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Numerical Mathematics II

Exercise Problems 09

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.
Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Linear system of ordinary differential equations.* The restriction to scalar ordinary differential equations in most parts of the course is just for simplicity of presentation. All methods and most statements can be extended in a straightforward way to the vector-valued case, i.e., to systems of ordinary differential equations. In this problem, the analytic solution should be computed for one of the most simple cases for such systems.

   Find the general solution of the following linear system of ordinary differential equations

$$
\begin{aligned}
y'(x) &= x + y(x) + z(x) + u(x) \\
z'(x) &= x + z(x) + u(x) \\
u'(x) &= x + u(x)\ .
\end{aligned}
$$

   Hints:

   - Find first the solution of the homogeneous problem. To this end, insert equations in each other. Compute the other functions with a backward substitution.
   - Use an appropriate ansatz for finding a special solution of the inhomogeneous system.
   - Compare Appendix A, in particular A.1.3, if you are interested in the theory of linear systems of ordinary differential equations with constant coefficients.

2. *Calculation of a matrix exponential.* Consider the matrix

$$
A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix}
$$

   Write A as a sum of a diagonalizable matrix D and nilpotent matrix N and calculate the matrix exponential e^{tA}.

3. *Stability function of* `ode23s`. Consider the linearly implicit Runge–Kutta method `ode23s`

$$
\begin{aligned}
(I - ahJ)\,K_1 &= f(y_k), \quad a = \frac{1}{2+\sqrt{2}}, \\
(I - ahJ)\,K_2 &= f\left(y_k + \frac{1}{2}hK_1\right) - ahJK_1, \\
y_{k+1} &= y_k + hK_2
\end{aligned}
$$

with $J = f_y(y_k) = f'(y_k)$. Show that the stability function of this method, for sufficiently small step sizes h, is

$$R(z) = \frac{1 + (1 - 2a)z}{(1 - az)^2}.$$

Hint: It suffices to consider an autonomous equation.

4. *Moore–Penrose inverse, pseudo inverse.* Let $A \in \mathbb{R}^{m \times n}$. The pseudo inverse or Moore–Penrose inverse $A^+ \in \mathbb{R}^{n \times m}$ of A is determined uniquely by the so-called Moore–Penrose conditions

$$AA^+A = A, \ A^+AA^+ = A^+, \ (AA^+)^T = AA^+, \ (A^+A)^T = A^+A.$$

Compute on the basis of these conditions the pseudo inverse of $A = (2, 3, 4) \in \mathbb{R}^{1 \times 3}$.

5. *Conjugate gradient (CG) method.* Continue Problem 6 from Exercise sheet 04. In this problem, iterative schemes had to be implemented for solving a linear system of equations with symmetric positive definite matrix. Consider again the meshes with $h \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$.

   (a) Implement the Conjugate Gradient method as given in Algorithm 6.8. How does the number of iteration change with h?

   (b) If you use MATLAB, then apply the Conjugate Gradient method as provided by the routine `pcg`. Otherwise, find out if the used programming language provides a Conjugate Gradient method and use this method. Compare the number of iterations with your own implementation.

   (c) Measure the executing times for both methods. If you use MATLAB, use the commands `tic` and `toc`. Which implementation is more efficient?

The exercise problems will be discussed at the turorial on **Thursday, June 25, 2026, 12-14**.