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Numerical Mathematics IV

Exercise Sheet 03

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Local DMPs.* Consider a matrix $A = (a_{ij})_{j=1,\ldots,n}^{i=1,\ldots,m}$ of non-negative type with $0 < m < n$. Then the local DMPs

$$\sum_{j=1}^{n} a_{ij}\, u_j \le 0 \quad \Longrightarrow \quad u_i \le \max_{j\neq i,\, a_{ij}\neq 0} u_j\,, \tag{1}$$
$$\sum_{j=1}^{n} a_{ij}\, u_j \ge 0 \quad \Longrightarrow \quad u_i \ge \min_{j\neq i,\, a_{ij}\neq 0} u_j$$

hold for any $i \in \{1, \ldots, m\}$ and any $u_1, \ldots, u_n \in \mathbb{R}$ if and only if the conditions

$$a_{ij} \le 0, \quad a_{ii} > 0 \quad \forall\, i = 1, \ldots, m,\ j = 1, \ldots, n,\ i \neq j \tag{2}$$

and

$$\sum_{j=1}^{n} a_{ij} = 0 \quad \forall\, i = 1, \ldots, m \tag{3}$$

are satisfied.
Hint: it suffices to prove (1). From the classes it is known that (2) are necessary conditions. Show with a counter example that also (3) is necessary condition.

2. *Monotone matrices.* Prove the following statements:

   (a) A matrix $A \in \mathbb{R}^{n\times n}$ is monotone if and only if

   $$A\underline{v} \ge 0 \quad \Longrightarrow \quad \underline{v} \ge 0 \quad \forall\, \underline{v} \in \mathbb{R}^n. \tag{4}$$

   (b) Let $A \in \mathbb{R}^{n\times n}$ be a monotone matrix. If $A\underline{v} \le A\underline{w}$ for $\underline{v}, \underline{w} \in \mathbb{R}^n$, then it follows that $\underline{v} \le \underline{w}$.

   (c) The product of two monotone matrices is a monotone matrix.

3. *Properties of bubble functions.* Let $K \in \mathcal{T}^h$ be a simplex and let

$$v_{\mathrm{bub}}^h(\boldsymbol{x}) = \prod_{i=1}^{d+1} \lambda_i(\boldsymbol{x}), \quad \boldsymbol{x} \in K,$$

be a bubble function on K. Show that the following estimates hold

$$\left\| v_{\mathrm{bub}}^h \right\|_{L^2(K)} \ \le \ C h_K^d, \tag{5}$$
$$\left\| \nabla v_{\mathrm{bub}}^h \right\|_{L^2(K)} \ \le \ C h_K^{(d-2)/2}, \tag{6}$$

where the constants are independent of K.
Hint: (6) can be proved by using a transform to the reference mesh cell, applying the compatibility of the spectral matrix norm and the Euclidean vector norm, and using known properties of the reference map from Numerics III.

4. *Weak form of the viscous term with deformation tensor.* The viscous term with deformation tensor is given by $-2\nabla \cdot (\mathbb{D}(\boldsymbol{u}))$, which is equivalent to $-\Delta \boldsymbol{u}$ if $\boldsymbol{u}$ is divergence-free, see Remark 1.11 of the lecture notes. Show that the weak form of the viscous term with deformation tensor is $2(\mathbb{D}(\boldsymbol{u}), \mathbb{D}(\boldsymbol{v}))$.

5. *Equivalent weak formulations of the Stokes equations.* Consider the Stokes equations with homogeneous Dirichlet boundary conditions

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma.
\end{aligned}
$$

The standard weak formulation reads as follows: Given $\boldsymbol{f} \in H^{-1}(\Omega)$, find $(\boldsymbol{u}, p) \in H_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$
\begin{aligned}
(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} && \forall\, \boldsymbol{v} \in H_0^1(\Omega), \\
-(\nabla \cdot \boldsymbol{u}, q) &= 0 && \forall\, q \in L_0^2(\Omega).
\end{aligned} \tag{7}
$$

Show that this formulation is equalent to the following weak formulation: Given $\boldsymbol{f} \in H^{-1}(\Omega)$, find $(\boldsymbol{u}, p) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$
\begin{aligned}
(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} && \forall\, \boldsymbol{v} \in H_0^1(\Omega), \\
-(\nabla \cdot \boldsymbol{u}, q) + (p, 1)(q, 1) &= 0 && \forall\, q \in L^2(\Omega).
\end{aligned} \tag{8}
$$

The solutions of the exercise problems will be discussed in the afternoon class on February 07th, 2022.