2. *SUPG method in 1D for P_1 finite elements and the model problem.* Consider $\Omega = (0,1)$ and the model problem

$$-\varepsilon u'' + u' = 1 \quad \text{in } \Omega, \quad u(0) = u(1) = 1.$$

Let Ω be decomposed by an equidistant grid of mesh widht $h = 1/N$.

a. Derive the concrete formulation of the SUPG method for the model problem. Presenting the simplified bilinear forms is sufficient, the presentation of the matrix entries is not necessary. Assume that the stabilization parameter δ_K is the same for each mesh cell ($\delta_K = \delta$).
b. Consider $\varepsilon = 10^{-8}$. Write a code for solving the model problem for $N \in \{8, 16, 32, 64, 128, 256, 512, 1024\}$. For the implementation, the concrete formulation from 2a can be used. Use as stabilization parameter $\delta_K = h$. Compute the errors in l^∞ of the nodes. Any language can be used.
c. Formulate this method as a finite difference method. How has the stabilization parameter to be chosen such that the simple upwind FDM and the IAS scheme are obtained?
d. Use the stabilization parameters that correspond to the finite difference methods in the code and perform the same numerical studies as in 2b.

Solution:

In the derivation of the formulas, an arbitrary constant convection b is considered.

a. Consider $\Omega = (0,1)$ and $V_h = P_1$ on an equidistant grid with $h_i = h$, $i = 1, \ldots, N$. Since all coefficients are constant, $\sigma = 0$, and the SUPG parameter is also constant, the left-hand side of the SUPG method reduces to

$$\begin{aligned}
&\varepsilon(\left(u^h\right)', \left(v^h\right)') + (b\left(u^h\right)', v^h) \\
&\quad + \sum_{i=1}^{N} \delta \int_{x_{i-1}}^{x_i} \Big(-\varepsilon \cdot 0 + b\left(u^h\right)'(x)\Big)\Big(b\left(v^h\right)'(x)\Big)\, dx \\
&= \varepsilon\left(\left(u^h\right)', \left(v^h\right)'\right) + b\left(\left(u^h\right)', v^h\right) + \delta b^2\left(\left(u^h\right)', \left(v^h\right)'\right).
\end{aligned} \tag{1.3}$$

The right-hand side of the SUPG method is

$$\begin{aligned}
(f, v^h) + \sum_{i=1}^{N} \delta \int_{x_{i-1}}^{x_i} f b\left(v^h\right)'(x)\, dx &= (f, v^h) + \delta f b \underbrace{\sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left(v^h\right)'(x)\, dx}_{=0} \\
&= (f, v^h) = h f_i.
\end{aligned}$$

The sum vanishes, since each test function $\left(v^h\right)'(x)$ can be written as a linear combination of the basis functions $\{\phi_i(x)\}$ of P_1 and the integral of the derivative of each basis function vanishes. Alternatively, one can apply integration by parts to check this fact.

b. See 2d, results for 'SUPG standard'.
c. Expression (1.3) is of the form of a Galerkin finite element method for an equation with left-hand side

$$-\left(\varepsilon + \delta b^2\right) u''(x) + b u'(x).$$

It is known from Exercise Problem 1 that the Galerkin finite element method is in this case equivalent to a central finite difference scheme.
Altogether, the SUPG method with the conditions stated above is equivalent to the fitted finite difference scheme

$$-\varepsilon\left(1+\delta\frac{b^2}{\varepsilon}\right)D^+D^-u_i + bD^0u_i, = f_i,$$

i.e., $\kappa(\mathrm{Pe}) = 1+\delta b^2/\varepsilon = 1+\frac{2\delta b}{h}\mathrm{Pe}$ with $\mathrm{Pe} = bh/(2\varepsilon)$.
With $\delta = h/(2b)$, one gets the simple upwind scheme. Choosing the SUPG parameter by

$$\delta(\mathrm{Pe}) = \frac{h}{2b}\left(\coth(\mathrm{Pe}) - \frac{1}{\mathrm{Pe}}\right),$$

then it is

$$\kappa(\mathrm{Pe}) = 1 + \frac{hb^2}{2b\varepsilon}\left(\coth(\mathrm{Pe}) - \frac{1}{\mathrm{Pe}}\right) = 1 + \mathrm{Pe}\left(\coth(\mathrm{Pe}) - \frac{1}{\mathrm{Pe}}\right) = \mathrm{Pe}\coth(\mathrm{Pe}).$$

One obtains the Iljin–Allen–Southwell scheme.

d. The results computed by the code below are as follows:

```
Galerkin n =  8     781249.8750002801
Galerkin n =  16     195312.43750140007
Galerkin n =  32     48828.09375620007
Galerkin n =  64     12207.015651039968
Galerkin n =  128     3051.750106679735
Galerkin n =  256     762.9359786727381
Galerkin n =  512     190.73464753525425
Galerkin n =  1024     47.68970895011962
SUPG standard  n =  8     0.3332317427913126
SUPG standard  n =  16     0.3333333889573362
SUPG standard  n =  32     0.33333347555552517
SUPG standard  n =  64     0.33333361777765713
SUPG standard  n =  128     0.3333339022217401
SUPG standard  n =  256     0.33333447110917
SUPG standard  n =  512     0.33333560888112224
SUPG standard  n =  1024     0.3333378844134236
SUPG upwind  n =  8     7.999999351859799e-08
SUPG upwind  n =  16     1.5999997549087652e-07
SUPG upwind  n =  32     3.1999989769104786e-07
SUPG upwind  n =  64     6.399995903239031e-07
SUPG upwind  n =  128     1.2799983617473032e-06
SUPG upwind  n =  256     2.559993446560327e-06
SUPG upwind  n =  512     5.1199737856055805e-06
SUPG upwind  n =  1024     1.0239895256280995e-05
SUPG IAS  n =  8     0.0
SUPG IAS  n =  16     7.771561172376096e-16
SUPG IAS  n =  32     0.0
SUPG IAS  n =  64     0.0
SUPG IAS  n =  128     0.0
SUPG IAS  n =  256     0.0
SUPG IAS  n =  512     0.0
SUPG IAS  n =  1024     0.0
```

```python
import numpy as np
```

2. *Connection of M-matrices to diagonally dominant matrices.* A matrix $A = (a_{ij})_{j=1,\ldots,n}^{i=1,\ldots,m}$ is said to be a Minkowski matrix or a matrix of non-negative type if it satisfies the conditions

$$a_{ij} \leq 0 \quad \forall\, i \neq j,\, i = 1,\ldots,m,\, j = 1,\ldots,n\,, \tag{1.5}$$

$$\sum_{j=1}^{n} a_{ij} \geq 0 \quad \forall\, i = 1,\ldots,m. \tag{1.6}$$

A Minkowski matrix is called a proper Minkowski matrix if all row sums are positive, i.e., the matrix is diagonally dominant.
Show the following statement: Each M-matrix $A \in \mathbb{R}^{n\times n}$ can be obtained from a proper Minkowski matrix $\tilde{A}$ by scaling each column of $\tilde{A}$ with an appropriate positive number.
Hint: consider the system $A\underline{x} = \underline{1}$ for an arbitrary M-matrix A, where $\underline{1}$ is a vector where all entries are 1.

Solution:
Consider the system $A\underline{x} = \underline{1}$ for an arbitrary M-matrix A. Applying A^{-1}, one finds that the solution of this system is

$$x_i = \sum_{j=1}^{n} a_{ij}^{\text{inv}} > 0,$$

since $A^{-1} \geq 0$ and $a_{ii}^{\text{inv}} > 0$ by Exercise 1. Consider now the matrix $\tilde{A}$ that is obtained by multiplying the jth column of A with x_j, $j = 1,\ldots,n$. These multiplications do not change the signs, hence the properties of a Minkowski matrix with respect to the signs of the entries are satisfied for $\tilde{A}$. By construction, it holds for the row sums

$$\sum_{j=1}^{n} \tilde{a}_{ij} = \sum_{j=1}^{n} a_{ij} x_j = 1.$$

Consequently, the property of a proper Minkowski matrix with respect to the row sums is satisfied and $\tilde{A}$ is such a matrix. In turn, the M-matrix A can be obtained by the proper Minkowski matrix $\tilde{A}$ by multiplying the jth column of $\tilde{A}$ with $1/x_j$, $j = 1,\ldots,n$.
$\square$

3. *Estimating the $L^2(\Omega)$ norm of the divergence by the $L^2(\Omega)$ norm of the gradient for functions from $H_0^1(\Omega)$.* Let $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}), v_3(\boldsymbol{x}))^T$, $\boldsymbol{x} = (x, y, z)^T$, be a vector field in a domain $\Omega \subset \mathbb{R}^3$ which is sufficiently regular. Then the rotation or the curl of $\boldsymbol{v}(\boldsymbol{x})$ is defined by

$$\nabla \times \boldsymbol{v}(\boldsymbol{x}) = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ v_1(\boldsymbol{x}) & v_2(\boldsymbol{x}) & v_3(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix} (\boldsymbol{x}). \tag{2.2}$$

A vector field $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}))^T$, $\boldsymbol{x} = (x, y)^T$, in a two-dimensional domain Ω can be extended formally to a vector field with three values by $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}), 0)^T$. Then, the first two components in (2.2) vanish.

a. Show that in both cases and for sufficiently smooth functions

$$\nabla \times (\nabla \times \boldsymbol{v})(\boldsymbol{x}) = -\Delta \boldsymbol{v}(\boldsymbol{x}) + \nabla (\nabla \cdot \boldsymbol{v})(\boldsymbol{x}), \tag{2.3}$$

where a two-dimensional vector field is formally extended to a three-dimensional field.

b. Using (2.3), show the following statement: Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and let $\boldsymbol{v} \in H_0^1(\Omega)$, then it holds

$$\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}^2 = \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)}^2 + \|\nabla \times \boldsymbol{v}\|_{L^2(\Omega)}^2 \tag{2.4}$$

and consequently

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega). \tag{2.5}$$

Hint: use the integration by parts formula

$$(\nabla \times \boldsymbol{v}, \boldsymbol{\phi}) = (\boldsymbol{v}, \nabla \times \boldsymbol{\phi}) + \int_{\partial\Omega} ((\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{\phi})(\boldsymbol{s})\, d\boldsymbol{s} \quad \forall\, \boldsymbol{\phi} \in H^1(\Omega). \tag{2.6}$$

Solution:

a. A direct calculation, using the Theorem of Schwarz, gives

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{v})(\boldsymbol{x}) &= \nabla \times \begin{pmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix} = \begin{pmatrix} \partial_y (\partial_x v_2 - \partial_y v_1) - \partial_z (\partial_z v_1 - \partial_x v_3) \\ \partial_z (\partial_y v_3 - \partial_z v_2) - \partial_x (\partial_x v_2 - \partial_y v_1) \\ \partial_x (\partial_z v_1 - \partial_x v_3) - \partial_y (\partial_y v_3 - \partial_z v_2) \end{pmatrix} \\ &= \begin{pmatrix} -\partial_{yy} v_1 - \partial_{zz} v_1 + \partial_x (\partial_y v_2 + \partial_z v_3) \\ -\partial_{xx} v_2 - \partial_{zz} v_2 + \partial_y (\partial_x v_1 + \partial_z v_3) \\ -\partial_{xx} v_3 - \partial_{yy} v_3 + \partial_z (\partial_x v_1 + \partial_y v_2) \end{pmatrix} = -\Delta \boldsymbol{v} + \nabla (\nabla \cdot \boldsymbol{v}). \end{aligned}$$

b. The proof is based on the identity (2.3). Considering $\boldsymbol{v} \in H_0^1(\Omega)$, this identity can be transformed into a weak form by multiplication with a test function $\boldsymbol{w} \in H_0^1(\Omega)$ and applying integration by parts

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{w}) = (\nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{w}) + (\nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{w}) \quad \forall\, \boldsymbol{w} \in H_0^1(\Omega). \tag{2.7}$$

The derivation of the first two terms is standard. For deriving the last term, the integration by parts formula (2.6) can be applied. However, the derivation can be also checked with a straightforward calculation. Using (2.2) gives

$$\nabla \times \nabla \times \boldsymbol{v} = \begin{pmatrix} \partial_y \left(\partial_x v_2 - \partial_y v_1\right) - \partial_z \left(\partial_z v_1 - \partial_x v_3\right) \\ \partial_z \left(\partial_y v_3 - \partial_z v_2\right) - \partial_x \left(\partial_x v_2 - \partial_y v_1\right) \\ \partial_x \left(\partial_z v_1 - \partial_x v_3\right) - \partial_y \left(\partial_y v_3 - \partial_z v_2\right) \end{pmatrix}.$$

Applying integration by parts, boundary integrals will not appear for test functions $\boldsymbol{w} \in H_0^1(\Omega)$. One obtains, considering the individual terms,

$$\begin{aligned}
(\nabla \times \nabla \times \boldsymbol{v}, \boldsymbol{w}) &= -\left(\partial_x v_2 - \partial_y v_1, \partial_y w_1\right) + \left(\partial_z v_1 - \partial_x v_3, \partial_z w_1\right) - \left(\partial_y v_3 - \partial_z v_2, \partial_z w_2\right) \\
&\quad + \left(\partial_x v_2 - \partial_y v_1, \partial_x w_2\right) - \left(\partial_z v_1 - \partial_x v_3, \partial_x w_3\right) + \left(\partial_y v_3 - \partial_z v_2, \partial_y w_3\right) \\
&= \left(\partial_y v_3 - \partial_z v_2, \partial_y w_3 - \partial_z w_2\right) + \left(\partial_z v_1 - \partial_x v_3, \partial_z w_1 - \partial_x w_3\right) \\
&\quad + \left(\partial_x v_2 - \partial_y v_1, \partial_x w_2 - \partial_y w_1\right) \\
&= \left(\nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{w}\right).
\end{aligned}$$

Inserting now $\boldsymbol{w} = \boldsymbol{v} \in H_0^1(\Omega)$ in (2.7) gives (2.4).

$\square$

4. *The pair of finite element spaces* P_1/P_0. This pair of spaces approximates the velocity by a continuous piecewise linear function and the pressure by a piecewise constant function on simplicial grids. It is easily to implement and it has the favorable property that $\nabla \cdot V^h = \nabla \cdot P_1 = P_0 = Q^h$. Consider the two-dimensional domain $\Omega = (0,1)^2$ and a decomposition of Ω in rectangular triangles. To this end, Ω is first decomposed in n^2 squares and then each square is decomposed into triangles by choosing an arbitrary diagonal. Consider a problem with Dirichlet boundary conditions, such that the degrees of freedom for the velocity are not situated on the boundary. Show that in this situation the pair P_1/P_0 does not satisfy the discrete inf-sup condition.

Solution:
The dimension of V^h corresponds to twice the number of interior vertices, $\dim\left(V^h\right) = 2\left(n-1\right)^2$. The dimension of Q^h corresponds to the number of mesh cells reduced by one (because of the integral mean value condition), i.e., $\dim\left(Q^h\right) = 2n^2 - 1$. It follows that

$$\dim\left(V^h\right) - \dim\left(Q^h\right) = 2n^2 - 4n + 2 - 2n^2 + 1 = 3 - 4n < 0$$

for $n \geq 1$. Hence, the necessary condition for the unique solvability of the discrete system, $\dim\left(V^h\right) \geq \dim\left(Q^h\right)$, see Remark 2.7 of the lecture notes, is not satisfied. $\square$