Chapter 1

The Stokes Equations as Simplest Model for Incompressible Flows

Remark 1.1. Basic principles and variables. The basic equations of fluid dynamics are called Navier–Stokes equations. In the case of an isothermal flow, i.e., a flow at constant temperature, they represent two physical conservation laws: the conservation of mass and the conservation of linear momentum. There are various ways for deriving these equations. Here, the classical one of continuum mechanics will sketched.

The flow will be described with the variables

- $\rho(t, \boldsymbol{x})$: density $[{}^{\mathrm{kg}}/{}_{\mathrm{m}^3}]$,
- $\boldsymbol{v}(t, \boldsymbol{x})$: velocity $[{}^{\mathrm{m}}/{}_{\mathrm{s}}]$,
- $P(t, \boldsymbol{x})$: pressure $[\mathrm{Pa} = {}^{\mathrm{N}}/{}_{\mathrm{m}^2}]$,

which are assumed to be sufficiently smooth functions in the time interval $[0, T]$ and the domain $\Omega \subset \mathbb{R}^3$. □

1.1 The Conservation of Mass

Remark 1.2. General conservation law. Let ω be an arbitrary open volume in Ω with sufficiently smooth surface $\partial\omega$, which is constant in time, and with mass

$$m(t) = \int_\omega \rho(t, \boldsymbol{x}) \; d\boldsymbol{x} \; [\mathrm{kg}].$$

If mass in ω is conserved, the rate of change of mass in ω must be equal to the flux of mass $\rho\boldsymbol{v}(t, \boldsymbol{x})$ $[\mathrm{kg}/(\mathrm{m}^2\mathrm{s})]$ across the boundary $\partial\omega$ of ω

$$\frac{d}{dt} m(t) = \frac{d}{dt} \int_\omega \rho(t, \boldsymbol{x}) \; d\boldsymbol{x} = -\int_{\partial\omega} (\rho\boldsymbol{v})\,(t, \boldsymbol{s}) \cdot \boldsymbol{n}(\boldsymbol{s}) \; d\boldsymbol{s}, \tag{1.1}$$

where $\boldsymbol{n}(\boldsymbol{s})$ is the outward pointing unit normal on $\boldsymbol{s} \in \partial\omega$. Since all functions and $\partial\omega$ are assumed to be sufficiently smooth, the divergence theorem can be applied (integration by parts), which gives

$$\int_\omega \nabla \cdot (\rho \boldsymbol{v})(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_{\partial\omega} (\rho \boldsymbol{v})(t,\boldsymbol{s}) \cdot \boldsymbol{n}(\boldsymbol{s})\ d\boldsymbol{s}.$$

Inserting this identity in (1.1) and changing differentiation with respect to time and integration with respect to space leads to

$$\int_\omega \left(\partial_t \rho(t,\boldsymbol{x}) + \nabla \cdot (\rho \boldsymbol{v})(t,\boldsymbol{x})\right)\ d\boldsymbol{x} = 0.$$

Since ω is an arbitrary volume, it follows that

$$(\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}))(t,\boldsymbol{x}) = 0 \quad \forall\, t \in (0,T],\ \boldsymbol{x} \in \Omega. \tag{1.2}$$

This relation is the first equation of fluid dynamics, which is called continuity equation. □

Remark 1.3. Time-dependent domain. It is also possible to consider a time-dependent domain $\omega(t)$. In this case, the Reynolds transport theorem can be applied. Let $\phi(t,\boldsymbol{x})$ be a sufficiently smooth function defined on an arbitrary volume $\omega(t)$ with sufficiently smooth boundary $\partial\omega(t)$, then the Reynolds transport theorem has the form

$$\frac{d}{dt}\int_{\omega(t)} \phi(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_{\omega(t)} \partial_t \phi(t,\boldsymbol{x})\ d\boldsymbol{x} + \int_{\partial\omega(t)} (\phi \boldsymbol{v} \cdot \boldsymbol{n})(t,\boldsymbol{s})\ ds. \tag{1.3}$$

In the special case that $\phi(t,\boldsymbol{x})$ is the density, one gets for the change of mass

$$\frac{d}{dt}\int_{\omega(t)} \rho(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_{\omega(t)} \partial_t \rho(t,\boldsymbol{x})\ d\boldsymbol{x} + \int_{\partial\omega(t)} (\rho \boldsymbol{v} \cdot \boldsymbol{n})(t,\boldsymbol{s})\ ds.$$

Conservation of mass and the divergence theorem yields

$$0 = \int_{\omega(t)} (\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}))(t,\boldsymbol{x})\ d\boldsymbol{x}.$$

Since $\omega(t)$ is assumed to be arbitrary, equation (1.2) follows. □

Remark 1.4. Incompressible, homogeneous fluids. If the fluid is incompressible and homogeneous, i.e., composed of one fluid only, then $\rho(t,\boldsymbol{x}) = \rho > 0$ and (1.2) reduces to

$$(\partial_x v_1 + \partial_y v_2 + \partial_z v_3)(t,\boldsymbol{x}) = \nabla \cdot \boldsymbol{v}(t,\boldsymbol{x}) = 0 \quad \forall\, t \in (0,T],\ \boldsymbol{x} \in \Omega, \tag{1.4}$$

where

$$\boldsymbol{v}(t,\boldsymbol{x}) = \begin{pmatrix} v_1(t,\boldsymbol{x}) \\ v_2(t,\boldsymbol{x}) \\ v_3(t,\boldsymbol{x}) \end{pmatrix}.$$

Thus, the conservation of mass for an incompressible, homogeneous fluid imposes a constraint on the velocity only. □

1.2 The Conservation of Linear Momentum

Remark 1.5. Newton's second law of motion. The conservation of linear momentum is the formulation of Newton's second law of motion

$$\text{net force} = \text{mass} \ \times \ \text{acceleration} \tag{1.5}$$

for flows. It states that the rate of change of the linear momentum must be equal to the net force acting on a collection of fluid particles. □

Remark 1.6. Conservation of linear momentum. The linear momentum in an arbitrary volume ω is given by

$$\int_\omega \rho \boldsymbol{v}(t, \boldsymbol{x}) \ d\boldsymbol{x} \quad [\mathrm{Ns}].$$

Then, the conservation of linear momentum in ω can be formulated analogously to the conservation of mass in (1.1)

$$\frac{d}{dt} \int_\omega \rho \boldsymbol{v}(t, \boldsymbol{x}) \ d\boldsymbol{x} = -\int_{\partial\omega} (\rho \boldsymbol{v}) \left(\boldsymbol{v} \cdot \boldsymbol{n}\right)(t, \boldsymbol{s}) \ d\boldsymbol{s} + \int_\omega \boldsymbol{f}_{\text{net}}(t, \boldsymbol{x}) \ d\boldsymbol{x} \ [\mathrm{N}],$$

where the term on the left-hand side describes the change of the momentum in ω, the first term on the right-hand side models the flux of momentum across the boundary of ω, and $\boldsymbol{f}_{\text{net}}$ $[\mathrm{N}/\mathrm{m}^3]$ represents the force density in ω. It is

$$\boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{n}) = \begin{pmatrix} v_1 v_1 n_1 + v_1 v_2 n_2 + v_1 v_3 n_3 \\ v_2 v_1 n_1 + v_2 v_2 n_2 + v_2 v_3 n_3 \\ v_3 v_1 n_1 + v_3 v_2 n_2 + v_3 v_3 n_3 \end{pmatrix} = \boldsymbol{v}\boldsymbol{v}^T \boldsymbol{n}.$$

Applying integration by parts and changing differentiation with respect to time and integration on ω gives

$$\int_\omega \left(\partial_t (\rho \boldsymbol{v}) + \nabla \cdot \left(\rho \boldsymbol{v}\boldsymbol{v}^T\right)\right)(t, \boldsymbol{x}) \ d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\text{net}}(t, \boldsymbol{x}) \ d\boldsymbol{x}.$$

The divergence of a tensor is defined row-wise

$$\nabla \cdot \mathbb{A} = \begin{pmatrix} \partial_x a_{11} + \partial_y a_{12} + \partial_z a_{13} \\ \partial_x a_{21} + \partial_y a_{22} + \partial_z a_{23} \\ \partial_x a_{31} + \partial_y a_{32} + \partial_z a_{33} \end{pmatrix}.$$

The product rule yields

$$\int_\omega \left(\partial_t \rho \boldsymbol{v} + \rho \partial_t \boldsymbol{v} + \boldsymbol{v}\boldsymbol{v}^T \nabla \rho + \rho(\nabla\cdot\boldsymbol{v})\boldsymbol{v} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)(t,\boldsymbol{x})\ d\boldsymbol{x}$$
$$= \int_\omega \boldsymbol{f}_{\mathrm{net}}(t,\boldsymbol{x})\ d\boldsymbol{x}. \tag{1.6}$$

In the usual notation $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$, one can think of $\boldsymbol{v}\cdot\nabla = v_1\partial_x + v_2\partial_y + v_3\partial_z$ acting on each component of $\boldsymbol{v}$. In the literature, one often finds the notation $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$.

In the case of incompressible fluids, i.e., ρ is constant, it is known that $\nabla\cdot\boldsymbol{v} = 0$, see (1.4), such that (1.6) simplifies to

$$\int_\omega \rho\left(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right)(t,\boldsymbol{x})\ d\boldsymbol{x} = \int_\omega \boldsymbol{f}_{\mathrm{net}}(t,\boldsymbol{x})\ d\boldsymbol{x}.$$

Since ω was chosen to be arbitrary, one gets the conservation law

$$\rho\left(\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = \boldsymbol{f}_{\mathrm{net}} \quad \forall\, t\in(0,T],\ \boldsymbol{x}\in\Omega.$$

The same conservation law can be derived for a time-dependent volume $\omega(t)$ using the Reynolds transport theorem (1.3). □

Remark 1.7. The Navier–Stokes equations. The modeling of the net forces shall be only sketched here. In short, there might be external forces $\boldsymbol{f}_{\mathrm{ext}}$ acting on the fluid and there are internal forces, i.e., forces that exerts the fluid on itself. The latter forces lead, via the so-called stress principle of Cauchy, to the inclusion of the pressure P [Pa] into the equation and a viscous stress tensor with viscosity μ [$^{\mathrm{kg}}/_{(\mathrm{m\ s})}$]. For a detailed derivation, it is referred to (John, 2016, Chapter 1.2).

If the fluid is incompressible and homogeneous, such that μ and ρ are positive constants, the Navier–Stokes equations read as follows:

$$\begin{aligned} \partial_t \boldsymbol{v} - 2\nu\nabla\cdot\mathbb{D}(\boldsymbol{v}) + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla\frac{P}{\rho} &= \frac{\boldsymbol{f}_{\mathrm{ext}}}{\rho} && \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{v} &= 0 && \text{in } (0,T]\times\Omega. \end{aligned} \tag{1.7}$$

Here, $\nu = \mu/\rho$ [$^{\mathrm{m}^2}/_{\mathrm{s}}$] is the kinematic viscosity of the fluid and $\mathbb{D}(\boldsymbol{v})$ is the symmetric part of the velocity gradient, the so-called velocity rate-of-deformation tensor or shortly velocity deformation tensor

$$\mathbb{D}(\boldsymbol{v}) = \frac{\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T}{2} \quad [^1/_{\mathrm{s}}].$$

The gradient of the velocity is a tensor with the components

$$(\nabla\boldsymbol{v})_{ij} = \partial_j v_i = \frac{\partial v_i}{\partial x_j}, \quad i,j = 1,2,3.$$

□

1.3 The Dimensionless Navier–Stokes Equations

Remark 1.8. Characteristic scales. Mathematical analysis and numerical simulations are based on dimensionless equations. To derive dimensionless equations from system (1.7), the quantities

- L [m] – a characteristic length scale of the flow problem,
- U [m/s] – a characteristic velocity scale of the flow problem,
- T^* [s] – a characteristic time scale of the flow problem,

are introduced. □

Remark 1.9. The Navier–Stokes equations in dimensionless form. Denote by $(t', \boldsymbol{x}')$ $[\mathrm{s}, \mathrm{m}]$ the old variables. Applying the transform of variables

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{L}, \quad \boldsymbol{u} = \frac{\boldsymbol{v}}{U}, \quad t = \frac{t'}{T^*}, \tag{1.8}$$

one obtains from (1.7) and a rescaling

$$\begin{aligned} \frac{L}{UT^*}\partial_t \boldsymbol{u} - \frac{2\nu}{UL}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\frac{P}{\rho U^2} &= \frac{L}{\rho U^2}\boldsymbol{f}_{\mathrm{ext}} && \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } (0,T]\times\Omega, \end{aligned}$$

where all derivatives are with respect to the new variables. Without having emphasized this issue in the notation, also the domain and the time interval are now dimensionless. Defining

$$p = \frac{P}{\rho U^2}, \quad \mathrm{Re} = \frac{UL}{\nu}, \quad \mathrm{St} = \frac{L}{UT^*}, \quad \boldsymbol{f} = \frac{L}{\rho U^2}\boldsymbol{f}_{\mathrm{ext}}, \tag{1.9}$$

the incompressible Navier–Stokes equations in dimensionless form

$$\begin{aligned} \mathrm{St}\partial_t \boldsymbol{u} - \frac{2}{\mathrm{Re}}\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } (0,T]\times\Omega, \end{aligned} \tag{1.10}$$

are obtained. The constant Re is called Reynolds number and the constant St Strouhal number. These numbers allow the classification and comparison of different flows. □

Remark 1.10. Inherent difficulties of the dimensionless Navier–Stokes equations. To simplify the notations, one uses the characteristic time scale $T^* = L/U$ such that (1.10) simplifies to

$$\begin{aligned} \partial_t \boldsymbol{u} - 2\nu\nabla\cdot\mathbb{D}(\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } (0,T]\times\Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \text{ in } (0,T]\times\Omega, \end{aligned} \tag{1.11}$$

with the dimensionless viscosity $\nu = \mathrm{Re}^{-1}$. Here, with an abuse of notation, the same symbol is used as for the kinematic viscosity.

This transform and the resulting equations (1.11) are the basic equations for the mathematical analysis of the incompressible Navier–Stokes equations and the numerical simulation of incompressible flows. System (1.11) comprises two important difficulties:
- the coupling of velocity and pressure,
- the nonlinearity of the convective term.

Additionally, difficulties for the numerical simulation occur if
- the convective term dominates the viscous term, i.e., if ν is small.

□

Remark 1.11. Different form of the viscous term. With the help of the divergence constraint, i.e., the second equation in (1.11), the viscous term of the Navier–Stokes equations can be reformulated equivalently.

Assume that $\boldsymbol{u}$ is sufficiently smooth with $\nabla \cdot \boldsymbol{u} = 0$. Then, straightforward calculations, using the Theorem of Schwarz and the second equation of (1.11), give

$$\nabla \cdot (\nabla \boldsymbol{u}) = \Delta \boldsymbol{u}, \quad \nabla \cdot \left(\nabla \boldsymbol{u}^T\right) = \nabla \left(\nabla \cdot \boldsymbol{u}\right) = \begin{pmatrix} \partial_x \left(\nabla \cdot \boldsymbol{u}\right) \\ \partial_y \left(\nabla \cdot \boldsymbol{u}\right) \\ \partial_z \left(\nabla \cdot \boldsymbol{u}\right) \end{pmatrix} = \boldsymbol{0}. \qquad (1.12)$$

Thus, the viscous term becomes

$$-2\nu \nabla \cdot \mathbb{D}\left(\boldsymbol{u}\right) = -\nu \Delta \boldsymbol{u}. \qquad (1.13)$$

□

Remark 1.12. Special cases of incompressible flow models.
- In a stationary flow, the velocity and the pressure do not change in time. Hence $\partial_t \boldsymbol{u} = \boldsymbol{0}$ and these flows are modeled by the so-called stationary or steady-state Navier–Stokes equations

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \text{ in } \Omega. \end{aligned} \qquad (1.14)$$

  A necessary condition for the time-independence of a flow field is that the data of the problem, i.e., the right-hand side and the boundary conditions, are time-independent. But this condition is not sufficient.
- If in a stationary flow the viscous transport dominates the convective transport, i.e., if the fluid moves very slowly, the nonlinear convective term of the Navier–Stokes equations (1.14) can be neglected. This situation leads to a linear system of equations, the so-called Stokes equations

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \text{ in } \Omega. \end{aligned} \qquad (1.15)$$

Finite element methods for the Stokes equations will be the topic of this course. In order to define a well-posed problem, (1.15) has to be equipped with appropriate boundary conditions. For simplicity, in this course always homogeneous Dirichlet boundary conditions are considered:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \mathit{\Gamma}, \tag{1.16}$$

where $\mathit{\Gamma}$ is the boundary of $\mathit{\Omega}$.

□