Chapter 6
Outlook

Remark 6.1. Further linear residual-based stabilizations. Besides the SUPG method, there are other residual-based stabilizations for finite element methods were proposed. As already mentioned, one can take the full residual in the stabilization term also for the test function, which gives the Galerkin least squares (GLS) method. Another proposal is the so-called unusual finite element method. The numerical analysis of these methods is similar to the numerical analysis of the SUPG method. However, the numerical results with the other methods are generally not better than with the SUPG method. Since the SUPG method is easiest to implement, it is preferred and the other methods are only of little importance in practice.

The idea of residual-based stabilization techniques is also used for other problems, e.g., for (turbulent) incompressible flow problems governed by the incompressible Navier–Stokes equations. □

Remark 6.2. Spurious Oscillations at Layers Diminishing (SOLD) methods. The applicability of the SUPG method in practice is, however, restricted by the appearance of generally considerable spurious oscillations in the numerical solutions. These oscillations correspond to unphysical values, like negative concentrations, which do not appear in practical problems. In addition, if a numerical solution with unphysical values is used in coupled problems as a parameter in other equations, one can get easily instabilities in the numerical simulations.

There has been much effort to reduce the spurious oscillations in residual-based stabilizations. A detailed investigation of the solutions computed with the SUPG method shows that the spurious oscillations appear orthogonal to the streamline direction. Recall that the SUPG method only introduces numerical diffusion in streamline direction. The basic idea consists now in extending the SUPG method by introducing some numerical diffusion orthogonal to the streamline direction. To achieve methods of higher order, this numerical diffusion has to depend on the finite element solution. Hence, one obtains a nonlinear term. There are many proposals for such terms and this class of methods is called Spurious Oscillations at Layers Diminishing

(SOLD) methods or shock capturing methods. Altogether, one has to solve a nonlinear discrete problem for a linear boundary value problem. The questions of existence and uniqueness of a solution of the nonlinear problem arise. There are only answers for very few SOLD methods. A competitive study John & Knobloch (2007) showed that most of the SOLD methods in fact reduce the size of the spurious oscillations of the SUPG method. However, even the reduced oscillations are still considerable large. Altogether, none of the SOLD methods proposed so far cures the drawback of the SUPG method. □

Remark 6.3. Other stabilized finite element methods. In the last decade, several approaches for stabilized finite element methods have been proposed which do not rely on the residual, like Local Projection Stabilization (LPS) methods and Continuous Interior Penalty (CIP) methods. A numerical analysis for these methods can be performed, which is, however, more complicated than for the SUPG method. In simulations, the results are generally not better than for the SUPG method, often even worse. □

Remark 6.4. Algebraic flux correction schemes. All mentioned stabilizations so far modify the bilinear form of the Galerkin finite element method to introduce some numerical diffusion. A much different approach are algebraic flux correction schemes, see Kuzmin (2007), which start with the matrix-vector equation obtained with the Galerkin finite element method. Then, the matrix is modified such that one gets an M-matrix. This modification introduces numerical diffusion and the resulting scheme satisfies the discrete maximum principle, but the solutions are smeared very much. Thus, the next step consists in modifying the right-hand side to remove the numerical diffusion where it is not necessary. This step is a nonlinear step. First results on the existence and uniqueness of the solution of the nonlinear problem and on the convergence of the discrete solution are proved in Barrenechea *et al.* (2016). In practice, one observes sometimes difficulties in the convergence to compute the solution by a fixed point iteration. This class of method is only well defined for linear and bilinear finite elements. □

Remark 6.5. Summary. There are a lot of proposals for stabilized discretizations for linear convection-dominated convection-diffusion equations. But none of the proposals can be recommended for all practical purposes. For many methods, there are open questions concerning their numerical analysis. Therefore, the design and analysis of numerical methods for convection-dominated problems is still an active field of research. □

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