Chapter 1
The Navier–Stokes Equations as Model for Incompressible Flows

Remark 1.1. Basic principles and variables. The basic equations of fluid dynamics are called Navier–Stokes equations. In the case of an isothermal flow, i.e., a flow at constant temperature, they represent two physical conservation laws: the conservation of mass and the conservation of linear momentum. There are various ways for deriving these equations. Here, the classical one of continuum mechanics will be outlined. There are some heuristic steps in this approach.

The flow will be described with the variables

- $\rho(t, \boldsymbol{x})$: density [kg/m³],
- $\boldsymbol{v}(t, \boldsymbol{x})$: velocity [m/s],
- $P(t, \boldsymbol{x})$: pressure [Pa = N/m²],

which are assumed to be sufficiently smooth functions in the time interval $[0, T]$ and the domain $\Omega \subset \mathbb{R}^3$. $\Box$

1.1 The Conservation of Mass

Remark 1.2. General conservation law. Let ω be an arbitrary open volume in Ω with sufficiently smooth surface $\partial\omega$, which is constant in time, and with mass

$$m(t) = \int_\omega \rho(t, \boldsymbol{x})\ d\boldsymbol{x}\ [\text{kg}].$$

If mass in ω is conserved, the rate of change of mass in ω must be equal to the flux of mass $\rho\boldsymbol{v}(t, \boldsymbol{x})$ [kg/(m^{2}s)] across the boundary $\partial\omega$ of ω

$$\frac{d}{dt}m(t) = \frac{d}{dt}\int_\omega \rho(t, \boldsymbol{x})\ d\boldsymbol{x} = -\int_{\partial\omega} (\rho\boldsymbol{v})(t, \boldsymbol{s}) \cdot \boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s}, \tag{1.1}$$

where $\boldsymbol{n}(\boldsymbol{s})$ is the outward pointing unit normal on $\boldsymbol{s} \in \partial\omega$. Since all functions and $\partial\omega$ are assumed to be sufficiently smooth, the divergence theorem can be applied (integration by parts), which gives

$$\int_\omega \nabla \cdot (\rho \boldsymbol{v})(t, \boldsymbol{x}) \, d\boldsymbol{x} = \int_{\partial\omega} (\rho \boldsymbol{v})(t, \boldsymbol{s}) \cdot \boldsymbol{n}(\boldsymbol{s}) \, d\boldsymbol{s}.$$

Inserting this identity into (1.1) leads to

$$\int_\omega \left(\partial_t \rho(t, \boldsymbol{x}) + \nabla \cdot (\rho \boldsymbol{v})(t, \boldsymbol{x}) \right) \, d\boldsymbol{x} = 0.$$

Since ω is an arbitrary volume, it follows that

$$\left(\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) \right)(t, \boldsymbol{x}) = 0 \quad \forall \, t \in (0, T], \; \boldsymbol{x} \in \Omega. \tag{1.2}$$

This relation is the first equation of fluid dynamics, which is called continuity equation. □

Remark 1.3. Incompressible, homogeneous fluids. If the fluid is incompressible and homogeneous, i.e., composed of one fluid only, then $\rho(t, \boldsymbol{x}) = \rho > 0$ and (1.2) reduces to

$$\left(\partial_x v_1 + \partial_y v_2 + \partial_z v_3 \right)(t, \boldsymbol{x}) = \nabla \cdot \boldsymbol{v}(t, \boldsymbol{x}) = 0 \quad \forall \, t \in (0, T], \; \boldsymbol{x} \in \Omega, \tag{1.3}$$

where

$$\boldsymbol{v}(t, \boldsymbol{x}) = \begin{pmatrix} v_1(t, \boldsymbol{x}) \\ v_2(t, \boldsymbol{x}) \\ v_3(t, \boldsymbol{x}) \end{pmatrix}.$$

Thus, the conservation of mass for an incompressible, homogeneous fluid imposes a constraint on the velocity only. □

1.2 The Conservation of Linear Momentum

Remark 1.4. Newton's second law of motion. The conservation of linear momentum is the formulation of Newton's second law of motion

$$\text{net force} = \text{mass} \times \text{acceleration} \tag{1.4}$$

for flows. It states that the rate of change of the linear momentum must be equal to the net force acting on a collection of fluid particles. □

Remark 1.5. Conservation of linear momentum. The linear momentum in an arbitrary volume ω is given by

$$\int_\omega \rho \boldsymbol{v}(t, \boldsymbol{x}) \, d\boldsymbol{x} \quad [\mathrm{Ns}].$$

Then, the conservation of linear momentum in ω can be formulated analogously to the conservation of mass in (1.1)

$$\frac{d}{dt}\int_\omega \rho \boldsymbol{v}(t,\boldsymbol{x})\ d\boldsymbol{x} = -\int_{\partial\omega} (\rho\boldsymbol{v})\,(\boldsymbol{v}\cdot\boldsymbol{n})\,(t,\boldsymbol{s})\ d\boldsymbol{s} + \int_\omega \boldsymbol{f}_{\text{net}}(t,\boldsymbol{x})\ d\boldsymbol{x}\ [\text{N}],$$

where the term on the left-hand side describes the change of the momentum in ω, the first term on the right-hand side models the flux of momentum across the boundary of ω and $\boldsymbol{f}_{\text{net}}$ [$^{\text{N}}/_{\text{m}^3}$] represents the force density in ω. It is

$$\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n}) = \begin{pmatrix} v_1v_1n_1 + v_1v_2n_2 + v_1v_3n_3 \\ v_2v_1n_1 + v_2v_2n_2 + v_2v_3n_3 \\ v_3v_1n_1 + v_3v_2n_2 + v_3v_3n_3 \end{pmatrix} = \boldsymbol{v}\boldsymbol{v}^T\boldsymbol{n}.$$

Applying integration by parts and changing differentiation with respect to time and integration on ω gives

$$\int_\omega \left(\partial_t\left(\rho\boldsymbol{v}\right) + \nabla\cdot\left(\rho\boldsymbol{v}\boldsymbol{v}^T\right)\right)(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\text{net}}(t,\boldsymbol{x})\ d\boldsymbol{x}.$$

The product rule yields

$$\int_\omega \left(\partial_t\rho\boldsymbol{v} + \rho\partial_t\boldsymbol{v} + \boldsymbol{v}\boldsymbol{v}^T\nabla\rho + \rho(\nabla\cdot\boldsymbol{v})\boldsymbol{v} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)(t,\boldsymbol{x})\ d\boldsymbol{x} \\ = \int_\omega \boldsymbol{f}_{\text{net}}(t,\boldsymbol{x})\ d\boldsymbol{x}. \tag{1.5}$$

In the usual notation $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$, one can think of $\boldsymbol{v}\cdot\nabla = v_1\partial_x + v_2\partial_y + v_3\partial_z$ acting on each component of $\boldsymbol{v}$. In the literature, one often finds the notation $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$.

In the case of incompressible flows, i.e., ρ is constant, it is known that $\nabla\cdot\boldsymbol{v} = 0$, see (1.3), such that (1.5) simplifies to

$$\int_\omega \rho\left(\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\text{net}}(t,\boldsymbol{x})\ d\boldsymbol{x}.$$

Since ω was chosen to be arbitrary, one gets the conservation law

$$\rho\left(\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = \boldsymbol{f}_{\text{net}} \quad \forall\ t\in(0,T],\ \boldsymbol{x}\in\Omega.$$

□

Remark 1.6. External forces. The forces acting on ω are composed of external (body) forces and internal forces. External forces include, e.g., gravity, buoyancy, and electromagnetic forces (in liquid metals). These forces are collected in a body force term

$$\int_\omega \boldsymbol{f}_{\text{ext}}(t,\boldsymbol{x})\ d\boldsymbol{x}.$$

□

Remark 1.7. Internal forces, Cauchy's principle, and the stress tensor. Internal forces are forces which a fluid exerts on itself. These forces include the pressure and the viscous drag that a 'fluid element' exerts on the adjacent element. The internal forces of a fluid are contact forces, i.e., they act on the surface of the fluid element ω. Let $\boldsymbol{t}$ $[\mathrm{N}/\mathrm{m}^2]$ denote this internal force vector, which is called Cauchy stress vector or torsion vector, then the contribution of the internal forces on ω is

$$\int_{\partial\omega} \boldsymbol{t}(t,\boldsymbol{s})\ d\boldsymbol{s}.$$

Adding the external and internal forces, the equation for the conservation of linear momentum is, for an arbitrary constant-in-time volume ω,

$$\int_{\omega} \rho(t,\boldsymbol{x})\left(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_{\omega} \boldsymbol{f}_{\mathrm{ext}}(t,\boldsymbol{x})\ d\boldsymbol{x} + \int_{\partial\omega} \boldsymbol{t}(t,\boldsymbol{s})\ d\boldsymbol{s}. \quad (1.6)$$

The right-hand side of (1.6) describes the net force acting on and inside ω. Now, a detailed description of the internal forces represented by $\boldsymbol{t}(t,\boldsymbol{s})$ is necessary.

The foundation of continuum mechanics is the stress principle of Cauchy. The idea of Cauchy on internal contact forces was that on any (imaginary) plane on $\partial\omega$ there is a force that depends (geometrically) only on the orientation of the plane. Thus, it is $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{n})$, where $\boldsymbol{n}$ is the outward pointing unit normal vector of the imaginary plane.

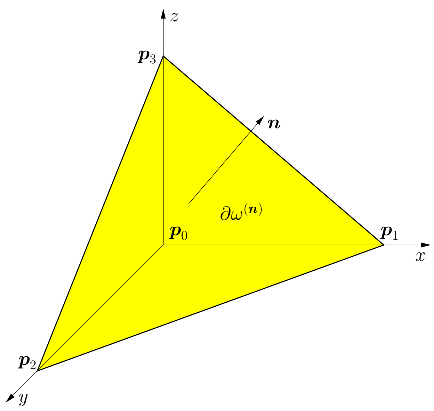

Fig. 1.1 Illustration of the tetrahedron used for discussing the linear dependency of the Cauchy stress vector on the normal.

Next, it will be discussed that $\boldsymbol{t}$ depends linearly on $\boldsymbol{n}$. To this end, consider a tetrahedron ω with the vertices $\boldsymbol{p}_0 = (0,0,0)^T$, $\boldsymbol{p}_1 = (x_1,0,0)^T$, $\boldsymbol{p}_2 = (0,y_2,0)^T$, $\boldsymbol{p}_3 = (0,0,z_3)^T$, and with $x_1, y_2, z_3 > 0$, see Figure 1.1 for an illustration. Denote the plane defined by $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ by $\partial\omega^{(\boldsymbol{n})}$. The unit outward pointing normal of $\partial\omega^{(\boldsymbol{n})}$ is given by

$$\begin{aligned}\boldsymbol{n} &= \frac{(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)}{\|(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)\|_2} \\ &= \frac{1}{\|(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)\|_2} \begin{pmatrix} y_2 z_3 \\ z_3 x_1 \\ x_1 y_2 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \end{aligned} \tag{1.7}$$

The face of the tetrahedron with the normal $-\boldsymbol{e}_i$ will be denoted by $\partial\omega^{(\boldsymbol{e}_i)}$, $i = 1,2,3$. Let $\boldsymbol{t}^{(\boldsymbol{n})}$ be the Cauchy stress vector at $\partial\omega^{(\boldsymbol{n})}$. Assuming that the Cauchy stress vectors depend only on the normal of the respective face, they are constant on each face of the tetrahedron and the integrals on the faces can be computed easily. Applying in addtion Newton's second law of motion (1.4) and the formula for the volume of a tetrahedron leads to

$$\underbrace{\boldsymbol{t}^{(\boldsymbol{n})} \left|\partial\omega^{(\boldsymbol{n})}\right| - \sum_{i=1}^{3} \boldsymbol{t}^{(\boldsymbol{e}_i)} \left|\partial\omega^{(\boldsymbol{e}_i)}\right|}_{\text{internal force}} = \underbrace{\rho \frac{h^{(\boldsymbol{n})}}{3} \left|\partial\omega^{(\boldsymbol{n})}\right|}_{\text{mass}} \boldsymbol{a}, \quad [\mathrm{N}] \tag{1.8}$$

where $|\cdot|$ is the area of the faces, $\boldsymbol{t}^{(\boldsymbol{e}_i)}$ the stress vector at face $\partial\omega^{(\boldsymbol{e}_i)}$, $\boldsymbol{a}$ [m/s^2] is an acceleration, and $h^{(\boldsymbol{n})}$ is the distance of the face $\partial\omega^{(\boldsymbol{n})}$ to the origin. The area of $\partial\omega^{(\boldsymbol{n})}$ can be calculated with the cross product, giving

$$\left|\partial\omega^{(\boldsymbol{n})}\right| = \frac{1}{2} \left|(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)\right| = \frac{1}{2} \left\| \begin{pmatrix} y_2 z_3 \\ z_3 x_1 \\ x_1 y_2 \end{pmatrix} \right\|_2 .$$

Using the representation (1.7) of the normal leads to

$$\left|\partial\omega^{(\boldsymbol{e}_1)}\right| = \frac{1}{2} y_2 z_3 = \frac{1}{2} n_1 \left\|(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)\right\|_2 = \left|\partial\omega^{(\boldsymbol{n})}\right| n_1.$$

Analogous formulas are derived for $\left|\partial\omega^{(\boldsymbol{e}_2)}\right|$ and $\left|\partial\omega^{(\boldsymbol{e}_3)}\right|$. Inserting these formulas into (1.8) gives

$$\boldsymbol{t}^{(\boldsymbol{n})} - \sum_{i=1}^{3} \boldsymbol{t}^{(\boldsymbol{e}_i)} n_i = \rho \frac{h^{(\boldsymbol{n})}}{3} \boldsymbol{a}. \tag{1.9}$$

Shrinking now the tetrahedron to the origin, where $\partial\omega^{(\boldsymbol{n})}$ moves in the direction $\boldsymbol{n}$, the left-hand side of (1.9) stays constant whereas the right-hand side vanishes since $h^{(\boldsymbol{n})} \to 0$. Hence, one obtains in the limit that

$$\boldsymbol{t}^{(\boldsymbol{n})} = \sum_{i=1}^{3} \boldsymbol{t}^{(\boldsymbol{e}_i)} n_i = \left(\boldsymbol{t}^{(\boldsymbol{e}_1)}\ \boldsymbol{t}^{(\boldsymbol{e}_2)}\ \boldsymbol{t}^{(\boldsymbol{e}_3)}\right) \boldsymbol{n},$$

where $(\cdot, \cdot, \cdot)$ denotes a tensor with the respective columns. This relation means that $\boldsymbol{t}^{(\boldsymbol{n})}$ depends linearly on $\boldsymbol{n}$.

Thus, the model for the Cauchy stress vector is

$$\boldsymbol{t} = \mathbb{S}\boldsymbol{n}, \tag{1.10}$$

where $\mathbb{S}(t, \boldsymbol{x})$ [N/m^2] is a 3×3-tensor which is called stress tensor. The stress tensor represents all internal forces of the flow. Inserting (1.10) into the term representing the internal forces in (1.6) and applying the divergence theorem gives

$$\int_{\partial\omega} \boldsymbol{t}(t, \boldsymbol{s})\ d\boldsymbol{s} = \int_{\omega} \nabla \cdot \mathbb{S}(t, \boldsymbol{x})\ d\boldsymbol{x},$$

where the divergence of a tensor is defined row-wise

$$\nabla \cdot \mathbb{A} = \begin{pmatrix} (a_{11})_x + (a_{12})_y + (a_{13})_z \\ (a_{21})_x + (a_{22})_y + (a_{23})_z \\ (a_{31})_x + (a_{32})_y + (a_{33})_z \end{pmatrix} = \begin{pmatrix} \partial_x a_{11} + \partial_y a_{12} + \partial_z a_{13} \\ \partial_x a_{21} + \partial_y a_{22} + \partial_z a_{23} \\ \partial_x a_{31} + \partial_y a_{32} + \partial_z a_{33} \end{pmatrix}.$$

Since (1.6) holds for every volume ω, it follows that

$$\rho\left(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right) = \nabla \cdot \mathbb{S} + \boldsymbol{f}_{\text{ext}} \quad \forall\ t \in (0, T],\ \boldsymbol{x} \in \Omega. \tag{1.11}$$

This relation is the momentum equation. □

Remark 1.8. Symmetry of the stress tensor. Let ω be an arbitrary volume with sufficiently smooth boundary $\partial\omega$ and let the net force given by the right-hand side of (1.6). The torque in ω with respect to the origin $\mathbf{0}$ of the coordinate system is then defined by

$$\boldsymbol{M}_{\mathbf{0}} = \int_{\omega} \boldsymbol{r} \times \boldsymbol{f}_{\text{ext}}\ d\boldsymbol{x} + \int_{\partial\omega} \boldsymbol{r} \times (\mathbb{S}\boldsymbol{n})\ d\boldsymbol{s} \quad [\mathrm{Nm}], \tag{1.12}$$

where (1.10) was used. In (1.12), $\boldsymbol{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$ is the vector pointing from $\mathbf{0}$ to a point $\boldsymbol{x} \in \overline{\omega}$. A straightforward calculation shows

$$\boldsymbol{r} \times (\mathbb{S}\boldsymbol{n}) = \left(\boldsymbol{r} \times \mathbb{S}_{*1}\ \boldsymbol{r} \times \mathbb{S}_{*2}\ \boldsymbol{r} \times \mathbb{S}_{*3}\right) \boldsymbol{n},$$

where $\mathbb{S}_{*i}$ is the i-th column of $\mathbb{S}$ and $(\cdot)$ denotes here the tensor with the respective columns. Inserting this expression into (1.12), applying integration by parts and using the product rule leads to

$$\begin{aligned}\boldsymbol{M_0} &= \int_\omega \boldsymbol{r}\times\boldsymbol{f}_{\text{ext}}\ d\boldsymbol{x} + \int_\omega \nabla\cdot\left(\left(\boldsymbol{r}\times\mathbb{S}_{*1}\ \boldsymbol{r}\times\mathbb{S}_{*2}\ \boldsymbol{r}\times\mathbb{S}_{*3}\right)\right)d\boldsymbol{x}\\ &= \int_\omega \boldsymbol{r}\times\left(\boldsymbol{f}_{\text{ext}}+\nabla\cdot\mathbb{S}\right)d\boldsymbol{x}\\ &\quad + \int_\omega \partial_x\boldsymbol{r}\times\mathbb{S}_{*1}+\partial_y\boldsymbol{r}\times\mathbb{S}_{*2}+\partial_z\boldsymbol{r}\times\mathbb{S}_{*3}\ d\boldsymbol{x}. \end{aligned}\tag{1.13}$$

Consider now a fluid in equilibrium state, i.e., the net forces acting on this fluid are zero. Hence, the right-hand side of (1.11) vanishes and so the first integral of (1.13). In addition, equilibrium requires in particular that $\boldsymbol{M_0} = \boldsymbol{0}$. Thus, from (1.13) it follows that

$$\boldsymbol{0} = \int_\omega \partial_x\boldsymbol{r}\times\mathbb{S}_{*1}+\partial_y\boldsymbol{r}\times\mathbb{S}_{*2}+\partial_z\boldsymbol{r}\times\mathbb{S}_{*3}\ d\boldsymbol{x}. \tag{1.14}$$

Using now

$$\partial_x\boldsymbol{r} = \lim_{\Delta x\to 0}\frac{(x+\Delta x)\boldsymbol{e}_1 - x\boldsymbol{e}_1}{\Delta x} = \boldsymbol{e}_1,$$

$\partial_y\boldsymbol{r} = \boldsymbol{e}_2$, $\partial_z\boldsymbol{r} = \boldsymbol{e}_3$, and inserting these equations into (1.14) leads finally to

$$\boldsymbol{0} = \int_\omega \begin{pmatrix}\mathbb{S}_{32}-\mathbb{S}_{23}\\ \mathbb{S}_{13}-\mathbb{S}_{31}\\ \mathbb{S}_{21}-\mathbb{S}_{12}\end{pmatrix}(t,\boldsymbol{x})\ d\boldsymbol{x}$$

for an arbitrary volume ω. From this relation, one deduces that $\mathbb{S}$ has to be symmetric, $\mathbb{S}=\mathbb{S}^T$, and $\mathbb{S}$ possesses six unknown components. □

Remark 1.9. Decomposition of the stress tensor. To model the stress tensor in the basic variables introduced in Remark 1.1, this tensor is decomposed into

$$\mathbb{S} = \mathbb{V} - P\mathbb{I}. \tag{1.15}$$

Here, $\mathbb{V}$ [N/m^2] is the so-called viscous stress tensor, representing the forces coming from the friction of the particles, and P [Pa] is the pressure, describing the forces acting on the surface of each fluid volume ω. The viscous stress tensor will be modeled in terms of the velocity, see Remark 1.11. □

Remark 1.10. The pressure. The pressure P acts on a surface of a fluid volume ω only normal to that surface and it is directed into ω. This property is reflected by the negative sign in the ansatz (1.15) since

$$-\int_{\partial\omega} P\boldsymbol{n}\ d\boldsymbol{s} = -\int_\omega \nabla P\ d\boldsymbol{x} = -\int_\omega \nabla\cdot(P\mathbb{I})\ d\boldsymbol{x}.$$

□

Remark 1.11. The viscous stress tensor. Friction between fluid particles can only occur if the particles move with different velocities. For this reason, the viscous stress tensor is modeled to depend on the gradient of the velocity. For the reason of symmetry, Remark 1.8, it is modeled to depend on the symmetric part of the gradient, the so-called velocity deformation tensor or rate-of-deformation tensor

$$\mathbb{D}\left(\boldsymbol{v}\right) = \frac{\nabla \boldsymbol{v} + \left(\nabla \boldsymbol{v}\right)^T}{2} \quad [1/\mathrm{s}].$$

The gradient of the velocity is a tensor with the components

$$(\nabla \boldsymbol{v})_{ij} = \partial_j v_i = \frac{\partial v_i}{\partial x_j}, \quad i, j = 1, 2, 3.$$

If the velocity gradients are not too large, one can assume that first the dependency is linear and second that higher order derivatives can be neglected. Since there is no friction for a flow with constant velocity, such that $\mathbb{V}$ vanishes in this case, lower order terms than first order derivatives of the velocity should not appear in the model. The most general form of a tensor which satisfies all conditions is

$$\mathbb{V} = a\mathbb{D}\left(\boldsymbol{v}\right) + b\left(\nabla \cdot \boldsymbol{v}\right)\mathbb{I},$$

where a and b do not depend on the velocity. Introducing the first order viscosity μ $[\mathrm{kg}/(\mathrm{m\ s})]$ and the second order viscosity ζ $[\mathrm{kg}/(\mathrm{m\ s})]$, one writes this tensor in fluid dynamics in the form

$$\mathbb{V} = 2\mu\mathbb{D}\left(\boldsymbol{v}\right) + \left(\zeta - \frac{2\mu}{3}\right)\left(\nabla \cdot \boldsymbol{v}\right)\mathbb{I} \quad [\mathrm{N}/\mathrm{m}^2]. \tag{1.16}$$

The viscosity μ is also called dynamic or shear viscosity. The law (1.16) is for fluids the analog of Hooke's law for solids. □

Example 1.12. Steady rotation. There is no viscous stress, i.e., $\mathbb{V} = \mathbb{0}$, if the fluid is rotating steadily. In this situation, the velocity is given by

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{x} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \omega_3 y - \omega_2 z \\ \omega_1 z - \omega_3 x \\ \omega_2 x - \omega_1 y \end{pmatrix},$$

where $\boldsymbol{\omega}$ $[1/\mathrm{s}]$ is a constant angular velocity. One has obviously $\nabla \cdot \boldsymbol{v} = 0$ and

$$\nabla \boldsymbol{v} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathbb{D}\left(\boldsymbol{v}\right) = \mathbb{0}.$$

Hence, (1.16) is an appropriate model in this case. □

Remark 1.13. Newtonian fluids. The linear relation (1.16) is only an approximation for a real fluid. In general, the relation will be non-linear. Only for small stresses, a linear approximation of the general stress-deformation relation can be used. A linear stress-deformation relation was postulate by Newton. For this reason, a fluid satisfying assumption (1.16) is called Newtonian fluid. More general relations than (1.16) exist, however they are less well understood from the mathematical point of view. □

Remark 1.14. Normal and shear stresses, trace of the stress tensor. The diagonal components $\mathbb{S}_{11}, \mathbb{S}_{22}, \mathbb{S}_{33}$ of the stress tensor are called normal stresses and the off-diagonal components shear stresses.

For incompressible flows one gets with (1.3), (1.15), and (1.16)

$$\mathbb{S} = 2\mu\mathbb{D}(\boldsymbol{v}) - P\mathbb{I}. \tag{1.17}$$

The trace of the stress tensor is the sum of the normal stresses

$$\begin{aligned}\operatorname{tr}(\mathbb{S}) &= \mathbb{S}_{11} + \mathbb{S}_{22} + \mathbb{S}_{33} \\ &= 2\mu(\partial_x v_1 + \partial_y v_2 + \partial_z v_3) + 3\left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \boldsymbol{v}) - 3P \\ &= 3\zeta\,(\nabla \cdot \boldsymbol{v}) - 3P.\end{aligned}$$

For incompressible fluids, it follows that

$$P(t, \boldsymbol{x}) = -\frac{1}{3}\left(\mathbb{S}_{11} + \mathbb{S}_{22} + \mathbb{S}_{33}\right)(t, \boldsymbol{x}). \tag{1.18}$$

□

Remark 1.15. The Navier–Stokes equations. Now, the pressure part of the stress tensor and the model (1.16) of the viscous stress tensor can be inserted into (1.11) giving the general Navier–Stokes equations (including the conservation of mass)

$$\begin{aligned}\rho\left(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right) - 2\nabla \cdot (\mu\mathbb{D}(\boldsymbol{v})) & \\ -\nabla \cdot \left(\left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \boldsymbol{v})\,\mathbb{I}\right) + \nabla P = \boldsymbol{f}_{\text{ext}} & \text{ in } (0, T] \times \Omega, \\ \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad & \text{ in } (0, T] \times \Omega.\end{aligned} \tag{1.19}$$

If the fluid is incompressible and homogeneous, such that μ and ρ are positive constants, the Navier–Stokes equations simplify to

$$\begin{aligned}\partial_t \boldsymbol{v} - 2\nu\nabla \cdot \mathbb{D}(\boldsymbol{v}) + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \nabla\frac{P}{\rho} = \frac{\boldsymbol{f}_{\text{ext}}}{\rho} & \text{ in } (0, T] \times \Omega, \\ \nabla \cdot \boldsymbol{v} = 0 \quad & \text{ in } [0, T] \times \Omega.\end{aligned} \tag{1.20}$$

Here, $\nu = \mu/\rho$ $[\mathrm{m}^2/\mathrm{s}]$ is the kinematic viscosity of the fluid. □

1.3 The Dimensionless Navier–Stokes Equations

Remark 1.16. Characteristic scales. Mathematical analysis and numerical simulations are based on dimensionless equations. To derive dimensionless equations from system (1.20), the quantities

- L [m] – a characteristic length scale of the flow problem,
- U [m/s] – a characteristic velocity scale of the flow problem,
- T^* [s] – a characteristic time scale of the flow problem,

are introduced. □

Remark 1.17. The Navier–Stokes equations in dimensionless form. Denote by $(t', \boldsymbol{x}')$ [s, m] the old variables. Applying the transform of variables

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{L}, \quad \boldsymbol{u} = \frac{\boldsymbol{v}}{U}, \quad t = \frac{t'}{T^*}, \tag{1.21}$$

one obtains from (1.20) and a rescaling

$$\begin{aligned} \frac{L}{UT^*}\partial_t \boldsymbol{u} - \frac{2\nu}{UL}\nabla \cdot \mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\frac{P}{\rho U^2} &= \frac{L}{\rho U^2}\boldsymbol{f}_{\text{ext}} && \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } [0,T]\times\Omega, \end{aligned}$$

where all derivatives are with respect to the new variables. Without having emphasized this issue in the notation, also the domain and the time interval are now dimensionless. Defining

$$p = \frac{P}{\rho U^2}, \quad \text{Re} = \frac{UL}{\nu}, \quad \text{St} = \frac{L}{UT^*}, \quad \boldsymbol{f} = \frac{L}{\rho U^2}\boldsymbol{f}_{\text{ext}}, \tag{1.22}$$

the incompressible Navier–Stokes equations in dimensionless form

$$\begin{aligned} \text{St}\partial_t \boldsymbol{u} - \frac{2}{\text{Re}}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } [0,T]\times\Omega, \end{aligned} \tag{1.23}$$

are obtained. The constant Re is called Reynolds number and the constant St Strouhal number. These numbers allow the classification and comparison of different flows. □

Remark 1.18. Inherent difficulties of the dimensionless Navier–Stokes equations. To simplify the notations, one uses the characteristic time $T^* = L/U$ such that (1.23) simplifies to

$$\begin{aligned} \partial_t \boldsymbol{u} - 2\nu\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \text{ in } [0,T]\times\Omega, \end{aligned} \tag{1.24}$$

with the dimensionless viscosity $\nu = Re^{-1}$. Here, with an abuse of notation, the same symbol is used as for the kinematic viscosity.

This transform and the resulting equations (1.24) are the basis equations for the mathematical analysis of the incompressible Navier–Stokes equations and the numerical simulation of incompressible flows. System (1.24) comprises two important difficulties:

- the coupling of velocity and pressure,
- the nonlinearity of the convective term.

Additionally, difficulties for the numerical simulation occur if

- the convective term dominates the viscous term, i.e., if ν is small.

□

Remark 1.19. Different forms of terms in (1.24)*.* With the help of the divergence constraint, i.e., the second equation in (1.24), the viscous and the convective term of the Navier–Stokes equations can be reformulated equivalently.

Assume that $\boldsymbol{u}$ is sufficiently smooth with $\nabla \cdot \boldsymbol{u} = 0$. Then, straightforward calculations, using the theorem of Schwarz and the second equation of (1.24), give

$$\nabla \cdot (\nabla \boldsymbol{u}) = \Delta \boldsymbol{u}, \quad \nabla \cdot \left(\nabla \boldsymbol{u}^T\right) = \begin{pmatrix} \partial_x \left(\nabla \cdot \boldsymbol{u}\right) \\ \partial_y \left(\nabla \cdot \boldsymbol{u}\right) \\ \partial_z \left(\nabla \cdot \boldsymbol{u}\right) \end{pmatrix} = \boldsymbol{0}.$$

Thus, the viscous term becomes

$$-2\nu \nabla \cdot \mathbb{D}(\boldsymbol{u}) = -\nu \Delta \boldsymbol{u}. \tag{1.25}$$

For the convective term, ones uses the identity (product rule)

$$\begin{aligned} \nabla \cdot \left(\boldsymbol{u}\boldsymbol{v}^T\right) &= \begin{pmatrix} \partial_x (u_1 v_1) + \partial_y (u_1 v_2) + \partial_z (u_1 v_3) \\ \partial_x (u_2 v_1) + \partial_y (u_2 v_2) + \partial_z (u_2 v_3) \\ \partial_x (u_3 v_1) + \partial_y (u_3 v_2) + \partial_z (u_3 v_3) \end{pmatrix} \\ &= \begin{pmatrix} u_1 (\partial_x v_1 + \partial_y v_2 + \partial_z v_3) \\ u_2 (\partial_x v_1 + \partial_y v_2 + \partial_z v_3) \\ u_3 (\partial_x v_1 + \partial_y v_2 + \partial_z v_3) \end{pmatrix} + \begin{pmatrix} v_1 \partial_x u_1 + v_2 \partial_y u_1 + v_3 \partial_z u_1 \\ v_1 \partial_x u_2 + v_2 \partial_y u_2 + v_3 \partial_z u_2 \\ v_1 \partial_x u_3 + v_2 \partial_y u_3 + v_3 \partial_z u_3 \end{pmatrix} \\ &= (\nabla \cdot \boldsymbol{v})\boldsymbol{u} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}. \end{aligned} \tag{1.26}$$

In the case $\boldsymbol{v} = \boldsymbol{u}$ with $\nabla \cdot \boldsymbol{u} = 0$, it follows that

$$(\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = \nabla \cdot \left(\boldsymbol{u}\boldsymbol{u}^T\right). \tag{1.27}$$

A detailed presentation and discussion of different forms of the convective term is given in Section 5.1.2. □

Remark 1.20. Two-dimensional Navier–Stokes equations. Even if real flows occur only in three dimensions, the consideration of the Navier–Stokes equations (1.24) in two dimensions is also of interest. There are applications where the flow is, for instance, constant in the third direction and it behaves virtually two-dimensional. □

Remark 1.21. Special cases of incompressible flow models.

- In a stationary flow, the velocity and the pressure do not change in time. Hence $\partial_t \boldsymbol{u} = \boldsymbol{0}$ and these flows are modeled by the so-called stationary or steady-state Navier–Stokes equations

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \text{ in } \Omega. \end{aligned} \tag{1.28}$$

  A necessary condition for the stationarity of a flow field is that the data of the problem, i.e., the right-hand side and the boundary conditions, see Section 1.4, are time-independent. But this condition is not sufficient, cf. Example D.10.
- If in a stationary flow the viscous transport dominates the convective transport, i.e., if the fluid moves very slowly, the nonlinear convective term of the Navier–Stokes equations (1.28) can be neglected. This situation leads to a linear system of equations, the so-called Stokes equations

$$\begin{aligned} -\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \text{ in } \Omega. \end{aligned} \tag{1.29}$$

  Here, the momentum equation was divided by ν, defining a new pressure and a new right-hand side.
- In the numerical analysis, often the so-called Oseen equations are considered. Given a divergence-free flow field $\boldsymbol{b}$, the Oseen equations are a system of linear equations of the form

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \nabla p + c\boldsymbol{u} &= \boldsymbol{f} \text{ in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \text{ in } \Omega, \end{aligned}$$

  with a scalar-valued function $c(\boldsymbol{x}) \ge 0$.

□

1.4 Initial and Boundary Conditions

Remark 1.22. General considerations. The Navier–Stokes equations (1.24) are a first order partial differential equation with respect to time and a second order partial differential equation with respect to space. Thus, they have to be equipped with an initial condition for the velocity at $t = 0$ and with boundary conditions on the boundary $\Gamma = \partial\Omega$ of Ω, if Ω is a bounded domain. There are several kinds of boundary conditions which can be prescribed for incompressible flows. Of course, a compatibility condition should be fulfilled between the boundary conditions of the initial velocity field and the limit of the prescribed boundary conditions for $t \to 0, t > 0$.

In applications, the initial and boundary conditions are given in terms of quantities with dimensions. For the analysis and the simulation of the Navier–Stokes equations, these conditions have to be non-dimensionalized with the characteristic scales from Remark 1.16. Here, only conditions for the dimensionless equations will be discussed. □

Remark 1.23. Initial condition. Concerning the initial condition, an initial velocity field, which has to be divergence-free, is prescribed at $t = 0$

$$\boldsymbol{u}(0, \boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{with} \quad \nabla \cdot \boldsymbol{u}_0 = 0 \quad \text{in} \quad \Omega.$$

□

Remark 1.24. Dirichlet boundary conditions, no-slip boundary conditions, essential boundary conditions. Often used boundary conditions describe the velocity field at a part of the boundary

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{g}(t, \boldsymbol{x}) \quad \text{in} \quad (0, T] \times \Gamma_{\text{diri}},$$

with $\Gamma_{\text{diri}} \subset \Gamma$. This boundary condition is called Dirichlet boundary condition. It models in particular prescribed inflows into Ω and outflows from Ω.

In the special case $\boldsymbol{g}(t, \boldsymbol{x}) = \boldsymbol{0}$ in $(0, T] \times \Gamma_{\text{diri}}$, this boundary condition is called no-slip boundary condition. The no-slip condition is usually applied at fixed walls. Let $\boldsymbol{n}$ be the unit normal vector in $\boldsymbol{x} \in \Gamma_{\text{nosl}} \subset \Gamma_{\text{diri}}$ and $\{\boldsymbol{t}_1, \boldsymbol{t}_2\}$ unit tangential vectors such that $\{\boldsymbol{n}, \boldsymbol{t}_1, \boldsymbol{t}_2\}$ is an orthonormal system of vectors. Then, the no-slip boundary condition can be decomposed into three parts:

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{0} \iff \boldsymbol{u}(t, \boldsymbol{x}) \cdot \boldsymbol{n} = 0, \ \boldsymbol{u}(t, \boldsymbol{x}) \cdot \boldsymbol{t}_1 = 0, \ \ \boldsymbol{u}(t, \boldsymbol{x}) \cdot \boldsymbol{t}_2 = 0$$

in $\boldsymbol{x} \in \Gamma_{\text{nosl}}$. The condition $\boldsymbol{u}(t, \boldsymbol{x}) \cdot \boldsymbol{n} = 0$ states that the fluid does not penetrate the wall. The other two conditions describe that the fluid does not slip along the wall.

If Dirichlet boundary conditions are prescribed on the whole boundary of Ω, there are two additional issues. First, the pressure is determined only up to an additive constant. An additional condition for fixing the constant has to be introduced, e.g., that the integral mean value of the pressure should vanish

$$\int_\Omega p(t, \boldsymbol{x}) \, d\boldsymbol{x} = 0 \quad t \in (0, T].$$

Second, from the divergence-free constraint and integration by parts it follows that the boundary condition has to satisfy the compatibility condition

$$0 = \int_\Omega \nabla \cdot \boldsymbol{u}(t, \boldsymbol{x}) \, d\boldsymbol{x} = \int_\Gamma (\boldsymbol{u} \cdot \boldsymbol{n}) (t, \boldsymbol{s}) \, d\boldsymbol{s} = \int_\Gamma (\boldsymbol{g} \cdot \boldsymbol{n})(t, \boldsymbol{s}) \, d\boldsymbol{s} \qquad (1.30)$$

for all times.

In the case of the Navier–Stokes equations and their special cases, Dirichlet boundary conditions are so-called essential boundary conditions. Such boundary conditions enter the definition of appropriate function spaces for the study of the equations in the framework of functional analysis, see Section 2.2. □

Remark 1.25. Free slip boundary conditions, slip with friction boundary conditions. The free slip boundary condition is applied on boundaries without friction. It has the form

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{n} &= g \quad \text{in } (0,T]\times\mathit{\Gamma}_{\text{slip}}\subset\mathit{\Gamma}, \\ \boldsymbol{n}^T\mathbb{S}\boldsymbol{t}_k &= 0 \quad \text{in } (0,T]\times\mathit{\Gamma}_{\text{slip}}, \quad 1\le k\le d-1, \end{aligned} \tag{1.31}$$

where the dimensionless stress tensor is given by

$$\mathbb{S} = 2\nu\mathbb{D}\left(\boldsymbol{u}\right) - p\mathbb{I}. \tag{1.32}$$

There is no penetration through the wall if $g=0$ on $\mathit{\Gamma}_{\text{slip}}$.

The slip with linear friction and no penetration boundary condition has the form

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{in } (0,T]\times\mathit{\Gamma}_{\text{slfr}}\subset\mathit{\Gamma}, \\ \boldsymbol{u}\cdot\boldsymbol{t}_k + \beta^{-1}\boldsymbol{n}^T\mathbb{S}\boldsymbol{t}_k &= 0 \quad \text{in } (0,T]\times\mathit{\Gamma}_{\text{slfr}}, \quad 1\le k\le d-1. \end{aligned} \tag{1.33}$$

This boundary condition states that the fluid does not penetrate the wall and it slips along the wall while loosing energy. The loss of energy is given by the friction parameter β. In the limit case $\beta^{-1}\to 0$, the no-slip condition is recovered and in the limit case $\beta^{-1}\to\infty$ the free slip condition. Slip with friction boundary conditions were studied already by Maxwell (1879) and Navier (1823). The difficulty in the application of this boundary condition consists in the determination of the friction parameter β, which might depend, e.g., on the local flow field and on the roughness of the wall.

Since $\boldsymbol{n}$ and $\boldsymbol{t}_k$ are orthogonal vectors, the values of the pressure do not play any role in the boundary conditions (1.31) and (1.33). Hence, an additional condition for the pressure is needed to fix the additive constant. □

Remark 1.26. Outflow or do-nothing boundary conditions, natural boundary conditions, directional do-nothing condition. For numerical simulations, the so-called outflow boundary condition or do-nothing boundary condition

$$\mathbb{S}\boldsymbol{n} = \boldsymbol{0} \quad \text{in} \quad (0,T]\times\mathit{\Gamma}_{\text{outf}}\subset\mathit{\Gamma} \tag{1.34}$$

is often applied. This boundary condition models the situation that the normal stress, which is equal to the Cauchy stress vector (1.10), vanishes on the boundary part $\mathit{\Gamma}_{\text{outf}}$. The do-nothing boundary condition is often used if no other outflow boundary condition is available.

From the mathematical point of view, the do-nothing boundary conditions are natural boundary conditions. Deriving from the strong form of the equations (1.24) a so-called weak form, natural boundary conditions appear in the arising integrals on the boundary. With the do-nothing boundary conditions, the integral on Γ_{outf} vanishes.

The boundary condition (1.34) contains also a contribution from the pressure. This issue fixes the problem of the additive constant, i.e., if on a part of the boundary the do-nothing boundary condition is prescribed, it is not necessary to introduce an additional condition for the pressure.

However, there might be also some inflow at Γ_{outf}, e.g., if a vortex crosses Γ_{outf}. It was observed that an inflow which is too strong can destabilize numerical simulations. In addition, a mathematical theory of the Navier–Stokes equations with the boundary condition (1.34) cannot be performed since a possible inflow across Γ_{outf} cannot be controlled, e.g., see **?**. For these reasons, a modified do-nothing condition was suggested which cures these problems

$$\left.\begin{aligned} \mathbb{S}\boldsymbol{n} = \boldsymbol{0} \text{ if } \boldsymbol{u}\cdot\boldsymbol{n} \ge 0, \\ \mathbb{S}\boldsymbol{n} - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{u} = \boldsymbol{0} \text{ if } \boldsymbol{u}\cdot\boldsymbol{n} < 0, \end{aligned}\right\} \text{ in } (0,T]\times\Gamma_{\mathrm{outf}} \subset \Gamma. \tag{1.35}$$

This condition is called directional do-nothing condition. □

Remark 1.27. Conditions for an infinite domain, periodic boundary conditions. The case $\Omega = \mathbb{R}^3$ is also considered in analytical and numerical studies of the Navier–Stokes equations. There are two situations in this case. In the first one, the decay of the velocity field as $\|\boldsymbol{x}\|_2 \to \infty$ is prescribed. The second situation consists of applying periodic boundary conditions. These boundary conditions do not posses any physical meaning. They are used to simulate an infinite extension of Ω in one or more directions. Let, e.g., this direction be $\boldsymbol{e}_i$. It is assumed that the flow is periodic in this direction with the length l of the period. In computations, e.g., the cube $\Omega = (0,l)^d$ is used and the periodic boundary conditions are given by

$$\boldsymbol{u}(t,\boldsymbol{x}+l\boldsymbol{e}_i) = \boldsymbol{u}(t,\boldsymbol{x}) \quad \forall\ (t,\boldsymbol{x}) \in (0,T]\times\Gamma.$$

From the point of view of the finite computational domain, all appearing functions have to be extended periodically in the periodic direction to return to the original problem.

The use of space periodic boundary conditions may also facilitate analytical investigations, see (Temam, 1995, p. 4). □