Department for Mathematics and Computer Science
Freie Universität Berlin
Prof. Dr. V. John, john@wias-berlin.de

Berlin, 02.11.2021

Numerical Mathematics IV

Exercise Sheet 01

Attention: The approach for getting a solution has to be clearly presented. All statements have to be proved, auxiliary calculations have to be written down. Statements given in the lectures can be used without proof.

1. *Analytic solutions of two-point boundary value problems with negative reaction term.* Compute the solution $u(x)$ of the differential equation

$$-u'' + u' - u = 1,$$

with respect to the following intervals and boundary values:

$$\begin{array}{rclrclrcl} \Omega &=& \left(0, \frac{2\pi}{\sqrt{3}}\right) & u(0) &=& 0 & u\left(\frac{2\pi}{\sqrt{3}}\right) &=& 0, \\ \Omega &=& \left(0, \frac{2\pi}{\sqrt{3}}\right) & u(0) &=& 1 & u\left(\frac{2\pi}{\sqrt{3}}\right) &=& -2e^{\frac{\pi}{\sqrt{3}}} - 1, \\ \Omega &=& (0,1) & u(0) &=& 1 & u(1) &=& 1. \end{array}$$

2. *Central finite difference method vs. Galerkin finite element method for a two-point boundary value problem.* Let $\Omega = (0,1)$ and consider

$$-\varepsilon u'' + bu' = f \quad \text{in } \Omega, \quad u(0) = u(1) = 0.$$

Let the coefficients be constant, i.e., $b(x) = b$ and $f(x) = f$. Consider an equidistant decomposition of the interval in subintervals of length h.

   (a) Compute the equation that is obtained for the node i if one applies a finite difference scheme with the second order finite difference for the term u'' and the central finite difference for the term u'.

   (b) Compute the equation that is obtained for the node i if a finite element method with P_1 (piecewise linear and continuous) functions is applied.

   (c) Compare both equations.

3. *M-matrices: definition and simple properties.* A matrix $A = (a_{ij})_{i,j=1}^n$ is an M-matrix if:

   i) The off-diagonal entries are non-positive

$$a_{ij} \le 0, \quad i,j = 1,\ldots,n, \ i \ne j.$$

   ii) A is non-singular.

   iii) It holds $A^{-1} \ge 0$, i.e., all entries of A^{-1} are non-negative.

Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix. Prove the following statements:

   (a) It is $a_{ii} > 0$.

   (b) Show that $a_{ii}^{\text{inv}} > 0$, $i = 1,\ldots,n$, where a_{ii}^{inv} are the diagonal entries of A^{-1}.

(c) Construct an example which shows that the sum of two M-matrices is not necessarily an M-matrix.

4. *Equivalent formulation of the convective term.* Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be sufficiently smooth. Show that

$$\nabla \cdot \left(\boldsymbol{u}\boldsymbol{v}^T\right) = (\nabla \cdot \boldsymbol{v})\boldsymbol{u} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}.$$

Which equivalent form follows for the convective term of the Navier–Stokes equations?

5. *Estimating the $L^2(\Omega)$ norm of the divergence by the $L^2(\Omega)$ norm of the gradient for functions from $H^1(\Omega)$.* Prove the following statements: Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and let $\boldsymbol{v} \in H^1(\Omega)$, then it holds

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \sqrt{d}\,\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H^1(\Omega). \tag{1}$$

This estimate is sharp.

The solutions of the exercise problems will be discussed in the afternoon class on November 15th, 2021.