Chapter 7
The Time-Dependent Navier-Stokes Equations – Turbulent Flows

Remark 7.1. Turbulent flows. The usually used model for turbulent incompressible flows are the incompressible Navier–Stokes equations (1.24). There is no mathematical definition of what is "turbulence". From the mathematical point of view, turbulent flows occur at high Reynolds numbers. From the physical point of view, these flows are characterized by possessing flow structures (eddies, scales) of very different sizes. Consider, e.g., a tornado. This tornado has some very large flow structures (large eddies) but also millions of very small flow structures.

The small scales are important for the physics of turbulent flows. A numerical scheme which simply neglects them, e.g., by introducing sufficient artificial viscosity, computes a solution which is laminar and lacks important properties of the turbulent solution. A prominent example is the mean velocity profile of a channel flow: it is of parabolic form for a laminar flow and it tends to become of a step profile for turbulent flows, see Figure **??**. The way to treat the small scales, which cannot be resolved, in simulations consists in modeling their influence onto the resolved scales. With other words, a turbulence model has to be applied. A turbulence model has to contain less scales than the Navier–Stokes equations, which means, it has to be less complex than the Navier–Stokes equations. □

Remark 7.2. Contents. This chapter presents some approaches for turbulence modeling. The emphasis will be on models that allow a mathematical or numerical analysis or whose derivation is based on mathematical arguments. Some remarks concerning the application of these models in practical simulations are given. However, the use of turbulence models in practice is a wide field of research and it is not the goal of to give a comprehensive presentation here. There are several monographs on this topic, like Sagaut (2006); Ferziger and Perić (1999); **?**.

With respect to the presentation of mathematical and numerical analysis there will be the emphasis on results which are obtained for the case of bounded domains. One can find in the literature a number of results for the space-periodic case. As already mentioned in Remark 1.27, the space-periodic case mimics the situation of a domain without boundary and the periodic boundary conditions do not posses a physical meaning. From the analytical point of view, the absence of a boundary might simplify the analysis considerably and a somewhat different mathematical setup is used than in the case of a bounded domain. However, the interaction of a flow with the boundary of the domain is often of utmost importance in practice, such that results obtained for the case of a bounded domain seem to be of more interest. □

7.1 Some Physical and Mathematical Characteristics of Turbulent Incompressible Flows

Remark 7.3. Monographs. The physics of turbulent flows is the topic of a number of monographs, e.g., Pope (2000) and Davidson (2004). Mathematical aspects of turbulence flows are studied in Foias et al. (2001); Chacón Rebollo and Lewandowski (2014). □

Remark 7.4. The incompressible Navier–Stokes equations. For describing the physical properties of turbulent flows, it is sometimes convenient to use the incompressible Navier–Stokes equations with the the Reynolds number in the viscous term

$$\begin{aligned} \partial_t \boldsymbol{u} - 2\mathrm{Re}^{-1}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } (0,T]\times\Omega, \\ \boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_0 \text{ in } \Omega. \end{aligned} \tag{7.1}$$

If $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$, is a bounded domain, (7.1) has to be equipped with boundary conditions.

Turbulent flows are characterized by a high Reynolds number. In applications, the range of the Reynolds number for flows of this type starts around several thousand. Often, it is even larger by some orders of magnitude. In the case of high Reynolds numbers, the stabilizing forces in the momentum balance (the viscous term $2\mathrm{Re}^{-1}\nabla\cdot\mathbb{D}(\boldsymbol{u})$) are small compared with the destabilizing forces (the convective term $\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$, right-hand side of Newton's second law of motion (1.4)). □

Remark 7.5. Instantaneous flow and statistics. Considering turbulent flows, one has the idea that the flow behaves in some sense chaotic and hardly predictable. In fact, small changes of the data might lead to a large change in the instantaneous flow field, i.e., in the flow field at a certain time $t\in(0,T]$ and in a certain point $\boldsymbol{x}\in\Omega$. In practice, one has never complete information on the data, e.g., on the initial condition in the whole domain Ω or on the boundary conditions at the complete boundary for all times. For instance, boundary conditions are available from measurement data only at some points at the boundary and at some times. Then, the needed boundary conditions for the equations are defined by an interpolation. Since the boundary conditions obtained in this way most probably differ from the actual boundary conditions, it is to be expected that the computed instantaneous flow field is also different from the actual one. From this point of view, the consideration of instantaneous turbulent flow fields is not of practical interest. Instead, one is interested in practice in statistics of the flow fields, often defined by averages in space and time. For different problems, different statistics are of importance, e.g., see Examples D.15 and D.16. □

Remark 7.6. On the concept of isotropic turbulence. Much of the physical turbulence theory, e.g., the determination of the size of the smallest scales, is based on the concept of isotropic turbulence. A field $u(t,x)$ is called statistically stationary if all statistics of $u(t,x)$ are invariant under a shift of time. It is called statistically homogeneous if all statistics are invariant under a shift of position. If the field is also statistically invariant under rotations and reflections of the coordinate system, it is called (statistically) isotropic.

Wind tunnel experiments have been performed on (approximately) isotropic turbulence. However, isotropic turbulence is in general an idealization. □

Remark 7.7. The Richardson energy cascade. Let $\Omega \subset \mathbb{R}^3$. In Richardson (1922), a description of the physical mechanisms was given which work in turbulent flows. Kinetic energy enters the flow at the largest eddies. Large eddies are unstable and break up into smaller ones. Thereby energy is transferred in the mean to the smaller eddies. This transfer in the mean does not exclude a local (in time and space) transfer in the opposite direction from smaller to larger eddies, a so-called backscatter. The smaller eddies undergo a similar process. This process is continued until the Reynolds number $\mathrm{Re}(l) = u(l)l/\nu$ of the eddies of size l is sufficiently small (of order one) such that the eddy motion is stable and the molecular viscosity is effective in dissipating the kinetic energy. This process is called energy cascade.

The size of the smallest eddies will be denoted by λ. □

Remark 7.8. The rate of dissipation of turbulent energy. Denote by ε $[\mathrm{m}^2/\mathrm{s}^3]$ the rate of dissipation of turbulent energy which is defined in the following way. Consider $\boldsymbol{u}$ as a random variable and let $\langle \boldsymbol{u} \rangle$ be the mean value (expectation) of $\boldsymbol{u}$. The difference $\boldsymbol{u}' := \boldsymbol{u} - \langle \boldsymbol{u} \rangle$ is called the fluctuations. The rate of dissipation of turbulence energy is now defined by

$$\varepsilon := 2\nu \langle \mathbb{D}(\boldsymbol{u}') : \mathbb{D}(\boldsymbol{u}') \rangle.$$

The detailed theoretical and experimental study of particular flows shows that

$$\varepsilon = \mathcal{O}\left(\frac{U^3}{L}\right) \tag{7.2}$$

independently of Re, (Pope, 2000, p. 183). □

Remark 7.9. The Kolmogorov hypotheses. In the fundamental paper Kolmogorov (1941) three hypotheses about turbulent flows were postulated:

1. At sufficiently high Reynolds numbers, the small scale turbulent motions are isotropic.
2. In every turbulent flow at sufficiently high Reynolds number, the statistics of the small scale motions have a universal form which is uniquely given by ν and ε.
3. In every turbulent flow at sufficiently high Reynolds number, the statistics of motions of scale of size l in the range $L \gg l \gg \lambda$ have a universal form uniquely determined by ε and independent of ν.

For describing the size of the smallest scales, the first and second hypothesis are of importance. □

Remark 7.10. The size of the Kolmogorov scales. Let ε and ν be given. Then, there are unique length, velocity, and time scales which can be defined, the so-called Kolmogorov scales

$$\lambda = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \text{ [m]}, \quad u_\lambda = (\varepsilon\nu)^{1/4} \text{ [m/s]}, \quad t_\lambda = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \text{ [s]}. \tag{7.3}$$

The scale λ is also called viscous length scale. The Reynolds number of eddies of size λ is

$$\mathrm{Re}(\lambda) = \frac{\lambda u_\lambda}{\nu} = \frac{\nu\varepsilon^{1/4}}{\nu\varepsilon^{1/4}} = 1, \tag{7.4}$$

such that it is sufficiently small for the dissipation to be effective. The motion of eddies with $\mathrm{Re}(\lambda) = 1$ is isotropic and the first hypothesis is met.

In addition, using (7.3), the rate of dissipation is given by

$$\varepsilon^{1/2} = \frac{u_\lambda^2}{\nu^{1/2}},\ \varepsilon^{1/2} = \frac{\nu^{3/2}}{\lambda^2} \implies \varepsilon = \nu\frac{u_\lambda^2}{\lambda^2} \text{ and } \varepsilon = \nu\frac{1}{t_\lambda^2}, \tag{7.5}$$

such that

$$\frac{u_\lambda}{\lambda} = \frac{1}{t_\lambda} = \left(\frac{\varepsilon}{\nu}\right)^{1/2} \quad [1/\mathrm{s}]. \tag{7.6}$$

The left-hand side is an approximation to the spatial derivative of the Kolmogorov velocity, which describes the change of the velocity gradient, since λ is small. For the large eddies in turbulent flows, the velocity gradient increases with the Reynolds number since the flow field varies rapidly in space and time. Equation (7.6) shows that for the Kolmogorov scales, the velocity gradient is bounded uniformly with respect to the Reynolds number. It depends only on ν and ε. This dependency is required by the second Kolmogorov hypothesis. Altogether, (7.4) and (7.6) characterize the Kolmogorov scales as dissipative scales.

Now, one can estimate the size of the Kolmogorov scales. Using (7.2) and (7.3) gives

$$\frac{\lambda}{L} = \mathcal{O}\left(\left(\frac{\nu^3}{L^3U^3}\right)^{1/4}\right) = \mathcal{O}\left(\mathrm{Re}^{-3/4}\right) \quad \Longleftrightarrow \quad \lambda = \mathcal{O}\left(\mathrm{Re}^{-3/4}\right), \qquad (7.7)$$

where $L = \mathcal{O}(1\ \mathrm{m})$ was assumed. □

Remark 7.11. On the impact of the size of the Kolmogorov scales in numerical simulations. A standard discretization of the Navier–Stokes equations (7.1), like the Galerkin finite element method studied in Section 6.3.1, seeks to simulate the behavior of all scales, including the Kolmogorov scales.

Consider as example the domain $\Omega = (0,1)^3$, such that $L = 1$, and a mesh of roughly 10^8 cubic mesh cells ($\approx 464^3$). Assuming that the mesh width is equal to the resolution of the discretization, as for low order finite elements, then scales of size $1/464$ can be represented on this mesh. Only those scales can be simulated which can be represented. Hence, a necessary condition for flows to be simulated on this grid is that for its Kolmogorov length it holds $\lambda \gtrapprox 1/464$. Assuming additionally that equality holds in (7.2), then it follows from (7.7) that flows up to a Reynolds number of $\mathrm{Re} \approx 464^{4/3} \approx 3590$ can be simulated. This is far less than the Reynolds number of turbulent flows in most applications.

Arguing the same way for a general situation, one finds with (7.7) that the number of degrees of freedom to resolve the Kolmogorov scales behaves like

$$\mathcal{O}\left(\left(\frac{L}{\lambda}\right)^3\right) = \mathcal{O}\left(\mathrm{Re}^{9/4}\right).$$

The application of the Galerkin finite element method (or any other standard discretization) is called Direct Numerical Simulation (DNS) in the context of turbulent flow simulations. The considerations in this remark show that a DNS is generally not feasible for the simulation of turbulent flows and it will not be feasible in the foreseeable future. □

Remark 7.12. The kinetic energy spectrum. The energy cascade, Remark 7.7, and the third hypothesis of Kolmogorov, Remark 7.9 can be expressed with the so-called kinetic energy spectrum.

Consider $\Omega = \mathbb{R}^3$ and and a velocity field $\boldsymbol{u}(t,\boldsymbol{x})$. Applying the Fourier transform (A.6) gives

$$\mathcal{F}(\boldsymbol{u})(t,\boldsymbol{k}) = \int_{\mathbb{R}^3} \boldsymbol{u}(t,\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\ d\boldsymbol{x},$$

where $\boldsymbol{k} = (k_1,k_2,k_3)^T$ is the wave number vector and the Fourier space is called wave number space. The kinetic energy for the wave number $\boldsymbol{k}$ is given by

$$E(t,\boldsymbol{k}) = \frac{1}{2}\langle \mathcal{F}(\boldsymbol{u})(t,\boldsymbol{k}) \cdot \mathcal{F}(\boldsymbol{u})^*(t,\boldsymbol{k})\rangle,$$

with $\mathcal{F}(\boldsymbol{u})^*$ denoting the conjugate complex of $\mathcal{F}(\boldsymbol{u})$ and $\langle \cdot, \cdot, \rangle$ denotes an appropriate mean value. The case of homogeneous isotropic turbulence with mean value zero will be considered. For wave numbers with the same absolute value $k = \|\boldsymbol{k}\|_2$, one defines

$$E(t,k) = \int_{\|\boldsymbol{k}\|_2 = k} E(t, \boldsymbol{k}) \; d\boldsymbol{k}.$$

Kolmogorov's third hypothesis implies a universal form of scales of size l with $L \gg l \gg \lambda$.

The kinetic energy spectrum for turbulent flows is sketched in Figure 7.1, see (Pope, 2000, pp. 229) for a detailed derivation. In the so-called inertial subrange one finds that

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3},$$

where C_K is called Kolmogorov constant. This relation is often called Kolmogorov $-5/3$ spectrum.

Flows at high Reynolds number show a distinct inertial subrange which is absent for laminar flows. □

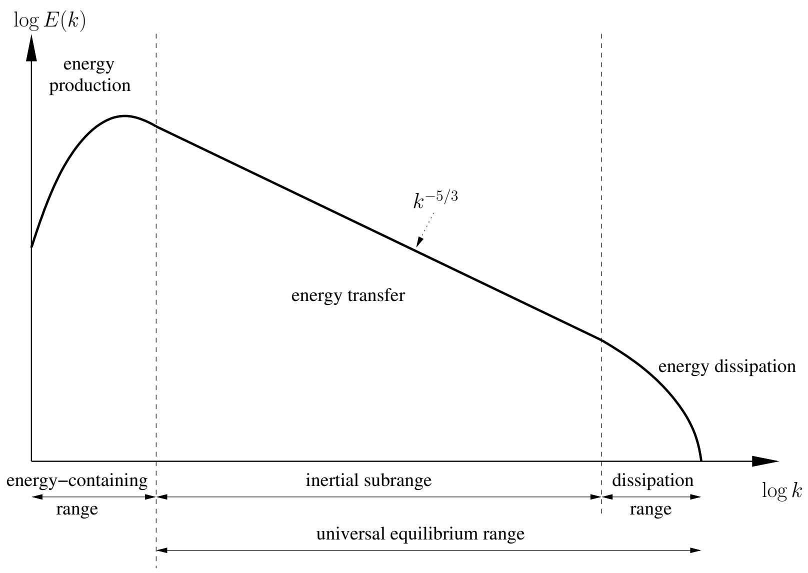

Fig. 7.1 Sketch of the kinetic energy spectrum of turbulent flows.

Remark 7.13. Boundary layers. Consider an example like a turbulent channel flow, see Example D.15. Then, there is a planar solid wall as boundary which is situated at $y = 0$, the domain at this boundary is given for $y > 0$, and a no-slip condition $\boldsymbol{u} = \mathbf{0}$ is applied at this wall. In this situation, there is a layer at the boundary. For the mean velocity field it holds that $\langle u_1 \rangle = \langle u_1 \rangle(y)$, $\langle u_2 \rangle = \langle u_3 \rangle = 0$. Then, the viscous stress (1.16) for the mean velocity is given by

$$\rho\nu \frac{\nabla\langle \boldsymbol{u} \rangle + (\nabla\langle \boldsymbol{u} \rangle)^T}{2} = \frac{\rho\nu}{2} \begin{pmatrix} 0 & \partial_y \langle u_1 \rangle & 0 \\ \partial_y \langle u_1 \rangle & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The mean viscous stress at the wall, the so-called wall shear stress is the non-vanishing component

$$\tau_{\mathrm{w}} = \rho\nu\partial_y \langle u_1 \rangle\big|_{y=0} \quad [\mathrm{N}/\mathrm{m}^2],$$

where the unit assumes a velocity with dimension. With τ_{w} one can define an appropriate velocity and length scale for the near-wall region, which enable a characterization of the layer independently of the Reynolds number. The velocity scale is the so-called friction velocity given by

$$U_\tau = \sqrt{\frac{\tau_{\mathrm{w}}}{\rho}} \quad [\mathrm{m}/\mathrm{s}]$$

and the viscous length scale is

$$\delta_\nu = \nu\sqrt{\frac{\rho}{\tau_{\mathrm{w}}}} = \frac{\nu}{U_\tau} \quad [\mathrm{m}].$$

Then, the distance from the wall can be measured in viscous length or wall units

$$y^+ = \frac{y}{\delta_\nu} = \frac{U_\tau}{\nu} y.$$

Depending on y^+, several regions for near-wall flows are distinguished. The region $y^+ < 50$ is called viscous wall region since effects of the molecular viscosity are of importance. For $y^+ < 5$, the viscous stress τ_{w} even dictates the behavior of the flow in this region and it is called viscous sublayer. The region $y^+ > 50$ is called outer layer. □

Remark 7.14. Differences between two- and three-dimensional flows. There are at least two fundamental differences between two- and three-dimensional flows.

First, the smallest scales in two-dimensional flows behave differently than (7.7). In Kraichnan (1967) it was shown that they are of order $\lambda_{2\mathrm{d}} = \mathcal{O}\left(\mathrm{Re}^{-1/2}\right)$ in two dimensions, where $\lambda_{2\mathrm{d}}$ is the Kraichnan dissipation length.

A second difference is the so-called vortex stretching. The vorticity is defined by $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, see Definition 2.153. Applying the curl operator to the Navier–Stokes equations (7.1) and assuming sufficiently smooth functions gives for the viscous term, applying (1.25), the Theorem of Schwarz, and again (1.25), using $\nabla \cdot \boldsymbol{\omega} = 0$ which follows from (2.135),

$$\begin{aligned}
2\nabla \times (\nabla \cdot \mathbb{D}(\boldsymbol{u})) = \nabla \times (\Delta \boldsymbol{u}) &= \begin{pmatrix} \partial_y \Delta u_3 - \partial_z \Delta u_2 \\ \partial_z \Delta u_1 - \partial_x \Delta u_3 \\ \partial_x \Delta u_2 - \partial_y \Delta u_1 \end{pmatrix} \\
&= \begin{pmatrix} \Delta(\partial_y u_3 - \partial_z u_2) \\ \Delta(\partial_z u_1 - \partial_x u_3) \\ \Delta(\partial_x u_2 - \partial_y u_1) \end{pmatrix} = \Delta(\nabla \times \boldsymbol{u}) \\
&= \Delta \boldsymbol{\omega} = 2\nabla \times (\nabla \cdot \mathbb{D}(\boldsymbol{w})).
\end{aligned}$$

For the convective term, one obtains with (5.6), (2.134), and (2.137), using again that $\nabla \cdot \boldsymbol{\omega} = 0$,

$$\begin{aligned}
\nabla \times ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) &= \nabla \times \left((\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \frac{1}{2}\nabla\left(\boldsymbol{u}^T \boldsymbol{u}\right)\right) = \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) \\
&= (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}.
\end{aligned}$$

Thus, one gets for the Navier–Stokes equations with $\boldsymbol{f} = \boldsymbol{0}$ the following equation for the vorticity

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - 2\mathrm{Re}^{-1}\nabla \cdot \mathbb{D}(\boldsymbol{\omega}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}.$$

The viscous term is small for high Reynolds numbers and can be neglected. Thus

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} \approx (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}. \tag{7.8}$$

This relation is the equation of an infinitesimal line element of material. If $\nabla \boldsymbol{u}$ acts to stretch the line element, then $|\boldsymbol{\omega}|$ will be stretched, too. Thus, in turbulent three-dimensional flows, vortex stretching occurs, which is an important feature of such flows. Vortex stretching causes a change of the local length scale and it leads to a hierarchy of vorticial structures of different size. Thus, vortex stretching is responsible for the multiscale character of turbulent flows.

In two-dimensional flows, the right-hand side of (7.8) vanishes, which induces that vortex stretching cannot occur.

Because of the absence of vortex stretching and the different size of the viscous length scales, two-dimensional flows at high Reynolds numbers are qualitatively different to three-dimensional turbulent flows. For this reason, one can share the point of view that in two dimensions there are no turbulent flows. However, from the point of view of numerical mathematics it is legitimate to check new methods for high Reynolds number flows also at two-dimensional problems. If they fail, their success at three-dimensional flows, which possess additional complex features, is very unlikely. On the other hand, if they are successful, it cannot be concluded without numerical studies that they will work well in three dimensions, too. □

Remark 7.15. A mathematical approach for studying turbulent flows. A mathematical concept for studying turbulent flows was developed within the theory of dynamical systems. A dynamical system is given by

$$\frac{d\boldsymbol{u}}{dt} = F(\boldsymbol{u}), \quad \boldsymbol{u}_0 = \boldsymbol{u}(0). \tag{7.9}$$

The incompressible Navier–Stokes equations in the weakly divergence-free space (6.3) fit into this concept with

$$F(\boldsymbol{u}) = \boldsymbol{f} - \nu A\boldsymbol{u} - N(\boldsymbol{u}, \boldsymbol{u}),$$

where $A : V_{\mathrm{div}} \to V'$ is the Laplace operator in the distributional sense defined by

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = \langle A\boldsymbol{u}, \boldsymbol{v}\rangle_{V',V} \quad \forall\, \boldsymbol{v} \in V_{\mathrm{div}}$$

and $N : V_{\mathrm{div}} \times V_{\mathrm{div}} \to V'$ is the bilinear convective operator

$$\left(N\left(\boldsymbol{u}, \boldsymbol{v}\right), \boldsymbol{w}\right) = n\left(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\right) \quad \forall\, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V_{\mathrm{div}}.$$

In the theory for the Navier–Stokes equations, it is assumed that the body force is independent of time, i.e., $\boldsymbol{f}(t, \boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$. In this case, one obtains an autonomous dynamical system.

To study sets of solutions of dynamical systems, one can define attractors, see Remark 7.18 and in particular the dimension of these attractors. In (Temam, 1997, genauer) it is written “It is our understanding here that the number of degrees of freedom of a turbulent phenomenon is the dimension of the attractor which represents it”.

Here, a short introduction into this concept and a brief review of some results for the incompressible Navier–Stokes equations will be given. More detailed presentations can be found, e.g., in (Temam, 1997, genauer), (Foias et al., 2001, Chapter III) or (Marion and Temam, 1998, Chapter 13). □

Remark 7.16. A family of operators describing the evolution of a dynamical system. Consider the autonomous dynamical system (7.9) which is assumed to posses a unique solution. Then, a family of operators $\{S(t)\}_{t\ge 0}$ is defined by

$$S(t) : \boldsymbol{u}(0) \mapsto \boldsymbol{u}(t) \quad \Longleftrightarrow \quad \boldsymbol{u}(t) = S(t)\boldsymbol{u}(0),$$

i.e., the initial condition is mapped to the solution at time t. Basic properties are the followings:

- $S(0) = \mathrm{Id}$.
- One obtains the same solution at time $t + s$, $t, s \ge 0$,
  - if the system is started at time 0 and evolved until time $t + s$ or
  - if the system is started at time 0, evolved until time t and then evolved further until time $t + s$ is reached.

  Mathematically, this property is

$$\boldsymbol{u}(t+s) = S(t+s)\boldsymbol{u}(0) = S(s)\left(S(t)\boldsymbol{u}(0)\right) = \left(S(s)\circ S(t)\right)\boldsymbol{u}(0) \quad \forall\, s,t \geq 0$$

or

$$S(t+s) = S(s)\circ S(t) = S(t)\circ S(s) \quad \forall\, s,t \geq 0.$$

Therefore, the operators $\{S(t)\}_{t\geq 0}$ form a semi-group. There are no inverse elements. □

Remark 7.17. Setup for the incompressible Navier–Stokes equations. For the incompressible Navier–Stokes equations, it is assumed that $\boldsymbol{u}_0, \boldsymbol{f} \in L^2_{\mathrm{div}}(\Omega)$, see (2.38). Note that the assumption of unique solvability for the incompressible Navier–Stokes equations is known so far only for two dimensions, see Remark 6.26. In this case, one can show that $S(t)$ is a continuous operator in $L^2_{\mathrm{div}}(\Omega)$, (Marion and Temam, 1998, p. 554), (Foias et al., 2001, p. 138). □

Remark 7.18. Global attractor, (Foias et al., 2001, p. 138). The global attractor of $\{S(t)\}_{t\geq 0}$ is a set $\mathcal{A} \subset L^2_{\mathrm{div}}(\Omega)$ with the following properties:

- The set $\mathcal{A}$ is compact in $L^2_{\mathrm{div}}(\Omega)$.
- The set $\mathcal{A}$ is invariant for the semi-group, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.
- The set $\mathcal{A}$ attracts all bounded sets in $L^2_{\mathrm{div}}(\Omega)$, i.e., for every bounded set $B \subset L^2_{\mathrm{div}}(\Omega)$ it is

$$\mathrm{dist}_{L^2_{\mathrm{div}}}\left(S(t)B, \mathcal{A}\right) = \sup_{b\in S(t)B}\inf_{a\in\mathcal{A}} \|a-b\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t\to\infty,$$

  i.e., the distant between these two sets goes to zero for $t \to \infty$.

If the global attractor exists, it can be shown that it is unique. □

Remark 7.19. The Hausdorff dimension of the global attractor, (Foias et al., 2001, p. 117). The so-called Hausdorff dimension of a global attractor is of particular interest for the incompressible Navier–Stokes equations. Given a set X and $\varepsilon > 0$. This set is covered by balls of dimension $\hat{d} \in \mathbb{R}$ with radii which are not larger than $\varepsilon > 0$. The volume of these balls is proportional to $\varepsilon^{\hat{d}}$. Defining the quantity

$$\mu_{\mathrm{H}}(X,\hat{d},\varepsilon) = \inf \sum_{i\in I} r_i^{\hat{d}},$$

where the infimum is for all coverings of X with a family $\{B_i\}_{i\in I}$ of balls with $r_i \leq \varepsilon$. Reducing the maximal radius, the volume of the covering will not increase, thus $\mu_{\mathrm{H}}(X,\hat{d},\varepsilon)$ is a non-increasing function with respect to ε. Then,

$$\mu_{\mathrm{H}}(X,\hat{d}) = \lim_{\varepsilon\to 0}\mu_{\mathrm{H}}(X,\hat{d},\varepsilon) = \sup_{\varepsilon>0}\mu_{\mathrm{H}}(X,\hat{d},\varepsilon)$$

is the $\hat{d}$-dimensional Hausdorff measure of X. One can show that there is a number $d_0 \in [0,\infty]$ such that $\mu_{\mathrm{H}}(X,\hat{d}) = \infty$ for $\hat{d} < d_0$ and $\mu_{\mathrm{H}}(X,\hat{d}) = 0$ for $\hat{d} > d_0$. This number d_0 is called the Hausdorff dimension of X and it is denoted by $d_{\mathrm{H}}(X)$. □

Remark 7.20. Results for the two-dimensional Navier–Stokes equations. Since there exists a unique solution of the incompressible Navier–Stokes equations in two dimensions, one can define the semi-group $\{S(t)\}_{t\geq 0}$ and study its properties. In Foiaş and Temam (1979), the existence of a global attractor and the finiteness of its Hausdorff dimension were proved in the case of a smooth boundary of Ω. There are several extensions of these results, in particular to the periodic case, see (Foias et al., 2001, p. 140) for an overview. An estimate of the Hausdorff dimension was given in Temam (1986). It can be shown that

$$d_{\mathrm{H}}\left(\mathcal{A}\right) = \mathcal{O}\left(\left(\frac{L}{\lambda_{2\mathrm{d}}}\right)^2\right),$$

see (Foias et al., 2001, p. 141), where $\lambda_{2\mathrm{d}}$ is given in Remark 7.14. □

Remark 7.21. Extensions to the Navier–Stokes equations in three dimensions. For the three-dimensional Navier–Stokes equations, one can consider so-called invariant sets which are bounded in V_{div}, see (Foias et al., 2001, pp. 147). A set X in V_{div} is invariant if, for any initial condition $\boldsymbol{u}_0 \in X$, the corresponding unique local solution say something about local solutions extends globally in time to a unique solution $\boldsymbol{u}(t)$ that is defined for all $t \geq 0$ with values in X. It was shown in Constantin et al. (1985a) and Constantin et al. (1985b) that the Hausdorff dimension of any invariant bounded set $X \subset V_{\mathrm{div}}$ has the same order of the number of degrees of freedom as predicted from Kolmogorov's theory, see Remark 7.11, i.e.,

$$d_{\mathrm{H}}\left(X\right) = \mathcal{O}\left(\left(\frac{L}{\lambda}\right)^3\right). \qquad (7.10)$$

□

Remark 7.22. Smaller complexity of turbulence models. As already mentioned in Remark 7.1, a turbulence model has to be less complex than the incompressible Navier–Stokes equations. Each turbulence model has a parameter which determines the scales that should be simulated. This parameter is usually related to the (local) mesh width, e.g., as in (7.64), since the mesh width determines which scales at least represented. Thus, in practice, the parameter becomes smaller on finer meshes and asymptotically the turbulence model converges to the Navier–Stokes equations as the mesh width tends to zero. To study if a turbulence model is less complex than the Navier–Stokes equations, one fixes the parameter, like in Section 7.3.2, and considers the countinous counterpart of the turbulence model. Indicators for a reduced complexity are the existence and uniqueness of a weak solution for long times since the uniqueness is not proved for the Navier–Stokes equations so far, see Remark 6.27, or that the dimension of the global attractor is smaller than (7.10). □