Fachbereich Mathematik und Informatik/ Institut für Mathematik
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Berlin, 08.12.2025

Numerik I

English translation of Übungsserie 08

Attention: Only solutions which provide a comprehensible reasoning will be graded. Every statement has to be argued. You can use results from the lecture. Statments without reasoning won't get any points.

1. *Roots of* p_{n+1}. Let p_{n+1} be the polynomial defined in Lemma 4.17. Show that this polynomial has $n+1$ mutually different roots that are all contained in (a, b).
   Hint: Start with showing that there is at least one root in (a, b). Then consider a polynomial that has the same roots as p_{n+1}. **3 points**

2. *Quadrature rule for tensor product domains.* Let $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$, then a tensor product domain in $\mathbb{R}^2$ determined by I_1 and I_2 is given by
   $$\Omega = I_1 \times I_2 = \left\{(x, y) \in \mathbb{R}^2 \ : \ x \in I_1, y \in I_2\right\}.$$
   Consider quadrature rules on I_1, I_2, given by grid points $\{x_i \in I_1\}_{i=0}^m$, $\{y_j \in I_2\}_{j=0}^n$ and and weights $\{\lambda_i\}_{i=0}^m$,$\{\kappa_j\}_{j=0}^n$ respectively. Compare formula (4.11) in the lecture notes.

   i) On Ω one can derive a quadrature rule with grid points $\{x_i, y_j\}$, $i = 0, \ldots, m$, $j = 0, \ldots, n$. Determine the (form of the) weights for this rule.
   ii) Let $I_1 = I_2 = [0, 1]$ and choose the above mentioned quadrature rules to be the trapezoidal rule. Compute the resulting quadrature rule on Ω. Subsequently approximate
   $$\int_\Omega x^3 y^3 \ dxdy$$
   and compute the absolute value of the error.
   iii) Consider the same task as in ii), but choose the quadrature rules on both intervals to be the Gauß-Legendre rule for $n = 1$. For this purpose derive the grid points and weights for this rule on the interval $[0, 1]$ similarly to Beispiel 4.24 in the lecture notes.

   All computations have to be in exact arithmetic. **6 points**

3. *Romberg and Simpson rule.* Every element $P_{k,j}$, $k = 0, \ldots, m$, $j = 0, \ldots, k$, of the Romberg method can be interpreted as result of a quadrature rule. Determine k and j in such a way that $P_{k,j}$ gives the same value as the summed up Simpson rule. **3 points**

4. *Programming exercise: Romberg method.* Approximate
   $$\int_{-1}^{1} \frac{(x-0.5)^3}{\sqrt{x+8}} \ dx$$

by the Romberg method introduced in the lecture notes. For this task implement a program.

The domain of integration has to be decomposed into $1, 2, 4, \ldots, 64$ subintervals. Determine and report the approximation error of the exact solution

$$\frac{12371}{20}\sqrt{7} - \frac{229179}{140}.$$

Furthermore display the Romberg method analogous to Beispiel 4.34 in the lecture notes. **4 points**

The exercises should be solved in groups of two students. They have to be submitted until Sie **Wednesday, 17.12.2025, 10:00** electronically via whiteboard.