Chapter 4

Weak Solution Theory

Remark 4.1 *Motivation.* This chapter presents an extension of the notation of a solution of partial differential equations, the so-called weak or variational solution. This extension is necessary for the following reasons:

- In general, one cannot expect that a partial differential equation has a classical solution. For the existence of a classical solution, all parameters have to be sufficiently smooth. In higher dimensions, also the domain has to satisfy certain regularity conditions. Such smoothness or regularity conditions are often not satisfied in applications. Nevertheless, the processes which are modeled with the partial differential equations occur and there is obviously a solution. However, this solution will not possess the (regularity) properties of the classical solution and therefore one needs an extension of the notation of the solution.
- It is already known from Numerical Mathematics 3 that finite element methods are based on a weak or variational formulation of the partial differential equation.

□

Remark 4.2 *Tools from functional analysis.* The study of variational equations and of finite element methods requires many tools from functional analysis, like Lebesgue spaces, Sobolev spaces, and a number of inequalities, like the Poincaré–Friedrichs inequality. For a detailed introduction to the these tools, it will be referred to the lecture notes on Numerical Mathematics 3. □

4.1 Variational Formulation

Remark 4.3 *Convection-diffusion-reaction equation.* Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a bounded domain with Lipschitz boundary $\partial\Omega$. A linear convection-diffusion-reaction equation with homogeneous Dirichlet boundary conditions is given by

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

In (4.1), $\mathbf{b}$ is the convection field. □

Remark 4.4 *Derivation of the variational or weak formulation.* Consider problem (4.1). Multiplication of the differential equation with an appropriate function $v(\mathbf{x})$, with $v = 0$ on $\partial\Omega$, integration of the resulting equation on Ω, and integration by

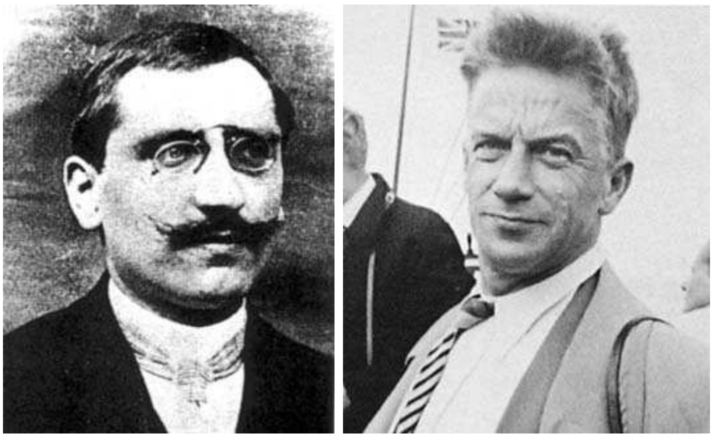

Figure 4.1: Left: Henri Lebesgue (1875 – 1941), right: Serge Leewards Sobolev (1908 – 1989).

Figure 4.2: David Hilbert (1862 – 1943).

parts (Gaussian theorem) yields

$$
\begin{aligned}
&\int_\Omega \left(-\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu\right)(\mathbf{x})v(\mathbf{x})\ d\mathbf{x} \\
&\quad = \int_{\partial\Omega} -\varepsilon(\nabla u\cdot\mathbf{n})(\mathbf{s})v(\mathbf{s})\ d\mathbf{s} + \int_\Omega \left(\varepsilon\nabla u\cdot\nabla v + \left(\mathbf{b}\cdot\nabla u + cu\right)v\right)(\mathbf{x})\ d\mathbf{x} \\
&\quad = \int_\Omega \left(\varepsilon\nabla u\cdot\nabla v + \left(\mathbf{b}\cdot\nabla u + cu\right)v\right)(\mathbf{x})\ d\mathbf{x} \\
&\quad = \int_\Omega f(\mathbf{x})v(\mathbf{x})\ d\mathbf{x}.
\end{aligned}
$$

Here, $\mathbf{n}$ is the outward pointing unit normal vector on $\partial\Omega$. The integral on the boundary vanishes because of the boundary condition of the test function. The highest order derivative of $u(\mathbf{x})$ has been transferred to $v(\mathbf{x})$. Let $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$, then this equation can be written in a more compact form. $\square$


Figure 4.3: Left: Jules Henri Poincaré (1854 – 1912), right: Kurt Otto Friedrichs (1901 – 1982).

Definition 4.5 Variational or weak formulation. Let $\mathbf{b}, c \in L^\infty(\Omega)$ and $f \in H^{-1}(\Omega)$. The variational or weak formulation of the convection-diffusion-reaction equation (4.1) is: Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$\varepsilon\,(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v) = (f, v), \tag{4.2}$$

where (with some abuse of notation, the dual pairing of $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is also denoted by $(\cdot,\cdot)$. A solution of (4.2) is called variational or weak solution. The space in which the solution is searched is called solution or ansatz space. The functions $v(\mathbf{x})$ are called test functions and the space from which they come is the test space.
□

Remark 4.6 *The variational formulation.*
- The space $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ and not of $H^1(\Omega)$.
- With the given assumptions, all terms are well defined.
- For the weak solution, only the first derivative, in the weak sense, is required.
- Each classical solution is a weak solution. The other direction holds only for sufficiently regular coefficients, right-hand side, and domain.

□

Remark 4.7 *Other boundary conditions.*
- Consider first inhomogeneous Dirichlet boundary conditions

  $$u(\mathbf{x}) = g(\mathbf{x}) \quad \text{on } \partial\Omega.$$

  These are so-called essential boundary conditions. Such boundary conditions are included into the definition of the ansatz space

  $$V_a = \left\{ v \in H^1(\Omega)\ :\ v|_{\partial\Omega} = g \right\},$$

  where the restriction to the boundary is understood in the sense of traces. The test space is still $V = H_0^1(\Omega)$. Then, the weak formulation reads as follows: Find $u \in V_a$ such that

  $$\varepsilon\,(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v) = (f, v) \quad \forall\, v \in V.$$

A different way of writing the variational problem uses an extension $u_g \in H^1(\Omega)$ of $g(\mathbf{x})$ to Ω. Then, one seeks $u \in H^1(\Omega)$ such that $u - u_g \in V$ and

$$\varepsilon (\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v) = (f, v) \quad \forall\, v \in V.$$

- Neumann boundary conditions appear in a straightforward way in the variational formulation since the integral on Neumann boundaries appears in the integration by parts. They are called natural boundary conditions. Let $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ with $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, and assume for simplicity that $u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \partial\Omega_D$. Let $V_0 = \{v \in H^1(\Omega) \ : \ v|_{\partial\Omega_D} = 0\}$, then the variational formulation has the form: Find $u \in V_0$ such that

$$\varepsilon (\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v) = (f, v) + \int_{\partial\Omega_N} \varepsilon (\nabla u \cdot \mathbf{n})(s) v(\mathbf{s})\, d\mathbf{s} \quad \forall\, v \in V_0.$$

□

Definition 4.8 Properties of bilinear forms. Let $(V, \|\cdot\|_V)$ be a Banach space. A map $a \ : \ V \times V \to \mathbb{R}$ is called

1. bilinear, if $a(\cdot, \cdot)$ is linear in both arguments,
2. symmetric, if $a(u, v) = a(v, u)$ for all $u, v \in V$,
3. positive, if $a(v, v) \geq 0$ for all $v \in V$,
4. strictly positive or coercive or V-elliptic or positive definite if there is a $m > 0$ such that $a(v, v) \geq m \|v\|_V^2$ for all $v \in V$,
5. bounded if there is a $M > 0$ such that

$$|a(u, v)| \leq M \|u\|_V \|v\|_V$$

   for all $u, v \in V$.

□

Figure 4.4: Stefan Banach (1892 – 1945).

Example 4.9 *Properties of the bilinear form of problem* (4.2). Consider the boundary value problem (4.2).

- It is
$$a(u,v) := \int_\Omega \Big(\varepsilon \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) + \mathbf{b}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) v(\mathbf{x}) + c(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x})\Big)\, d\mathbf{x} \quad (4.3)$$
a bilinear form in the space $V = H_0^1(\Omega)$. This property follows directly from the linearity of integration and differentiation. It will be assumed that $u, v \in H_0^1(\Omega)$ for the remainder of this example.
- If $\mathbf{b}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \Omega$, then $a(u,v)$ is symmetric.
- Let $\mathbf{b} \in C^1(\overline{\Omega})$ and $c \in C(\overline{\Omega})$. Applying integration by parts and the product rule, one obtains
$$\begin{aligned}\frac{1}{2}\int_\Omega \mathbf{b}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} &= -\frac{1}{2}\int_\Omega \nabla \cdot (\mathbf{b}(\mathbf{x}) v(\mathbf{x})) v(\mathbf{x})\, d\mathbf{x} \\ &= -\frac{1}{2}\int_\Omega (\nabla \cdot \mathbf{b}(\mathbf{x}))\, v(\mathbf{x}) v(\mathbf{x})\, dx - \frac{1}{2}\int_\Omega \mathbf{b}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) v(\mathbf{x})\, dx.\end{aligned}$$
It follows that
$$\int_\Omega \mathbf{b}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} = -\frac{1}{2}\int_\Omega \nabla \cdot \mathbf{b}(\mathbf{x}) v(\mathbf{x}) v(\mathbf{x})\, dx.$$
Inserting this relation into (4.3) with $u(\mathbf{x}) = v(\mathbf{x})$ yields
$$a(v,v) = \int_\Omega \left(\varepsilon \left(\nabla v(\mathbf{x})\right)^2 + \left(-\frac{\nabla \cdot \mathbf{b}}{2}(\mathbf{x}) + c(\mathbf{x})\right) \left(v(\mathbf{x})\right)^2\right)\, d\mathbf{x}.$$
If
$$-\frac{1}{2}\nabla \cdot \mathbf{b}(\mathbf{x}) + c(\mathbf{x}) \ge 0 \quad (4.4)$$
for all $\mathbf{x} \in \overline{\Omega}$, then it is for all $v \in H_0^1(\Omega)$
$$a(v,v) \ge \int_\Omega \varepsilon \left(\nabla v(\mathbf{x})\right)^2\, d\mathbf{x} = \varepsilon \left\|\nabla v\right\|_{L^2(\Omega)}^2 = \varepsilon \left\|v\right\|_V^2.$$
Hence, $a(\cdot,\cdot)$ is coercive under condition (4.4) since $\|\nabla v\|_{L^2(\Omega)}$ is a norm in $H_0^1(\Omega)$.
Considerations of this form can be performed with the weaker assumptions $\mathbf{b}, \nabla\mathbf{b}, c \in L^\infty(\Omega)$, since all integrals are still well defined under these assumptions. Then, condition (4.4) for coercivity has to hold almost everywhere in Ω.
- Let $\mathbf{b}, c \in L^\infty(\Omega)$. One gets, using the Cauchy–Schwarz inequality, Hölder's inequality, and the Poincaré–Friedrichs inequality
$$\begin{aligned}|a(u,v)| &\le \varepsilon \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|\mathbf{b}\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\quad + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\le \varepsilon \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + C_{PF} \|\mathbf{b}\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\quad + C_{PF}^2 \|c\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &= C \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}.\end{aligned}$$
Hence, the bilinear form is bounded.

□

Theorem 4.10 Theorem of Lax–Milgram. *Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a bounded and coercive bilinear form on the Hilbert space V. Then, for each bounded linear functional $f \in V'$ there is exactly one $u \in V$ with*
$$a(u,v) = f(v) \quad \forall\, v \in V. \quad (4.5)$$

Figure 4.5: Left: Augustin Louis Cauchy (1789 – 1857), right: Hermann Amandus Schwarz (1843 – 1921).

Proof: The proof can be found in the lecture notes of Numerical Mathematics 3. ■

Corollary 4.11 Existence and uniqueness of a solution of the weak problem (4.3)**.** *Let* $V = H_0^1(\Omega)$ *and assume* $f \in V'$, $\mathbf{b}, \nabla\mathbf{b}, c \in L^\infty(\Omega)$ *and* (4.4) *almost everywhere in* Ω*. Then,* (4.3) *has a unique solution.*

Proof: The statement follows directly from the Theorem of Lax–Milgram and the properties of the bilinear form which were proved in Example 4.9. ■