac{6r\|\boldsymbol{x}-\boldsymbol{s}\|_2^2}{\delta^2}\right) \le \exp\left(3r\frac{-\|\boldsymbol{x}\|_2^2 + 2\|\boldsymbol{s}\|_2^2}{\delta^2}\right).$$

Inserting this estimate into (7.41) yields

$$\begin{aligned}
&\left|\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\ d\boldsymbol{s}\right| \\
&\le \left(\frac{6}{\delta^2\pi}\right)^{d/2} \|\psi\|_{L^q(\partial\Omega)} \left(\int_{\partial\Omega} \exp\left(\frac{6r\|\boldsymbol{s}\|_2^2}{\delta^2}\right)\ d\boldsymbol{s}\right)^{1/r} \exp\left(-\frac{3\|\boldsymbol{x}\|_2^2}{\delta^2}\right) \\
&< \infty, \qquad (7.42)
\end{aligned}$$

since $\partial\Omega$ has finite measure in $\mathbb{R}^{d-1}$ and the exponential is a bounded function on $\partial\Omega$. This estimate proves the statement for $L^\infty(\mathbb{R}^d)$ and $q > 1$.

ii) $p \in [1,\infty), q > 1$. The proof for $p \in [1,\infty)$ and $q > 1$ is obtained by raising both sides of (7.42) to the power p, integrating on $\mathbb{R}^d$ and using (7.15), from what follows that

$$\int_{\mathbb{R}^d} \exp\left(-\frac{3p\|\boldsymbol{x}\|_2^2}{\delta^2}\right)\ d\boldsymbol{x} < \infty.$$

iii) $p \in [1,\infty), q = 1$. If $q = 1$, one has for $1 \le p < \infty$ with Hölder's inequality (A.11) and the monotonicity of the Gaussian filter (7.22), such that the largest values are taken if the absolute value of the argument is as small is possible,

$$\begin{aligned}
\int_{\mathbb{R}^d}\left|\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\,d\boldsymbol{s}\right|^p\ d\boldsymbol{x} &\le \int_{\mathbb{R}^d}\sup_{\boldsymbol{s}\in\partial\Omega} g^p_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\ d\boldsymbol{x}\left(\int_{\partial\Omega}\psi(\boldsymbol{s})\ d\boldsymbol{s}\right)^p \\
&= \int_{\mathbb{R}^d} g^p_{\text{Gauss}}(d(\boldsymbol{x},\partial\Omega))\ d\boldsymbol{x}\ \|\psi\|^p_{L^1(\partial\Omega)}.
\end{aligned}$$

Choosing a ball $B(\mathbf{0},R)$ with sufficiently large radius R such that $d(\boldsymbol{x},\partial\Omega) > \|\boldsymbol{x}\|_2/2$ for all $\boldsymbol{x} \notin B(\mathbf{0},R)$. Then, the integral on $\mathbb{R}^d$ is split into a sum of two integrals. The first integral is computed on $B(\mathbf{0},R)$. It is finite since the term in the integral is a continuous function on $\overline{B}(\mathbf{0},R)$. The second integral on $\mathbb{R}^d \setminus B(\mathbf{0},R)$ is also finite because

$$\int_{\mathbb{R}^d\setminus B(\mathbf{0},R)} g^p_{\text{Gauss}}(d(\boldsymbol{x},\partial\Omega))\ d\boldsymbol{x} \le \int_{\mathbb{R}^d} g^p_{\text{Gauss}}\left(\frac{\|\boldsymbol{x}\|_2}{2}\right)\ d\boldsymbol{x}$$

and the integrability of the Gaussian filter (7.20). Hence, the statement is proved for $p < \infty$.

iv) $p = \infty, q = 1$. For $p = \infty$ and $q = 1$, one finds

$$\begin{aligned}
\operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^d}\left|\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\ d\boldsymbol{s}\right| &\le \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^d}\operatorname*{ess\,sup}_{\boldsymbol{s}\in\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\|\psi\|_{L^1(\partial\Omega)} \\
&\le g_{\text{Gauss}}(\mathbf{0})\,\|\psi\|_{L^1(\partial\Omega)} < \infty,
\end{aligned}$$

where in the second estimate, it was used that the Gaussian filter takes its largest value at the origin, which follows from its monotonicity (7.22). This estimate finishes the proof of the lemma. ■

Theorem 7.45. Behavior of the $L^p(\mathbb{R}^d)$ norm of the commutation error for $\delta \to 0$. *Let $\psi \in L^p(\partial\Omega)$, $1 \le p \le \infty$. A necessary and sufficient condition for*

$$\lim_{\delta\to 0}\left\|\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\ d\boldsymbol{s}\right\|_{L^p(\mathbb{R}^d)} = 0, \qquad (7.43)$$

$1 \le p \le \infty$, is that ψ vanishes almost everywhere on $\partial\Omega$.

Proof. It is obvious that the condition is sufficient.

Let (7.43) hold. From Hölder's inequality (A.11), one obtains for an arbitrary function $\varphi \in C_0^\infty\left(\mathbb{R}^d\right)$

$$\lim_{\delta\to 0}\left|\int_{\mathbb{R}^d}\varphi(\boldsymbol{x})\left(\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\ d\boldsymbol{s}\right)\ d\boldsymbol{x}\right|$$
$$\le \lim_{\delta\to 0}\|\varphi\|_{L^q(\mathbb{R}^d)}\left\|\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\ d\boldsymbol{s}\right\|_{L^p(\mathbb{R}^d)} = 0, \tag{7.44}$$

where $p^{-1}+q^{-1}=1$. By Fubini's theorem and the symmetry of the Gaussian filter (7.21), one gets

$$\lim_{\delta\to 0}\int_{\mathbb{R}^d}\varphi(\boldsymbol{x})\left(\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\ d\boldsymbol{s}\right)\ d\boldsymbol{x}$$
$$= \lim_{\delta\to 0}\int_{\partial\Omega}\psi(\boldsymbol{s})\left(\int_{\mathbb{R}^d} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\varphi(\boldsymbol{x})\ d\boldsymbol{x}\right)\ d\boldsymbol{s}$$
$$= \int_{\partial\Omega}\psi(\boldsymbol{s})\lim_{\delta\to 0}\left(\int_{\mathbb{R}^d} g_{\text{Gauss}}(\boldsymbol{s}-\boldsymbol{x})\,\varphi(\boldsymbol{x})\ d\boldsymbol{x}\right)\ d\boldsymbol{s} = \int_{\partial\Omega}\psi(\boldsymbol{s})\,\varphi(\boldsymbol{s})\ d\boldsymbol{s},$$

where the last step is a consequence of Lemma 7.26 since $\varphi \in L^\infty\left(\mathbb{R}^d\right)$ and φ is uniformly continuous on the compact set $\partial\Omega$. Thus, with (7.44) it follows that

$$0 = \int_{\mathbb{R}^d}\varphi(\boldsymbol{x})\left(\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\ d\boldsymbol{s}\right)\ d\boldsymbol{x} = \int_{\partial\Omega}\psi(\boldsymbol{s})\,\varphi(\boldsymbol{s})\ d\boldsymbol{s}\quad \forall\,\varphi \in C_0^\infty\left(\mathbb{R}^d\right).$$

This statement is true if and only if ψ vanishes almost everywhere on $\partial\Omega$. ■

Remark 7.46. Interpretation of Lemma 7.44 and Theorem 7.45. Lemma 7.44 states that the commutation error $A_\delta(\mathbb{S})$ is a function in $L^p\left(\mathbb{R}^d\right)$, $1 \le p \le \infty$. Then, in Theorem 7.44 it is shown that $A_\delta(\mathbb{S})$ vanishes in $L^p\left(\mathbb{R}^d\right)$ as $\delta \to 0$ if and only if the normal stress is identically zero almost everywhere on $\partial\Omega$. This condition means that the wall has zero influence on the wall-bounded turbulent flow. This situation is not expected to be satisfied in any interesting flow problem ! If the commutation error term is simply dropped and then the strong form of the space-averaged Navier–Stokes equations is discretized, as, e.g., by a finite difference method, this result shows that the commutation error committed is $\mathcal{O}(1)$! □

Remark 7.47. Further results on the $L^p\left(\mathbb{R}^d\right)$ norm of the commutation error. With a quite technical proof, a bound for the $L^p\left(\mathbb{R}^d\right)$ norm of the commutation error in terms of δ can be derived, see Dunca et al. (2004) or (John, 2004, Theorem 3.13). An inspection of the proof shows that the commutation error is largest at the boundary and it decays away from the boundary. □

Remark 7.48. Motivation for considering the $H^{-1}(\Omega)$ norm of the commutation error. Variational methods, such as finite element methods, discretize the weak form of the considered equations. For these methods the $H^{-1}(\Omega)$ norm of any omitted term is of interest. □

Lemma 7.49. Estimate of $\|\overline{v} - v\|_{H^{1/2}(\mathbb{R}^d)}$. *There exists a constant C which does not depend on v and δ such that*

$$\|\overline{v} - v\|_{H^{1/2}(\mathbb{R}^d)} \le C\delta^{1/2} \|v\|_{H^1(\mathbb{R}^d)} \tag{7.45}$$

for any $v \in H^1(\mathbb{R}^d)$ and any $\delta > 0$.

Proof. The norm in $H^k(\mathbb{R}^d)$ can be expressed with the Fourier transform more details?

$$\|v\|^2_{H^k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left(1 + \|\boldsymbol{x}\|_2^2\right)^k |\mathcal{F}(v)|^2 \ d\boldsymbol{x}.$$

Using this definition for $\|\cdot\|_{H^{1/2}(\mathbb{R}^d)}$, the linearity of the Fourier transform, and (7.13) yields

$$\begin{aligned}
\|\overline{v} - v\|^2_{H^{1/2}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2} |\mathcal{F}(g_{\text{Gauss}} * v - v)|^2 \ d\boldsymbol{x} \\
&= \int_{\mathbb{R}^d} \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2} |\mathcal{F}(g_{\text{Gauss}} * v) - \mathcal{F}(v)|^2 \ d\boldsymbol{x} \\
&= \int_{\mathbb{R}^d} \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2} |\mathcal{F}(g_{\text{Gauss}})\mathcal{F}(v) - \mathcal{F}(v)|^2 \ d\boldsymbol{x} \\
&= \int_{\mathbb{R}^d} \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2} |1 - \mathcal{F}(g_{\text{Gauss}})|^2 |\mathcal{F}(v)|^2 \ d\boldsymbol{x} \\
&= \int_{\{\|\boldsymbol{x}\|_2 > \pi/\delta\}} \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2} |1 - \mathcal{F}(g_{\text{Gauss}})|^2 |\mathcal{F}(v)|^2 \ d\boldsymbol{x} \\
&\quad + \int_{\{\|\boldsymbol{x}\|_2 \le \pi/\delta\}} \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2} |1 - \mathcal{F}(g_{\text{Gauss}})|^2 |\mathcal{F}(v)|^2 \ d\boldsymbol{x}.
\end{aligned}$$

The integrals on the right-hand side will be bounded separately.

The bound of the first integral relies on the fact that $\|\boldsymbol{x}\|_2$ is sufficiently large. There exists a constant $C > 0$, which does not depend on δ and v, such that for $\|\boldsymbol{x}\|_2 > \pi/\delta$

$$\frac{1}{\left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2}} < \frac{1}{\left(1 + \pi^2/\delta^2\right)^{1/2}} = \delta \frac{1}{\left(\delta^2 + \pi^2\right)^{1/2}} \le \frac{\delta}{\pi^{1/2}} = C\delta.$$

From (7.16) it follows the pointwise estimate

$$|1 - \mathcal{F}(g_{\text{Gauss}})(\boldsymbol{x})| \le 1 \quad \forall\, \boldsymbol{x} \in \mathbb{R}^d.$$

Thus, the first integral can be bounded by

$$\begin{aligned}
&\left| \int_{\{\|\boldsymbol{x}\|_2 > \pi/\delta\}} \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2} |1 - \mathcal{F}(g_{\text{Gauss}})|^2 |\mathcal{F}(v)|^2 \ d\boldsymbol{x} \right| \\
&\le \int_{\{\|\boldsymbol{x}\|_2 > \pi/\delta\}} \left(1 + \|\boldsymbol{x}\|_2^2\right) \left(1 + \|\boldsymbol{x}\|_2^2\right)^{-1/2} |\mathcal{F}(v)|^2 \ d\boldsymbol{x} \\
&\le C\delta \int_{\{\|\boldsymbol{x}\|_2 > \pi/\delta\}} \left(1 + \|\boldsymbol{x}\|_2^2\right) |\mathcal{F}(v)|^2 \ d\boldsymbol{x}.
\end{aligned} \tag{7.46}$$

To bound the second integral, it is used that the Fourier transform of the Gaussian filter is close to one at the origin. Applying a Taylor series expansion of (7.16) at $\|\boldsymbol{x}\|_2 = 0$ for fixed δ yields

$$\mathcal{F}(g_{\text{Gauss}})(\boldsymbol{x}) = 1 - \frac{\delta^2 \|\boldsymbol{x}\|_2^2}{24} + \mathcal{O}\left(\delta^4 \|\boldsymbol{x}\|_2^4\right).$$

One obtains for all $\boldsymbol{x}$ with $\|\boldsymbol{x}\|_2 \leq \pi/\delta$ the pointwise bound

$$|1 - \mathcal{F}(g_{\text{Gauss}})(\boldsymbol{x})|^2 \leq C\delta^4 \|\boldsymbol{x}\|_2^4 \leq C\delta^4 \frac{\pi^3}{\delta^3} \|\boldsymbol{x}\|_2 = C\delta \|\boldsymbol{x}\|_2,$$

where C does not depend on δ or $\boldsymbol{x}$. Continuing this estimate with $\|\boldsymbol{x}\|_2 \leq \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2}$ shows that the second integral can be bounded as follows

$$\left| \int_{\{\|\boldsymbol{x}\|_2 \leq \pi/\delta\}} \left(1 + \|\boldsymbol{x}\|_2^2\right)^{1/2} |1 - \mathcal{F}(g_{\text{Gauss}})|^2 |\mathcal{F}(v)|^2 \, d\boldsymbol{x} \right| \tag{7.47}$$
$$\leq C\delta \int_{\{\|\boldsymbol{x}\|_2 \leq \pi/\delta\}} \left(1 + \|\boldsymbol{x}\|_2^2\right) |\mathcal{F}(v)|^2 \, d\boldsymbol{x}.$$

Combining (7.46) and (7.47) gives

$$\|\,\overline{v} - v\|_{H^{1/2}(\mathbb{R}^d)}^2 \leq C\delta \int_{\mathbb{R}^d} \left(1 + \|\boldsymbol{x}\|_2^2\right) |\mathcal{F}(v)|^2 \, d\boldsymbol{x} = C\delta \|v\|_{H^1(\mathbb{R}^d)}^2,$$

which is the statement of the lemma. ■

Theorem 7.50. Convergence of the commutation error in $H^{-1}(\Omega)$. *Let $\psi \in L^2(\partial\Omega)$, then there exists a constant $C > 0$ which depends only on Ω such that*

$$\left\| \int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x} - \boldsymbol{s})\, \psi(\boldsymbol{s}) \, d\boldsymbol{s} \right\|_{H^{-1}(\Omega)} \leq C\delta^{1/2} \|\psi\|_{L^2(\partial\Omega)} \tag{7.48}$$

for every $\delta > 0$.

Proof. It is

$$\|\varphi\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)} \frac{\int_\Omega v\varphi(\boldsymbol{x}) \, d\boldsymbol{x}}{\|\nabla v\|_{L^2(\Omega)}}. \tag{7.49}$$

The numerator of the right-hand side will be estimated for the commutation error.

Let $v \in H_0^1(\Omega)$. Extending v by zero outside Ω, applying Fubini's theorem, utilizing the symmetry of the Gaussian filter (7.21), using that v vanishes on $\partial\Omega$, applying the Cauchy–Schwarz inequality (A.12), the trace theorem (**??**), Lemma 7.49, that v vanishes off Ω, and the Poincaré inequality (A.13) yields

$$
\begin{aligned}
\int_{\Omega}\left(\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\ d\boldsymbol{s}\right) v(\boldsymbol{x})\ d\boldsymbol{x} &= \int_{\partial\Omega}\psi(\boldsymbol{s})\left(\int_{\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,v(\boldsymbol{x})\ d\boldsymbol{x}\right) d\boldsymbol{s}\\
&= \int_{\partial\Omega}\psi(\boldsymbol{s})\left(\int_{\Omega} g_{\text{Gauss}}(\boldsymbol{s}-\boldsymbol{x})\,v(\boldsymbol{x})\ d\boldsymbol{x}\right) d\boldsymbol{s}\\
&= \int_{\partial\Omega}\psi(\boldsymbol{s})\,\overline{v}\,(\boldsymbol{s})\ d\boldsymbol{s}\\
&= \int_{\partial\Omega}\psi(\boldsymbol{s})\,(\overline{v}\,(\boldsymbol{s})-v\,(\boldsymbol{s}))\ d\boldsymbol{s}\\
&\le \|\,\overline{v}\,-v\|_{L^2(\partial\Omega)}\,\|\psi\|_{L^2(\partial\Omega)}\\
&\le C\,\|\,\overline{v}\,-v\|_{H^{1/2}(\Omega)}\,\|\psi\|_{L^2(\partial\Omega)}\\
&\le C\,\|\,\overline{v}\,-v\|_{H^{1/2}(\mathbb{R}^d)}\,\|\psi\|_{L^2(\partial\Omega)}\\
&\le C\delta^{1/2}\,\|v\|_{H^1(\mathbb{R}^d)}\,\|\psi\|_{L^2(\partial\Omega)}\\
&= C\delta^{1/2}\,\|v\|_{H^1(\Omega)}\,\|\psi\|_{L^2(\partial\Omega)}\\
&\le C\delta^{1/2}\,\|\nabla v\|_{L^2(\Omega)}\,\|\psi\|_{L^2(\partial\Omega)}\,.
\end{aligned}
$$

Inserting this estimate into (7.49) gives the result of the theorem. ■

Remark 7.51. Interpretation of Theorem 7.50. Theorem 7.50 shows that the commutation error tends to zero in $H^{-1}(\Omega)$ as $\delta \to 0$. The order of convergence is at least $\mathcal{O}\left(\delta^{1/2}\right)$. Thus, using variational methods, like the finite element method, leads to the expected asymptotic vanishing of the commutation error. It is not known if the estimate (7.48) is optimal. □

Remark 7.52. On a weak form of the commutation error. One can derive an estimate for a weak form of the commutation error, i.e., the commutation error is multiplied with a test function and integrated. Considering any $v \in H^1\left(\mathbb{R}^d\right)$ with $v|_\Omega \in H^2(\Omega) \cap V$ and $v(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin \overline{\Omega}$, then the estimate is of the form

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\left|\overline{v}\,(\boldsymbol{x})\int_{\partial\Omega} g_{\text{Gauss}}(\boldsymbol{x}-\boldsymbol{s})\,\psi(\boldsymbol{s})\,d\boldsymbol{s}\right|^k d\boldsymbol{x}\\
&\quad\le C\delta^{1+\left(-d+\frac{(d-1)\theta}{q}+\beta\theta\right)k}\,\|\psi\|_{L^p(\partial\Omega)}^k\,\|v\|_{H^2(\Omega)}^k\,, \qquad (7.50)
\end{aligned}
$$

where $\delta \in (0,\epsilon)$, $\epsilon > 0$, $\theta \in (0,1)$, $k \in [1,\infty)$, $\beta \in (0,1)$ if $d = 2$ and $\beta = 1/2$ if $d = 3$, $p^{-1} + q^{-1} = 1$, $p > 1$, and C and ϵ depend on θ, k and $|\partial\Omega|$. The proof can be found in Dunca et al. (2004) and (John, 2004, Section 3.7).

For $d = 2$, $k = 1$, and $p < \infty$ arbitrary large, i.e., ψ is sufficiently smooth one finds that q is arbitrary close to 1. Choosing θ and β also arbitrary close to 1 leads to the following power of δ on the right-hand side of (7.50)

$$
1 + (-2 + (1-\epsilon_1) + (1-\epsilon_2)) = 1 - (\epsilon_1 + \epsilon_2) = 1 - \epsilon_3
$$

for arbitrary small $\epsilon_1, \epsilon_2, \epsilon_3 > 0$. In this case, the convergence is almost of first order.

Estimate (7.50) does not provide convergence for $d = 3$. Lemma 7.36 suggests choosing $p = 4$, i.e., $q = 4/3$. Then, for $k = 1$, the power of δ on the right-hand side of (7.50) becomes $2(\theta - 1)$, which is negative for $\theta < 1$. □

Remark 7.53. Non-constant filter width and skewed filtering. The application of a filter is called skewed if the point in which a function is filtered and the center of the filter kernel do not coincide. Possible advantages of studying skewed filters are discussed in van der Bos and Geurts (2005). The skewed version of the Gaussian filter with skewness $\tilde{z}(x)$ and with variable filter width $\delta(x)$, in one dimension and neglecting the dependency on time, reads as follows

$$\overline{v}\ (x) = \frac{\sqrt{6}}{\delta(x)\pi} \int_{-\infty}^{\infty} \exp\left(-6\left(\frac{z + \tilde{z}(x)}{\delta(x)}\right)^2\right) v(x - z)\ dz.$$

Both, the variable filter width and the skewness introduce in general commutation errors. Estimates for the error $\left(\partial_i \overline{v} - \overline{\partial_i v}\right)(\boldsymbol{x})$ can be found in Berselli et al. (2007). □

7.2.5 Analysis of the Commutation Error for the Box Filter

Remark 7.54. Motivation for using the box filter. The box filter, see Example 7.27, is a filter whose kernel has a compact support. It will be required that the application of this filter leads to integrals whose domain of integration is a subset of $\overline{\Omega}$, i.e., the filter width at a point $\boldsymbol{x}$ in any direction is not allowed to be larger than the distance of $\boldsymbol{x}$ to the boundary in that direction. This situation has the appealing property that an extension of the function v to be filtered outside Ω is not necessary. Note that the non-smooth extension of the functions was the origin of the commutation error studied in Section 7.2.4. But the requirement that the domain of filtering is always in $\overline{\Omega}$ also implies that the filter width has to tend to zero (at least in one direction) as the point $\boldsymbol{x}$ in which v is filtered tends to the boundary of Ω. Thus, necessarily, the filter width is not constant but it is a function of $\boldsymbol{x}$. This property leads to a commutation error. □

Remark 7.55. Normalized box filter. The filter kernel of the normalized box filter is given by

$$g_{\text{box}}(x) = \begin{cases} 1 & \text{for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \\ 0 & \text{else.} \end{cases}$$

It follows that

$$\int_{-1/2}^{1/2} g_{\text{box}}(x)\ dx = 1\ , \quad \int_{-1/2}^{1/2} g_{\text{box}}(x) x\ dx = 0\ , \quad \int_{-1/2}^{1/2} g_{\text{box}} x^2\ dx = \frac{1}{12}.$$

□

Remark 7.56. The box filter with variable filter width. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $v \in C^1(\overline{\Omega})$, $\delta_l(\boldsymbol{x})$ be scalar filter width functions with $\delta_l(\boldsymbol{x}) \in C^1(\overline{\Omega})$, $\delta_l(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \overline{\Omega}$, and $\delta_l(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \Omega$, $l = 1, \ldots, d$. The support of the the filter will be denoted by $B(\boldsymbol{x}) = [-\delta_1(\boldsymbol{x}), \delta_1(\boldsymbol{x})] \times \cdots \times [-\delta_d(\boldsymbol{x}), \delta_d(\boldsymbol{x})]$ and it is assumed that for each point the filtering is applied such that this support is in the closure of the domain, i.e., that

$$\boldsymbol{x} + B(\boldsymbol{x}) := [x_1 - \delta_1(\boldsymbol{x}), x_1 + \delta_1(\boldsymbol{x})] \times \cdots \times [x_d - \delta_d(\boldsymbol{x}), x_d + \delta_d(\boldsymbol{x})] \subset \overline{\Omega}$$

for all $\boldsymbol{x} = (x_1, \ldots, x_d) \in \overline{\Omega}$. The filter of v is defined by

$$\overline{v}(\boldsymbol{x}) = \frac{1}{a(\boldsymbol{x})} \int_{B(\boldsymbol{x})} \prod_{l=1}^{d} g_{\text{box}} \left(\frac{x_l}{2\delta_l(\boldsymbol{x})} \right) v(\boldsymbol{x} - \boldsymbol{z}) \, d\boldsymbol{z}, \tag{7.51}$$

with

$$a(\boldsymbol{x}) = \prod_{l=1}^{d} 2\delta_l(\boldsymbol{x}).$$

Note that with this definition, the filter width in $\boldsymbol{x}$ in the direction x_l is $2\delta_l(\boldsymbol{x})$. □

Lemma 7.57. Representation formula for the commutation error. *Let $v \in C^1(\overline{U(\boldsymbol{x})})$, where $U(\boldsymbol{x})$ is a neighborhood of $\boldsymbol{x}$ such that $\boldsymbol{x} + B(\boldsymbol{x}) \subset U(\boldsymbol{x})$, and $\delta_l \in C^1(\overline{U(\boldsymbol{x})})$, $l = 1, \ldots, d$. Then, the i-th component of the commutation error has the form*

$$\left(\partial_i \overline{v} - \overline{\partial_i v}\right)(\boldsymbol{x}) = \sum_{l=1}^{d} \frac{\partial_i \delta_l(\boldsymbol{x})}{\delta_l(\boldsymbol{x})} \left(\overline{x_l \partial_l v} - x_l \overline{\partial_l v}\right)(\boldsymbol{x}). \tag{7.52}$$

Proof. For the sake of simplifying the presentation, the proof will be given in one dimension. Because of the tensor product structure of the multi-dimensional filter, the proof for multiple dimensions uses the same techniques, see Berselli et al. (2007).

With the product rule of differentiation and the Leibniz rule, denoting with the prime the derivative, one gets

$$
\begin{aligned}
\frac{d\,\overline{v}}{dx}(x) &= \frac{d}{dx}\left(\frac{1}{2\delta(x)}\int_{-\delta(x)}^{\delta(x)} v(x-z)\ dz\right)\\
&= -\frac{\delta'(x)}{2\delta^2(x)}\int_{-\delta(x)}^{\delta(x)} v(x-z)\ dz\\
&\quad +\frac{1}{2\delta(x)}\left[v(x-\delta(x))\delta'(x) - v(x+\delta(x))(-\delta'(x)\right] + \frac{1}{2\delta(x)}\int_{-\delta(x)}^{\delta(x)} v'(x-z)\ dz\\
&= -\frac{\delta'(x)}{2\delta^2(x)}\int_{-\delta(x)}^{\delta(x)} v(x-z)\ dz\\
&\quad +\frac{\delta'(x)}{2\delta(x)}\left[v(x-\delta(x)) + v(x+\delta(x))\right] + \overline{\frac{dv}{dx}}(x). \qquad (7.53)
\end{aligned}
$$

Since the bounds of the integral do not depend on z, integration by parts and the chain rule yields

$$
\begin{aligned}
\int_{-\delta(x)}^{\delta(x)} zv'(x-z)\ dz &= -zv(x-z)\Big|_{z=-\delta(x)}^{z=\delta(x)} + \int_{-\delta(x)}^{\delta(x)} v(x-z)\ dz\\
&= -\delta(x)v(x-\delta(x)) - \delta(x)v(x+\delta(x)) + \int_{-\delta(x)}^{\delta(x)} v(x-z)\ dz.
\end{aligned}
$$

Multiplying this identity by $\delta'(x)/(2\delta^2(x))$ and adding it to (7.53) gives

$$
\begin{aligned}
&\left(\frac{d\,\overline{v}}{dx} - \overline{\frac{dv}{dx}}\right)(x)\\
&= -\frac{\delta'(x)}{2\delta^2(x)}\int_{-\delta(x)}^{\delta(x)} zv'(x-z)\ dz \qquad (7.54)\\
&= \frac{\delta'(x)}{\delta(x)}\left(\frac{1}{2\delta(x)}\int_{-\delta(x)}^{\delta(x)}(x-z)v'(x-z)\ dz - x\frac{1}{2\delta(x)}\int_{-\delta(x)}^{\delta(x)} v'(x-z)\ dz\right)\\
&= \frac{\delta'(x)}{\delta(x)}\left(\overline{xv'(x)} - x\,\overline{v'}\right).
\end{aligned}
$$

This representation is exactly (7.52) for $d = 1$. ■

Theorem 7.58. Pointwise error estimate of the commutation error. *Let $v \in C^2(\overline{U(\boldsymbol{x})})$ where $U(\boldsymbol{x})$ is defined in Lemma 7.57, and let $\delta_l(\boldsymbol{x}) \in C^1(\overline{U(\boldsymbol{x})})$, $l = 1,\ldots,d$, then*

$$
\left|\partial_i\overline{v} - \overline{\partial_i v}\right|(\boldsymbol{x}) \le \frac{\|v\|_{C^2(\overline{U(\boldsymbol{x})})}}{3}\sum_{l=1}^{d}\left|\partial_i\delta_l(\boldsymbol{x})\right|\left|\delta_l(\boldsymbol{x})\right|. \qquad (7.55)
$$

Proof. For the same reasons as in Lemma 7.57, the proof will be performed in one dimension. A Taylor series expansion with Lagrangian remainder gives

$$
v'(x-z) = v'(x) - zv''(\xi), \quad \text{with some } \xi \in U(x).
$$

Inserting this expression into (7.54) and using the definition of the norm in $C^2(\overline{U(x)})$, see Remark A.24, gives

$$
\begin{aligned}
\left|\frac{d\,\overline{v}}{dx} - \overline{\frac{dv}{dx}}\right|(x) &= \left|\frac{\delta'(x)}{2\delta^2(x)} \int_{-\delta(x)}^{\delta(x)} zv'(x) - z^2 v''(\xi)\ dz\right| \\
&= \left|\frac{\delta'(x)}{2\delta^2(x)} \left(v'(x)\int_{-\delta(x)}^{\delta(x)} z\ dz - v''(\xi)\int_{-\delta(x)}^{\delta(x)} z^2\ dz\right)\right| \\
&\le \left|\frac{\delta'(x)}{2\delta^2(x)}\right| \|v\|_{C^2(\overline{U(\boldsymbol{x})})} \left|0 - \frac{2}{3}\delta^3(x)\right| \\
&= \frac{|\delta'(x)|\,\delta(x)}{3} \|v\|_{C^2(\overline{U(x)})}.
\end{aligned}
$$

This estimate is the one-dimensional version of (7.55). ■

Remark 7.59. Interpretation of Theorem 7.58. Estimate (7.55) shows that the commutation error vanishes if the filter width is constant in all directions, i.e., $\partial_i \delta_l(\boldsymbol{x}) = 0$, $i = 1, \ldots, d$. These conditions cannot be satisfied (or only trivially) if one considers a bounded domain and requires that the filter kernel should always be inside the closure of this domain. For simplicity, let $\Omega = (a, b) \subset \mathbb{R}$. If the filter kernel should be contained in $[a, b]$ then necessarily $\delta(x) \to 0$ as $x \to a$ and $\delta(x) \to 0$ as $x \to b$. Thus, either one has $\delta(x) = 0$, i.e., no filtering, or the filter width is not constant.

If the derivatives $\partial_i \delta_l(\boldsymbol{x}) = 0$, $i = 1, \ldots, d$, are bounded, then the commutation error tends to zero as $\delta_l(\boldsymbol{x}) \to 0$. □

Remark 7.60. Further results on commutation errors for filters with compact support.

- In Berselli et al. (2007), the case of a general filter with compact support, variable filter width, and non-vanishing skewness is studied. A representation formula of the commutation error and a pointwise error estimate are derived.
- The analysis presented in this section requires a certain regularity of v, namely $v \in C^2(\overline{U(\boldsymbol{x})})$. In the case of turbulent flows, however, one cannot expect smooth solutions. In Berselli et al. (2007) also a pointwise estimate for the commutation error for functions with low regularity, concretely for Hölder-continuous functions, is proved.
- Since the filter width cannot be constant near the boundary, see Remark 7.59, a commutation error will be committed especially near the boundary. Estimates of this error for certain models of the mean velocity and pressure can be also found in Berselli et al. (2007). For the velocity, these models are called wall laws. An asymptotic analysis in Berselli and John (2006) shows for the turbulent channel flow, Example D.15, that the commutation error near the wall is at least as important as the divergence of the sgs stress tensor (7.27). Note that modelling the sgs stress tensor is the main issue in LES.

□

7.2.6 Summary of the Results Concerning Commutation Errors

Remark 7.61. Summary of the results concerning the commutation of convolution and differentiation. There are important situations in which the assumed commutation of filtering and differentiation is generally not true, e.g., if Ω is a bounded domain or if the filter width is not constant. Commuting these operators in these situation leads to extra terms. Omitting these terms, so-called commutation errors are committed.

Sections 7.2.3 and 7.2.4 considered the case of a bounded domain and the convolution with the Gaussian filter. First, the commutation error, which is caused by extensions of functions from Ω to $\mathbb{R}^d$ that are not sufficiently smooth, was derived in Section 7.2.3. In the following sections, an analysis of the commutation error in various norms for an arbitrary but fixed time was presented. In practical computations, the commutation error term is always neglected, expecting that it is small and vanishes if the filter width δ tends to zero. In Section 7.2.4, it was shown that this is not always true. The commutation error is asymptotically negligible in $L^p\left(\mathbb{R}^d\right)$, i.e., it vanishes as the averaging radius $\delta \to 0$, if and only if the normal stress vanishes almost everywhere on the boundary. In other words, it is asymptotically negligible in $L^p\left(\mathbb{R}^d\right)$ if and only if the fluid and the boundary exert zero force on each other. The expected convergence of the commutation error as $\delta \to 0$ was shown in the $H^{-1}(\Omega)$ norm and for a weak form of the commutation error.

When using the box filter, one can avoid the extension of functions off the domain. However, the required non-constant filter width causes also a commutation error. This error was analyzed in Section 7.2.5. It was shown that it depends on the variation of the filter width. □

Remark 7.62. Practical simulations. In practice, the commutation error is not of importance. By experience, it is known that LES models do not behave appropriately near boundaries. Thus, they are either modified, like by using the van Driest damping in the Smagorinsky model, see Remark 7.126, or even completely different approaches are used near the boundary. An overview on such approaches is given in Piomelli and Balaras (2002). Two possibilities are

- imposing some form of wall law,
- solving numerically a set of simplified equations in the boundary layer region, which is the so-called zonal approach.

□

7.3 Large Eddy Simulation – The Smagorinsky Model

Remark 7.63. Motivation and contents of this section. The Smagorinsky model, proposed in Smagorinsky (1963), is one of the most popular turbulence