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Numerical Mathematics II

Exercise Problems 10 (final exercise sheet)

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.
Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *3-step Adams–Bashforth method.* Derive the 3-step Adams–Bashforth method ($q = 3$).

2. *BDF2 with variable step size.* Derive a method by using the following approach. Consider three subsequent nodes x_{k-1}, x_k, x_{k+1} with the mesh widths $h_{k+1} = x_{k+1} - x_k$ and $h_k = x_k - x_{k-1}$. Denote the numerical approximations of the solutions by y_{k-1}, y_k, and y_{k+1}. Now, take the interpolation polynomial $p(x)$ through these points and require that $p'(x_{k+1}) = f(x_{k+1}, y_{k+1})$. This gives a method that is called BDF2. Give this method by expressing as much terms as possible with $\sigma_{k+1} = h_{k+1}/h_k$ and order the terms with respect to y_{k-1}, y_k, and y_{k+1}.

3. *Compressed sparse row storage format.* Sparse matrices are stored usually in the so-called Compressed Sparse Row (CSR) format.
   (a) Read the the file csr.pdf (from book by Saad (1996)) about this topic.
   (b) Give two CSR storages of the matrix

$$\begin{pmatrix} 4 & 0 & 0 & -1 & 0 & 0 & 8 & 10 & 0 \\ 0 & 10 & -3 & 0 & 0 & 8 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 6 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 17 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 & 11 & 0 & 0 & 7 \end{pmatrix}.$$

4. *Preconditioned conjugate gradient (PCG) method.* Continue Problem 5 from Exercise sheet 09. Consider meshes with

$$h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256, 1/512, 1/1024\}.$$

   Implement the preconditioned conjugate gradient method (Algorithm 8.8) for solving these equations. Use as preconditioner
   (a) $M = I$, i.e., no preconditioning, same as Problem 3 from Exercise sheet 09,
   (b) $M = \mathrm{diag}(A)$, Jacobi preconditioner, see Remark 8.2,
   (c) $M = SSOR(A)$, see Remark 8.2,
   (d) $M = L^T L$, where L is the incomplete Cholesky factorization of A with zero fill-in. The incomplete Cholesky decomposition is a variant of ILU for symmetric matrices for which $L = U^T$. If you use MATLAB, then use the command `ichol`, otherwise try to figure out whether a routine for the incomplete Cholesky factorization is available for the programming language that you use.

Give the number of iterations for solving the system. What can be observed? Find an explanation for the behavior of the Jacobi preconditioner.

Hint. Since only the number of iterations is of interest, the solution of the linear systems with the preconditioner can be implemented with the backslash command. Also the SSOR preconditioner can be implemented in the form given in Remark 8.2.

The exercise problems will be discussed at the turorial on **Thursday, July 02, 2026, 12-14**.