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Numerical Mathematics II

Exercise Problems 08

The solutions have to be presented in the tutorial by participants of the course. In order to fulfill the tutorial requirements, each student has to present two correct solutions (depending on the number of subproblems, a 'solution' might cover only a part of the subproblems) and obtain a total of 4 points from their presentations. A fully correct solution is awarded 2 points, a partially correct solution is awarded 1 point, and an incorrect solution is awarded 0 points.
Prepare these presentations! All statements have to be proved, auxiliary calculations have to be presented. Statements given in the lectures can be used without proof.

1. *Stability function at* ∞. Consider a Runge–Kutta method $(A, \boldsymbol{b}, \boldsymbol{c})$ with s stages and assume that A is non-singular. Prove the following statement: It holds $R(\infty) = 0$ if one of the following conditions is satisfied:

   (a) $a_{si} = b_i$ for $i = 1, \ldots, s$, or

   (b) $a_{i1} = b_1$ for $i = 1, \ldots, s$, $b_1 \neq 0$.

   Hint: Derive first a formula for $R(\infty)$.

2. *L-stability vs. A-stability.* If a one-step method that is only A-stable is applied to a stiff IVP, then oscillations around a smooth solution are not quickly damped out. This behavior is in contrast to a L-stable method. These properties shall be illustrated at the linear IVP

   $$y'(x) = 2000\left(\cos(x) - y(x)\right), \quad y(0) = 0, \quad x \in 0, 1.5].$$

   i) Find the analytic solution of this problem.
   ii) Consider an equidistant grid with 40 intervals. Derive a formula for the implicit Euler scheme and perform a simulation.
   iii) Do the same as in b) for the trapezoidal rule.
   iv) Present pictures of the results and discuss them briefly.

3. *Matrix exponential.* Show the properties of the matrix exponential given in Lemma 2.25, lecture notes *Numerical Methods for Ordinary Differential Equations.*

4. *Second order boundary value problem with first order term and upwind discretization, GMRES with restart.* In addition to the methods from Exercise 07, Problem 04, the method GMRES(restart) with

   $$\text{restart} \ \in \{5, 10, 20, 30, 40, 50\}$$

   should be applied for solving the arising linear systems of equations. If you use MATLAB, you can use the build-in routine with the stopping criterion $tol = 10^{-10}$. For other languages, stop the iteration with the same criterion as in Exercise 07, Problem 03. The maximal number of outer iterations should coincide with the dimension of the problem and the initial iterate should be the zero vector. How do the numbers of iterations of GMRES(restart) compare with the other methods from Exercise 07, Problem 03(d) ? Are there trends with respect to the parameter ε or with respect to the parameter restart ?

The exercise problems will be discussed at the turorial on **Thursday, June 18, 2026, 12-14**.