Chapter 2
Finite Element Spaces for Linear Saddle Point Problems

Remark 2.1. Motivation. This chapter deals with the first difficulty inherent to the incompressible Navier–Stokes equations, see Remark 1.10, namely the coupling of velocity and pressure. The characteristic feature of this coupling is the absence of a pressure contribution in the continuity equation. In fact, the continuity equation can be considered as a constraint for the velocity and the pressure in the momentum equation as a Lagrangian multiplier. This kind of coupling is called saddle point problem.

Appropriate finite element spaces for velocity and pressure have to satisfy the so-called discrete inf-sup condition. This condition is derived on the basis of the theory for an abstract linear saddle point problem. Techniques for proving the discrete inf-sup condition will be presented briefly and applied for concrete pairs of finite element spaces for velocity and pressure.

All special cases of models for incompressible flow problems given in Remark 1.12 possess the same coupling of velocity and pressure, in particular the linear model of the Stokes equations. Linear problems are also of interest in the numerical simulation of the Navier–Stokes equations. After having discretized these equations implicitly in time, a nonlinear saddle point problem has to be solved in each discrete time. The solution of this problem is performed iteratively, requiring in each iteration step the solution of a linear saddle point problem for velocity and pressure. These linear saddle point problems will be discretized with finite element spaces. The existence and uniqueness of a solution of these discrete linear problems is crucial for performing the iteration. Altogether, the theory of linear saddle problems plays an essential role for the theory of all models for incompressible flows from Chapter 1.

A comprehensive presentation of the theory of linear saddle point problems can be found in the monograph Boffi *et al.* (2013). $\square$

2.1 Existence and Uniqueness of a Solution of an Abstract Linear Saddle Point Problem

Remark 2.2. Contents. This section presents an abstract framework for studying the existence and uniqueness of solutions of those types of linear saddle point problems which are of interest for incompressible flow problems. The presentation follows (Girault & Raviart, 1986, Chapter I, § 4). □

Remark 2.3. Abstract linear saddle point problem. Let V and Q be two real Hilbert spaces with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$ and with induced norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Their corresponding dual spaces are given by V' and Q', with the dual pairing denoted by $\langle\cdot,\cdot\rangle_{V',V}$ and $\langle\cdot,\cdot\rangle_{Q',Q}$. The norms of the dual spaces are defined in the usual way by

$$\|\phi\|_{V'} := \sup_{v\in V, v\neq 0} \frac{\langle\phi, v\rangle_{V',V}}{\|v\|_V}, \quad \|\psi\|_{Q'} := \sup_{q\in Q, q\neq 0} \frac{\langle\psi, q\rangle_{Q',Q}}{\|q\|_Q}. \tag{2.1}$$

Two continuous bilinear forms are considered

$$a(\cdot,\cdot) \ : \ V\times V \to \mathbb{R}, \quad b(\cdot,\cdot) \ : \ V\times Q\to\mathbb{R}, \tag{2.2}$$

with the usual definition of their norms

$$\|a\| = \sup_{v,w\in V, v,w\neq 0} \frac{a(v,w)}{\|v\|_V \, \|w\|_V}, \quad \|b\| = \sup_{v\in V, q\in Q, v,q\neq 0} \frac{b(v,q)}{\|v\|_V \, \|q\|_Q}. \tag{2.3}$$

The following problem is studied: Find $(u,p)\in V\times Q$ such that for given $(f,r)\in V'\times Q'$

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle_{V',V} \quad \forall\, v\in V,\\ b(u,q) &= \langle r, q\rangle_{Q',Q} \quad \forall\, q\in Q. \end{aligned} \tag{2.4}$$

System (2.4) is called linear saddle point problem. Concrete choices of the spaces and bilinear forms for incompressible flow problems are discussed in Section 2.2. □

Remark 2.4. Operator form of the linear saddle point problem. Problem (2.4) can be transformed into an equivalent form using operators instead of bilinear forms. Linear operators can be defined which are associated with the bilinear forms given in (2.2):

$$\begin{aligned} &A\in\mathcal{L}(V,V') \ \text{ defined by } \ \langle Au, v\rangle_{V',V} = a(u,v) \quad \forall\, u,v\in V,\\ &B\in\mathcal{L}(V,Q') \ \text{ defined by } \ \langle Bu, q\rangle_{Q',Q} = b(u,q) \quad \forall\, u\in V, \ \forall\, q\in Q. \end{aligned}$$

Using the definition of the norms of the dual spaces (2.1), the norms of the operators are given by

$$\|Av\|_{V'} = \sup_{w\in V, w\neq 0} \frac{\langle Av, w\rangle_{V',V}}{\|w\|_V} \quad \Longrightarrow$$
$$\|A\|_{\mathcal{L}(V,V')} = \sup_{v\in V, v\neq 0} \frac{\|Av\|_{V'}}{\|v\|_V} = \sup_{v,w\in V, v,w\neq 0} \frac{a(v,w)}{\|v\|_V \, \|w\|_V} = \|a\| ,$$

and analogously

$$\|B\|_{\mathcal{L}(V,Q')} = \|b\| .$$

Let $B' \in \mathcal{L}(Q, V')$ be the adjoint (dual) operator of B defined by

$$\langle B'q, v\rangle_{V',V} = \langle Bv, q\rangle_{Q',Q} = b(v,q) \quad \forall\, v \in V,\ \forall\, q \in Q.$$

With these operators, Problem (2.4) can be written in the equivalent form: Find $(u,p) \in V \times Q$ such that

$$\begin{aligned} Au + B'p &= f \ \text{ in } V', \\ Bu &= r \ \text{ in } Q'. \end{aligned} \tag{2.5}$$

□

Definition 2.5. Well-posedness of Problem (2.5)**.** Let

$$\Phi \in \mathcal{L}\left(V \times Q, V' \times Q'\right) \ : \ \Phi(v,q) = (Av + B'q, Bv)$$

be a linear operator, where $(\cdot,\cdot)$ denotes a vector with two components. Problem (2.5) is said to be well-posed if $\Phi(\cdot,\cdot)$ is an isomorphism from $V \times Q$ onto $V' \times Q'$. □

Remark 2.6. On Definition 2.5. Definition 2.5 means that Problem (2.5) possesses for all possible right-hand sides a unique solution. The purpose of the following studies consists in deriving necessary and sufficient conditions for (2.5) to be well-posed. □

Remark 2.7. The finite-dimensional case. Consider for the moment that V and Q are finite-dimensional spaces of dimension n_V and n_Q, respectively. Then, the operators in (2.5) can be represented with matrices, with $B' = B^T$, and the functions with vectors. The well-posedness of (2.5) means that the linear system of equations

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{r} \end{pmatrix}, \quad \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \in \mathbb{R}^{(n_V+n_Q)\times(n_V+n_Q)}, \tag{2.6}$$

has a unique solution for all right-hand sides or, equivalently, that the system matrix is non-singular. Here, conditions will be derived such that this property is given. These considerations should provide an idea of the kind of conditions to be expected in the general case.

Separate consideration of velocity and pressure. A possible way to solve (2.6) starts by solving the first equation of (2.6) for $\underline{u}$

$$\underline{u} = A^{-1}\left(\underline{f} - B^T\underline{p}\right). \tag{2.7}$$

Inserting this expression in the second equation gives

$$\left(BA^{-1}B^T\right)\underline{p} = BA^{-1}\underline{f} - \underline{r}. \tag{2.8}$$

If (2.8) possesses a unique solution $\underline{p}$, this solution can be inserted in (2.7) and a unique solution $\underline{u}$ is obtained, too. This way to compute a unique solution works if

- $A \;:\; V \to V'$ is an isomorphism, i.e., A is non-singular,
- $BA^{-1}B^T \;:\; Q \to Q'$ is an isomorphism, i.e., $BA^{-1}B^T$ is non-singular.

Let $\underline{p}$ be a solution of (2.8). Then, also $\underline{p}+\tilde{\underline{p}}$ with $\tilde{\underline{p}} \in \ker\left(B^T\right)$ is a solution of (2.8). Thus, for $BA^{-1}B^T$ to be non-singular, it is necessary that $\ker\left(B^T\right) = \{\underline{0}\}$ or equivalently that $B^T \;:\; Q \to V' = V$ is injective. With a similar argument, one finds that B must be injective on the range of $A^{-1}B^T$, i.e., $\ker(B) \cap \operatorname{range}\left(A^{-1}B^T\right) = \{\underline{0}\}$.

Joint consideration of velocity and pressure. One can also consider the system matrix (2.6) as a whole. A first necessary condition for the matrix to be non-singular is $n_Q \le n_V$, since the last rows of the system matrix span a space of dimension at most n_V (only the first n_V entries of these rows might be non-zero). Assume that A is non-singular, then the system matrix is non-singular if and only if B has full rank, i.e., $\operatorname{rank}(B) = n_Q$. It will be shown now that $\operatorname{rank}(B) = n_Q$ if and only if

$$\inf_{\underline{q}\in\mathbb{R}^{n_Q},\underline{q}\neq\underline{0}} \sup_{\underline{v}\in\mathbb{R}^{n_V},\underline{v}\neq\underline{0}} \frac{\underline{v}^T B^T \underline{q}}{\|\underline{v}\|_2 \left\|\underline{q}\right\|_2} \ge \beta > 0. \tag{2.9}$$

Let (2.9) be satisfied and let $\operatorname{rank}(B) < n_Q$. Then, there is a $\underline{q} \in \mathbb{R}^{n_Q}$, $\underline{q} \neq \underline{0}$, such that $\underline{q} \in \ker\left(B^T\right)$, i.e., $B^T\underline{q} = \underline{0}$. For this vector, it is $\underline{v}^T B^T \underline{q} = 0$ for all $\underline{v} \in \mathbb{R}^{n_V}$ such that the supremum of (2.9) is zero and (2.9) cannot be satisfied. This result is a contradiction and hence $\operatorname{rank}(B) = n_Q$.

On the other hand, let $\operatorname{rank}(B) = n_Q$. Then, for each $\underline{q} \in \mathbb{R}^{n_Q}$, $\underline{q} \neq \underline{0}$, one has that $B^T\underline{q} \neq \underline{0}$ with $B^T\underline{q} \in \mathbb{R}^{n_V}$. Choosing $\underline{v} = B^T\underline{q}$ gives

$$\begin{aligned}\inf_{\underline{q}\in\mathbb{R}^{n_Q},\underline{q}\neq\underline{0}} \sup_{\underline{v}\in\mathbb{R}^{n_V},\underline{v}\neq\underline{0}} \frac{\underline{v}^T B^T \underline{q}}{\|\underline{v}\|_2 \left\|\underline{q}\right\|_2} &\ge \inf_{\underline{q}\in\mathbb{R}^{n_Q},\underline{q}\neq\underline{0}} \frac{\left\|B^T\underline{q}\right\|_2^2}{\left\|B^T\underline{q}\right\|_2 \left\|\underline{q}\right\|_2} \\ &= \inf_{\underline{q}\in\mathbb{R}^{n_Q},\underline{q}\neq\underline{0}} \frac{\left\|B^T\underline{q}\right\|_2}{\left\|\underline{q}\right\|_2}.\end{aligned} \tag{2.10}$$

It is

$$\frac{\left\|B^T\underline{q}\right\|_2^2}{\left\|\underline{q}\right\|_2^2} = \frac{\underline{q}^T B B^T \underline{q}}{\underline{q}^T\underline{q}}.$$

This expression is a Rayleigh quotient and it is known that

$$\inf_{\underline{q}\in\mathbb{R}^{n_Q},\underline{q}\neq\underline{0}} \frac{\underline{q}^T BB^T \underline{q}}{\underline{q}^T \underline{q}} = \lambda_{\min}\left(BB^T\right),$$

where $\lambda_{\min}\left(BB^T\right)$ is the smallest eigenvalue of BB^T, see Lemma A.19. Since B was assumed to have full rank, one has $\lambda_{\min}\left(BB^T\right) > 0$ and hence with (2.10)

$$\inf_{\underline{q}\in\mathbb{R}^{n_Q},\underline{q}\neq\underline{0}} \sup_{\underline{v}\in\mathbb{R}^{n_V},\underline{v}\neq\underline{0}} \frac{\underline{v}^T B^T \underline{q}}{\|\underline{v}\|_2 \|\underline{q}\|_2} \geq \lambda_{\min}^{1/2}\left(BB^T\right) > 0.$$

Altogether, under the assumption that

- A is non-singular, i.e., $A : V \to V'$ is an isomorphism,
- (2.9) is satisfied,

the system matrix (2.6) is non-singular.

The result presented here just states that the given problem has a unique solution because (2.9) is satisfied. In the finite element theory it turns out that there is another important aspect to study, namely the dependency of β on the dimension of the finite element spaces. To obtain optimal orders of convergence, β has to be independent of the dimension, e.g., compare Remark 3.26. This aspect can also be taken into account in the matrix-vector formulation of linear saddle point problems.Then, one has to solve a generalized eigenvalue problem.

It turns out that one gets similar conditions in the general case. Whether or not these conditions are satisfied depends finally on the spaces V and Q. $\square$

Remark 2.8. A manifold and a subspace in V. A manifold of V will be defined that contains all elements which fulfill the second equation of (2.5)

$$V(r) = \{v \in V \ : \ Bv = r\}, \quad V_0 := V(0) = \ker(B).$$

The manifold V_0 is even a subspace of V. From Hilbert space theory, it follows that there is an orthogonal decomposition, with respect to the inner product of V,

$$V = V_0^\perp \oplus V_0,$$

where $V_0^\perp$ is the orthogonal complement of V_0. $\square$

Lemma 2.9. Properties of V_0 and $V_0^\perp$. *The spaces V_0 and $V_0^\perp$ are closed subspaces of V.*

Proof. First, the closeness of V_0 will be proved. Let $\{v_n\}_{n=1}^\infty$ be an arbitrary Cauchy sequence with $v_n \in V_0$ for all n. Since V is complete, there exists a $v \in V$ with $\lim_{n\to\infty} v_n = v$. One has to show that $v \in V_0$. By the continuity of the linear operator B, it follows that

$$Bv = B\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} (Bv_n) = \lim_{n\to\infty} 0 = 0.$$

Hence $v \in V_0$ and V_0 is closed.

The closeness of $V_0^\perp$ follows from the fact that the orthogonal complement of every subspace is closed, see Lemma A.17. ■

Remark 2.10. Functionals vanishing on V_0. A subset of V' is defined for the following analysis:

$$\tilde{V}' = \{\phi \in V' \ : \ \langle \phi, v\rangle_{V',V} = 0 \ \ \forall\, v \in V_0\} \subset V'. \tag{2.11}$$

This subset, which is even a closed subspace of V', contains all linear functionals on V that vanish for all $v \in V_0 = \ker(B)$. □

Remark 2.11. Reduction of the system to a single equation in a subspace. In the next step, the following problem is associated with Problems (2.4) and (2.5): Find $u \in V(r)$ such that

$$a(u, v) = \langle f, v\rangle_{V',V} \quad \forall\, v \in V_0. \tag{2.12}$$

Clearly, if $(u, p) \in V \times Q$ is a solution of (2.4) or (2.5), then $u \in V(r)$. In addition, one obtains

$$\langle B'p, v\rangle_{V',V} = \langle Bv, p\rangle_{Q',Q} = b(v, p) = 0 \quad \forall\, v \in V_0. \tag{2.13}$$

Since the first equation of (2.4) holds for all $v \in V$, it holds in particular for all $v \in V_0$. With (2.13) it follows that u is a solution of (2.12).

The aim of the analysis consists now in finding conditions to ensure that the converse of this statement holds: if $u \in V(r)$ is a solution of (2.12), one can find a unique $p \in Q$ such that (u, p) is the unique solution of (2.4) or (2.5), respectively. □

Lemma 2.12. The inf-sup condition. *The three following properties are equivalent:*

i) There exists a constant $\beta_{\mathrm{is}} > 0$ *such that*

$$\inf_{q\in Q, q\neq 0} \sup_{v\in V, v\neq 0} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \geq \beta_{\mathrm{is}}. \tag{2.14}$$

ii) The operator B' *is an isomorphism from* Q *onto* $\tilde{V}'$ *and*

$$\|B'q\|_{V'} \geq \beta_{\mathrm{is}} \|q\|_Q \quad \forall\, q \in Q. \tag{2.15}$$

iii) The operator B *is an isomorphism from* $V_0^\perp$ *onto* Q' *and*

$$\|Bv\|_{Q'} \geq \beta_{\mathrm{is}} \|v\|_V \quad \forall\, v \in V_0^\perp. \tag{2.16}$$

Proof. For the proof, it is referred to Girault & Raviart (1986) or (John, 2016, Lemma 2.12). ■

Remark 2.13. Well-posedness of Problem (2.5). It is possible to derive a sufficient and necessary condition for the well-posedness of problem (2.5), e.g., see (John, 2016, Theorem 2.18). However, from the practical point of view, the following sufficient condition is more important. The relaxation in comparison with the necessary condition is with respect to the assumptions on the bilinear form $a(\cdot,\cdot)$. □

Lemma 2.14. Sufficient condition for the well-posedness of (2.5)**.** *Assume that the bilinear form $a(\cdot,\cdot)$ is V_0-elliptic, i.e., there is a constant $\alpha > 0$ such that*

$$a(v,v) \ge \alpha \|v\|_V^2 \quad \forall\, v \in V_0.$$

Then, Problem (2.5) *is well-posed if and only if the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition* (2.14)*.*

Proof. See (John, 2016, Lemma 2.19). ■

Remark 2.15. Formulation as an optimization problem, saddle point problem. Problems (2.5) and (2.12) can formulated as optimization problems under certain conditions. Let $J_0 : V \to \mathbb{R}$ and $J_1 : V \times Q \to \mathbb{R}$ be two quadratic functionals defined by

$$J_0(v) = \frac{1}{2} a(v,v) - \langle f, v\rangle_{V',V}, \quad J_1(v,q) = J_0(v) + b(v,q) - \langle r, q\rangle_{Q',Q}.$$

The functional J_0 is called energy functional associated with Problem (2.12) and J_1 is the Lagrangian functional associated with Problem (2.5).

Consider the following problem: Find a saddle point $(u,p) \in V \times Q$ of the Lagrangian functional J_1 over $V \times Q$, i.e., find a pair $(u,p) \in V \times Q$ such that

$$J_1(u,q) \le J_1(u,p) \le J_1(v,p) \quad \forall\, v \in V,\ \forall\, q \in Q. \tag{2.17}$$

This form is the classical formulation of a saddle point problem. The characterization (2.17) inspired the notation saddle point problem also for Problem (2.5). □

Theorem 2.16. Existence and uniqueness of a solution of (2.17)**.** *Assume the conditions of Lemma 2.14. Assume in addition that the bilinear form $a(\cdot,\cdot)$ is symmetric. Then, Problem* (2.17) *has a unique solution $(u,p) \in V \times Q$ that is precisely the solution of Problem* (2.5)*.*

Proof. It is referred to (Girault & Raviart, 1986, p. 62) for the proof. ■

2.2 Appropriate Function Spaces for Continuous Incompressible Flow Problems

Remark 2.17. Contents. The theory of Section 2.1 will now be applied to characterize appropriate function spaces for weak formulations of incompressible flow problems. Lemma 2.14 gives two conditions for the well-posedness of the linear saddle point problem. One condition concerns only the space V. It will be discussed for the individual incompressible flow models later, e.g., see Theorem 3.6 for the Stokes equations. The emphasis of this section is on the second condition, which establishes a connection between the spaces V and Q. These spaces have to satisfy the inf-sup condition (2.14). Note that the inf-sup condition guarantees the uniqueness of the pressure. □

Remark 2.18. The bilinear form $b(\cdot,\cdot)$ for incompressible flow problems. In the inf-sup condition (2.14), the velocity and pressure space are coupled by a bilinear form. A weak formulation of incompressible flow problems is obtained in the usual way by multiplying the momentum equation with a test function $\boldsymbol{v} \in V$ and the continuity equation with a test function $q \in Q$. Then, both equations are integrated on Ω. One obtains for the continuity equation

$$\int_\Omega (\nabla \cdot \boldsymbol{u})\, q \; d\boldsymbol{x} = (\nabla \cdot \boldsymbol{u}, q) = 0.$$

For the viscous term and the pressure term in the continuity equation, integration by parts is applied. Assuming that the functions are sufficiently smooth and that the integral on the boundary vanishes in performing the integration by parts, one gets the term

$$\int_\Omega \nabla p \cdot \boldsymbol{v} \; d\boldsymbol{x} = -\int_\Omega (\nabla \cdot \boldsymbol{v})\, p \; d\boldsymbol{x} = -(\nabla \cdot \boldsymbol{v}, p). \tag{2.18}$$

Thus, the framework of Section 2.1 can be used if one defines

$$b(\boldsymbol{v}, q) = -\int_\Omega (\nabla \cdot \boldsymbol{v})\, q \; d\boldsymbol{x} = -(\nabla \cdot \boldsymbol{v}, q) \quad \boldsymbol{v} \in V,\ q \in Q. \tag{2.19}$$

□

Remark 2.19. Function spaces for velocity and pressure for homogeneous Dirichlet boundary conditions. Let Ω be a bounded and connected domain in $\mathbb{R}^d$, $d \in \{2,3\}$, with Lipschitz boundary. To simplify the presentation, only problems with Dirichlet boundary conditions on the whole boundary will be considered. Since these are essential boundary conditions, they enter the definition of the velocity space. Define

$$V = H_0^1(\Omega) = \left\{\boldsymbol{v} \ : \ \boldsymbol{v} \in H^1(\Omega) \text{ with } \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\right\},$$

where the value of $\boldsymbol{v}$ on the boundary is to be understood in the sense of traces, and

$$Q = L_0^2(\Omega) = \left\{ q \ : \ q \in L^2(\Omega) \text{ with } \int_\Omega q(\boldsymbol{x}) \; d\boldsymbol{x} = 0 \right\}.$$

Both spaces are Hilbert spaces. The inner product in V and the induced norm are given by

$$(\boldsymbol{v}, \boldsymbol{w}) = \int_{\Omega} (\nabla \boldsymbol{v} : \nabla \boldsymbol{w}) (\boldsymbol{x}) \ d\boldsymbol{x}, \quad \|\boldsymbol{v}\|_V = \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} . \tag{2.20}$$

Poincaré's inequality (A.12) shows that (2.20) defines in fact an inner product and a norm in V. The inner product and the induced norm in Q are given by

$$(q, r) = \int_{\Omega} (qr)(\boldsymbol{x}) \ d\boldsymbol{x}, \quad \|q\|_Q = \|q\|_{L^2(\Omega)} .$$

The dual space of V is $V' = H^{-1}(\Omega)$ and the dual of the pressure space is $Q' = Q$.

For $\boldsymbol{v} \in V$, it follows that $\nabla \boldsymbol{v} \in L^2(\Omega)$ and with estimate (2.26) proved below, one obtains that $\nabla \cdot \boldsymbol{v} \in L^2(\Omega)$. Thus, the definition of the spaces implies that all terms in (2.18) are well defined and that this equality holds. □

Remark 2.20. Notation for spaces of vector-valued and tensor-valued functions. For simplicity of notation, spaces of vector-valued or tensor-valued are denoted with the same symbol as the corresponding space for scalar functions. This notation has to be understood in the sense that each component of the vector-valued or tensor-valued function belongs to this space. □

Remark 2.21. The divergence operator. The divergence operator is defined by

$$\mathrm{div} \ : \ V \to \mathrm{range(div)}, \quad \boldsymbol{v} \mapsto \nabla \cdot \boldsymbol{v}.$$

From (2.26) below, one gets for $\boldsymbol{v} \in V$ that $\nabla \cdot \boldsymbol{v} \in L^2(\Omega)$. Integration by parts gives

$$\int_{\Omega} (\nabla \cdot \boldsymbol{v}) (\boldsymbol{x}) \ d\boldsymbol{x} = 0 \quad \forall \ \boldsymbol{v} \in V,$$

such that the integral mean value is zero and hence $\mathrm{range(div)} \subseteq Q = Q'$ can be concluded. In Lemma 2.34, it will be shown that even equality holds: $\mathrm{range(div)} = Q'$. It follows from Lemma 2.12 iii) that this condition necessarily holds if the inf-sup condition is satisfied.

Altogether, the operator $B \in \mathcal{L}(V, Q')$ from Section 2.1 can be characterized in incompressible flow problems as the negative divergence operator. □

Remark 2.22. The gradient operator. The gradient operator will be defined on Q

$$\mathrm{grad} \ : \ Q \to \mathrm{range(grad)}, \quad q \mapsto \nabla q.$$

Since the gradient of a function from $L^2(\Omega)$ is in $H^{-1}(\Omega)$, one obtains $\mathrm{range(grad)} \subset V'$. The range of grad will be characterized more precisely in Lemma 2.32, accordingly to the condition from Lemma 2.12 ii).

Integration by parts gives

$$\langle -\mathrm{div}(\boldsymbol{v}), q \rangle_{Q',Q} = -\int_{\Omega} (\nabla \cdot \boldsymbol{v}) q \ d\boldsymbol{x} = \int_{\Omega} \nabla q \cdot \boldsymbol{v} \ d\boldsymbol{x}$$

$$= \langle \operatorname{grad}(q), \boldsymbol{v} \rangle_{V',V} \quad \forall\, \boldsymbol{v} \in V, q \in Q. \tag{2.21}$$

From this identity, it follows that $-\mathrm{div}$ and grad are dual operators and grad represents the operator $B' \in \mathcal{L}(Q, V')$ from Section 2.1. □

Definition 2.23. Distributional and weak divergence. For a vector field $\boldsymbol{v} \in L^1(\Omega)$, the mapping

$$C_0^\infty(\Omega) \to \mathbb{R}, \quad \psi \mapsto \int_\Omega \nabla\psi \cdot \boldsymbol{v}\ d\boldsymbol{x}$$

is called the distributional divergence of $\boldsymbol{v}$.

If for a vector field $\boldsymbol{v} \in L^p(\Omega)$ with $p \geq 1$ there exists a function $\theta \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$-\int_\Omega \nabla\psi \cdot \boldsymbol{v}\ d\boldsymbol{x} = \int_\Omega \psi\theta\ d\boldsymbol{x} \quad \forall\, \psi \in C_0^\infty(\Omega),$$

then the function θ is called the weak divergence of $\boldsymbol{v}$. □

Remark 2.24. A space of functions with weak divergence. For incompressible flow problems, the space of vector fields in $L^2(\Omega)$ where the divergence belongs also to $L^2(\Omega)$

$$H(\mathrm{div}, \Omega) = \left\{ \boldsymbol{v} \in L^2(\Omega)\ :\ \nabla \cdot \boldsymbol{v} \in L^2(\Omega) \right\} \tag{2.22}$$

is important. The space $H(\mathrm{div}, \Omega)$ is a Hilbert space with the inner product and the induced norm, respectively,

$$\begin{aligned} (\boldsymbol{v}, \boldsymbol{w})_{H(\mathrm{div},\Omega)} &= (\boldsymbol{v}, \boldsymbol{w}) + (\nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{w}), \\ \|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)} &= \left(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\nabla \cdot \boldsymbol{v}\|^2_{L^2(\Omega)} \right)^{1/2}. \end{aligned}$$

□

Definition 2.25. Divergence-free vector field. In view of Definition 2.23, a vector field $\boldsymbol{v} \in L^p(\Omega)$, $p \geq 1$, is called to be weakly divergence-free if

$$\int_\Omega \nabla\psi \cdot \boldsymbol{v}\ d\boldsymbol{x} = 0 \quad \forall\, \psi \in C_0^\infty(\Omega).$$

□

Remark 2.26. Spaces of weakly divergence-free functions. It became clear in Section 2.1, Remark 2.8, that the kernel of the operator B is of importance. This kernel is the space of weakly divergence-free functions in V

$$V_0 = V_{\mathrm{div}} = \{ \boldsymbol{v} \in V\ :\ (\nabla \cdot \boldsymbol{v}, q) = 0\ \forall\, q \in Q \}. \tag{2.23}$$

Thus, the divergence of the functions from V_{div} vanishes in the sense of $L^2(\Omega)$, i.e., it is $(\nabla \cdot \boldsymbol{v})(\boldsymbol{x}) = 0$ almost everywhere in Ω.

Another space of divergence-free functions is defined by

$$\begin{aligned} H_{\mathrm{div}}(\Omega) = \{\boldsymbol{v} \in H\,(\mathrm{div}, \Omega) \ : \ \nabla \cdot \boldsymbol{v} = 0 \text{ and } \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \\ \text{in the sense of traces}\}\,. \end{aligned} \tag{2.24}$$

The regularity requirement for functions from $H_{\mathrm{div}}(\Omega)$ is weaker than for functions from V_{div}.

For bounded domains with Lipschitz boundary, it can be shown that $H_{\mathrm{div}}(\Omega)$ is the closure of $C_{0,\mathrm{div}}^{\infty}(\Omega)$, see (A.7), in the norm $\|\cdot\|_{L^2(\Omega)}$, e.g., see (Constantin & Foias, 1988, Proposition 1.8) or (Sohr, 2001, Chapter II, Lemma 2.5.3). □

Lemma 2.27. Estimating the $L^2(\Omega)$ norm of the divergence by the $L^2(\Omega)$ norm of the gradient for functions from $H^1(\Omega)$. *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, and let $\boldsymbol{v} \in H^1(\Omega)$, then it holds*

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \sqrt{d}\, \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H^1(\Omega). \tag{2.25}$$

This estimate is sharp.

Proof. Exercise. ■

Remark 2.28. Improvement of (2.25) *for functions from $H_0^1(\Omega)$.* It can be shown, e.g., (John, 2016, Lemma 3.179), that for functions from $V = H_0^1(\Omega)$, it holds

$$\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in H_0^1(\Omega). \tag{2.26}$$

□

Remark 2.29. Estimating the norm of the deformation tensor by the norm of the gradient. Using the triangle inequality and that the norm of a tensor is defined component-by-component, one obtains readily

$$\begin{aligned} \|\mathbb{D}\,(\boldsymbol{u})\|_{L^2(\Omega)} &= \left\| \frac{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T}{2} \right\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\, \|\nabla \boldsymbol{u}\|_{L^2(\Omega)} + \frac{1}{2} \left\| (\nabla \boldsymbol{u})^T \right\|_{L^2(\Omega)} = \|\nabla \boldsymbol{u}\|_{L^2(\Omega)}\,. \end{aligned}$$

Thus, the norm of the symmetric part of the gradient can be estimated by the norm of the gradient. There is also an estimate in the other direction, which is called Korn's inequality, *exercise.* □

Lemma 2.30. Boundedness, continuity, and norm of the bilinear form $b(\cdot,\cdot)$. *The bilinear form $b(\cdot,\cdot)$ from* (2.19) *is bounded*

$$|b(\boldsymbol{v}, q)| \le \|\boldsymbol{v}\|_V\, \|q\|_Q$$

and consequently it is continuous. In addition, it holds $\|b\| = 1$.

Proof. The boundedness follows with the Cauchy–Schwarz inequality (A.10) and (2.26)

$$|b(\boldsymbol{v},q)| = \left| - \int_\Omega (\nabla\cdot\boldsymbol{v})\, q \, d\boldsymbol{x} \right| \le \|\nabla\cdot\boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)} \le \|\nabla\boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}. \quad (2.27)$$

Continuity follows from boundedness.

The statement concerning the norm of $b(\cdot,\cdot)$ follows from the definition (2.3) of this norm and (2.27). ■

Lemma 2.31. V_{div} is a closed subspace of V. *The subspace of weakly divergence-free functions* V_{div} *is closed in* V.

Proof. For interested students only, not presented in the class.

The proof is essentially the same as in the general case, see Lemma 2.9. It is given here for completeness of presentation.

Since $b(\cdot,\cdot)$ is a bilinear form, it follows that

$$(\alpha\nabla\cdot\boldsymbol{v}_1 + \beta\nabla\cdot\boldsymbol{v}_2, q) = \alpha\,(\nabla\cdot\boldsymbol{v}_1, q) + \beta\,(\nabla\cdot\boldsymbol{v}_2, q) = 0 \quad \forall\, \alpha,\beta\in\mathbb{R},\ \boldsymbol{v}_1,\boldsymbol{v}_2\in V_{\mathrm{div}},\ q\in Q.$$

Hence, any linear combination of weakly divergence-free functions is weakly divergence-free and therefore V_{div} is a subspace of V.

Let $\boldsymbol{v}\in V$ be arbitrary such that a sequence $\boldsymbol{v}_n \to \boldsymbol{v}$, $\boldsymbol{v}_n \in V_{\mathrm{div}}$, $n = 1,2,\ldots$, exists which converges to $\boldsymbol{v}$ in V, i.e., $\|\boldsymbol{v}-\boldsymbol{v}_n\|_V \to 0$ as $n\to\infty$. To prove that V_{div} is closed, one has to show that $\boldsymbol{v}\in V_{\mathrm{div}}$. Let $q\in Q$ be arbitrary but fixed, then it follows from the continuity of $b(\cdot,\cdot)$ that

$$b(\boldsymbol{v},q) = b\left(\lim_{n\to\infty}\boldsymbol{v}_n, q\right) = \lim_{n\to\infty} b\left(\boldsymbol{v}_n, q\right) = \lim_{n\to\infty} 0 = 0.$$

Since $q\in Q$ was arbitrary, one gets $b(\boldsymbol{v},q) = 0$ for all $q\in Q$, i.e., $\boldsymbol{v}\in V_{\mathrm{div}}$. ■

Lemma 2.32. Isomorphism of the gradient operator. *If* $\boldsymbol{f}\in V'$ *satisfies*

$$\langle \boldsymbol{f},\boldsymbol{v}\rangle_{V',V} = 0 \quad \forall\, \boldsymbol{v}\in V_{\mathrm{div}},$$

then there exists a unique $q\in Q$ *such that*

$$\boldsymbol{f} = grad(q).$$

That means, the range of the gradient operator consists of the functionals in V' *that vanish on* V_{div}

$$\tilde{V}' = \left\{\boldsymbol{f}\in V' \ :\ \langle \boldsymbol{f},\boldsymbol{v}\rangle_{V',V} = 0, \quad \forall\,\boldsymbol{v}\in V_{\mathrm{div}}\right\},$$

compare (2.11)*, and this operator is an isomorphism from* Q *onto* $\tilde{V}'$.

Proof. For interested students only, not presented in the class.

It is known that the range of grad is a subspace of V', see Remark 2.22. It can be even shown, see (Girault & Raviart, 1986, p. 20) on the basis of results from Carroll *et al.* (1966) or Duvaut & Lions (1972), that range (grad) is a closed subspace of V'. The operators $-\text{div}$ and grad are dual operators. From the Closed Range Theorem of Banach, Theorem A.71 iv), it follows that range (grad) is the subspace of functionals from V' which vanish on the kernel of div, i.e., range (grad) $= \tilde{V}'$.

To prove uniqueness, consider $q_1, q_2 \in Q$ with

$$\boldsymbol{f} = \text{grad}(q_1) = \text{grad}(q_2).$$

Then, one has

$$\mathbf{0} = \text{grad}(q_1) - \text{grad}(q_2) = \text{grad}(q_1 - q_2).$$

Hence $q_1 - q_2 \in \ker(\text{grad})$, i.e., $q_1 - q_2 \in Q$ is almost everywhere a constant function. The only function that is constant almost everywhere in Q is $q = 0$. It follows that $q_1 = q_2$ in the sense of $L^2(\Omega)$. ∎

Remark 2.33. Orthogonal decomposition of V. The space V can be decomposed into orthogonal subspaces

$$V = V_{\text{div}} \oplus V_{\text{div}}^{\perp},$$

where the orthogonality is based on the inner product (2.20) of V. □

Lemma 2.34. Isomorphism of the divergence operator. *The operator div is an isomorphis from $V_{\text{div}}^{\perp}$ onto Q.*

Proof. For interested students only, not presented in the class.

The operator $-\text{div}$ is the dual of grad. From Lemma 2.32 it follows that $-\text{div}$, and with that the operator div, is an isomorphism from the dual space of $\tilde{V}'$ onto Q'. It will be shown that the dual space of $\tilde{V}'$ is $V_{\text{div}}^{\perp}$, which is equivalent to show that $\tilde{V}' = \left(V_{\text{div}}^{\perp}\right)'$. To this end, an isomorphism $\left(V_{\text{div}}^{\perp}\right)' \to \tilde{V}'$ will be constructed.

Let $\tilde{\boldsymbol{g}} \in \left(V_{\text{div}}^{\perp}\right)'$, then a functional $\boldsymbol{g} \in V'$ can be defined by setting

$$\langle \boldsymbol{g}, \boldsymbol{v} \rangle_{V',V} = \left\langle \tilde{\boldsymbol{g}}, \boldsymbol{v}^{\perp} \right\rangle_{V',V} \quad \forall\, \boldsymbol{v} \in V,$$

where $\boldsymbol{v}^{\perp}$ is the orthogonal projection of $\boldsymbol{v}$ onto $V_{\text{div}}^{\perp}$. In particular, it holds for all $\boldsymbol{v} \in V_{\text{div}}$ that

$$\langle \boldsymbol{g}, \boldsymbol{v} \rangle_{V',V} = \langle \tilde{\boldsymbol{g}}, \mathbf{0} \rangle_{V',V} = 0.$$

Hence, $\boldsymbol{g} \in \tilde{V}'$. In this way, a linear mapping

$$\left(V_{\text{div}}^{\perp}\right)' \to \tilde{V}', \quad \tilde{\boldsymbol{g}} \mapsto \boldsymbol{g}$$

is defined.

First, it will be shown that this mapping is injective. Let $\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2 \in \left(V_{\text{div}}^{\perp}\right)'$ with

$$\langle \boldsymbol{g}, \boldsymbol{v} \rangle_{V',V} = \left\langle \tilde{\boldsymbol{g}}_1, \boldsymbol{v}^{\perp} \right\rangle_{V',V} = \left\langle \tilde{\boldsymbol{g}}_2, \boldsymbol{v}^{\perp} \right\rangle_{V',V} \quad \forall\, \boldsymbol{v} \in V,$$

then it is

$$\left\langle \tilde{\boldsymbol{g}}_1 - \tilde{\boldsymbol{g}}_2, \boldsymbol{v}^{\perp} \right\rangle_{V',V} = 0 \quad \forall\, \boldsymbol{v} \in V.$$

This equality holds in particular for all $\boldsymbol{v} \in V_{\text{div}}^{\perp}$, from which it follows that the functionals $\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2$ are identical.

Next, the surjectivity of the mapping will be proved. Let $\boldsymbol{g} \in \tilde{V}'$, i.e., $\langle \boldsymbol{g}, \boldsymbol{v} \rangle_{V',V} = 0$ for all $\boldsymbol{v} \in V_{\text{div}}$. Consider an arbitrary $\boldsymbol{v} \in V$. This function can be decomposed into $\boldsymbol{v} = \boldsymbol{v}_{\text{div}} + \boldsymbol{v}_{\text{div}}^{\perp}$ with $\boldsymbol{v}_{\text{div}} \in V_{\text{div}}$, $\boldsymbol{v}_{\text{div}}^{\perp} \in V_{\text{div}}^{\perp}$. Since $\boldsymbol{v}$ is arbitrary, also $\boldsymbol{v}_{\text{div}}^{\perp}$ is arbitrary. It follows that

$$\langle \boldsymbol{g}, \boldsymbol{v} \rangle_{V',V} = \langle \boldsymbol{g}, \boldsymbol{v}_{\mathrm{div}} \rangle_{V',V} + \left\langle \boldsymbol{g}, \boldsymbol{v}_{\mathrm{div}}^{\perp} \right\rangle_{V',V} = \left\langle \boldsymbol{g}, \boldsymbol{v}_{\mathrm{div}}^{\perp} \right\rangle_{V',V} \quad \forall\, \boldsymbol{v}_{\mathrm{div}}^{\perp} \in V_{\mathrm{div}}^{\perp}.$$

This relation defines a functional on $V_{\mathrm{div}}^{\perp}$ which is mapped onto $\boldsymbol{g}$. Consequently, the mapping is surjective. ∎

Corollary 2.35. Each pressure is the divergence of a velocity field. *For each $q \in Q$ there is a unique $\boldsymbol{v} \in V_{\mathrm{div}}^{\perp} \subset V$ such that*

$$\nabla \cdot \boldsymbol{v} = q \quad \textit{and} \quad \|q\|_Q \le \|\boldsymbol{v}\|_V \,, \quad \|\boldsymbol{v}\|_V \le C \|q\|_Q \,, \tag{2.28}$$

with C independent of $\boldsymbol{v}$ and q. In the proof of Theorem 2.37 below, it will be shown that $C = \beta_{\mathrm{is}}^{-1}$.

Proof. The existence of a unique $\boldsymbol{v} \in V_{\mathrm{div}}^{\perp}$ with $\nabla \cdot \boldsymbol{v} = q$ follows from the isomorphism of the divergence operator, see Lemma 2.34. Then, one gets with (2.26)

$$\|q\|_Q = \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} \le \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} = \|\boldsymbol{v}\|_V \,.$$

The inverse map of the divergence operator is an isomorphism, too. In particular, it is bounded, see Theorem A.70. Hence there is a $C > 0$ such that $\|\boldsymbol{v}\|_V = \left\|\mathrm{div}^{-1} q\right\|_V \le C \|q\|_Q$ for all $q \in Q$ and all $\boldsymbol{v} \in V_{\mathrm{div}}^{\perp}$. ∎

Remark 2.36. Forms of the inf-sup condition (2.14) *found in the literature.* Since with each function which can be inserted in the inf-sup condition also its negative can be inserted, one has

$$\begin{aligned}
&\inf_{q \in Q, q \ne 0} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \ne \boldsymbol{0}} \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V \|q\|_Q} \\
&= \inf_{q \in Q, q \ne 0} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \ne \boldsymbol{0}} \frac{-(\nabla \cdot \boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \\
&= \inf_{q \in Q, q \ne 0} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \ne \boldsymbol{0}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \ge \beta_{\mathrm{is}} > 0.
\end{aligned}$$

The last line is a form that can be found often in the literature.

Another form is that for each $q \in Q$, it holds that

$$\sup_{\boldsymbol{v} \in V, \boldsymbol{v} \ne \boldsymbol{0}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)}} \ge \beta_{\mathrm{is}} \|q\|_{L^2(\Omega)} \,. \tag{2.29}$$

□

Theorem 2.37. Inf-sup condition for V and Q. *The spaces V and Q satisfy the inf-sup condition* (2.14)*, i.e., there is a $\beta_{\mathrm{is}} > 0$ such that*

$$\inf_{q \in Q, q \ne 0} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \ne \boldsymbol{0}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V \|q\|_Q} \ge \beta_{\mathrm{is}}. \tag{2.30}$$

Proof. Let $q \in Q$ be arbitrary. By Corollary 2.35 there exists a unique $\boldsymbol{v} \in V_{\mathrm{div}}^{\perp}$ such that

$$\nabla \cdot \boldsymbol{v} = q, \quad \|\boldsymbol{v}\|_V \le C \|q\|_Q .$$

It follows that

$$\frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V} = \frac{(q,q)}{\|\boldsymbol{v}\|_V} = \frac{\|q\|_Q^2}{\|\boldsymbol{v}\|_V} \ge \frac{1}{C} \|q\|_Q .$$

Hence

$$\sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq \boldsymbol{0}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V} \ge \frac{1}{C} \|q\|_Q ,$$

and because $q \in Q$ is arbitrary, one obtains

$$\inf_{q \in Q, q \neq 0} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq \boldsymbol{0}} \frac{(\nabla \cdot \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_V \|q\|_Q} \ge \frac{1}{C} =: \beta_{\mathrm{is}}.$$

■

Corollary 2.38. Estimating the norm of the gradient by the norm of the divergence for functions from $V_{\mathrm{div}}^{\perp}$. *For all $\boldsymbol{v} \in V_{\mathrm{div}}^{\perp}$, it holds*

$$\|\boldsymbol{v}\|_V \le \frac{1}{\beta_{\mathrm{is}}} \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} , \tag{2.31}$$

cf. Lemma 2.12 and (2.16)*.*

Proof. From (2.28) and the specification of C, it follows that

$$\|\boldsymbol{v}\|_V \le C \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} = \frac{1}{\beta_{\mathrm{is}}} \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)} .$$

■

Lemma 2.39. Upper bound for the inf-sup constant. *It is $\beta_{\mathrm{is}} \le 1$.*

Proof. Using Corollary 2.35, one can take $q = \nabla \cdot \boldsymbol{v}$ in the inf-sup condition (2.30). Applying then estimate (2.26) yields

$$\begin{aligned}
\beta_{\mathrm{is}} &\le \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq \boldsymbol{0}} \frac{(\nabla \cdot \boldsymbol{v}, \nabla \cdot \boldsymbol{v})}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)}} = \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq \boldsymbol{0}} \frac{\|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)}^2}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)}} \\
&\le \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq \boldsymbol{0}} \frac{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|\nabla \cdot \boldsymbol{v}\|_{L^2(\Omega)}} = 1.
\end{aligned}$$

■

2.3 General Considerations on Appropriate Function Spaces for Finite Element Discretizations

Remark 2.40. On finite element methods. A brief introduction to finite element methods is provided in Appendix B. The main idea of using finite element methods consists in replacing the infinite-dimensional spaces V and Q by a finite-dimensional velocity space V^h and a finite-dimensional pressure space Q^h and to apply the Galerkin method, see Remark B.10. If $V^h \subset V$ and $Q^h \subset Q$, the finite element method is called conforming, otherwise it is called non-conforming.

For incompressible flow problems, the pair of velocity-pressure finite element spaces is denoted by V^h/Q^h. It is usual that it will not be emphasized in the notation that V^h consists of vector-valued functions and that Q^h is possibly intersected with $L_0^2(\Omega)$, depending on the boundary condition. □

Remark 2.41. Application of the abstract theory, the discrete inf-sup condition. Clearly, the finite-dimensional spaces are Hilbert spaces and the theory developed in Section 2.1 can be applied for the investigation of the existence and the uniqueness of a solution of the finite element problems arising in the discretization of incompressible flow models. In particular, the spaces V^h and Q^h have to satisfy an inf-sup condition of the form

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\|\boldsymbol{v}^h\|_{V^h} \|q^h\|_{Q^h}} \geq \beta_{\text{is}}^h > 0 \tag{2.32}$$

or equivalently that there is a $\beta_{\text{is}}^h > 0$ such that

$$\sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\|\boldsymbol{v}^h\|_{V^h}} \geq \beta_{\text{is}}^h \left\|q^h\right\|_{Q^h} \quad \forall\, q^h \in Q^h. \tag{2.33}$$

This condition is called discrete inf-sup condition or discrete Babuška–Brezzi or discrete Ladyzhenskaya–Babuška–Brezzi (LBB) condition. In (2.32) and (2.33), the bilinear form $b^h \ : \ V^h \times Q^h \to \mathbb{R}$ is defined by

$$b^h\left(\boldsymbol{v}^h, q^h\right) = -\sum_{K \in \mathcal{T}^h} \left(\nabla \cdot \boldsymbol{v}^h, q^h\right)_K, \tag{2.34}$$

where $\mathcal{T}^h$ is a triangulation of Ω and $K \in \mathcal{T}^h$ are the mesh cells. For conforming finite element spaces, the bilinear form $b^h(\cdot,\cdot)$ can be written in the same form as the bilinear form $b(\cdot,\cdot)$ with an integral on Ω, see (2.19). In this case, $b^h(\cdot,\cdot)$ is just the restriction of $b(\cdot,\cdot)$ from $V \times Q$ to $V^h \times Q^h$. The norms in the denominator are defined by

$$\left\|\boldsymbol{v}^h\right\|_{V^h} = \left(\sum_{K \in \mathcal{T}^h} \left(\nabla \boldsymbol{v}^h, \nabla \boldsymbol{v}^h\right)_K\right)^{1/2}, \quad \left\|q^h\right\|_{Q^h} = \left\|q^h\right\|_{L^2(\Omega)}. \tag{2.35}$$

For a conforming velocity finite element space, it is $\left\|\boldsymbol{v}^h\right\|_{V^h} = \left\|\nabla \boldsymbol{v}^h\right\|_{L^2(\Omega)}$.

In the same way as in the proof of Lemma 2.39, one finds for conforming finite element spaces that $\beta_{\text{is}}^h \leq 1$. □

Remark 2.42. Non-inheritance of the inf-sup condition from V and Q. Consider a conforming finite element method, then

$$\sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h, q\right)}{\|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \le \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}, q\right)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}$$

since the supremum in V^h is searched in a smaller set. In general, the strong inequality will hold. Hence

$$\begin{aligned} &\inf_{q \in Q \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h, q\right)}{\|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \\ &\le \inf_{q \in Q \setminus \{0\}} \sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}, q\right)}{\|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}} \end{aligned} \tag{2.36}$$

and the continuous inf-sup parameter β_{is}, which is a lower bound of the right-hand side of (2.36), cannot be expected to be a lower bound of the left-hand side, too. In fact, the left-hand side is zero since V^h and Q do not satisfy an inf-sup condition, see Remark 2.7. In this remark, it was discussed that the dimension of the pressure space should not exceed the dimension of the velocity space in order to get a well-posed problem.

Turning to a finite element method, the infinite-dimensional space Q has to be replaced by a finite-dimensional space Q^h. This replacement might lead to an increase of the left-hand side of (2.36) since now the infimum is taken in a smaller set. Eventually, Q^h becomes sufficiently small such that

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{\boldsymbol{v}^h \in V^h \setminus \{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h, q^h\right)}{\|\nabla \boldsymbol{v}^h\|_{L^2(\Omega)} \|q^h\|_{L^2(\Omega)}}$$

becomes positive. Then, V^h and Q^h satisfy a discrete inf-sup condition.

These considerations give a rough idea about appropriate choices for the finite element spaces with respect to the discrete inf-sup condition. The velocity space V^h should be sufficiently large such that the supremum of $\boldsymbol{v}^h \in V^h$ becomes large and the pressure space Q^h should be sufficiently small such that the infimum of $q^h \in Q^h$ becomes large, too. A condition in this direction can be found already in Remark 2.7, where $n_Q \le n_V$ was required. However, there is a conflicting requirement for the pressure finite element space. For obtaining accurate results, this space has to be large enough such that it is possible to approximate the continuous pressure sufficiently well. Also an accurate conservation of mass requires a large discrete pressure space compared with the discrete velocity space, see Remark 2.45 for details. □

Lemma 2.43. $\beta_{\text{is}}^h \le \beta_{\text{is}}$ for conforming finite element spaces. *Consider a family of finite element spaces $\left\{V^h \times Q^h\right\}$ with $V^h \subset V$, $Q^h \subset Q$, and let this family satisfy the discrete inf-sup condition* (2.32) *independently of h. Assume that for each $q \in Q \cap H^1(\Omega)$ there is a $q^h \in Q^h$ such that*

$$\left\|q - q^h\right\|_{L^2(\Omega)} \le Ch \|q\|_{H^1(\Omega)}, \tag{2.37}$$

with C independent of q and h. Then, it holds $\beta_{\text{is}}^h \le \beta_{\text{is}}$, where both values are the largest possible values in (2.32) *and* (2.30), *respectively.*

Proof. For interested students only, not presented in the class.
The proof follows Chizhonkov & Olshanskii (2000). Since $q \in Q \cap H^1(\Omega) \subset Q$, it is

$$\inf_{q\in(Q\cap H^1(\Omega))\setminus\{0\}} \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{(\nabla\cdot\boldsymbol{v},q)}{\|\boldsymbol{v}\|_V \|q\|_Q} \ge \inf_{q\in Q\setminus\{0\}} \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{(\nabla\cdot\boldsymbol{v},q)}{\|\boldsymbol{v}\|_V \|q\|_Q}.$$

On the other hand, because of the density of $Q \cap H^1(\Omega)$ in Q, which follows from Theorem A.38, even the equal sign holds, such that

$$\beta_{\text{is}} = \inf_{q\in(Q\cap H^1(\Omega))\setminus\{0\}} \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{(\nabla\cdot\boldsymbol{v},q)}{\|\boldsymbol{v}\|_V \|q\|_Q}. \tag{2.38}$$

Consider an arbitrary $q \in Q \cap H^1(\Omega)$ and $\varepsilon \in (0,1)$. For sufficiently small h, one has

$$Ch\,\|q\|_{H^1(\Omega)} \le \varepsilon\,\|q\|_{L^2(\Omega)}\,,$$

such that (2.37) gives

$$\left\|q-q^h\right\|_{L^2(\Omega)} \le \varepsilon\,\|q\|_{L^2(\Omega)}\,. \tag{2.39}$$

By the triangle inequality, one obtains from this relation

$$\|q\|_{L^2(\Omega)} \le \left\|q-q^h\right\|_{L^2(\Omega)} + \left\|q^h\right\|_{L^2(\Omega)} \le \varepsilon\,\|q\|_{L^2(\Omega)} + \left\|q^h\right\|_{L^2(\Omega)}\,,$$

which is equivalent to

$$(1-\varepsilon)\,\|q\|_{L^2(\Omega)} \le \left\|q^h\right\|_{L^2(\Omega)}\,. \tag{2.40}$$

For each $q^h \in Q^h$, one gets with the discrete inf-sup condition (2.32), the property that the supremum of a sum is lower or equal than the sum of the suprema, the Cauchy–Schwarz inequality (A.10), estimates (2.26), (2.39), (2.40) with $q \in Q \cap H^1(\Omega)$, and the inclusion $V^h \subset V$

$$\begin{aligned}
\beta_{\text{is}}^h &\le \sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h,q^h\right)}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|q^h\right\|_{L^2(\Omega)}} \\
&\le \sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h,q\right)}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|q^h\right\|_{L^2(\Omega)}} + \sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h,q^h-q\right)}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|q^h\right\|_{L^2(\Omega)}} \\
&\le \sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h,q\right)}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|q^h\right\|_{L^2(\Omega)}} + \sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}} \frac{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|q-q^h\right\|_{L^2(\Omega)}}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\left\|q^h\right\|_{L^2(\Omega)}} \\
&\le \frac{1}{1-\varepsilon} \sup_{\boldsymbol{v}^h\in V^h\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h,q\right)}{\left\|\nabla\boldsymbol{v}^h\right\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}} + \frac{\varepsilon}{1-\varepsilon} \\
&\le \frac{1}{1-\varepsilon} \sup_{\boldsymbol{v}\in V\setminus\{\boldsymbol{0}\}} \frac{b(\boldsymbol{v},q)}{\|\nabla\boldsymbol{v}\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}} + \frac{\varepsilon}{1-\varepsilon}.
\end{aligned}$$

Taking the infimum with respect to q on both sides of this inequality gives with (2.38)

$$\beta_{\text{is}}^h \le \frac{1}{1-\varepsilon}\left(\beta_{\text{is}}+\varepsilon\right) \quad\Longleftrightarrow\quad \beta_{\text{is}}^h - \beta_{\text{is}} \le \varepsilon\left(1+\beta_{\text{is}}^h\right).$$

Since the right-hand side of the last inequality is arbitrarily close to zero for sufficiently small ε, the relation $\beta_{\text{is}}^h > \beta_{\text{is}}$ cannot hold, which proves the statement of the lemma. ■

Remark 2.44. The space of discretely divergence-free functions. Exactly as in Section 2.1, a linear operator B^h can be associated with the bilinear form $b^h(\cdot,\cdot)$

$$B^h \ : \ V^h \to \left(Q^h\right)', \quad \left\langle B^h \boldsymbol{v}^h, q^h \right\rangle_{(Q^h)',Q^h} = b^h\left(\boldsymbol{v}^h, q^h\right). \tag{2.41}$$

Thus, B^h is a discrete (negative) divergence operator div^h. Note that by the representation theorem of Riesz, Theorem B.3, the space $\left(Q^h\right)'$ can be identified with Q^h. Usually, it is $\nabla \cdot \boldsymbol{v}^h \notin Q^h$. Thus, definition (2.41) strictly speaking uses the $L^2(\Omega)$ projection of $\nabla \cdot \boldsymbol{v}^h$ into Q^h, which reads for a conforming finite element method

$$\left(B^h \boldsymbol{v}^h, q^h\right) = -\left(P_{L^2}^h\left(\nabla \cdot \boldsymbol{v}^h\right), q^h\right) = -\left(\nabla \cdot \boldsymbol{v}^h, q^h\right) \quad \forall\, q^h \in Q^h.$$

From Section 2.1, it is known that the kernel of B^h plays an important role in the theory. This kernel is called the space of discretely divergence-free functions

$$V_{\text{div}}^h = \left\{\boldsymbol{v}^h \in V^h \ : \ b^h\left(\boldsymbol{v}^h, q^h\right) = 0 \ \forall\, q^h \in Q^h\right\}. \tag{2.42}$$

The dual operator of the discrete divergence is a discrete gradient operator

$$\left(B^h\right)^T \ : \ Q^h \to \left(V^h\right)' \quad \left\langle \left(B^h\right)^T q^h, \boldsymbol{v}^h \right\rangle_{(V^h)',V^h} = b^h\left(\boldsymbol{v}^h, q^h\right), \tag{2.43}$$

which will be denoted by grad^h.

Since the discrete divergence B^h is a linear operator between finite-dimensional spaces, it can be represented by a matrix, once bases in V^h and Q^h have been chosen. This matrix has the dimension $\dim Q^h \times \dim V^h$. The notation in (2.43) for the discrete gradient is used because it can be represented with the transposed matrix. By the Riesz representation theorem, Theorem B.3, $\left(V^h\right)'$ can be identified with V^h. In particular it holds that $\dim\left(V^h\right) = \dim\left(\left(V^h\right)'\right)$. $\square$

Remark 2.45. On discretely divergence-free functions, violation of mass conservation. Let $Q^h \subsetneq Q$, then the functions from V_{div}^h need to satisfy less conditions than the functions from V_{div}. Consequently, there is no injection, i.e., in general $V_{\text{div}}^h \not\subset V_{\text{div}}$. In particular, one finds that discretely divergence-free functions are in general neither weakly nor pointwise divergence-free. Thus, the conservation of mass, which was modeled by the divergence-free constraint, Section 1.1, is not satisfied exactly, but only in some approximate or mean sense.

When applying finite element methods for the simulation of incompressible flows, one has to be aware that the conservation of mass might be violated. The extent of the violation depends on the concrete choice of the finite element spaces. Note that there are some pairs of finite element spaces which are mass conservative, like the Scott–Vogelius finite element, see Remark 2.75.

Consider the case $V^h \subset V$. Note that a finite element function $\boldsymbol{v}^h \in V^h_{\mathrm{div}}$ is weakly divergence-free if $\nabla \cdot V^h \subseteq Q^h$. In this case, it is $\nabla \cdot \boldsymbol{v}^h \in Q^h$ such that from the definition (2.42) of V^h_{div}, it follows that

$$0 = -b\left(\boldsymbol{v}^h, \nabla \cdot \boldsymbol{v}^h\right) = \left\|\nabla \cdot \boldsymbol{v}^h\right\|^2_{L^2(\Omega)} .$$

Thus, the divergence vanishes in the sense of $L^2(\Omega)$. For the condition $\nabla \cdot V^h \subseteq Q^h$ to be hold, Q^h has to be sufficiently large or V^h should be sufficiently small. These requirements are just contrary to the requirements for the fulfillment of the discrete inf-sup condition, see the discussion at the end of Remark 2.42. Thus, one might suspect that the enforcement of the discrete inf-sup condition (2.32) probably has to be paid with a relaxation of the continuity constraint, as it is in fact the case for most inf-sup stable pairs of finite element spaces. □

Remark 2.46. The discrete inf-sup parameter β^h_{is}. A standard approach of discretizing partial differential equations consists in starting with a coarse triangulation of Ω, solving the considered problem on this triangulation, refining the grid, and repeating this process until, e.g., a finest grid is reached on which the solution is sufficiently accurate, or on which memory restrictions prevent a further refinement. On all grid levels, finite element spaces which satisfy the discrete inf-sup condition (2.32) should be used, where the corresponding inf-sup parameters β^h_{is} might be different.

Finite element error analysis will reveal that the inf-sup parameters enter the error estimates, e.g., see Theorem 3.18 for the Stokes equations. The error bounds depend on inverse of powers of β^h_{is}. Thus, a behavior of the form $\beta^h_{\mathrm{is}} \to 0$ for successive refinements leads to a deterioration of the order of convergence in the error estimates, e.g., compare Remark 3.26. For this reason, it is important that the used finite element spaces satisfy (2.32) with a parameter $\beta^h_{\mathrm{is}} > 0$ that is independent of the refinement level of the grid or, equivalently, independent of the mesh size parameter h. □

Lemma 2.47. Each discrete pressure is the divergence of a discrete velocity field. *Let $V^h \subset V$ with $V^h = V^h_{\mathrm{div}} \oplus \left(V^h_{\mathrm{div}}\right)^\perp$ and let the discrete inf-sup condition* (2.32) *be satisfied. Then there is for each $q^h \in Q^h$ a unique $\boldsymbol{v}^h \in \left(V^h_{\mathrm{div}}\right)^\perp$ such that*

$$\nabla \cdot \boldsymbol{v}^h = q^h, \quad \left\|\boldsymbol{v}^h\right\|_V \leq \frac{1}{\beta^h_{\mathrm{is}}} \left\|q^h\right\|_Q . \tag{2.44}$$

Proof. By Lemma 2.12, it is known that the satisfaction of the discrete inf-sup condition, point i) in Lemma 2.12, is equivalent with the existence of an isomorphism between $\left(V^h_{\mathrm{div}}\right)^\perp$ and Q^h and with the inequality from (2.44), point iii) of Lemma 2.12. ■

Remark 2.48. Importance of the best approximation error. In the Galerkin method, the error of the finite element solution $u^h \in V^h$ to the solution of the continuous problem $u \in V$, in the norm of V, can be estimated with the best approximation error, see the Lemma of Cea, Lemma B.12. For incompressible flow problems, it is often convenient to perform the finite element error analysis in the space V_{div}^h, since in V_{div}^h the problem is only an equation for the velocity and not a coupled system. Sometimes, it turns out that the best approximation error in V_{div}^h can be estimated directly, e.g., by constructing a sequence of elements in V_{div}^h which have the optimal order of convergence. One example where this can be done is the non-conforming Crouzeix–Raviart element P_1^{nc}/P_0, see (John, 2016, Lemma 4.53). However, estimates of the best approximation error are generally known only for standard finite element spaces, which can be used for V^h, e.g., see the interpolation error estimates in Appendix C. With the help of the discrete inf-sup condition, it is possible to estimate the best approximation error in V_{div}^h with the best approximation error in V^h. □

Lemma 2.49. Best approximation estimate for V_{div}^h. *Let $V^h \subset V$, $\boldsymbol{v} \in V_{\text{div}}$, and let the discrete inf-sup condition* (2.32) *hold. Then*

$$\inf_{\boldsymbol{v}^h \in V_{\text{div}}^h} \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{v}^h\right) \right\|_{L^2(\Omega)} \le \left(1 + \frac{1}{\beta_{\text{is}}^h}\right) \inf_{\boldsymbol{w}^h \in V^h} \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{w}^h\right) \right\|_{L^2(\Omega)}. \quad (2.45)$$

Proof. Let $\boldsymbol{w}^h \in V^h$ be arbitrary. Since the discrete inf-sup condition holds, V_{div}^h is not empty. It follows from Hilbert space theory that there is a unique decomposition of $\boldsymbol{w}^h = \boldsymbol{v}^h - \boldsymbol{z}^h$ into a component $\boldsymbol{v}^h \in V_{\text{div}}^h$ and a component $-\boldsymbol{z}^h \in \left(V_{\text{div}}^h\right)^{\perp}$. Hence, one gets, with $b\left(\boldsymbol{v}^h, q^h\right) = 0$,

$$b\left(\boldsymbol{z}^h, q^h\right) = b\left(\boldsymbol{v}^h - \boldsymbol{w}^h, q^h\right) = b\left(\boldsymbol{v} - \boldsymbol{w}^h, q^h\right) \quad \forall\, q^h \in Q^h. \quad (2.46)$$

Note that $b\left(\boldsymbol{v}, q^h\right) = 0$ since $\boldsymbol{v}$ is weakly divergence-free. From Lemma 2.47, it follows that there is a $q^h = \nabla \cdot \boldsymbol{z}^h \in Q^h$. Inserting this function in (2.46) gives, together with the Cauchy–Schwarz inequality (A.10) and (2.26),

$$\begin{aligned} \left\| \nabla \cdot \boldsymbol{z}^h \right\|_{L^2(\Omega)}^2 &\le \left\| \nabla \cdot \left(\boldsymbol{v} - \boldsymbol{w}^h\right) \right\|_{L^2(\Omega)} \left\| \nabla \cdot \boldsymbol{z}^h \right\|_{L^2(\Omega)} \\ &\le \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{w}^h\right) \right\|_{L^2(\Omega)} \left\| \nabla \cdot \boldsymbol{z}^h \right\|_{L^2(\Omega)}. \end{aligned}$$

With (2.44), one obtains

$$\left\| \nabla \boldsymbol{z}^h \right\|_{L^2(\Omega)} \le \frac{1}{\beta_{\text{is}}^h} \left\| q^h \right\|_{L^2(\Omega)} = \frac{1}{\beta_{\text{is}}^h} \left\| \nabla \cdot \boldsymbol{z}^h \right\|_{L^2(\Omega)} \le \frac{1}{\beta_{\text{is}}^h} \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{w}^h\right) \right\|_{L^2(\Omega)}.$$

Applying the triangle inequality and inserting this estimate yields

$$\begin{aligned} \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{v}^h\right) \right\|_{L^2(\Omega)} &\le \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{w}^h\right) \right\|_{L^2(\Omega)} + \left\| \nabla \boldsymbol{z}^h \right\|_{L^2(\Omega)} \\ &\le \left(1 + \frac{1}{\beta_{\text{is}}^h}\right) \left\| \nabla \left(\boldsymbol{v} - \boldsymbol{w}^h\right) \right\|_{L^2(\Omega)}. \end{aligned}$$

Since $\boldsymbol{w}^h$ was chosen to be arbitrary, one can find for each $\boldsymbol{w}^h \in V^h$ a function $\boldsymbol{v}^h \in V_{\text{div}}^h$ such that this estimate holds, which finishes the proof of the lemma. ■

Remark 2.50. On estimate (2.45). Estimate (2.45) is a worst case estimate. Taking $\boldsymbol{v}^h = \mathbf{0} \in V_{\mathrm{div}}^h$ shows that the left-hand side is always bounded, even if $\beta_{\mathrm{is}}^h = 0$. In contrast, the right-hand side becomes unbounded for $\beta_{\mathrm{is}}^h = 0$ or if β_{is}^h converges sufficiently fast to 0 as $h \to 0$. □

Remark 2.51. Jumps across faces and averages on faces of functions. Consider a triangulation $\mathcal{T}^h$ and let $K_1, K_2 \in \mathcal{T}^h$ be two mesh cells with a common $(d-1)$ face $E = K_1 \cap K_2$. Without loss of generality, the unit normal $\boldsymbol{n}_E$ at E should be the outward normal with respect to K_1. Then, the jump of a function v across the face E in the point $\boldsymbol{x} \in E$ is defined by

$$[|v|]_E = \lim_{\boldsymbol{y}\to\boldsymbol{x},\boldsymbol{y}\in K_1} v(\boldsymbol{y}) - \lim_{\boldsymbol{y}\to\boldsymbol{x},\boldsymbol{y}\in K_2} v(\boldsymbol{y}), \quad \boldsymbol{x} \in E, \tag{2.47}$$

if both limits are well defined. Changing the direction of $\boldsymbol{n}_E$ changes the sign of the jump.

The average is defined by

$$\{\{v\}\}_E = \frac{\lim_{\boldsymbol{y}\to\boldsymbol{x},\boldsymbol{y}\in K_1} v(\boldsymbol{y}) + \lim_{\boldsymbol{y}\to\boldsymbol{x},\boldsymbol{y}\in K_2} v(\boldsymbol{y})}{2}, \quad \boldsymbol{x} \in E.$$

Straightforward calculations, using these definitions, show

$$\begin{aligned} [|v+w|]_E &= [|v|]_E + [|w|]_E\,, \\ \{\{v+w\}\}_E &= \{\{v\}\}_E + \{\{w\}\}_E\,, \\ [|vw|]_E &= [|v|]_E\,\{\{w\}\}_E + \{\{v\}\}_E\,[|w|]_E\,. \end{aligned} \tag{2.48}$$

If w is continuous almost everywhere on E, then it follows from (2.48)

$$[|vw|]_E = [|v|]_E\, w.$$

□

Remark 2.52. Sets of $(d-1)$ faces. The set of all $(d-1)$ faces will be denoted by $\overline{\mathcal{E}}^h$ and the set of all faces which are not part of the boundary of Ω will be denoted by $\mathcal{E}^h$. □

Lemma 2.53. Sufficient and necessary condition for a finite element function to be in $H(\mathrm{div}, \Omega)$, i.e., to possess a divergence in $L^2(\Omega)$. *Let $\mathcal{T}^h$ be a regular triangulation of Ω. A finite element function $\boldsymbol{v}^h \in L^2(\Omega)$, i.e., a piecewise polynomial function belongs to $H(\mathrm{div}, \Omega)$, see* (2.22)*, if and only if $\boldsymbol{v}^h \cdot \boldsymbol{n}_E$ is continuous for all faces E of the triangulation.*

Proof. It has to be shown that $\nabla \cdot \boldsymbol{v}^h \in L^2(\Omega)$ if and only if the normal component of $\boldsymbol{v}^h$ is continuous for all faces. By definition, $\nabla \cdot \boldsymbol{v}^h \in L^2(\Omega)$ if and only if there exists a function $w \in L^2(\Omega)$ such that

$$-\int_\Omega \boldsymbol{v}^h(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x})\; d\boldsymbol{x} = \int_\Omega w(\boldsymbol{x})\varphi(\boldsymbol{x})\; d\boldsymbol{x} \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{2.49}$$

Integration by parts yields

$$
\begin{aligned}
& -\int_{\Omega} \boldsymbol{v}^h(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x}) \ d\boldsymbol{x} \\
&= -\sum_{K\in\mathcal{T}^h} \int_K \boldsymbol{v}^h(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x}) \ d\boldsymbol{x} \\
&= \sum_{K\in\mathcal{T}^h} \left(\int_K \nabla\cdot \boldsymbol{v}^h(\boldsymbol{x})\varphi(\boldsymbol{x}) \ d\boldsymbol{x} - \int_{\partial K} \varphi(\boldsymbol{s})\boldsymbol{v}^h(\boldsymbol{s}) \cdot \boldsymbol{n}_{\partial K} \ d\boldsymbol{s} \right) \\
&= \sum_{K\in\mathcal{T}^h} \int_K \nabla\cdot \boldsymbol{v}^h(\boldsymbol{x})\varphi(\boldsymbol{x}) \ d\boldsymbol{x} - \sum_{K\in\mathcal{T}^h} \sum_{E\in\partial K} \int_E \varphi(\boldsymbol{s})\boldsymbol{v}^h(\boldsymbol{s}) \cdot \boldsymbol{n}_E \ d\boldsymbol{s} \\
&= \int_{\Omega} \nabla\cdot \boldsymbol{v}^h(\boldsymbol{x})\varphi(\boldsymbol{x}) \ d\boldsymbol{x} - \sum_{E\in\mathcal{E}^h} \int_E \varphi(\boldsymbol{s}) \left[\left|\boldsymbol{v}^h \cdot \boldsymbol{n}_E\right|\right]_E (\boldsymbol{s}) \ d\boldsymbol{s} \\
&\quad - \sum_{E\in\overline{\mathcal{E}}^h\setminus\mathcal{E}^h} \int_E \varphi(\boldsymbol{s})\boldsymbol{v}^h(\boldsymbol{s}) \cdot \boldsymbol{n}_E \ d\boldsymbol{s} \quad \forall\, \varphi \in C_0^\infty(\Omega).
\end{aligned} \tag{2.50}
$$

The normal $\boldsymbol{n}_E$ on the interior faces can be chosen arbitrarily. Using the opposite normal $-\boldsymbol{n}_E$, also the sign of the jump has to be changed, i.e., one obtains

$$
-\left[\left|\boldsymbol{v}^h \cdot (-\boldsymbol{n}_E)\right|\right]_E (\boldsymbol{s}) = \left[\left|\boldsymbol{v}^h \cdot \boldsymbol{n}_E\right|\right]_E (\boldsymbol{s}),
$$

where (2.48) was applied. The last term in (2.50) vanishes since the test function vanishes at the boundary of Ω. Thus, (2.49) is satisfied if and only if all integrals on the interior faces vanish for all test functions. Therefore, the jumps $\left[\left|\boldsymbol{v}^h \cdot \boldsymbol{n}_E\right|\right]_E$ have to vanish on all interior faces, which is equivalent with the requirement that the normal component of $\boldsymbol{v}^h$ is continuous across all faces of the mesh cells. ■

2.4 The Discrete Inf-Sup Condition and Finite Element Spaces

Remark 2.54. Contents of this section. Some simple pairs of finite element spaces do not satisfy the discrete inf-sup condition (2.32). Here, one example will be discussed in detail. Then, a technique for proving the inf-sup condition will be presented. Finally, a number of popular pairs of finite element spaces will be introduced that satisfy the discrete inf-sup condition. □

2.4.1 Examples of Pairs of Finite Element Spaces Violating the Discrete Inf-Sup Condition

Remark 2.55. A condition for the violation of the discrete inf-sup condition. The violation of the discrete inf-sup condition (2.32) is proved, e.g., if one finds a non-trivial $q^h \in Q^h$ such that

$$b^h\left(\boldsymbol{v}^h, q^h\right) = 0 \quad \forall\, \boldsymbol{v}^h \in V^h. \tag{2.51}$$

In this case, it holds

$$\sup_{\boldsymbol{v}^h \in V^h, \boldsymbol{v}^h \neq \boldsymbol{0}} \frac{b^h\left(\boldsymbol{v}^h, q^h\right)}{\left\|\boldsymbol{v}^h\right\|_{V^h}} = 0,$$

from what follows, by dividing by $\left\|q^h\right\|_Q > 0$ and taking on both sides the infimum of the finite element pressure functions, that the discrete inf-sup condition (2.32) cannot be satisfied. Such non-trivial $q^h \in Q^h$ are called spurious pressure modes. □

Example 2.56. The P_1/P_1 pair of finite element spaces. Probably every finite element code which uses simplicial grids can apply the P_1 finite element. If it would be possible to choose P_1/P_1 for velocity and pressure finite elements, the extension of such codes to the simulation of incompressible flows would be straightforward. However, this example shows that P_1/P_1 does not satisfy the discrete inf-sup condition (2.32).

Let $\Omega = (0,1)^2$ and consider a triangulation of Ω with equally sized triangles with measure $|K| > 0$. Both, the finite element velocity and the finite element pressure are continuous and piecewise linear functions. The nodes of the finite element functions are their values in the vertices of the triangles, see Example B.38.

Consider first the integral mean value condition for the pressure, $Q^h \subset L_0^2(\Omega)$. Let K be a mesh cell and $q_{1,K}^h, q_{2,K}^h, q_{3,K}^h$ be the values of the pressure in the vertices of K. Then, the integral of q^h on K can be evaluated exactly by a quadrature rule which uses only the values at the vertices of K

$$\int_K q^h(\boldsymbol{x})\ d\boldsymbol{x} = \frac{|K|}{3}\left(q_{1,K}^h + q_{2,K}^h + q_{3,K}^h\right).$$

Hence, the integral mean value condition for the finite element pressure reads as follows

$$0 = \int_\Omega q^h(\boldsymbol{x})\ d\boldsymbol{x} = \sum_{K \in \mathcal{T}^h} \int_K q^h(\boldsymbol{x})\ d\boldsymbol{x} = \frac{|K|}{3} \sum_{K \in \mathcal{T}^h} \left(q_{1,K}^h + q_{2,K}^h + q_{3,K}^h\right). \tag{2.52}$$

Now, a function $q^h \in Q^h$ will be constructed that satisfies (2.51). On each mesh cell K, it is

$$\left.\boldsymbol{v}^h\right|_K(\boldsymbol{x}) = \begin{pmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \gamma_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \gamma_2 \end{pmatrix},$$

from what follows that

$$\nabla \cdot \left.\boldsymbol{v}^h\right|_K(\boldsymbol{x}) = \alpha_{11} + \alpha_{22} = c_K.$$

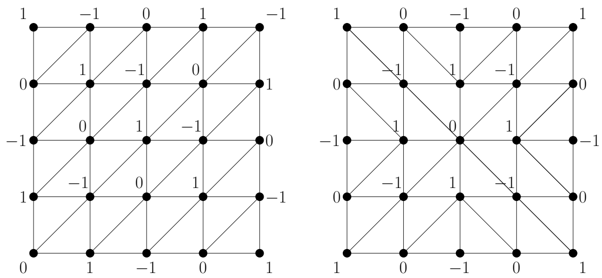

Fig. 2.1 Checkerboard instabilities for the P_1/P_1 finite element.

Then, (2.51) becomes

$$\begin{aligned} 0 &= -\sum_{K\in\mathcal{T}^h}\int_K \left(\left(\nabla\cdot\boldsymbol{v}^h\right)q^h\right)(\boldsymbol{x})\ d\boldsymbol{x} = -\sum_{K\in\mathcal{T}^h} c_K\int_K q^h(\boldsymbol{x})\ d\boldsymbol{x} \\ &= -\frac{|K|}{3}\sum_{K\in\mathcal{T}^h} c_K\left(q^h_{1,K}+q^h_{2,K}+q^h_{3,K}\right). \end{aligned} \tag{2.53}$$

From (2.52) and (2.53), it follows that a counterexample for the fulfillment of the discrete inf-sup condition (2.32) is found, if a non-trivial function q^h with

$$q^h_{1,K}+q^h_{2,K}+q^h_{3,K}=0$$

for all $K\in\mathcal{T}^h$ can be constructed. In this case, the integral mean value condition and (2.51) are satisfied both. Two examples of such functions are given in Figure 2.1. The form of the spurious modes led to the name checkerboard-type instabilities. □

Remark 2.57. Other pairs of finite element spaces that do not satisfy the discrete inf-sup condition. Other pairs of finite element spaces that do not satisfy the discrete inf-sup condition are

- P_1/P_0 (*exercise*), P_k/P_k, $k\ge 2$, on simplicial grids,
- Q_1/Q_0, Q_k/Q_k, $k\ge 1$, on quadrilateral/hexahedral grids,
- $P_k/P_{k-1}^{\mathrm{disc}}$, $k\ge 2$, on some commonly used types of simplicial grids.

Details of these examples can be found in (John, 2016, Chapter 3.4). □

2.4.2 A Technique for Checking the Discrete Inf-Sup Condition

Remark 2.58. Checking the discrete inf-sup condition. In the literature, one can find several approaches that have been used for proving that certain pairs of finite element spaces satisfy the discrete inf-sup condition (2.32) or equivalently (2.33). A comprehensive overview of techniques for proving the discrete inf-sup condition and corresponding results can be found, e.g., in Boffi *et al.* (2008) and (Boffi *et al.*, 2013, Sections 8.4 and 8.5). Here, only one approach will be presented. □

Remark 2.59. A connection between the continuous and the discrete inf-sup condition. For conforming finite element spaces, it is possible to check the discrete inf-sup condition (2.32) with the help of the continuous inf-sup condition (2.30). The connection of both conditions is shown in the following lemma. The result is due to Fortin (1977). Also the generalization to some non-conforming cases is possible, see (John, 2016, Section 3.6.5).

For conforming finite element spaces, it is $b^h(\cdot,\cdot) = b(\cdot,\cdot)$. □

Lemma 2.60. Fortin criterion for checking the discrete inf-sup condition. *Let V, Q, and $b(\cdot,\cdot)$ fulfill the assumptions of Remark 2.3 and let the inf-sup condition* (2.30) *be satisfied. Consider conforming spaces $V^h \subset V$ and $Q^h \subset Q$. Then, V^h and Q^h satisfy the discrete inf-sup condition* (2.32) *if and only if there exists a constant $\gamma^h > 0$, which is independent of h, such that for all $\boldsymbol{v} \in V$ there is an element $P_{\mathrm{For}}^h \boldsymbol{v} \in V^h$ with*

$$b\left(\boldsymbol{v}, q^h\right) = b\left(P_{\mathrm{For}}^h \boldsymbol{v}, q^h\right) \quad \forall\, q^h \in Q^h \quad \textit{and} \quad \left\|P_{\mathrm{For}}^h \boldsymbol{v}\right\|_V \le \gamma^h \left\|\boldsymbol{v}\right\|_V . \quad (2.54)$$

Proof. • *Assume that* (2.54) *holds.*
Let $q^h \in Q^h$ be arbitrary. From span $\left\{P_{\mathrm{For}}^h \boldsymbol{v}\right\} \subseteq V^h$, it follows, using also (2.54) and (2.30), that

$$\begin{aligned}\sup_{\boldsymbol{v}^h \in V^h\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}^h, q^h\right)}{\left\|\boldsymbol{v}^h\right\|_V} &\ge \sup_{\boldsymbol{v} \in V\setminus\{\boldsymbol{0}\}} \frac{b\left(P_{\mathrm{For}}^h\boldsymbol{v}, q^h\right)}{\left\|P_{\mathrm{For}}^h\boldsymbol{v}\right\|_V} = \sup_{\boldsymbol{v} \in V\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}, q^h\right)}{\left\|P_{\mathrm{For}}^h\boldsymbol{v}\right\|_V} \\ &\ge \sup_{\boldsymbol{v} \in V\setminus\{\boldsymbol{0}\}} \frac{b\left(\boldsymbol{v}, q^h\right)}{\gamma^h \left\|\boldsymbol{v}\right\|_V} \ge \frac{\beta_{\mathrm{is}}}{\gamma^h}\left\|q^h\right\|_Q .\end{aligned}$$

This inequality is just the discrete inf-sup condition (2.32) with $\beta_{\mathrm{is}}^h = \beta_{\mathrm{is}}/\gamma^h$.

•*For interested students only, not presented in the class. Assume that* (2.32) *is satisfied.* Consider the restriction of $b(\cdot,\cdot)$ from $V \times Q$ to $V \times Q^h$. This restriction defines a continuous linear operator

$$\tilde{B} \in \mathcal{L}\left(V, \left(Q^h\right)'\right), \quad \left\langle \tilde{B}\boldsymbol{v}, q^h\right\rangle_{(Q^h)',Q^h} = b\left(\boldsymbol{v}, q^h\right).$$

By definition, it is $\tilde{B}\boldsymbol{v} \in \left(Q^h\right)'$ for all $\boldsymbol{v} \in V$. Since the discrete inf-sup condition (2.32) holds, it follows from Lemma 2.12 iii) that the operator B^h defined in (2.41), the discrete divergence operator, is an isomorphism from $(V_{\mathrm{div}}^h)^\perp$ onto $\left(Q^h\right)'$. In particular, B^h is surjective. Since $\tilde{B}\boldsymbol{v} \in \left(Q^h\right)'$, there must be an element $\tilde{\boldsymbol{v}}^h$ from $(V_{\mathrm{div}}^h)^\perp$ such that

$$\left\langle B^h \tilde{\boldsymbol{v}}^h, q^h\right\rangle_{(Q^h)',Q^h} = \left\langle \tilde{B}\boldsymbol{v}, q^h\right\rangle_{(Q^h)',Q^h} \quad \forall\, q^h \in Q^h.$$

Consequently, for all $\boldsymbol{v}^h \in V^h$ whose projection into $(V_{\mathrm{div}}^h)^\perp$ is equal to $\tilde{\boldsymbol{v}}^h$, it holds

$$\left\langle B^h \boldsymbol{v}^h, q^h\right\rangle_{(Q^h)',Q^h} = \left\langle \tilde{B}\boldsymbol{v}, q^h\right\rangle_{(Q^h)',Q^h} \quad \forall\, q^h \in Q^h. \quad (2.55)$$

One of these elements can be chosen to be $P_{\mathrm{For}}^h \boldsymbol{v}$. Then, (2.55) is equivalent to

$$b\left(P_{\mathrm{For}}^h \boldsymbol{v}, q^h\right) = b\left(\boldsymbol{v}, q^h\right) \quad \forall\, q^h \in Q^h.$$

With these relations, it follows from Lemma 2.12 iii), the definition of the norm in Q^h, the definition of the norm of $b(\cdot,\cdot)$ from (2.3), and the estimate of this norm from Lemma 2.30 that

$$\begin{aligned}
\left\|P_{\mathrm{For}}^h \boldsymbol{v}\right\|_V &\le \frac{1}{\beta_{\mathrm{is}}^h} \left\|B^h\left(P_{\mathrm{For}}^h \boldsymbol{v}\right)\right\|_Q = \frac{1}{\beta_{\mathrm{is}}^h} \sup_{q^h\in Q^h\setminus\{0\}} \frac{\left\langle B^h\left(P_{\mathrm{For}}^h \boldsymbol{v}\right), q^h\right\rangle_{(Q^h)',Q^h}}{\left\|q^h\right\|_Q} \\
&= \frac{1}{\beta_{\mathrm{is}}^h} \sup_{q^h\in Q^h\setminus\{0\}} \frac{b\left(P_{\mathrm{For}}^h \boldsymbol{v}, q^h\right)}{\left\|q^h\right\|_Q} = \frac{1}{\beta_{\mathrm{is}}^h} \sup_{q^h\in Q^h\setminus\{0\}} \frac{b(\boldsymbol{v}, q^h)}{\left\|q^h\right\|_Q} \\
&\le \frac{1}{\beta_{\mathrm{is}}^h} \sup_{q^h\in Q^h\setminus\{0\}} \frac{\|b\|\, \|\boldsymbol{v}\|_V \left\|q^h\right\|_Q}{\left\|q^h\right\|_Q} = \frac{\|b\|}{\beta_{\mathrm{is}}^h} \|\boldsymbol{v}\|_V = \gamma^h \|\boldsymbol{v}\|_V .
\end{aligned}$$

Since $\boldsymbol{v}$ was chosen to be arbitrary, (2.54) is proved. ∎

Remark 2.61. On condition (2.51)*.* This condition, which implies that the discrete inf-sup condition is violated, cannot be fulfilled if (2.54) holds. Assume that there is a $q^h \in Q^h$ such that $b(\boldsymbol{v}^h, q^h) = 0$ for all $\boldsymbol{v}^h \in V^h$. From (2.54), it follows that then $b(\boldsymbol{v}, q^h) = 0$ for all $\boldsymbol{v} \in V$, since $P_{\mathrm{For}}^h \boldsymbol{v} \in V^h$. Because $q^h \in Q$ and V and Q satisfy the inf-sup condition (2.30), it follows that $q^h = 0$. Hence, there is no non-trivial $q^h \in Q^h$ for which (2.51) holds. □

Remark 2.62. A possible construction of a Fortin operator. Sometimes, it is possible to construct a linear Fortin operator P_{For}^h with the help of two linear operators $P_1^h, P_2^h \in \mathcal{L}(V, V^h)$. Assume that

$$\left\|P_1^h \boldsymbol{v}\right\|_V \le C_1 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V, \tag{2.56}$$

$$\left\|P_2^h\left(I - P_1^h\right)\boldsymbol{v}\right\|_V \le C_2 \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V, \tag{2.57}$$

$$b\left(\boldsymbol{v} - P_2^h \boldsymbol{v}, q^h\right) = 0 \quad \forall\, \boldsymbol{v} \in V,\ \forall\, q^h \in Q^h, \tag{2.58}$$

where C_1, C_2 are independent of h. Then, a Fortin operator is defined by

$$P_{\mathrm{For}}^h \in \mathcal{L}(V, V^h) \quad \boldsymbol{v} \mapsto P_1^h \boldsymbol{v} + P_2^h\left(\boldsymbol{v} - P_1^h \boldsymbol{v}\right). \tag{2.59}$$

□

Lemma 2.63. A property of the operator (2.59)**.** *The operator defined in* (2.59) *satisfies* (2.54).

Proof. Applying (2.59), (2.58) for $P_1^h \boldsymbol{v}$, and once again (2.58) for $\boldsymbol{v}$, one obtains for all $q^h \in Q^h$

$$\begin{aligned}
b\left(P_{\mathrm{For}}^h \boldsymbol{v}, q^h\right) &= b\left(P_1^h \boldsymbol{v} + P_2^h\left(\boldsymbol{v} - P_1^h \boldsymbol{v}\right), q^h\right) \\
&= b\left(P_1^h \boldsymbol{v} - P_2^h P_1^h \boldsymbol{v}, q^h\right) + b\left(P_2^h \boldsymbol{v}, q^h\right) \\
&= b\left(P_2^h \boldsymbol{v}, q^h\right) = b\left(\boldsymbol{v}, q^h\right).
\end{aligned}$$

The boundedness of P_{For}^h is obtained by applying the triangle inequality and using (2.56) and (2.57)

$$\left\|P_{\mathrm{For}}^h \boldsymbol{v}\right\|_V \le \left\|P_1^h \boldsymbol{v}\right\|_V + \left\|P_2^h \left(\boldsymbol{v} - P_1^h \boldsymbol{v}\right)\right\|_V \le C_1 \left\|\boldsymbol{v}\right\|_V + C_2 \left\|\boldsymbol{v}\right\|_V = \gamma^h \left\|\boldsymbol{v}\right\|_V ,$$

with $\gamma^h = C_1 + C_2$. ■

Remark 2.64. A more detailed construction of the Fortin operator. Often, the Clément operator P_{Cle}^h (C.18), with the modification that preserves homogeneous Dirichlet bounary conditions, see Remark C.22, plays the role of P_1^h. Then, condition (2.57) for P_2^h can be replaced with

$$\left\|P_2^h \boldsymbol{v}\right\|_{H^1(K)} \le C \left(h_K^{-1} \left\|\boldsymbol{v}\right\|_{L^2(K)} + \left|\boldsymbol{v}\right|_{H^1(K)}\right), \quad \forall\, K \in \mathcal{T}^h,\ \forall\, \boldsymbol{v} \in V, \tag{2.60}$$

where the constant C does not depend on h_K. □

Lemma 2.65. A property of the Fortin operator constructed with (C.18)**,** (2.58)**, and** (2.60)**.** *Consider a family of quasi-uniform triangulations* $\left\{\mathcal{T}^h\right\}$. *Let* $P_1^h = P_{\mathrm{Cle}}^h$ *be the modified Clément interpolation operator* (C.18), *which preserves homogeneous Dirichlet boundary conditions, and let* P_2^h *satisfy* (2.58) *and* (2.60). *Then,* P_{For}^h *defined by* (2.59) *is a Fortin operator.*

Proof. For interested students only, not presented in the class. The first property of (2.54) is proved analogously as in the proof of Lemma 2.63, since the proof used only (2.58) and (2.59). It remains to show the second property with γ^h independent of h.

From the quasi-uniformity of the family of triangulations, it follows that for each K there is a maximal number of mesh cells in ω_K, see Figure C.1, which is independent of the triangulation and that the diameter of ω_K can be estimated by Ch_K with a constant C independent of $\mathcal{T}^h$. Using (2.59), the triangle inequality, (2.60), and (C.19) for $k = 0, l = 1$ and $k = l = 1$, one obtains

$$\begin{aligned}
\left\|P_{\mathrm{For}}^h \boldsymbol{v}\right\|_V^2 &= \left|P_{\mathrm{For}}^h \boldsymbol{v}\right|_{H^1(\Omega)}^2 \\
&\le 2\left|P_{\mathrm{Cle}}^h \boldsymbol{v}\right|_{H^1(\Omega)}^2 + 2\left|P_2^h\left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right)\right|_{H^1(\Omega)}^2 \\
&\le C\left(\left|P_{\mathrm{Cle}}^h \boldsymbol{v} - \boldsymbol{v}\right|_{H^1(\Omega)}^2 + \left|\boldsymbol{v}\right|_{H^1(\Omega)}^2\right) + 2\sum_{K\in\mathcal{T}^h} \left|P_2^h\left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right)\right|_{H^1(K)}^2 \\
&\le C\left(\sum_{K\in\mathcal{T}^h} \left|P_{\mathrm{Cle}}^h \boldsymbol{v} - \boldsymbol{v}\right|_{H^1(K)}^2 + \left|\boldsymbol{v}\right|_{H^1(\Omega)}^2\right) \\
&\quad + C\left(\sum_{K\in\mathcal{T}^h} (h_K^{-2} \left\|\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right\|_{L^2(K)}^2 + \left|\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right|_{H^1(K)}^2\right) \\
&\le C\left(\left|\boldsymbol{v}\right|_{H^1(\Omega)}^2 + \sum_{K\in\mathcal{T}^h}\left(\left|\boldsymbol{v}\right|_{H^1(\omega_K)}^2 + \left|\boldsymbol{v}\right|_{H^1(\omega_K)}^2 + \left|\boldsymbol{v}\right|_{H^1(\omega_K)}^2\right)\right) \\
&\le C\left\|\nabla \boldsymbol{v}\right\|_{L^2(\Omega)}^2 = C\left\|\boldsymbol{v}\right\|_V^2 .
\end{aligned}$$

■

2.4.3 Inf-sup Stable Pairs of Finite Element Spaces

2.4.3.1 The MINI Element

Remark 2.66. The MINI element. The MINI element is defined on simplicial grids and it is given by

$$V^h = P_1 \oplus V_{\mathrm{bub}}^h, \quad Q^h = P_1, \tag{2.61}$$

where V_{bub}^h is a space consisting of local bubble functions

$$V_{\mathrm{bub}}^h = \left\{ v_{\mathrm{bub}}^h \ : \ \mathrm{supp}\left(v_{\mathrm{bub}}^h\right) = K, \, v_{\mathrm{bub}}^h\big|_K = \alpha \prod_{i=1}^{d+1} \lambda_i, K \in \mathcal{T}^h, \alpha \in \mathbb{R} \right\},$$

where λ_i are the barycentric coordinates of the simplex K, see Definition B.31. It follows that

$$v_{\mathrm{bub}}^h\big|_K \in P_{d+1}(K) \cap H_0^1(K).$$

This pair of finite element spaces was introduced by Arnold *et al.* (1984). It is the lowest order conforming inf-sup stable pair of finite element spaces.

The basic idea for the construction of the MINI element consists in starting with standard finite element spaces for velocity and pressure and then enriching the velocity space such that the discrete inf-sup condition (2.32) is satisfied. The fulfillment of the discrete inf-sup condition will be proved with the construction of a Fortin operator, see Lemma 2.60. □

Lemma 2.67. Properties of bubble functions. *Let $K \in \mathcal{T}^h$ be a simplex and let*

$$v_{\mathrm{bub}}^h(\boldsymbol{x}) = \prod_{i=1}^{d+1} \lambda_i(\boldsymbol{x}), \quad \boldsymbol{x} \in K,$$

be a bubble function on K. Then, the following estimates hold

$$\left\| v_{\mathrm{bub}}^h \right\|_{L^2(K)} \le C h_K^d, \tag{2.62}$$

$$\left\| \nabla v_{\mathrm{bub}}^h \right\|_{L^2(K)} \le C h_K^{(d-2)/2}, \tag{2.63}$$

$$\int_K v_{\mathrm{bub}}^h(\boldsymbol{x}) \ d\boldsymbol{x} \ge C\left|K\right|, \tag{2.64}$$

where the constants are independent of K.

Proof. • *Estimates* (2.62) *and* (2.63). *exercise problems*

• *Estimate* (2.64). The bubble functions are polynomials of degree $d+1$ in K. Hence, there are quadrature rules with positive weights and nodes in the interior of K such that they can be integrated exactly

$$\int_K v_{\mathrm{bub}}^h(\boldsymbol{x})\ d\boldsymbol{x} = |K| \sum_{i=1}^{N_0} \omega_i v_{\mathrm{bub}}^h(\boldsymbol{x}_i),$$

see Cools & Rabinowitz (1993), i.e., $\omega_i > 0$, $v_{\mathrm{bub}}^h(\boldsymbol{x}_i) > 0$, $i = 1, \ldots, N_0$. It follows that

$$\begin{aligned}
\left| \int_K v_{\mathrm{bub}}^h(\boldsymbol{x})\ d\boldsymbol{x} \right| &= |K| \left| \sum_{i=1}^{N_0} \omega_i v_{\mathrm{bub}}^h(\boldsymbol{x}_i) \right| = |K| \sum_{i=1}^{N_0} \omega_i v_{\mathrm{bub}}^h(\boldsymbol{x}_i) \\
&\geq |K| \min_{i=1,\ldots,N_0} v_{\mathrm{bub}}^h(\boldsymbol{x}_i) \sum_{i=1}^{N_0} \omega_i = C\,|K|\,.
\end{aligned}$$

■

Remark 2.68. Generalization of the Fortin criterion (2.54)*.* The first part of the Fortin criterion (2.54) can be written in the form

$$-\int_\Omega \nabla \cdot \left(\boldsymbol{v} - P_{\mathrm{For}}^h \boldsymbol{v}\right) q^h\ d\boldsymbol{x} = 0 \quad \forall\, \boldsymbol{v} \in V,\ \forall\, q^h \in Q^h.$$

It follows, for conforming finite element spaces and a continuous finite element pressure space, using integration by parts, that

$$\int_\Omega \left(\boldsymbol{v} - P_{\mathrm{For}}^h \boldsymbol{v}\right) \cdot \nabla q^h\ d\boldsymbol{x} = 0 \quad \forall\, \boldsymbol{v} \in V,\ \forall\, q^h \in Q^h. \tag{2.65}$$

The first step of the construction of the Fortin operator consists in replacing the global criterion (2.65) by a set of local criteria

$$\int_K \left(\boldsymbol{v} - P_{\mathrm{For}}^h \boldsymbol{v}\right) \cdot \nabla q^h\ d\boldsymbol{x} = 0 \quad \forall\, \boldsymbol{v} \in V,\ \forall\, q^h \in Q^h,\ \forall\, K \in \mathcal{T}^h. \tag{2.66}$$

Clearly, (2.66) induces (2.65), but not vice versa. □

Remark 2.69. Enrichment of the velocity space. Let $Q^h(K) = P_k(K)$, then it follows that $\nabla q^h \in P_{k-1}(K)$. It is clear that (2.66) can be satisfied, for fixed Q^h, the easier the larger the space $V^h(K)$ is, since for a larger space $V^h(K)$ there are more possibilities to define $P_{\mathrm{For}}^h \boldsymbol{v}$. The idea of Arnold *et al.* (1984) was to start for $V^h(K)$ also with polynomials of order k and then to extend this space locally, i.e., with functions whose support is restricted to K, until the velocity space is sufficiently large to satisfy (2.66). □

Remark 2.70. Local condition (2.66) *for the MINI element.* For the MINI element (2.61), condition (2.66) simplifies to

$$\int_K \left(\boldsymbol{v} - P_{\mathrm{For}}^h \boldsymbol{v}\right)\ d\boldsymbol{x} = \boldsymbol{0} \quad \forall\, \boldsymbol{v} \in V,\ \forall\, K \in \mathcal{T}^h, \tag{2.67}$$

since the gradient of the local discrete pressure is a constant. □

Remark 2.71. Construction of the Fortin operator. The construction of the Fortin operator is based on the Clément interpolation operator P_{Cle}^h defined in (C.18), with the modification to preserve homogeneous Dirichlet boundary conditions, see Remark 2.64. This operator satisfies the interpolation estimate (C.19). Consider a quasi-uniform family of triangulations. Then, the number of mesh cells in the set ω_K from (C.19) is bounded uniformly from above and one gets the global estimates

$$\sum_{K\in\mathcal{T}^h} h_K^{-2} \left\|v - P_{\mathrm{Cle}}^h v\right\|_{L^2(K)}^2 \le C \left\|v\right\|_{H^1(\Omega)}^2 \quad \forall\, v \in H^1(\Omega), \tag{2.68}$$

$$\sum_{K\in\mathcal{T}^h} \left\|\nabla\left(v - P_{\mathrm{Cle}}^h v\right)\right\|_{L^2(K)}^2 \le C \left\|v\right\|_{H^1(\Omega)}^2 \quad \forall\, v \in H^1(\Omega). \tag{2.69}$$

From the triangle inequality and (2.69), one gets in particular the stability estimate

$$\begin{aligned} &\sum_{K\in\mathcal{T}^h} \left\|\nabla P_{\mathrm{Cle}}^h v\right\|_{L^2(K)}^2 \qquad\qquad (2.70)\\ &\le 2\left(\sum_{K\in\mathcal{T}^h} \left\|\nabla\left(v - P_{\mathrm{Cle}}^h v\right)\right\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}^h} \left\|\nabla v\right\|_{L^2(K)}^2\right) \le C \left\|v\right\|_{H^1(\Omega)}^2 . \end{aligned}$$

Now, the Fortin operator is defined by

$$P_{\mathrm{For}}^h \boldsymbol{v}(\boldsymbol{x}) = P_{\mathrm{Cle}}^h \boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{\alpha}_K v_{\mathrm{bub}}^h(\boldsymbol{x}), \tag{2.71}$$

with

$$\boldsymbol{\alpha}_K = \frac{\int_K \left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right)(\boldsymbol{x})\ d\boldsymbol{x}}{\int_K v_{\mathrm{bub}}^h(\boldsymbol{x})\ d\boldsymbol{x}}. \tag{2.72}$$

This construction is of form (2.59) with P_2^h just being the integral operator on K equipped with some scaling. □

Theorem 2.72. The discrete inf-sup condition for the MINI element. *Consider a quasi-uniform family of triangulations. Then, the operator* (2.71) *is a Fortin operator. Hence, the MINI element* (2.61) *satisfies the discrete inf-sup condition* (2.32) *or equivalently* (2.33).

Proof. One has to verify the conditions stated in (2.54). Instead of the first of these conditions, the more general condition (2.67) will be considered.

• *Condition* (2.67). Inserting (2.71) and (2.72) in (2.67) yields for an arbitrary mesh cell K

$$\begin{aligned} &\int_K \left(\boldsymbol{v} - P_{\mathrm{For}}^h \boldsymbol{v}\right)(\boldsymbol{x})\ d\boldsymbol{x}\\ &= \int_K \left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v} - \boldsymbol{\alpha}_K v_{\mathrm{bub}}^h\right)(\boldsymbol{x})\ d\boldsymbol{x}\\ &= \int_K \left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right)(\boldsymbol{x})\ d\boldsymbol{x} - \boldsymbol{\alpha}_K \int_K v_{\mathrm{bub}}^h(\boldsymbol{x})\ d\boldsymbol{x}\\ &= \int_K \left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right)(\boldsymbol{x})\ d\boldsymbol{x} - \frac{\int_K \left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right)(\boldsymbol{x})\ d\boldsymbol{x}}{\int_K v_{\mathrm{bub}}^h(\boldsymbol{x})\ d\boldsymbol{x}} \int_K v_{\mathrm{bub}}^h(\boldsymbol{x})\ d\boldsymbol{x} = \boldsymbol{0}. \end{aligned}$$

• *Second condition of* (2.54). The triangle inequality and the homogeneity of a norm, Definition A.6, gives

$$\left\|\nabla P_{\mathrm{For}}^h \boldsymbol{v}\right\|_{L^2(K)} \leq \left\|\nabla P_{\mathrm{Cle}}^h \boldsymbol{v}\right\|_{L^2(K)} + \|\boldsymbol{\alpha}_K\|_2 \left\|\nabla v_{\mathrm{bub}}^h\right\|_{L^2(K)}, \tag{2.73}$$

where $\|\boldsymbol{\alpha}_K\|_2$ is the Euclidean norm of the vector-valued constant $\boldsymbol{\alpha}_K$. One obtains for the second term, using (2.72), (2.63), the Cauchy–Schwarz inequality (A.10), (2.64), and $|K| = Ch_K^d$

$$\begin{aligned}
\|\boldsymbol{\alpha}_K\|_2 \left\|\nabla v_{\mathrm{bub}}^h\right\|_{L^2(K)} &\leq C \frac{\left\|\int_K \left(\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right)(\boldsymbol{x})\ d\boldsymbol{x}\right\|_2}{\left|\int_K v_{\mathrm{bub}}^h(\boldsymbol{x})\ d\boldsymbol{x}\right|} h_K^{(d-2)/2} \\
&\leq C \frac{\left\|\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right\|_{L^2(K)} |K|^{1/2}}{|K|} h_K^{(d-2)/2} \\
&\leq C \left\|\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right\|_{L^2(K)} h_K^{d/2-1-d/2} \\
&= C h_K^{-1} \left\|\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right\|_{L^2(K)}.
\end{aligned}$$

Inserting this estimate in (2.73), taking the square, using Young's inequality (A.5), and summing over all mesh cells gives

$$\left\|P_{\mathrm{For}}^h \boldsymbol{v}\right\|_V^2 \leq C \left(\sum_{K \in \mathcal{T}^h} \left\|\nabla P_{\mathrm{Cle}}^h \boldsymbol{v}\right\|_{L^2(K)}^2 + h_K^{-2} \left\|\boldsymbol{v} - P_{\mathrm{Cle}}^h \boldsymbol{v}\right\|_{L^2(K)}^2 \right).$$

Now the proof is finished by inserting (2.70), (2.68), and applying Poincaré's inequality (A.12). ■

Remark 2.73. To MINI-type elements.

- Using the MINI element is quite popular.
- The construction of the MINI element can be extended to higher order finite elements, see Arnold *et al.* (1984). But to the best of our knowledge, the use of these higher order elements is not popular.
- It is mentioned in (Boffi *et al.*, 2008, Section 4.6) that almost any pair of finite element spaces can be stabilized by enriching the velocity space with bubble functions.

□

2.4.3.2 Other Inf-Sup Stable Pairs of Finite Element Spaces

Remark 2.74. The family of Taylor–Hood finite element spaces. The family of Taylor–Hood finite element spaces on triangular and tetrahedral grids is given by P_k/P_{k-1}, $k \geq 2$, and on quadrilateral and hexahedral grids by Q_k/Q_{k-1}, $k \geq 2$. That means, the pressure is approximated by a continuous function. Hence, it is $b^h(\cdot,\cdot) = b(\cdot,\cdot)$ and $\|\cdot\|_{V^h} = \|\cdot\|_V$.

In Hood & Taylor (1974), actually the use of the $Q_2^{(8)}/Q_1$ pair of finite element spaces was proposed for solving the Navier–Stokes equations on quadri-

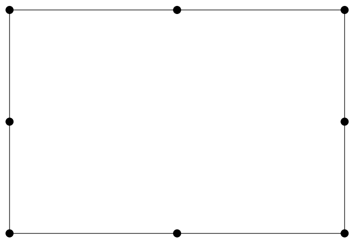

Fig. 2.2 The finite element $Q_2^{(8)}$.

lateral meshes, where the $Q_2^{(8)}$ finite element is the Q_2 finite element without internal degree of freedom, see Figure 2.2.

The pairs of Taylor–Hood finite element spaces are among the most popular pairs for discretizing equations modeling incompressible flows, in particular the pairs for $k = 2$. A reason for this popularity is certainly that the implementation of the P_2/P_1 and Q_2/Q_1 finite element pairs is comparatively easy compared with other inf-sup stable pairs of finite elements. Proving the discrete inf-sup condition is quite complicated, see Boffi (1994, 1997) □

Remark 2.75. The Scott–Vogelius pair of finite element spaces. This pair of finite element spaces is given by $P_k/P_{k-1}^{\mathrm{disc}}$, $k \geq 2$. Since

$$\nabla \cdot V^h = \nabla \cdot P_k = P_{k-1}^{\mathrm{disc}} = Q^h,$$

finite element velocities from this pair are weakly divergence-free, which is a desirable property. However, as already mentioned in Remark 2.57, the Scott–Vogelius finite element generally does not satisfy the discrete inf-sup condition (2.32). But it can be proved that the pair $P_k/P_{k-1}^{\mathrm{disc}}$ satisfies the discrete inf-sup condition in special situations, i.e., on special meshes. □

Remark 2.76. $P_k/P_{k-1}^{\mathrm{disc}}$ in two dimensions. The fulfillment of the discrete inf-sup condition (2.32) was proved already in Scott & Vogelius (1985) in the two-dimensional case for $k \geq 4$ if there is no so-called singular vertex in the mesh. An internal vertex is said to be singular if edges which meet at the vertex fall onto two straight lines.

The basic idea to overcome this problem consists in using meshes without singular vertices. To this end, so-called barycentric-refined grids are constructed. Starting from any admissible triangular mesh, new edges are introduced by connecting all vertices of a mesh cell with the barycenter of this mesh cell. This step creates smaller triangles, see Figure 2.3 for an example. On barycentric-refined meshes, the $P_k/P_{k-1}^{\mathrm{disc}}$, $k \in \{2, 3\}$, pair of finite element spaces was shown to satisfy the discrete inf-sup condition in Qin (1994), see also (John, 2016, Example 4.144) for a proof in the case $k = 2$. Note that the case $k \geq 4$ is covered by the analysis from Scott & Vogelius (1985).

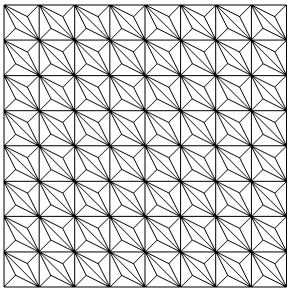

Fig. 2.3 Barycentric-refined simplicial grid on the unit square.

The use of the P_2/P_1^{disc} pair of finite element spaces on barycentric-refined meshes can be found occasionally in the literature, in particular to demonstrate the advantages of using pairs of finite element spaces which provide weakly divergence-free velocity solutions, e.g., see John *et al.* (2015) and the references therein. □

Remark 2.77. $P_k/P_{k-1}^{\text{disc}}$ in three dimensions. In three dimensions, the use of barycentric-refined meshes avoids singular vertices and singular edges. In Zhang (2005), it was shown that the pair $P_k/P_{k-1}^{\text{disc}}$, $k \geq 3$, satisfies the discrete inf-sup condition on such meshes. □

Remark 2.78. The spaces $Q_k/P_{k-1}^{\text{disc}}$. The most common pairs of spaces with conforming velocity and discontinuous pressure on quadrilateral and hexahedral meshes are the spaces $Q_k/P_{k-1}^{\text{disc}}$, $k \geq 2$. It was already mentioned in Remark 2.57 that $Q_1/P_0 = Q_1/Q_0$ is in general not inf-sup stable. For $k \geq 2$, one has to distinguish two cases, the so-called mapped and the unmapped $Q_k/P_{k-1}^{\text{disc}}$ spaces.

In the unmapped case, the local space $Q_k(K)$ is defined by a mapping from a reference cell $\hat{K}$ but the space $P_{k-1}^{\text{disc}}(K)$ is defined directly on the mesh cell K. The mapped version defines both spaces with the reference transformation. Since the reference transformation from a quadrilateral or hexahedral reference cell is in general a bilinear or trilinear mapping, it gives rise to mesh cells with curved boundaries. In addition, in general it does not preserve the type of mapped functions, i.e., the images of polynomials are in general not polynomials. Thus, the mapped and unmapped version of $Q_k/P_{k-1}^{\text{disc}}$ are generally different on arbitrary meshes.

All simulations with $Q_k/P_{k-1}^{\text{disc}}$, $k \geq 2$, presented in this manuscript were performed with the mapped version. □

Remark 2.79. Non-conforming finite element spaces. In the most general sense, non-conforming finite element methods are all methods where the finite element space is not a subspace of the function space used in the variational problem. This property might be caused, e.g., if for a problem, the domain Ω with curvilinear parts of the boundary is approximated by a domain Ω^h with polygonal or polyhedral boundary. But usually, one speaks of non-conforming finite element methods only if the non-inclusion of the spaces comes from the construction of the finite element space and it is independent of the special problem.

For incompressible flow models, the consideration of non-conforming discretizations will allow to define pairs of lowest order finite element spaces that satisfy the discrete inf-sup condition (2.32). The non-conformity is present only for the velocity but not for the pressure, i.e., $V^h \not\subset V$ and $Q^h \subset Q$.

Here, only lowest order non-conforming discretizations will be discussed because these are the most important non-conforming methods for incompressible flow problems. On simplicial meshes, this discretization is the so-called Crouzeix–Raviart finite element P_1^{nc}/P_0. That means, the velocity is approximated by a piecewise linear function that is continuous at the barycenters of the faces of the mesh cells, see Example B.43 for a detailed description, and the pressure is approximated by a piecewise constant function, see Example B.37.

The extension of this approach to quadrilateral and hexahedral meshes is the Rannacher–Turek element Q_1^{rot}/Q_0. For this element, the velocity approximation is achieved by rotated d-linear functions that have continuous degrees of freedom on the faces of the mesh cells, see Example B.53. The pressure is discretized by a piecewise constant function, see Example B.49.

Besides the possibility of using lowest order spaces, non-conforming finite element of lowest order possess some additional advantages. They can be used for the construction of efficient multigrid solvers or preconditioners for higher order discretizations of incompressible flow problems, see (John, 2016, Section 9.2.2). Implementing the code for solving the Navier–Stokes equations on parallel computers, non-conforming discretizations generally require less communication overhead than conforming finite element methods. However, non-conforming finite elements are often more complicated from the point of view of numerical analysis.

The discrete inf-sup condition is proved with the construction of an operator $V \to V^h$ such that for each function $\boldsymbol{v} \in V$ analogs to the conditions (2.54) of a Fortin operator are satisfied. $\square$

2.5 The Helmholtz Decomposition

Theorem 2.80. Helmholtz decomposition of a vector field in $L^2(\Omega)$. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. Then, each $\boldsymbol{v} \in L^2(\Omega)$ has a unique decomposition*

$$\boldsymbol{v} = \boldsymbol{w} + \nabla r, \tag{2.74}$$

with $\boldsymbol{w} \in H_{\text{div}}(\Omega)$ *and* $\nabla r \in G(\Omega)$*, where the space* $H_{\text{div}}(\Omega)$ *is defined in* (2.24) *and*

$$G(\Omega) = \left\{ \boldsymbol{z} \in L^2(\Omega) \ : \ \exists r \in L^2(\Omega) \ : \ \boldsymbol{z} = \nabla r \right\}.$$

The spaces $H_{\text{div}}(\Omega)$ *and* $G(\Omega)$ *are orthogonal in* $L^2(\Omega)$*, i.e.,*

$$G(\Omega) = H_{\text{div}}(\Omega)^{\perp}.$$

Consequently, it is $(\boldsymbol{w}, \nabla r) = 0$ *and it holds*

$$\|\boldsymbol{v}\|_{L^2(\Omega)}^2 = \|\boldsymbol{w}\|_{L^2(\Omega)}^2 + \|\nabla r\|_{L^2(\Omega)}^2 . \tag{2.75}$$

Proof. For the proof it is referred to (Sohr, 2001, pp. 82). ■

Definition 2.81. Helmholtz projection. Using the Helmholtz decomposition (2.74), the Helmholtz projection is defined by

$$P_{\text{helm}} \ : \ L^2(\Omega) \to H_{\text{div}}(\Omega), \quad \boldsymbol{v} \mapsto \boldsymbol{w}.$$

□

Lemma 2.82. Properties of the Helmholtz projection. *The Helmholtz projection is a uniquely determined, bounded linear operator with* $\|P_{\text{helm}}\| \le 1$*, i.e.,*

$$\|P_{\text{helm}}\boldsymbol{v}\|_{L^2(\Omega)} \le \|\boldsymbol{v}\|_{L^2(\Omega)} \quad \forall\, \boldsymbol{v} \in L^2(\Omega). \tag{2.76}$$

It has the following properties

$$\begin{aligned} P_{\text{helm}}(\nabla r) &= \boldsymbol{0}, & (I - P_{\text{helm}})\, \boldsymbol{v} &= \nabla r, \\ P_{\text{helm}}^2 \boldsymbol{v} &= P_{\text{helm}}\boldsymbol{v}, & (I - P_{\text{helm}})^2\, \boldsymbol{v} &= (I - P_{\text{helm}})\, \boldsymbol{v}, \end{aligned}$$

for all $\boldsymbol{v} \in L^2(\Omega)$*. Furthermore, the operator* P_{helm} *is selfadjoint, i.e.,*

$$(P_{\text{helm}}\boldsymbol{v}, \boldsymbol{g}) = (\boldsymbol{v}, P_{\text{helm}}\boldsymbol{g}) \quad \forall\, \boldsymbol{v}, \boldsymbol{g} \in L^2(\Omega).$$

Proof. By Hilbert space theory, the projection operator P_{helm} is uniquely determined. The boundedness (2.76) follows directly from (2.75)

$$\|\boldsymbol{v}\|_{L^2(\Omega)}^2 \ge \|\boldsymbol{w}\|_{L^2(\Omega)}^2 = \|P_{\text{helm}}\boldsymbol{v}\|_{L^2(\Omega)}^2 .$$

Then, the next four properties follow from (2.74) and the uniqueness of the Helmholtz decomposition. Finally, the last property follows from the orthogonality of $H_{\text{div}}(\Omega)$ and $G(\Omega)$. Let $\boldsymbol{v} = \boldsymbol{w} + \nabla r$ and $\boldsymbol{g} = \boldsymbol{w}_{\boldsymbol{g}} + \nabla r_{\boldsymbol{g}}$ with $\boldsymbol{w}, \boldsymbol{w}_{\boldsymbol{g}} \in H_{\text{div}}(\Omega)$, $r, r_{\boldsymbol{g}} \in G(\Omega)$, be the Helmholtz decompositions of $\boldsymbol{v}$ and $\boldsymbol{g}$, respectively. Then, it follows that

$$\begin{aligned} (P_{\text{helm}}\boldsymbol{v}, \boldsymbol{g}) &= (\boldsymbol{w}, \boldsymbol{w}_{\boldsymbol{g}} + \nabla r_{\boldsymbol{g}}) = (\boldsymbol{w}, \boldsymbol{w}_{\boldsymbol{g}}) + (\boldsymbol{w}, \nabla r_{\boldsymbol{g}}) = (\boldsymbol{w}, \boldsymbol{w}_{\boldsymbol{g}}) \\ &= (\boldsymbol{w} + \nabla r, \boldsymbol{w}_{\boldsymbol{g}}) = (\boldsymbol{v}, P_{\text{helm}}\boldsymbol{g}) . \end{aligned}$$

■