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Numerical Mathematics IV

Exercise Sheet 02

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *SUPG method in 1D for P_1 finite elements and the model problem.* Consider $\Omega = (0,1)$ and the model problem

$$-\varepsilon u'' + u' = 1 \quad \text{in } \Omega, \quad u(0) = u(1) = 1.$$

Let Ω be decomposed by an equidistant grid of mesh widht $h = 1/N$.

   (a) Derive the concrete formulation of the SUPG method for the model problem.

   (b) Consider $\varepsilon = 10^{-8}$. Write a code for solving the model problem for $N \in \{8, 16, 32, 64, 128, 256, 512, 1024\}$. For the implementation, the concrete formulation from 1a can be used. Use as stabilization parameter $\delta_K = h$. Compute the errors in l^∞ of the nodes. Any language can be used.

   (c) Formulate this method as a finite difference method. How has the stabilization parameter to be chosen such that the upwind FDM and the IAS scheme are obtained?

   (d) Use the stabilization parameters that correspond to the finite difference methods in the code and perform the same numerical studies as in 1b.

2. *Connection of M-matrices to diagonally dominant matrices.* A matrix $A = (a_{ij})_{j=1,\ldots,n}^{i=1,\ldots,m}$ is said to be a Minkowski matrix or a matrix of non-negative type if it satisfies the conditions

$$a_{ij} \leq 0 \quad \forall\, i \neq j,\, i = 1,\ldots,m,\, j = 1,\ldots,n\,, \tag{1}$$

$$\sum_{j=1}^{n} a_{ij} \geq 0 \quad \forall\, i = 1,\ldots,m. \tag{2}$$

A Minkowski matrix is called a proper Minkowski matrix if all row sums are positive, i.e., the matrix is diagonally dominant.

Show the following statement: Each M-matrix $A \in \mathbb{R}^{n\times n}$ can be obtained from a proper Minkowski matrix $\tilde{A}$ by scaling each column of $\tilde{A}$ with an appropriate positive number.
Hint: consider the system $A\underline{x} = \underline{1}$ for an arbitrary M-matrix A, where $\underline{1}$ is a vector where all entries are 1.

3. *Estimating the $L^2(\Omega)$ norm of the divergence by the $L^2(\Omega)$ norm of the gradient for functions from $H_0^1(\Omega)$.* Let $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}), v_3(\boldsymbol{x}))^T$, $\boldsymbol{x} =$

$(x, y, z)^T$, be a vector field in a domain $\Omega \subset \mathbb{R}^3$ which is sufficiently regular. Then the rotation or the curl of $\boldsymbol{v}(\boldsymbol{x})$ is defined by

$$\nabla \times \boldsymbol{v}(\boldsymbol{x}) = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ v_1(\boldsymbol{x}) & v_2(\boldsymbol{x}) & v_3(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix} (\boldsymbol{x}). \qquad (3)$$

A vector field $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}))^T$, $\boldsymbol{x} = (x, y)^T$, in a two-dimensional domain Ω can be extended formally to a vector field with three values by $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}), 0)^T$. Then, the first two components in (3) vanish.

(a) Show that in both cases and for sufficiently smooth functions

$$\nabla \times (\nabla \times \boldsymbol{v})(\boldsymbol{x}) = -\Delta \boldsymbol{v}(\boldsymbol{x}) + \nabla (\nabla \cdot \boldsymbol{v})(\boldsymbol{x}), \qquad (4)$$

where a two-dimensional vector field is formally extended to a three-dimensional field.

(b) Using (4), show the following statement: Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and let $\boldsymbol{v} \in H_0^1(\Omega)$, then it holds

$$\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 = \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)}^2 + \|\nabla \times \boldsymbol{v}\|_{L^2(\Omega)}^2 \qquad (5)$$

and consequently

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega). \qquad (6)$$

Hint: use the integration by parts formula

$$(\nabla \times \boldsymbol{v}, \boldsymbol{\phi}) = (\boldsymbol{v}, \nabla \times \boldsymbol{\phi}) + \int_{\partial\Omega} ((\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{\phi})(\boldsymbol{s})\, d\boldsymbol{s} \quad \forall\, \boldsymbol{\phi} \in H^1(\Omega). \quad (7)$$

4. *The pair of finite element spaces P_1/P_0.* This pair of spaces approximates the velocity by a continuous piecewise linear function and the pressure by a piecewise constant function on simplicial grids. It is easily to implement and it has the favorable property that $\nabla \cdot V^h = \nabla \cdot P_1 = P_0 = Q^h$.

   Consider the two-dimensional domain $\Omega = (0, 1)^2$ and a decomposition of Ω in rectangular triangles. To this end, Ω is first decomposed in n^2 squares and then each square is decomposed into triangles by choosing an arbitrary diagonal. Consider a problem with Dirichlet boundary conditions, such that the degrees of freedom for the velocity are not situated on the boundary. Show that in this situation the pair P_1/P_0 does not satisfy the discrete inf-sup condition.

The solutions of the exercise problems will be discussed in the afternoon class on January 03rd, 2022.