Chapter 1
Convection-Diffusion-Reaction Equations and Maximum Principles

1.1 A Model for Conservation Laws

1.1.1 Conservation of Energy in a Fluid

Let $\Omega \subset \mathbb{R}^3$ be a fixed domain that is occupied by a fluid, let $\boldsymbol{x} = (x_1, x_2, x_3)^T \in \Omega$ denote the points in Ω, and let $t \in \mathbb{R}$ denote the time. Then, the conservation of energy can be modeled in terms of the temperature

- $u(t, \boldsymbol{x})$ – temperature at time t and at the point $\boldsymbol{x}$ with unit [K].

To derive this model, one needs several physical properties of the fluid and of the given problem:

- $\rho(t, \boldsymbol{x})$ – density of the fluid with unit $[\mathrm{kg}/\mathrm{m}^3]$,
- $c(t, \boldsymbol{x})$ – specific heat capacity of the fluid with unit $[\mathrm{J}/\mathrm{kg\ K}] = [\mathrm{W\ s}/\mathrm{kg\ K}]$,
- $k(t, \boldsymbol{x})$ – thermal conductivity of the fluid with unit $[\mathrm{W}/\mathrm{m\ K}]$,
- $\boldsymbol{v}(t, \boldsymbol{x}) = (v_1, v_2, v_3)^T(t, \boldsymbol{x})$ – velocity of the motion of the fluid with unit $[\mathrm{m}/\mathrm{s}]$,
- $F(t, \boldsymbol{x})$ – intensity of heat sources or sinks with unit $[\mathrm{W}/\mathrm{m}^3]$.

The goal consists now in deriving a model for the conservation of energy in an arbitrary volume $V \subset \Omega$ with sufficiently smooth boundary ∂V and in an arbitrary time interval $(t, t + \Delta t)$. For simplicity of presentation, this volume will be considered to be fixed. It is also possible to derive the model for a volume that changes in time, using the Reynolds transport theorem. Both ways lead finally to the same model.

First, energy can be produced or absorbed in V. Then, the change of energy in the time interval is given by

$$Q_1 = \int_t^{t+\Delta t} \int_V F(t, \boldsymbol{x}) \, d\boldsymbol{x} \, dt, \ \ [\mathrm{W\ s}] = [\mathrm{J}]. \tag{1.1}$$

And second, energy heat can enter or leave V through ∂V by molecular motion and by the convective motion of the fluid with the velocity field. Let $\boldsymbol{n}(\boldsymbol{x})$ be the unit outer normal at $\boldsymbol{x} \in \partial V$. Using Fourier's law for the molecular motion, one finds that the energy

$$\begin{aligned} Q_2 &= \int_t^{t+\Delta t} \int_{\partial V} \left(k \frac{\partial u}{\partial \boldsymbol{n}} - c\rho \left(\boldsymbol{v} \cdot \boldsymbol{n} \right) u \right) (t, \boldsymbol{s}) \ d\boldsymbol{s} \ dt \\ &= \int_t^{t+\Delta t} \int_{\partial V} \left(k \nabla u \cdot \boldsymbol{n} - c\rho \left(\boldsymbol{v} \cdot \boldsymbol{n} \right) u \right) (t, \boldsymbol{s}) \ d\boldsymbol{s} \ dt, \ \text{[J]}, \end{aligned}$$

penetrates through ∂V. Since ∂V is assumed to be sufficiently smooth, one can apply integration by parts (Gaussian theorem), and obtains equivalently

$$Q_2 = \int_t^{t+\Delta t} \int_V \nabla \cdot (k \nabla u - c\rho u \boldsymbol{v})(t, \boldsymbol{x}) \ d\boldsymbol{x} \ dt, \ \text{[J]}. \tag{1.2}$$

On the other hand, a law for the change of the temperature in V has to be derived. Assuming that u is sufficiently smooth and using a Taylor series expansion with respect to time, one gets that the temperature at $\boldsymbol{x}$ changes in $(t, t + \Delta t)$ by

$$u(t + \Delta t, \boldsymbol{x}) - u(t, \boldsymbol{x}) = \frac{\partial u}{\partial t}(t, \boldsymbol{x}) \Delta t + \mathcal{O}\left((\Delta t)^2 \right).$$

For the simplest model, a linear ansatz is utilized, i.e.,

$$u(t + \Delta t, \boldsymbol{x}) - u(t, \boldsymbol{x}) = \frac{\partial u}{\partial t}(t, \boldsymbol{x}) \Delta t.$$

Then, one finds that for the change of the temperature in V and for a sufficiently small Δt, the energy

$$\begin{aligned} Q_3 &= \int_t^{t+\Delta t} \int_V c\rho \frac{u(t + \Delta t, \boldsymbol{x}) - u(t, \boldsymbol{x})}{\Delta t} \ d\boldsymbol{x} \ dt \\ &= \int_t^{t+\Delta t} \int_V c\rho \frac{\partial u}{\partial t}(t, \boldsymbol{x}) \ d\boldsymbol{x} \ dt, \ \text{[J]}, \end{aligned} \tag{1.3}$$

is needed.

Conservation of energy means that the needed energy equals the sources of energy, i.e., it has to hold that $Q_3 = Q_1 + Q_2$, from what follows that

$$\int_t^{t+\Delta t} \int_V \left[c\rho \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u - c\rho u \boldsymbol{v}) - F \right] (t, \boldsymbol{x}) \ d\boldsymbol{x} \ dt = 0.$$

Since the volume V was chosen to be arbitrary and Δt was arbitrary as well, the term in the integral has to vanish. After having divided by $c\rho$, one obtains the following partial differential equation

$$\frac{\partial u}{\partial t} - \frac{1}{c\rho}\nabla \cdot (k\nabla u - c\rho u\boldsymbol{v}) = \frac{F}{c\rho} \quad \text{in } (0,T)\times\Omega. \tag{1.4}$$

All terms have the physical unit [$^{\mathrm{K}}/_{\mathrm{s}}$].

In this course, only the equation for stationary problems will be studied, i.e., all data of the problem are independent of time and a solution is sought that also does not depend on time. In particular, it holds that $\partial u/\partial t = 0$. For a homogeneous fluid, c, ρ, k are positive constants and if the fluid is incompressible, the convection field is divergence-free, i.e., $\nabla \cdot \boldsymbol{v} = 0$. Then, the partial differential equation simplifies to

$$-\eta\Delta u + \boldsymbol{v} \cdot \nabla u = f \quad \text{in } \Omega, \tag{1.5}$$

with $\eta = k/(c\rho)$, [$^{\mathrm{m}^2}/_{\mathrm{s}}$], and $f = F/(c\rho)$, [$^{\mathrm{K}}/_{\mathrm{s}}$].

To obtain a well-posed problem, (1.5) has to be equipped with appropriate boundary conditions on the boundary $\partial\Omega$ of Ω. For (1.5), one can prescribe the following types of boundary conditions:

- Dirichlet condition: The temperature $u(t, \boldsymbol{x})$ at a part of the boundary is prescribed
$$u = g_1 \text{ on } \partial\Omega_D,$$
with $\partial\Omega_D \subset \partial\Omega$.
- Neumann condition: The heat flux is prescribed at a part of the boundary
$$k\frac{\partial u}{\partial \boldsymbol{n}} = g_2 \text{ on } \partial\Omega_{\mathrm{N}},$$
with $\partial\Omega_{\mathrm{N}} \subset \partial\Omega$.
- Mixed boundary condition, Robin boundary condition: At a part of the boundary, there is a heat exchange according to Newton's law
$$k\frac{\partial u}{\partial \boldsymbol{n}} + h(u - u_{\mathrm{env}}) = 0 \text{ on } \partial\Omega_{\mathrm{R}},$$
with $\partial\Omega_{\mathrm{R}} \subset \partial\Omega$, the heat exchange coefficient h, [$^{\mathrm{W}}/_{\mathrm{m}^2\mathrm{K}}$], and the temperature of the environment u_{env}, [K].

The next step for deriving a problem that is suited for purposes of mathematical analysis and numerical simulations consists in getting rid of the physical units and to obtain a so-called dimensionless problem. To this end, let

- L be a characteristic length scale of the problem, [m],
- U be a characteristic temperature scale of the problem, [K].

Denoting the coordinates and functions with dimensions with a prime and applying the transforms
$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{L}, \quad u = \frac{u'}{U}$$
to (1.5) yields the equation

$$-\eta \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} (Uu) \frac{\partial x_i}{\partial x_i'} \right) \frac{\partial x_i}{\partial x_i'} + \sum_{i=1}^{3} v_i' \frac{\partial (Uu)}{\partial x_i} \frac{\partial x_i}{\partial x_i'} = f' \quad \text{in } \Omega'$$
$$\Longleftrightarrow$$
$$-\frac{\eta U}{L^2} \sum_{i=1}^{3} \frac{\partial^2 u}{\partial (x_i)^2} + \frac{U}{L} \sum_{i=1}^{3} v_i' \frac{\partial u}{\partial x_i} = f' \quad \text{in } \Omega'.$$

Let V [m/s] be a characteristic velocity scale, a dimensionless equation of the following form is derived:

$$-\varepsilon \Delta u + \boldsymbol{b} \cdot \nabla u = f \quad \text{in } \Omega, \tag{1.6}$$

with $\varepsilon = \eta/(LV)$, $\boldsymbol{b} = \boldsymbol{v}'/V$, $f = Lf'/(UV)$, and the dimensionless domain Ω. Also the boundary conditions have to be converted into dimensionless expressions. One obtains, e.g.,

$$u = \frac{g_1}{U} \text{ on } \partial\Omega_D, \quad -\varepsilon \frac{\partial u}{\partial \boldsymbol{n}} = \frac{g_2}{c\rho VU} \text{ on } \partial\Omega_{\mathrm{N}}.$$

Remark 1.1. Modeling the transport of a species by molecular diffusion and by the movement of a fluid leads to a convection-diffusion equation for the concentration. If several species are present in the fluid, then one obtains even a system of convection-diffusion equations. A chemical reaction between the species couples the equations in the system. If this reaction is modeled with the law of mass action, then one obtains in each equation a coupling term that depends on the concentration, but not on derivatives, and the coefficient $\sigma \geq 0$ depends on the other concentrations. This zeroth order term σu is called reaction term. □

1.1.2 The Stationary Linear Convection-Diffusion-Reaction Equation

This section introduces the type of equation that will be studied in this course – the stationary linear convection-diffusion-reaction equation. As motivated in the previous section, such equations are used as model for certain physical processes. This issue will be explained at the end of this section.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with Lipschitz boundary $\partial\Omega$, then the considered stationary linear convection-diffusion-reaction boundary value problem is given by

$$\begin{aligned} -\varepsilon \Delta u + \boldsymbol{b} \cdot \nabla u + \sigma u &= f && \text{in } \Omega, \\ u &= u_{\mathrm{D}} && \text{on } \partial\Omega_{\mathrm{D}}, \\ \varepsilon \nabla u \cdot \boldsymbol{n} &= u_{\mathrm{N}} && \text{on } \partial\Omega_{\mathrm{N}}, \\ \varepsilon \nabla u \cdot \boldsymbol{n} + \kappa u &= u_{\mathrm{R}} && \text{on } \partial\Omega_{\mathrm{R}}. \end{aligned} \tag{1.7}$$

The dimensionless diffusion coefficient is assumed to be a positive constant, i.e., $\varepsilon > 0$. Appropriate requirements concerning the dimensionless convection field $\boldsymbol{b}(\boldsymbol{x})$ and reaction field $\sigma(\boldsymbol{x})$ will be formulated in the mathematical and numerical analysis. It is $\partial\Omega = \overline{\partial\Omega_{\mathrm{D}}} \cup \overline{\partial\Omega_{\mathrm{N}}} \cup \overline{\partial\Omega_{\mathrm{R}}}$ where all these sets are mutually disjoint and $\mathrm{meas}_{d-1}(\partial\Omega_{\mathrm{D}}) > 0$. The non-negative function $\kappa : \partial\Omega_{\mathrm{R}} \to \mathbb{R}$, $\kappa(\boldsymbol{x}) \geq 0$, and the data on the boundary $u_{\mathrm{D}}(\boldsymbol{x})$, $u_{\mathrm{N}}(\boldsymbol{x})$, and $u_{\mathrm{R}}(\boldsymbol{x})$ are given.

Let the closed set

$$\partial\Omega_{\mathrm{in}} = \overline{\{\boldsymbol{x} \in \partial\Omega \ : \ \boldsymbol{b}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) < 0\}} \tag{1.8}$$

denote the inlet boundary, i.e., the part of the boundary where u enters the domain driven by the convection field. It will be assumed that $\partial\Omega_{\mathrm{in}} \subset \overline{\partial\Omega_{\mathrm{D}}}$.

The results of the physical processes described in the previous sections possess some properties that become obvious by using physical understanding. Consider first the energy balance from Section 1.1.1. If there is a problem without (positive) energy sources in Ω and with a prescribed temperature at $\partial\Omega$, then it is not possible that the temperature at any point in Ω is larger than the highest temperature at the boundary. Concerning the concentration balance from Remark 1.1, it is clear by the definition of a concentration as the amount of species particles per volume that each concentration has to be non-negative. A well-posed problem for the concentration balance has to ensure non-negativity. These fundamental physical properties can be proved mathematically for the boundary value problem (1.7), which will be the contents of Section 1.2 on so-called Maximum Principles.

Usually, the solution of the boundary value problem (1.7) cannot be computed analytically and it has to be approximated by means of numerical methods. It turned out that the satisfaction of fundamental physical properties, which has the solution of (1.7), by the numerical solution is not guaranteed. Such physical properties are the range of admissible values or conservation properties. In fact, it is well known that many methods, also very popular ones, compute in important situations from applications numerical solutions that have (some) values which are not physically consistent, e.g., undershoots that lead to negative values for concentrations. The desired kind of physical consistency of solutions obtained with numerical methods can be described mathematically with so-called Discrete Maximum Principles (DMPs), see Chapter 4.

1.2 Classical Maximum Principles

Maximum principles are the mathematical formulation of the fundamental physical properties mentioned in the previous section. It is a classical topic of analysis of partial differential equations that is presented, e.g., in (Gilbarg

& Trudinger, 2001, Sec. 3.1, 3.2) and (Renardy & Rogers, 2004, Sec. 4.1). A consequence of a maximum principle is the uniqueness of the solution of the classical Dirichlet problem. This section contains a concise presentation of this topic.

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain and let us consider the operator $L : C^2(\Omega) \to C(\Omega)$ defined by

$$L\,u := -\mathscr{A} : D^2\,u + \boldsymbol{b} \cdot \nabla u + \sigma\,u\,, \quad u \in C^2(\Omega)\,, \tag{1.9}$$

where $D^2\,u$ is the Hessian matrix given by

$$D^2\,u := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1}^d$$

and $\mathscr{A} : \overline{\Omega} \to \mathbb{R}^{d\times d}$, $\boldsymbol{b} : \overline{\Omega} \to \mathbb{R}^d$, and $\sigma : \overline{\Omega} \to \mathbb{R}$ are given continuous functions. Moreover, we assume that the matrix $\mathscr{A}(\boldsymbol{x})$ is symmetric and positive definite for any $\boldsymbol{x} \in \overline{\Omega}$. If $d = 1$, then $D^2\,u = u''$ and $\mathscr{A} : \overline{\Omega} \to \mathbb{R}$ is only assumed to be positive in $\overline{\Omega}$. Consequently, L is an elliptic second order linear differential operator. The operator from problem (1.7) is a special case with $\mathscr{A} = \varepsilon I$, $I \in \mathbb{R}^{d\times d}$ being the identity tensor.

It is well known that, under suitable assumptions, solutions of boundary value problems with an elliptic partial differential equation attain there maximum or minimum on the boundary of the domain. The aim of this section is to formulate and prove assertions of this type and to explore their consequences. To assure that boundary values are well defined, it shall be always assumed that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ throughout this section. From the Theorem of Weierstrass, it follows that u possesses a minimum and a maximum in $\overline{\Omega}$ and also on $\partial\Omega$, since $\partial\Omega$ is a closed set ($\mathbb{R}^d \setminus \partial\Omega$ is open).

Theorem 1.2 (Maximum principle for $\sigma = 0$). *Let $\sigma = 0$ in Ω. Then*

$$L\,u \le 0 \quad \text{in } \Omega \quad \Longrightarrow \quad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u\,, \tag{1.10}$$

$$L\,u \ge 0 \quad \text{in } \Omega \quad \Longrightarrow \quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u\,. \tag{1.11}$$

Proof. It is sufficient to prove statement (1.10). Statement (1.11) follows by replacing u with $-u$.

First, let us assume that $L\,u < 0$ in Ω and let $\boldsymbol{x}_0 \in \overline{\Omega}$ satisfy $\max_{\overline{\Omega}} u = u(\boldsymbol{x}_0)$. Assume that $\boldsymbol{x}_0 \in \Omega$. Then $(\nabla u)(\boldsymbol{x}_0) = \boldsymbol{0}$ and the Hessian matrix $(D^2 u)(\boldsymbol{x}_0)$ is negative semi-definite ($(D^2 u)(x_0) = u''(x_0) \le 0$ if $d = 1$). If $d \ge 2$, the matrix $\mathscr{A}(\boldsymbol{x}_0)$ can be written in the form $\mathscr{A}(\boldsymbol{x}_0) = \sum_{k=1}^d \lambda_k \boldsymbol{q}_k \boldsymbol{q}_k^T$, where λ_k are the eigenvalues of $\mathscr{A}(\boldsymbol{x}_0)$ and $\boldsymbol{q}_k$ are the corresponding eigenvectors which are orthonormal. A direct calculation[1] shows that

[1] Let $A = (a_{ij})_{i,j=1}^d$, $B = (b_{ij})_{i,j=1}^d$ and $A = aa^T$ with $a = (\alpha_1, \ldots, \alpha_d)^T$, then

$$\mathscr{A}(\boldsymbol{x}_0) : (D^2u)(\boldsymbol{x}_0) = \sum_{k=1}^{d} \lambda_k \boldsymbol{q}_k \cdot (D^2u)(\boldsymbol{x}_0)\boldsymbol{q}_k \leq 0,$$

since the eigenvalues λ_k are positive. If $d = 1$, then the validity of $\mathscr{A}(\boldsymbol{x}_0) : (D^2u)(\boldsymbol{x}_0) \leq 0$ is obvious. Thus, for any $d \in \mathbb{N}$, one has $(Lu)(\boldsymbol{x}_0) \geq 0$, which is a contradiction. Therefore, $\boldsymbol{x}_0 \in \partial\Omega$ and hence the right-hand side of (1.10) holds.

Now, let $L\,u \leq 0$ in Ω. To prove (1.10), we introduce the function

$$u_\delta(\boldsymbol{x}) := u(\boldsymbol{x}) + \delta\, \mathrm{e}^{\alpha\, x_1}\,, \quad \boldsymbol{x} \in \overline{\Omega}\,,$$

for any $\delta > 0$ and suitable $\alpha > 0$ to be determined later. Then, one has

$$(L\,u_\delta)(\boldsymbol{x}) = (L\,u)(\boldsymbol{x}) + \delta\,\alpha\,\mathrm{e}^{\alpha\,x_1}\left[-(\mathscr{A})_{11}(\boldsymbol{x})\,\alpha + b_1(\boldsymbol{x})\right].$$

Since $(\mathscr{A})_{11} > 0$ in $\overline{\Omega}$, one can define α in such a way that $[-(\mathscr{A})_{11}(\boldsymbol{x})\,\alpha + b_1(\boldsymbol{x})] < 0$ for any $\boldsymbol{x} \in \Omega$. Then, $L\,u_\delta < 0$ in Ω and hence $\max_{\overline{\Omega}} u_\delta = \max_{\partial\Omega} u_\delta$ for any $\delta > 0$, according to the first part of the proof. Now, statement (1.10) follows from letting $\delta \to 0$. ∎

Definition 1.3 (Positive and negative part of a real number and a real-valued function). Let $\alpha \in \mathbb{R}$, then its positive part α^+ and negative part α^- are defined as follows

$$\alpha^+ := \max\{\alpha, 0\} \geq 0 \quad \text{and} \quad \alpha^- := \min\{\alpha, 0\} \leq 0\,. \tag{1.12}$$

The same notation is used to define the positive and negative parts of a real-valued function.

Theorem 1.4 (Maximum principle). *Let $\sigma \geq 0$ in Ω. Then*

$$L\,u \leq 0 \quad \text{in } \Omega \quad \Longrightarrow \quad \max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+\,, \tag{1.13}$$

$$L\,u \geq 0 \quad \text{in } \Omega \quad \Longrightarrow \quad \min_{\overline{\Omega}} u \geq \min_{\partial\Omega} u^-\,. \tag{1.14}$$

In particular, it is

$$L\,u = 0 \quad \text{in } \Omega \quad \Longrightarrow \quad \max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|\,. \tag{1.15}$$

Proof. Again, it is sufficient to prove the first statement since (1.14) follows from (1.13) by replacing u with $-u$, and (1.15) is a simple consequence of (1.13) and (1.14).

$$A : B = \sum_{i,j=1}^{d} a_{ij}b_{ij} = \sum_{i,j=1}^{d} \alpha_i\alpha_j b_{ij} = \sum_{i=1}^{d}\alpha_i \sum_{j=1}^{d}\alpha_j b_{ij} = a \cdot Ba = a^T Ba.$$

The proof of (1.13) can be divided into two cases. If $\max_{\overline{\Omega}} u \leq 0$, then (1.13) trivially holds. If $\max_{\overline{\Omega}} u > 0$, then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+ \quad \Longleftrightarrow \quad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u,$$

so that (1.13) is equivalent to (1.10). Then, the proof can be performed in the same way as in Theorem 1.2 since, due to the assumption on σ, one has $(Lu)(\boldsymbol{x}_0) \geq \sigma(\boldsymbol{x}_0)\, u(\boldsymbol{x}_0) \geq 0$, with $\boldsymbol{x}_0$ introduced in the proof of Theorem 1.2. ■

Corollary 1.5 (Inverse monotonicity, comparison principle). *Let $\sigma \geq 0$ in Ω and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$. Then*

$$L\,u \leq L\,v \quad \text{in } \Omega\,, \quad u \leq v \quad \text{on } \partial\Omega \quad \Longrightarrow \quad u \leq v \quad \text{in } \Omega\,, \tag{1.16}$$

$$L\,u = L\,v \quad \text{in } \Omega\,, \quad u = v \quad \text{on } \partial\Omega \quad \Longrightarrow \quad u = v \quad \text{in } \Omega\,. \tag{1.17}$$

Proof. The statements follow by applying (1.13) and (1.15) to the difference $u - v$. ■

Corollary 1.6 (Uniqueness of the solution of the fully homogeneous Dirichlet problem). *Consider the boundary value problem $L\,u = 0$ in Ω and $u = 0$ on $\partial\Omega$. If $\sigma \geq 0$ in Ω, then the problem has only the trivial solution $u \equiv 0$.*

Proof. This statement follows directly from (1.17) with $v = 0$. ■

Corollary 1.7 (Uniqueness of the solution of the inhomogeneous Dirichlet problem). *If $\sigma \geq 0$ in Ω, then the boundary value problem $L\,u = f$ in Ω and $u = u_{\mathrm{D}}$ on $\partial\Omega$ has at most one solution, a so-called classical solution.*

Proof. Assume that there are two different classical solutions and consider their difference. Then, it follows directly from (1.17) that this difference vanishes identically. ■

Theorem 1.8 (A priori estimates). *Let u satisfies $L\,u = f$ with $f \in C(\overline{\Omega})$. If $\sigma \geq 0$ in Ω, then*

$$\min_{\partial\Omega} u^- + C\, \min_{\overline{\Omega}} \frac{f^-}{\lambda} \leq u(\boldsymbol{x}) \leq \max_{\partial\Omega} u^+ + C\, \max_{\overline{\Omega}} \frac{f^+}{\lambda} \quad \forall\, \boldsymbol{x} \in \overline{\Omega}\,, \tag{1.18}$$

where C depends on $\mathscr{A}$, $\boldsymbol{b}$, and Ω, and $\lambda(\boldsymbol{x}) := \min\{\boldsymbol{v} \cdot \mathscr{A}(\boldsymbol{x})\, \boldsymbol{v}\,;\, \boldsymbol{v} \in \mathbb{R}^d, \|\boldsymbol{v}\|_2 = 1\}$ is the smallest eigenvalue of $\mathscr{A}(\boldsymbol{x})$, $\boldsymbol{x} \in \overline{\Omega}$.

If $\sigma > 0$ in $\overline{\Omega}$, then

$$\min\left\{\min_{\partial\Omega} u, \min_{\overline{\Omega}} \frac{f}{\sigma}\right\} \leq u(\boldsymbol{x}) \leq \max\left\{\max_{\partial\Omega} u, \max_{\overline{\Omega}} \frac{f}{\sigma}\right\} \quad \forall\, \boldsymbol{x} \in \overline{\Omega}\,. \tag{1.19}$$

Proof. For interested students only, not presented in the class. Let us prove first the upper bound in (1.18), i.e.,

$$u(\boldsymbol{x}) \le \max_{\partial\Omega} u^+ + C \max_{\overline{\Omega}} \frac{f^+}{\lambda} \quad \forall\, \boldsymbol{x} \in \overline{\Omega}\,. \tag{1.20}$$

Let $\Omega \subset (x_1^{\min}, x_1^{\max}) \times \mathbb{R}^{d-1}$ and set $s := x_1^{\max} - x_1^{\min}$. Let us introduce the function

$$v(\boldsymbol{x}) := \max_{\partial\Omega} u^+ + \left(\mathrm{e}^{\alpha\, s} - \mathrm{e}^{\alpha\,(x_1 - x_1^{\min})}\right) \max_{\overline{\Omega}} \frac{f^+}{\lambda}\,, \quad \boldsymbol{x} \in \overline{\Omega}\,,$$

where $\alpha > 0$ will be fixed later. Set

$$\beta := \max_{\overline{\Omega}} \frac{|\boldsymbol{b}|}{\lambda}\,.$$

Then, the operator

$$L_0\, u := -\mathscr{A} : D^2\, u + \boldsymbol{b} \cdot \nabla u\,, \quad u \in C^2(\Omega)\,,$$

satisfies

$$(L_0\, v)(\boldsymbol{x}) = \alpha\, \mathrm{e}^{\alpha\,(x_1 - x_1^{\min})} \left[(\mathscr{A})_{11}(\boldsymbol{x})\, \alpha - b_1(\boldsymbol{x})\right] \max_{\overline{\Omega}} \frac{f^+}{\lambda}\,.$$

One has $(\mathscr{A})_{11}(\boldsymbol{x})\, \alpha - b_1(\boldsymbol{x}) \ge (\alpha - \beta)\, \lambda(\boldsymbol{x})$ and hence, setting $\alpha := \beta + 1$, one obtains $(L_0\, v)(\boldsymbol{x}) \ge f^+(\boldsymbol{x})$ for any $\boldsymbol{x} \in \Omega$. Since $v \ge 0$ in Ω, one gets

$$L\, v \ge L_0\, v \ge f^+ \ge f = L\, u \quad \text{in } \ \Omega\,.$$

Furthermore, it is $v \ge u^+ \ge u$ on $\partial\Omega$. Thus, Corollary 1.5 implies that $u \le v$ in Ω, which proves (1.20) with $C = \mathrm{e}^{(\beta+1)s-1}$. The lower bound in (1.18) follows from (1.20) by replacing u and f with $-u$ and $-f$, respectively.

To prove the upper bound in (1.19), let us set

$$K := \max\left\{\max_{\partial\Omega} u, \max_{\overline{\Omega}} \frac{f}{\sigma}\right\}.$$

Then, for any $\boldsymbol{x} \in \Omega$, it is

$$L(u - K)(\boldsymbol{x}) = (L\, u)(\boldsymbol{x}) - \sigma(\boldsymbol{x})\, K \le f(\boldsymbol{x}) - \sigma(\boldsymbol{x}) \max_{\overline{\Omega}} \frac{f}{\sigma} \le 0\,.$$

In addition, it holds $u - K \le 0$ on $\partial\Omega$. Thus, $u \le K$ in Ω according to Corollary 1.5. The lower bound in (1.19) again follows from the upper bound by replacing u and f with $-u$ and $-f$, respectively. ∎

Corollary 1.9 (Stability of the solution, continuous dependency on the data). *Let u satisfies $L\,u = f$ with $f \in C(\overline{\Omega})$ and let $\sigma \ge 0$ in Ω. Then*

$$\|u\|_{C(\overline{\Omega})} \le \|u\|_{C(\partial\Omega)} + \Lambda\, \|f\|_{C(\overline{\Omega})}\,,$$

where Λ depends on $\mathscr{A}$, $\boldsymbol{b}$, σ, and Ω, but not on f.

Proof. The statement is an immediate consequence of the previous theorem. ∎

One can see that this estimate is in fact a stability estimate: if u and $\tilde{u}$ are solutions of $L\,u = f$ and $L\,\tilde{u} = \tilde{f}$, respectively, then

$$\|u - \tilde{u}\|_{C(\overline{\Omega})} \le \|u - \tilde{u}\|_{C(\partial\Omega)} + \Lambda\, \|f - \tilde{f}\|_{C(\overline{\Omega})}\,.$$

Hence, changes in the solution depend continuously, in the norm of $C(\overline{\Omega})$, on changes of the data of the problem.

In the previous statements, conditions were specified under which a solution of the Dirichlet problem attains its extremum at the boundary of Ω. However, these results do not exclude that the extremum is also attained at some interior point. The following theorem presents conditions under which this situation is not possible.

Theorem 1.10 (Strong maximum principle). *Let $L\,u \le 0$ in Ω and let u be not constant in Ω. Then*

$$\sigma = 0 \quad \text{in } \Omega \qquad\qquad \Longrightarrow \quad u(\boldsymbol{x}) < \max_{\overline{\Omega}} u \quad \forall\, \boldsymbol{x} \in \Omega\,, \tag{1.21}$$

$$\sigma \ge 0 \quad \text{in } \Omega \quad \& \quad \max_{\overline{\Omega}} u \ge 0 \quad \Longrightarrow \quad u(\boldsymbol{x}) < \max_{\overline{\Omega}} u \quad \forall\, \boldsymbol{x} \in \Omega\,, \tag{1.22}$$

$$u(\tilde{\boldsymbol{x}}) = 0 \quad \text{for some } \tilde{\boldsymbol{x}} \in \Omega \quad \Longrightarrow \quad u(\tilde{\boldsymbol{x}}) < \max_{\overline{\Omega}} u\,. \tag{1.23}$$

If $L\,u \ge 0$ in Ω and u is not constant in Ω, then

$$\sigma = 0 \quad \text{in } \Omega \qquad\qquad \Longrightarrow \quad u(\boldsymbol{x}) > \min_{\overline{\Omega}} u \quad \forall\, \boldsymbol{x} \in \Omega\,,$$

$$\sigma \ge 0 \quad \text{in } \Omega \quad \& \quad \min_{\overline{\Omega}} u \le 0 \quad \Longrightarrow \quad u(\boldsymbol{x}) > \min_{\overline{\Omega}} u \quad \forall\, \boldsymbol{x} \in \Omega\,,$$

$$u(\tilde{\boldsymbol{x}}) = 0 \quad \text{for some } \tilde{\boldsymbol{x}} \in \Omega \quad \Longrightarrow \quad u(\tilde{\boldsymbol{x}}) > \min_{\overline{\Omega}} u\,.$$

Proof. For interested students only, not presented in the class. Again, it suffices to prove the statements for $L\,u \le 0$ since the remaining statements follow by replacing u with $-u$.

Assume that u attains its maximum $M := \max_{\overline{\Omega}} u$ at an interior point. Since it is assumed that u is not constant, the set $G = \{\boldsymbol{x} \in \Omega\,;\ u(\boldsymbol{x}) < M\}$ is not empty. Consequently, $\Omega \cap \partial G$ is not empty as well. Let $\boldsymbol{y} \in G$ be any point satisfying

$$0 < \operatorname{dist}(\boldsymbol{y}, \partial G) < \operatorname{dist}(\boldsymbol{y}, \partial\Omega)\,.$$

Let us introduce the ball $B = \{\boldsymbol{x} \in \mathbb{R}^d\,;\ |\boldsymbol{x}-\boldsymbol{y}| < \operatorname{dist}(\boldsymbol{y}, \partial G)\}$. Then $\overline{B} \subset \Omega$ and $\partial B \cap \partial G$ contains at least one point $\boldsymbol{x}_0 \in \Omega$. Then, one has

$$u \in C^2(\overline{B})\,, \quad u(\boldsymbol{x}_0) = M > u(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in B\,.$$

Set

$$v(\boldsymbol{x}) = \mathrm{e}^{-\alpha\,|\boldsymbol{x}-\boldsymbol{y}|^2} - \mathrm{e}^{-\alpha\,R^2}\,, \quad \boldsymbol{x} \in \overline{B}\,,$$

with a positive constant α to be determined later. Then, v is a smooth function such that $v > 0$ in B and $v = 0$ on ∂B. One has

$$\begin{aligned}
&(L\,v)(\boldsymbol{x}) \\
&= \mathrm{e}^{-\alpha\,|\boldsymbol{x}-\boldsymbol{y}|^2}\big[- 4\,\alpha^2\,(\boldsymbol{x}-\boldsymbol{y})\cdot\mathscr{A}(\boldsymbol{x})\,(\boldsymbol{x}-\boldsymbol{y}) + 2\,\alpha\,\operatorname{tr}\mathscr{A}(\boldsymbol{x}) \\
&\qquad -2\,\alpha\,\boldsymbol{b}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{y})\big\} + \sigma(\boldsymbol{x})\,v(\boldsymbol{x})\big] \\
&\le \mathrm{e}^{-\alpha\,|\boldsymbol{x}-\boldsymbol{y}|^2}\left[-4\,\alpha^2\,\lambda(\boldsymbol{x})\,|\boldsymbol{x}-\boldsymbol{y}|^2 + 2\,\alpha\,(\operatorname{tr}\mathscr{A}(\boldsymbol{x}) + |\boldsymbol{b}(\boldsymbol{x})|\,|\boldsymbol{x}-\boldsymbol{y}|) + \sigma(\boldsymbol{x})\right],
\end{aligned}$$

where $\lambda(\boldsymbol{x})$ is the smallest eigenvalue of $\mathscr{A}(\boldsymbol{x})$. Since $\mathscr{A}(\boldsymbol{x})/\lambda(\boldsymbol{x})$, $|\boldsymbol{b}(\boldsymbol{x})|/\lambda(\boldsymbol{x})$, and $\sigma(\boldsymbol{x})/\lambda(\boldsymbol{x})$ are bounded in $\overline{B}$, the constant α can be chosen in such a way that

$$L\,v \le 0 \quad \text{in } \tilde{B} := \{\boldsymbol{x} \in B\,;\ |\boldsymbol{x}-\boldsymbol{y}| > R/2\}\,.$$

Then, for $\varepsilon > 0$ small enough, one has $u - u(\boldsymbol{x}_0) + \varepsilon\,v \le 0$ on $\partial\tilde{B}$. Assuming that $\sigma = 0$ in Ω, or $\sigma \ge 0$ in Ω and $M \ge 0$, one has $L\,(u - u(\boldsymbol{x}_0) + \varepsilon\,v) \le -\sigma\,u(\boldsymbol{x}_0) \le 0$ in $\tilde{B}$ and hence $u - u(\boldsymbol{x}_0) + \varepsilon\,v \le 0$ in $\tilde{B}$ according to Corollary 1.5. Thus, setting $\nu := (\boldsymbol{x}_0 - \boldsymbol{y})/|\boldsymbol{x}_0 - \boldsymbol{y}|$, one obtains

$$\frac{\partial u}{\partial \nu}(\boldsymbol{x}_0) \ge -\varepsilon\,\frac{\partial v}{\partial \nu}(\boldsymbol{x}_0) = 2\,\alpha\,\varepsilon\,|\boldsymbol{x}_0 - \boldsymbol{y}|\,\mathrm{e}^{-\alpha\,|\boldsymbol{x}_0-\boldsymbol{y}|^2} > 0\,,$$

which contradicts the assumption that u attains a maximum at $\boldsymbol{x}_0$. Therefore, the implications (1.21) and (1.22) hold.

Finally, if there is $\tilde{\boldsymbol{x}} \in \Omega$ such that $u(\tilde{\boldsymbol{x}}) = 0$ and $u(\tilde{\boldsymbol{x}}) = \max_{\overline{\Omega}} u$, then $M = 0$ and hence $(L - \sigma^-)u = L\,u - \sigma^-\,u \le 0$ in Ω. Then, using the operator $L - \sigma^-$ instead of L, the assumptions for the validity of (1.22) are satisfied, which proves (1.23). ∎

1.3 Variational Form of the Boundary Value Problem

Corollary 1.7 states that if a solution of the classical Dirichlet problem exists, this solution is unique. There are also results concerning the existence of such a solution, e.g., (Gilbarg & Trudinger, 2001, Thm. 6.14). Roughly speaking, if the domain has a sufficiently smooth boundary, if the coefficients of the differential operator, the right-hand side of the equation, and the Dirichlet conditions are sufficiently smooth, and if $\sigma \geq 0$ in Ω, then there exists a solution of the classical Dirichlet problem. However, in many situations from practice, the required smoothness of the data is not given, e.g., if the domain is a hexahedral box, but the physical process nevertheless happens, such that there is a solution. To study such solutions, a different notion than the classical (or strong) solution is necessary – the so-called variational or weak solution. This concept, which will be discussed in this section, is based on a corresponding form of the boundary value problem. In addition, this form is the basis of finite element methods for computing numerical approximations of the weak solution.

For the domain $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, it will be assumed that it is connected and it has a Lipschitz boundary $\partial\Omega$. With this property on $\partial\Omega$, the unit outer normal vector exists almost everywhere on $\partial\Omega$, see (Nečas, 2012, Ch. 2, Lem. 4.2), and the divergence theorem (Gauss theorem, Green's theorems) can be applied, see (Nečas, 2012, Ch. 3, Thm. 1.1). Furthermore, the Lipschitz boundary guarantees that several imbedding theorems can be applied for functions defined on Ω, e.g., see (Adams, 1975, Thm. 5.4) or (Demengel & Demengel, 2012, Thm. 2.72). In addition, it is $\partial\Omega = \overline{\partial\Omega_{\mathrm{D}}} \cup \overline{\partial\Omega_{\mathrm{N}}} \cup \overline{\partial\Omega_{\mathrm{R}}}$ with the Dirichlet boundary $\partial\Omega_{\mathrm{D}}$, the Neumann boundary $\partial\Omega_{\mathrm{N}}$, and the Robin boundary $\partial\Omega_{\mathrm{R}}$. The different parts of the boundary are mutually disjoint and it is assumed that $\mathrm{meas}_{d-1}(\partial\Omega_{\mathrm{D}}) > 0$.

Denote by $V_{\mathcal{D}} = \mathcal{D}\left(\Omega \cup \partial\Omega \setminus \overline{\partial\Omega_{\mathrm{D}}}\right) \cap C(\overline{\Omega})$, i.e., the functions of $V_{\mathcal{D}}$ are infinitly often differentiable in Ω, they vanish in a neighborhood of $\partial\Omega_{\mathrm{D}}$ and they can be extended continuously to the other parts of the boundary. Multiplying the strong form of the equation (1.7) with a so-called test function $v \in V_{\mathcal{D}}$, integrating on Ω, applying integration by parts to the diffusion term, and taking into account that the test function vanishes on $\partial\Omega_{\mathrm{D}}$ yields

$$\begin{aligned}
&\int_{\Omega} \left(-\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla u + cu\right)(\boldsymbol{x})v(\boldsymbol{x})\ d\boldsymbol{x} \\
&= \int_{\Omega} \left(\varepsilon\nabla u\cdot\nabla v + \left(\boldsymbol{b}\cdot\nabla u + cu\right)v\right)(\boldsymbol{x})\ d\boldsymbol{x} - \int_{\partial\Omega} \varepsilon(\nabla u\cdot\boldsymbol{n})(\boldsymbol{s})v(\boldsymbol{s})\ d\boldsymbol{s} \\
&= \int_{\Omega} \left(\varepsilon\nabla u\cdot\nabla v + \left(\boldsymbol{b}\cdot\nabla u + cu\right)v\right)(\boldsymbol{x})\ d\boldsymbol{x} \qquad (1.24)\\
&\quad - \int_{\partial\Omega_{\mathrm{N}}\cup\partial\Omega_{\mathrm{R}}} \varepsilon(\nabla u\cdot\boldsymbol{n})(\boldsymbol{s})v(\boldsymbol{s})\ d\boldsymbol{s} \quad \forall\, v \in V_{\mathcal{D}}.
\end{aligned}$$

In this way, derivatives have been transferred from u to the test function, which leads to a reduction of the smoothness requirements for u.

To obtain now a variational formulation of the boundary value problem, one has to incorporate that the solution should satisfy the prescribed boundary condition u_{D}. This information will be included as a constraint in an appropriate definition of the ansatz or trial space, see below. Boundary conditions that lead to such constraints are called essential boundary conditions. Note that also the space $D\left(\Omega \cup \partial\Omega \setminus \overline{\partial\Omega_{\mathrm{D}}}\right)$ contains conditions on the

Dirichlet boundary. In contrast, Neumann and Robin conditions appear in the variational form of the equation. Boundary conditions with this property are called natural conditions. For deriving the variational form, one has to insert the data on the Neumann and Robin boundary in (1.24).

In order to define a well-posed weak problem, functions spaces for the data of the problem, the solution, and the test function have to be specified such that all terms in (1.24) are well defined. An appropriate setup is as follows. Let

$$V = H^1_{0,\partial\Omega_{\mathrm{D}}}(\Omega) = \overline{\mathcal{D}\left(\Omega \cup \partial\Omega \setminus \overline{\partial\Omega_{\mathrm{D}}}\right)}^{H^1(\Omega)} \subset H^1(\Omega)$$

be the subspace of $H^1(\Omega)$ that contains the functions whose trace vanishes on $\overline{\partial\Omega_{\mathrm{D}}}$, let the data of the equation satisfy

$$\boldsymbol{b} \in \left(L^\infty(\Omega)\right)^d, \quad \sigma \in L^\infty(\Omega), \quad f \in V^*, \tag{1.25}$$

and let for the data of the boundary conditions hold

$$\begin{aligned} &u_{\mathrm{D}} \in H^{1/2}\left(\partial\Omega_{\mathrm{D}}\right), \quad u_{\mathrm{N}} \in \left(H^{1/2}_{00}\left(\partial\Omega_{\mathrm{N}}\right)\right)^*, \\ &u_{\mathrm{R}} \in \left(H^{1/2}_{00}\left(\partial\Omega_{\mathrm{R}}\right)\right)^*, \quad \kappa \in L^\infty\left(\partial\Omega_{\mathrm{R}}\right), \ \kappa \ge 0. \end{aligned} \tag{1.26}$$

Further, it will be assumed that the Dirichlet boundary $\partial\Omega_{\mathrm{D}}$ is a relatively open subset of $\partial\Omega$ and that the boundary of $\partial\Omega_{\mathrm{D}}$ is Lipschitz. Then, there exists a linear and continuous extension operator $E_{\partial\Omega_{\mathrm{D}}} : H^{1/2}(\partial\Omega_{\mathrm{D}}) \to H^1(\Omega)$, e.g., see (Wilbrandt, 2019, Thm. 4.2.4). Consequently, there is an extension $u_{\mathrm{D,ext}} = E_{\partial\Omega_{\mathrm{D}}} u_{\mathrm{D}} \in H^1(\Omega)$.

A variational or weak formulation of the convection-diffusion-reaction problem (1.7) reads as follows: Find $u \in H^1(\Omega)$ such that $u - u_{\mathrm{D,ext}} \in V$ and

$$a(u, v) = f(v) \quad \forall\, v \in V \tag{1.27}$$

with the bilinear form $a(\cdot,\cdot) : H^1(\Omega) \times V \to \mathbb{R}$,

$$a(u, v) = (\varepsilon \nabla u, \nabla v) + (\boldsymbol{b} \cdot \nabla u + \sigma u, v) + (\kappa u, v)_{\partial\Omega_{\mathrm{R}}} \tag{1.28}$$

and the linear form $f(\cdot) : V \to \mathbb{R}$,

$$\begin{aligned} f(v) = {} & \langle f, v\rangle_{V^*,V} + \langle u_{\mathrm{N}}, v\rangle_{\left(H^{1/2}_{00}(\partial\Omega_{\mathrm{N}})\right)^*, H^{1/2}_{00}(\partial\Omega_{\mathrm{N}})} \\ & + \langle u_{\mathrm{R}}, v\rangle_{\left(H^{1/2}_{00}(\partial\Omega_{\mathrm{R}})\right)^*, H^{1/2}_{00}(\partial\Omega_{\mathrm{R}})}. \end{aligned} \tag{1.29}$$

A function u that satisfies (1.27) is called variational or weak solution of the convection-diffusion-reaction boundary value problem.

If the data of the problem are more regular, i.e., $f \in L^2(\Omega)$, $u_{\mathrm{N}} \in L^2(\partial\Omega_{\mathrm{N}})$, or $u_{\mathrm{R}} \in L^2(\partial\Omega_{\mathrm{R}})$, then the dual pairings in (1.29) can be replaced by inner products in the respective Lebesgue spaces

$$(f, v)\,, \quad (u_{\mathrm{N}}, v)_{\partial\Omega_{\mathrm{N}}}\,, \quad (u_{\mathrm{R}}, v)_{\partial\Omega_{\mathrm{R}}}\,.$$

Theorem 1.11 (Existence and uniqueness of a variational solution). *Let the assumptions on the domain and its boundary stated above and let the regularity assumptions* (1.25) *and* (1.26) *on the data of the boundary value problem be satisfied. Assume in addition that* $\boldsymbol{b} \in \left(W^{1,\infty}(\Omega)\right)^d$*, that*

$$\left(-\frac{1}{2}\nabla\cdot\boldsymbol{b} + \sigma\right)(\boldsymbol{x}) \geq 0 \quad \text{in } \Omega, \tag{1.30}$$

and that for the inlet boundary defined in (1.8)*, it holds that* $\partial\Omega_{\mathrm{in}} \subset \overline{\partial\Omega_{\mathrm{D}}}$*. Then, the variational problem* (1.27) *possesses a unique solution.*

Proof. The proof will be utilize the Theorem of Lax–Milgram. To this end, one has first to reformulate (1.27) in an equivalent form such that the left-hand side of the new problem is a bilinear form defined on $V \times V$. Let $E_{\partial\Omega_{\mathrm{D}}} u_{\mathrm{D}}$ be an arbitrary but fixed extension of the Dirichlet data as introduced above, then the equivalent problem consists in finding $\tilde{u} = u - E_{\partial\Omega_{\mathrm{D}}} u_{\mathrm{D}} \in V$ such that

$$a\left(\tilde{u}, v\right) = f(v) - a\left(E_{\partial\Omega_{\mathrm{D}}} u_{\mathrm{D}}, v\right) \quad \forall\, v \in V. \tag{1.31}$$

Now, one has to show that the restriction of the bilinear form (1.28) to $V \times V$ is coercive, that it is bounded (or continuous), and that the linear form on the right-hand side of (1.31) is bounded (or continuous). Note that, since $\mathrm{meas}_{d-1}(\partial\Omega_{\mathrm{D}}) > 0$, $\|\nabla v\|_{L^2(\Omega)}$ is a norm in V that is equivalent to $\|v\|_{H^1(\Omega)}$ and a Poincaré inequality holds.

Coercivity of the bilinear form (1.28) *restricted to* $V \times V$. It is for all $v \in V$

$$a(v, v) = \varepsilon\, \|\nabla v\|^2_{L^2(\Omega)} + (\boldsymbol{b}\cdot\nabla v, v) + (\sigma v, v) + (\kappa v, v)_{\partial\Omega_{\mathrm{R}}}\,.$$

For the convective term, integration by parts and product rule yields

$$(\boldsymbol{b}\cdot\nabla v, v) = \int_{\partial\Omega} (\boldsymbol{b}\cdot\boldsymbol{n})\, v^2 \; d\boldsymbol{s} - (\boldsymbol{b}\cdot\nabla v, v) - ((\nabla\cdot\boldsymbol{b})v, v)\,,$$

such that

$$(\boldsymbol{b}\cdot\nabla v, v) = \frac{1}{2}\int_{\partial\Omega} (\boldsymbol{b}\cdot\boldsymbol{n})\, v^2 \; d\boldsymbol{s} - \frac{1}{2}\left((\nabla\cdot\boldsymbol{b})v, v\right).$$

Because v vanishes on $\partial\Omega_{\mathrm{D}}$ and from $\partial\Omega_{\mathrm{in}} \subset \overline{\partial\Omega_{\mathrm{D}}}$, one can conclude that $\boldsymbol{b}\cdot\boldsymbol{n} \geq 0$ on $\partial\Omega_{\mathrm{N}} \cup \partial\Omega_{\mathrm{R}}$, it follows that

$$\int_{\partial\Omega} (\boldsymbol{b}\cdot\boldsymbol{n})\, v^2 \; d\boldsymbol{s} = \int_{\partial\Omega_{\mathrm{N}}\cup\partial\Omega_{\mathrm{R}}} (\boldsymbol{b}\cdot\boldsymbol{n})\, v^2 \; d\boldsymbol{s} \geq 0.$$

Using (1.30) gives

$$(\boldsymbol{b}\cdot\nabla v, v) + (\sigma v, v) \geq -\frac{1}{2}\left((\nabla\cdot\boldsymbol{b})v, v\right) + (\sigma v, v) \geq 0.$$

In addition, thanks to the non-negativity of κ, it is

$$(\kappa v, v)_{\partial\Omega_{\mathrm{R}}} \geq \inf_{\boldsymbol{x}\in\partial\Omega_{\mathrm{R}}} \kappa(\boldsymbol{x}) \left\|v\right\|^2_{L^2(\partial\Omega_{\mathrm{R}})} \geq 0.$$

Altogether, one obtains

$$a(v,v) \geq \varepsilon \left\|\nabla v\right\|^2_{L^2(\Omega)} \geq C\varepsilon \left\|v\right\|^2_{H^1(\Omega)},$$

where the constant C depends on the square of the inverse of the constant from the Poincaré inequality.

Boundedness of the bilinear form (1.28) *restricted to* $V\times V$. Utilizing the Cauchy–Schwarz inequality, Hölder's inequality, the assumptions on the regularity of the coefficients, and that the trace operator is a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ yields

$$\begin{aligned}
a(v,w) &\leq \varepsilon \left\|\nabla v\right\|_{L^2(\Omega)} \left\|\nabla w\right\|_{L^2(\Omega)} + \left\|\boldsymbol{b}\right\|_{L^\infty(\Omega)} \left\|\nabla v\right\|_{L^2(\Omega)} \left\|w\right\|_{L^2(\Omega)} \\
&\quad + \left\|\sigma\right\|_{L^\infty(\Omega)} \left\|v\right\|_{L^2(\Omega)} \left\|w\right\|_{L^2(\Omega)} + \left\|\kappa\right\|_{L^\infty(\partial\Omega_{\mathrm{R}})} \left\|vw\right\|_{L^1(\partial\Omega_{\mathrm{R}})} \\
&\leq C \left\|v\right\|_{H^1(\Omega)} \left\|w\right\|_{H^1(\Omega)} + C \left\|vw\right\|_{L^1(\partial\Omega)} \\
&\leq C \left\|v\right\|_{H^1(\Omega)} \left\|w\right\|_{H^1(\Omega)} + C \left\|v\right\|_{L^2(\partial\Omega)} \left\|w\right\|_{L^2(\partial\Omega)} \\
&\leq C \left\|v\right\|_{H^1(\Omega)} \left\|w\right\|_{H^1(\Omega)} \quad \forall\, v,w\in V.
\end{aligned}$$

Note that, because Poincaré's inequality is not used in this estimate, the same bound holds for the bilinear form defined on $H^1(\Omega)\times H^1(\Omega)$.

Boundedness of the linear form on the right-hand side of (1.31). It is known that there are well defined trace operators from V to $H^{1/2}_{00}(\partial\Omega_{\mathrm{N}})$ and $H^{1/2}_{00}(\partial\Omega_{\mathrm{R}})$, respectively, which are linear and continuous, e.g., see (Wilbrandt, 2019, Thm. 4.3.4). Using the estimates for the dual pairings, the boundedness of the bilinear form defined on $H^1(\Omega)\times H^1(\Omega)$, the continuity of these trace operators, and the regularity assumptions on the data gives

$$\begin{aligned}
&f(v) - a\left(E_{\partial\Omega_{\mathrm{D}}} u_{\mathrm{D}}, v\right) \\
&\leq \left\|f\right\|_{V^*} \left\|v\right\|_{H^1(\Omega)} + \left\|u_{\mathrm{N}}\right\|_{\left(H^{1/2}_{00}(\partial\Omega_{\mathrm{N}})\right)^*} \left\|v\right\|_{H^{1/2}_{00}(\partial\Omega_{\mathrm{N}})} \\
&\quad + \left\|u_{\mathrm{R}}\right\|_{\left(H^{1/2}_{00}(\partial\Omega_{\mathrm{R}})\right)^*} \left\|v\right\|_{H^{1/2}_{00}(\partial\Omega_{\mathrm{R}})} + \left\|E_{\partial\Omega_{\mathrm{D}}} u_{\mathrm{D}}\right\|_{H^1(\Omega)} \left\|v\right\|_{H^1(\Omega)} \\
&\leq C\left(\left\|f\right\|_{V^*} + \left\|u_{\mathrm{N}}\right\|_{\left(H^{1/2}_{00}(\partial\Omega_{\mathrm{N}})\right)^*} + \left\|u_{\mathrm{R}}\right\|_{\left(H^{1/2}_{00}(\partial\Omega_{\mathrm{R}})\right)^*} + \left\|E_{\partial\Omega_{\mathrm{D}}} u_{\mathrm{D}}\right\|_{H^1(\Omega)}\right) \\
&\quad \times \left\|v\right\|_{H^1(\Omega)} \\
&= C \left\|v\right\|_{H^1(\Omega)} \quad \forall\, v\in V.
\end{aligned}$$

Summary. All assumptions of the Theorem of Lax–Milgram are satisfied. It follows that (1.31) has a unique solution $\tilde{u}$. Consequently, also (1.27) possesses a unique solution $u \in H^1(\Omega)$ that is given by $u = \tilde{u} + E_{\partial\Omega_{\mathrm{D}}} u_{\mathrm{D}}$. ∎